Master's Programme in Mathematics and Operations Research

Repairing the Universality Theorem for 4-Polytopes

Emil Verkama

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Supervisor

Prof. Kaie Kubjas

Advisor

Dr. Tobias Boege

Aalto University
School of Science

Aalto University
School of Science

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Author	Emil Verkama
Major	Mathematics, SCI3054
Supervisor	Prof. Kaie Kubjas
Advisor	Dr. Tobias Boege
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Abstract

The realization space of a polytope $\boldsymbol{P}$ is defined as the space of all polytopes realizing the combinatorial type of $\boldsymbol{P}$, modulo affine transformations. Answering several long-standing open questions, Richter-Gebert's Universality Theorem for 4-polytopes states that realization spaces of 4-polytopes can attain the homotopy type and algebraic complexity of any primary basic semialgebraic set.

Richter-Gebert's proof relies on a new notion of stable equivalence between semialgebraic sets. The term has been used earlier in various forms, and is meant to imply at least homotopy equivalence and equivalence of algebraic number type. However, due to a flaw in the definition, Richter-Gebert's version does not guarantee the former.

In this thesis, the flaw in Richter-Gebert's definition is repaired through the introduction of a stricter, homotopy type preserving version of stable equivalence. It is then demonstrated that the rest of Richter-Gebert's original proof still works with the new definition. Lastly, the scattered history of stable equivalence is reviewed and reconciled.

Keywords Polytopes, Discrete geometry, Real algebraic geometry, Topology

Aalto University
School of Science

Titel	Reparation av Universalitetsatsen för 4-Polytoper
Författare	Emil Verkama
Huvudämne	Matematik, SCI3054
Ansvarslärare	Prof. Kaie Kubjas
Handledare	Dr. Tobias Boege
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Sammandrag

Realisationsrymden av en polytop $\boldsymbol{P}$ are definierad som rymden av alla polytoper med samma kombinatoriella typ som $\boldsymbol{P}$, modulo affina avbildningar. I svar på ett flertal öppna frågor hävdar Richter-Geberts universalitetssats att realisationsrymderna av fyrdimensionella polytoper kan ha homotopitypen och algebraiska taltypen av en godtycklig primär semialgebraisk varietet.

Richter-Geberts bevis är baserat på en ny version av stabil ekvivalens för semialgebraiska varieteter. Termen har tidigare använts flera gånger, och är menad att medföra åtminstone homotopiekvivalens samt ekvivalens av algebraisk taltyp. Richter-Gebert's definiton är dock felaktig, i och med att den inte garanterar homotopiekvivalens.

I denna avhandling korrigeras felet i Richter-Gebert's bevis med hjälp av en ny, starkare stabil ekvivalens. Det bevisas att den nya definitionen implicerar homotopiekvivalens, och därefter att resten av Richter-Gebert's bevis fortfarande fungerar. Avslutningsvis recenseras och förenas den existerande litteraturen angående stabil ekvivalens.

Nyckelord Polytoper, Diskret geometri, Reell algebraisk geometri, Topologi

Preface

The work for this thesis was carried out in Prof. Kaie Kubjas' research group. I would like to extend my deep gratitude to Prof. Kubjas for her excellent supervision.

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Emil Verkama

Contents

1 Introduction

Polyhedral geometry as an area of mathematical research has roots reaching back thousands of years to ancient Greece. The topic experienced a resurgence of interest after the concept of two-dimensional polygons and three-dimensional polyhedra, the classical objects of study, was generalized to higher dimensions in the 19th century. Formally, a *polytope* $\boldsymbol{P}$ is the smallest convex set in euclidean space containing some finite point configuration $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \subseteq \mathbb{R}^d$, *i.e.*,

$$\boldsymbol{P} = \operatorname{conv}(\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}) = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{p}_i \mid \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n \right\}.$$

The primary application of these higher dimensional objects is in linear optimization: the maxima and minima of linear functions subjected to linear constraints occur on boundaries of n-dimensional polytopes.

Polytopes also connect purely combinatorial aspects with geometric structure. For example, two triangles share certain properties that are fundamentally different from the properties of a quadrilateral, even if the triangles are geometrically different. We are primarily interested in the combinatorial type of a polytope, which is an equivalence class defined by a structure that encodes the relationships between its faces. This structure is known as the *face lattice* of the polytope. Given a face lattice L, any polytope having L as its face lattice is called a *realization* of L.

It is natural to examine the topological and algebraic properties of the *realization space* $\mathcal{R}(\boldsymbol{P})$, defined as the geometric space containing all realizations of the face lattice of a given polytope $\boldsymbol{P}$, modulo affine transformations. In dimension three the situation is surprisingly simple as an indirect consequence of the well-known Steinitz's Theorem [36].

Theorem 1.1 (Steinitz's Theorem, 1922)**.** *An undirected simple graph G is the edge graph of some polyhedron if and only if G is three-connected and planar.*

It is possible to show that realization spaces of polyhedra hence must be topologically trivial with respect to homotopy type, and that every polyhedron has a realization with strictly rational coordinates for the vertices; see [28] for detailed discussion. Moreover, Steinitz's Theorem answers his question about the difficulty of determining whether a given lattice is realizable or not, in the three-dimensional case.

Following Steinitz's work, the question of whether any similar result would generalize to higher dimensions remained open. The answer was expected to be negative; *e.g.*, based on the following evidence:

- There exists an 8-polytope that is not realizable over $\mathbb{Q}$, as constructed by Perles in 1967 [17]. In particular, this proves that realization spaces of polytopes can be algebraically nontrivial;
- There exists a 4-polytope with a disconnected realization space, as constructed by Bokowski, Ewald and Kleinschmidt in 1984 [11]. Thus realization spaