Repairing the Universality Theorem for 4-polytopes

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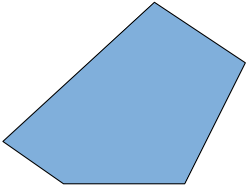
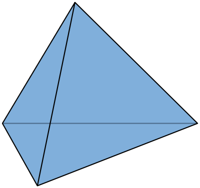
1 Introduction

Polytopes

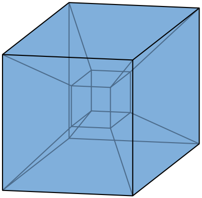
A *d-polytope* $\boldsymbol{P}$ is the *convex hull* of a d-dimensional point configuration $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \subseteq \mathbb{R}^d$.



5-gon



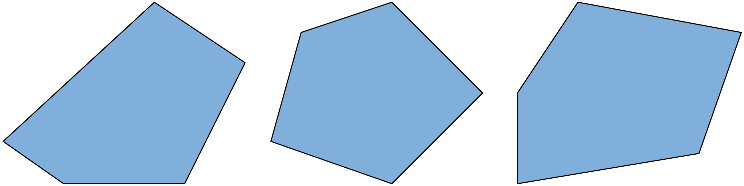
3-simplex



4-cube

Polytopes

A polytope $\boldsymbol{P}$ has a *combinatorial type* encoded by the relationships between its faces.



The *realization space* $\mathcal{R}(\boldsymbol{P})$ contains all polytopes of the same combinatorial type as $\boldsymbol{P}$, modulo affine transformations.

History

Theorem (Steinitz, 1922 [1])

An undirected simple graph G is the edge graph of some 3-polytope if and only if G is planar and three-connected.

Corollary

Let $\boldsymbol{P}$ be a 3-polytope.

1. $\mathcal{R}(\boldsymbol{P})$ *is contractible;*
2. $\boldsymbol{P}$ *can be realized with rational coordinates.*

History

- Perles, 1967: There exists an 8-polytope which is not realizable over $\mathbb{Q}$. [2]
- Bokowski, Ewald, Kleinschmidt, 1984: There exists a 4-polytope with a disconnected realization space. [3]

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Theorem (Mnëv's Universality Theorem, 1986 [4])

Let $V \subseteq \mathbb{R}^n$ be primary basic semialgebraic.

1. *There exists an oriented matroid whose realization space is stably equivalent to V;*
2. *There exists a polytope whose realization space is stably equivalent to V.*

Universality Theorem for 4-polytopes

Theorem (Richter-Gebert, 1996 [5])

For every primary basic semialgebraic set $V \subseteq \mathbb{R}^n$ there exists a 4-polytope $\boldsymbol{P}$ such that:

1. *V and the realization space $\mathcal{R}(\boldsymbol{P}) \subseteq \mathbb{R}^m$ are homotopy equivalent;*
2. *If A is a subfield of the real algebraic numbers, then*

$$V \cap A^n = \varnothing \iff \mathcal{R}(\boldsymbol{P}) \cap A^m = \varnothing;$$

3. *The face lattice of $\boldsymbol{P}$ can be computed in polynomial time from the defining equations and inequalities of V.*

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- Idea: stably equivalent sets differ only by “trivial fibration.”
- Boege, 2022: Richter-Gebert’s stable equivalence does not preserve homotopy type! [6]

Consequences

- The realizability problem for 4-polytopes is polynomial-time equivalent to the *existential theory of the reals*.
- There exists a nonrational 4-polytope, resolving a question in [7].
- The Universal Partition Theorem for 4-polytopes. [5, 8]

Goals

- Provide a satisfactory definition for stable equivalence.
- Verify that Richter-Gebert's proof works with the new definition.
- Review and clarify the existing literature on stable equivalence.

2 Polytopes

Hulls

The *affine*, *convex*, *linear* and *positive hulls* of a set $S \subseteq \mathbb{R}^n$ are, respectively,

$$\operatorname{aff}(S) = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \mid n \in \mathbb{N},\ \lambda_i \in \mathbb{R},\ \boldsymbol{x}_i \in S,\ \sum_{i=1}^{n} \lambda_i = 1 \right\},$$

$$\operatorname{conv}(S) = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \mid n \in \mathbb{N},\ \lambda_i \geqslant 0,\ \boldsymbol{x}_i \in S,\ \sum_{i=1}^{n} \lambda_i = 1 \right\},$$

$$\operatorname{lin}(S) = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \mid n \in \mathbb{N},\ \lambda_i \in \mathbb{R},\ \boldsymbol{x}_i \in S \right\} \text{ and}$$

$$\operatorname{pos}(S) = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \mid n \in \mathbb{N},\ \lambda_i \geqslant 0,\ \boldsymbol{x}_i \in S \right\}.$$

Polytopes and cones

Definition

Let $\boldsymbol{P} = (\boldsymbol{p}_i)_{i \in X} \in \mathbb{R}^{d \times |X|}$ be a finite point configuration in $\mathbb{R}^d$.

- If $\boldsymbol{P}$ has affine dimension d and $\operatorname{conv}\left(\boldsymbol{P}|_{X \setminus \{i\}}\right) \neq \operatorname{conv}(\boldsymbol{P})$ for all $i \in X$, then $\boldsymbol{P}$ is called a *d-polytope*.
- If $\boldsymbol{P}$ has linear dimension d and $\operatorname{pos}\left(\boldsymbol{P}|_{X \setminus \{i\}}\right) \neq \operatorname{pos}(\boldsymbol{P})$ for all $i \in X$, then $\boldsymbol{P}$ is called a *d-cone*.
- The *associated cone* of a d-polytope $\boldsymbol{P} \in \mathbb{R}^{d \times n}$ is the $(d+1)$-cone

$$\boldsymbol{P}^{\text{hom}} = \left(\boldsymbol{p}_i^{\text{hom}}\right)_{i \in X} = \boldsymbol{P} \times \{1\} \in \mathbb{R}^{(d+1) \times n}.$$

Faces

Definition

- The *faces* of a d-cone $\boldsymbol{P} = (\boldsymbol{p}_i)_{i \in X}$ are sets of the form $\{i \in X \mid h(\boldsymbol{p}_i) = 0\}$, where h is a linear form nonnegative on all points of $\boldsymbol{P}$.
- The *faces* of a polytope are the faces of its associated cone.

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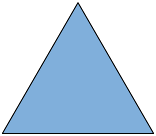
- The *faces* of a polytope are the faces of its associated cone.

- Idea: The faces of a polytope $\boldsymbol{P}$ are the intersections of $\boldsymbol{P}$ with affine hyperplanes external to $\boldsymbol{P}$.
- 0, 1 and $(d-1)$-dimensional faces are called vertices, edges and facets, respectively.

Example: Faces



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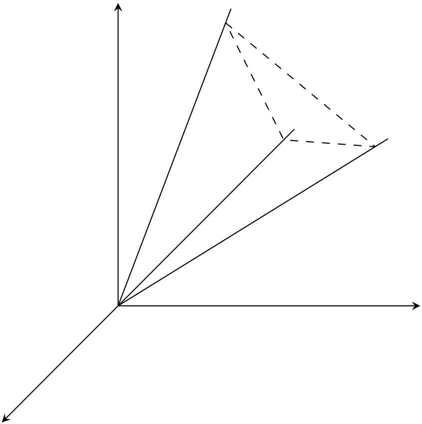

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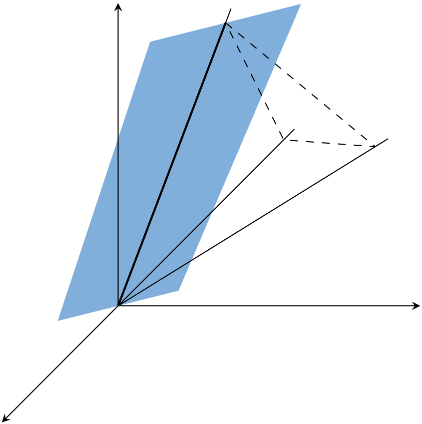


Face lattice

Definition

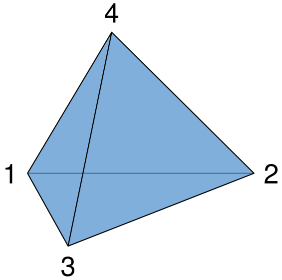
The *face lattice* $\mathrm{FL}(\boldsymbol{P})$ of a d-polytope $\boldsymbol{P}$ is given by

$$\mathrm{FL}(\boldsymbol{P}) = (\mathrm{faces}(\boldsymbol{P}), \subseteq),$$

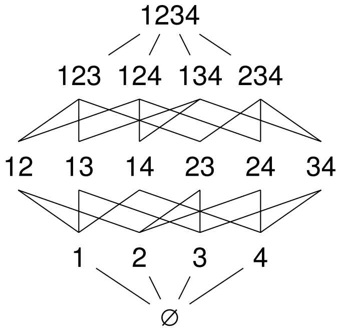
i.e. the set of faces partially ordered by inclusion.

- Idea: $\mathrm{FL}(\boldsymbol{P})$ gives the combinatorial type of $\boldsymbol{P}$.
- $\mathrm{FL}(\boldsymbol{P})$ is uniquely determined by the facets of $\boldsymbol{P}$.

Example: Face lattice



$\boldsymbol{P}$



$\mathrm{FL}(\boldsymbol{P})$

Realizations

Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be d-polytopes.

Definition

$\boldsymbol{Q}$ is a *realization* of $\boldsymbol{P}$ if $\mathrm{FL}(\boldsymbol{Q}) = \mathrm{FL}(\boldsymbol{P})$. Equivalently, we say that $\boldsymbol{Q}$ *realizes* $\boldsymbol{P}$ or $\mathrm{FL}(\boldsymbol{P})$.

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- Bijective affine transformations of $\boldsymbol{P}$ are realizations of $\boldsymbol{P}$.
- Which lattices are realizable?

Realization space

Definition

An *affine basis* of a d-polytope $\boldsymbol{P} = (\boldsymbol{p}_i)_{i \in X}$ is a set $B = \{b_1, \ldots, b_{d+1}\} \subseteq X$ such that the vertices corresponding to B are affinely independent in any realization of $\boldsymbol{P}$.

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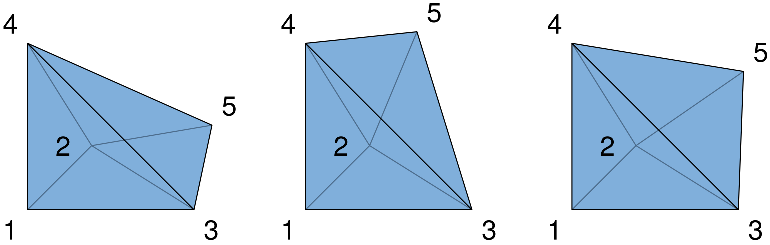
Definition

Let $\boldsymbol{P}$ be a d-polytope with a basis $B = \{b_1, \ldots, b_{d+1}\}$. The *realization space* $\mathcal{R}(\boldsymbol{P}, B)$ of $\boldsymbol{P}$ with respect to B is the set of all realizations $\boldsymbol{Q}$ of $\boldsymbol{P}$ such that $\boldsymbol{p}_i = \boldsymbol{q}_i$ for all $i \in B$.

Idea: Fixing $d + 1$ affinely independent points factors out affine transformations.

Example: Realizations



3 Semialgebraic sets

Semialgebraic sets

Definition

Let $V \subseteq \mathbb{R}^n$.

- V is *semialgebraic* if there are polynomials $f_{i,j} \in \mathbb{R}[x_1, \ldots, x_n]$ such that
$$V = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \left\{ \boldsymbol{v} \in \mathbb{R}^n \mid f_{i,j}(\boldsymbol{v}) \sim_{i,j} 0 \right\},$$
where $\sim_{i,j}$ is either $=$ or $>$.
- V is *primary basic semialgebraic* if there are polynomials $f_1, \ldots, f_s, g_1, \ldots, g_r \in \mathbb{Z}[x_1, \ldots, x_n]$ such that
$$V = \left\{ \boldsymbol{v} \in \mathbb{R}^n \mid f_i(\boldsymbol{v}) = 0,\ g_j(\boldsymbol{v}) > 0 \right\}.$$

Semialgebraic sets

Lemma

Realization spaces of polytopes are primary basic semialgebraic.

Semialgebraic sets

Lemma

Realization spaces of polytopes are primary basic semialgebraic.

Proof (sketch). Let $\boldsymbol{P} = (\boldsymbol{p}_i)_{i=1}^n$ be a d-polytope. Choose an affine basis B for $\boldsymbol{P}$, and an affine basis B_F for each facet F of $\boldsymbol{P}$.

For point configurations $\boldsymbol{Q} = (\boldsymbol{q}_i)_{i=1}^n \in \mathbb{R}^{d\times n}$ and $F \in \operatorname{facets}(\boldsymbol{P})$ we define $\varphi_{\boldsymbol{Q},F} \in \left(\mathbb{R}^{d+1}\right)^*$ by

$$\varphi_{\boldsymbol{Q},F}(\boldsymbol{x}) = \det\left[\boldsymbol{Q}^{\text{hom}}|_{B_F}, \boldsymbol{x}\right].$$

Semialgbraic sets

$\boldsymbol{Q} \in \mathcal{R}(\boldsymbol{P}, B)$ if and only if the following conditions hold:

1. $\boldsymbol{p}_i = \boldsymbol{q}_i$ for all $i \in B$;
2. If $F \in \text{facets}(\boldsymbol{P})$ and $i \in F$, then $\varphi_{\boldsymbol{Q},F}(\boldsymbol{q}_i^{\text{hom}}) = 0$;
3. If $F \in \text{facets}(\boldsymbol{P})$ and $i, j \in \{1, \ldots, n\} \setminus F$ are two (not necessarily distinct) labels, then

$$\varphi_{\boldsymbol{Q},F}\left(\boldsymbol{q}_i^{\text{hom}}\right) \cdot \varphi_{\boldsymbol{Q},F}\left(\boldsymbol{q}_j^{\text{hom}}\right) > 0.$$

The determinant is a polynomial in the entries of the matrix. □

Stable equivalence

Idea. If $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ are stably equivalent primary basic semialgebraic sets, then:

1. V and W are homotopy equivalent;
2. If A is a subfield of the real algebraic numbers, then

$$V \cap A^n = \varnothing \iff W \cap A^m = \varnothing.$$

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Many different versions: Mnëv [4], Günzel [9, 8], Richter-Gebert [10, 5], Boege [11], Wikipedia [12]...

Stable equivalence

Mnëv's original idea [4]:

Definition

Two primary basic semialgebraic sets V and W are *stably equivalent* if there exists a locally biregular homeomorphism f such that $W = f(V \times \mathbb{R}^k)$ for some k.

- $V \times \mathbb{R}^k$ deformation retracts onto V, so homotopy equivalence is implied.
- Local biregularity of f implies equivalence of algebraic number type.

Stable equivalence

Richter-Gebert's idea: move within one dimension with "nice" homeomorphisms, jump between dimensions with "nice" projections.

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Definition

Two primary basic semialgebraic sets V and W are *rationally equivalent* if there exists a homeomorphism $f : V \to W$ such that f and f^{-1} are rational functions with rational coefficients.

Rational equivalence preserves homotopy type and algebraic number type.

Stable projections

Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^{n+m}$ be primary basic semialgebraic such that $V = \pi(W)$.

Definition (preliminary)

The projection π is *stable* if we obtain

$$W = \left\{(\boldsymbol{v}, \boldsymbol{u}) \in \mathbb{R}^{n+m} \mid \boldsymbol{v} \in V,\ \varphi_i^{\boldsymbol{v}}(\boldsymbol{u}) > 0,\ \psi_j^{\boldsymbol{v}}(\boldsymbol{u}) = 0\right\}$$

for affine forms $\varphi_1^{\boldsymbol{v}}, \ldots, \varphi_r^{\boldsymbol{v}}, \psi_1^{\boldsymbol{v}}, \ldots, \psi_s^{\boldsymbol{v}}$ whose coefficients depend polynomially on $\boldsymbol{v}$.

The fibers are relative interiors of polyhedral sets: looks nice!

Stable equivalence

Definition

Stable equivalence is the equivalence relation generated by rational equivalence and stable projections. We denote stable equivalence between sets V and W by $V \approx W$.

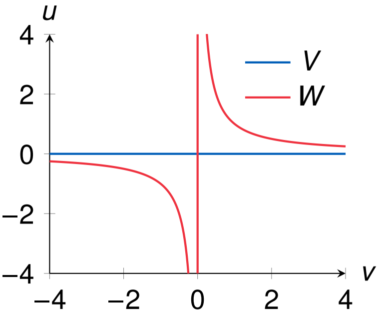
- If rational equivalence and stable projections preserve the desired properties, so does stable equivalence.
- Seemingly stricter than Mnëv's version.

Counterexample

Boege, 2022 [6]: Let $V = \mathbb{R}$ and

$$W = \left\{(v, u) \in \mathbb{R}^2 \mid v \in V,\ v(vu - 1) = 0\right\}.$$

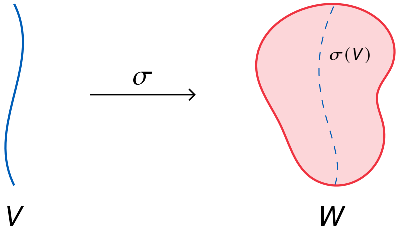
Then $V = \pi(W)$ is a stable projection, but W is disconnected!



Repairing stable projections

Suppose $V = \pi(W)$ and that each fiber is convex.

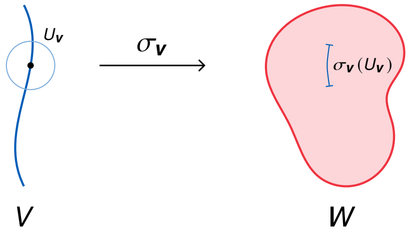
Idea. If there exists a continuous map $\sigma : V \to W$ with $\pi \circ \sigma = \mathrm{id}$ then W deformation retracts onto $\sigma(V)$, so V and W are homotopy equivalent.



Local sections

Suppose for each $\boldsymbol{v} \in V$ there exists a neighborhood $U_{\boldsymbol{v}} \subseteq V$ of $\boldsymbol{v}$ and a continuous map $\sigma_{\boldsymbol{v}} : U_{\boldsymbol{v}} \to W$ such that $\pi \circ \sigma_{\boldsymbol{v}} = \mathrm{id}$.

By paracompactness of V, this is sufficient to construct a global section $\sigma : V \to W$.



Constructing local sections

Suppose $V = \pi(W)$ is a stable projection. Strict inequalities are fine locally, so focus on equations.

We represent the equations $\psi_1^{\boldsymbol{v}}(\boldsymbol{u}) = \ldots = \psi_s^{\boldsymbol{v}}(\boldsymbol{u}) = 0$ as a linear system $A_{\boldsymbol{v}}\boldsymbol{u} = \boldsymbol{b}_{\boldsymbol{v}}$.

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- $A_{\boldsymbol{v}}$ has some rank r, and hence a submatrix $B_{\boldsymbol{v}} \in \mathbb{R}^{r \times r}$ with $\det(B_{\boldsymbol{v}}) \neq 0$.
- If the rank is constant locally around $\boldsymbol{v}$, we can extend the solution $(\boldsymbol{v}, \boldsymbol{u})$ uniquely to that neighborhood using Cramer's rule for $B_{\boldsymbol{v}}$.

New stable projections

Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^{n+m}$ be primary basic semialgebraic such that $V = \pi(W)$.

Definition (Boege, 2022 [6])

The projection π is *stable* if we obtain

$$W = \left\{ (\boldsymbol{v}, \boldsymbol{u}) \in \mathbb{R}^{n+m} \mid \boldsymbol{v} \in V,\ \varphi_i^{\boldsymbol{v}}(\boldsymbol{u}) > 0,\ \psi_j^{\boldsymbol{v}}(\boldsymbol{u}) = 0 \right\}$$

for affine forms $\varphi_1^{\boldsymbol{v}}, \ldots, \varphi_r^{\boldsymbol{v}}, \psi_1^{\boldsymbol{v}}, \ldots, \psi_s^{\boldsymbol{v}}$ whose coefficients depend polynomially on $\boldsymbol{v}$, and the fibers of π are locally constant dimensional.

4 Universality

Shor's normal form

Shor, 1991: Mnëv's universality theorem is easier if we simplify the semigalebraic set. [13]

Let V be primary basic semialgebraic.

Theorem

There exists a primary basic semialgebraic set $W \subseteq \mathbb{R}^n$ defined by

$$1 < x_1 < \ldots < x_n,$$

and equations of the form

$$x_i + x_j = x_k, \qquad x_i \cdot x_j = x_k,$$

such that $V \approx W$.

The universality theorem

Theorem

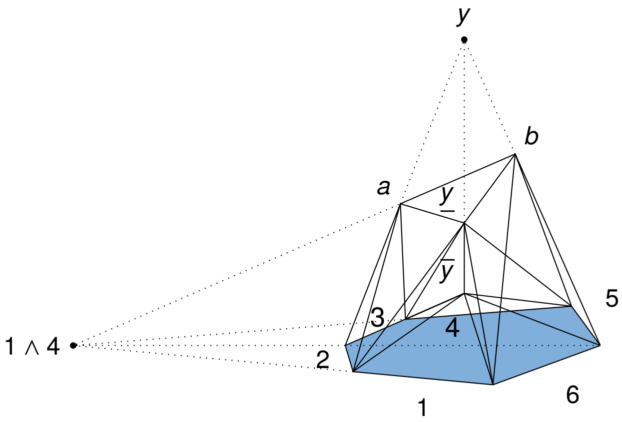
For every primary basic semialgebraic set V there exists a 4-polytope $\boldsymbol{P}$ such that $V \approx \mathcal{R}(\boldsymbol{P})$.

Idea. Assume V is in Shor's normal form. We construct a 4-polytope $\boldsymbol{P}$ with a 2-face $\boldsymbol{G}$ with line slopes $(s_1, \ldots, s_n)$ such that:

- $(s_1, \ldots, s_n) \in V$ in any realization of $\boldsymbol{P}$;
- For any $\boldsymbol{v} \in V$ there exists a realization of $\boldsymbol{P}$ with $\boldsymbol{v} = (s_1, \ldots, s_n)$.

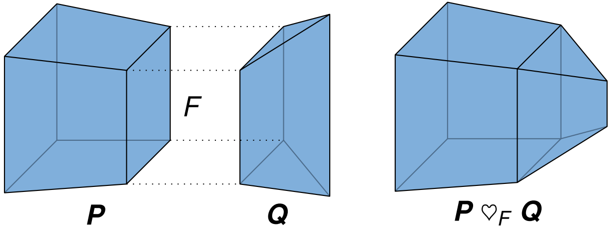
The heart

We need to add "structure" to $\mathcal{R}(\boldsymbol{P})$ in a controlled way. In the $\boldsymbol{X}$-polytope, $1 \wedge 4$, $2 \wedge 3$ and $5 \wedge 6$ are always collinear:



Building

To combine "structure" we use the connected sum:



If $\boldsymbol{P}$ and $\boldsymbol{Q}$ impose conditions on F and F is "necessarily flat," then $\boldsymbol{P} \heartsuit_F \boldsymbol{Q}$ imposes both sets of conditions.

5 Future work

Singularity structure

Richter-Gebert: Stable equivalence preserves “singularity structure.” [10, 14] (not claimed in [5].)

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- What are singularities of primary basic semialgebraic set V?
- What is the singularity structure of V?
- Does stable equivalence preserve said structure?

Singularity structure

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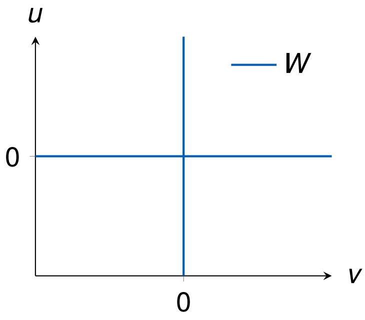
- What is the singularity structure of V?
- Does stable equivalence preserve said structure?

Typical approach: the singularities of V are the singularities of the Zariski closure of V [15, 16].

Singularity structure

Let $V = \mathbb{R}$ and $W = \left\{(v,u) \in \mathbb{R}^2 \mid v \in V,\ vu = 0\right\}$.

By Richter-Gebert's definition (not by ours!) $V \approx W$, but W has a singularity at $(0,0)$. [17]



Simplicial polytopes

A polytope is *simplicial* if all of its faces are simplices. Simplicial polytopes are universal by [18].

How about in dimension 4?

Conjecture [5]

For every open primary basic semialgebraic set V there exists a simplicial 4-polytope $\boldsymbol{P}$ such that $V \approx \mathcal{R}(\boldsymbol{P})$.

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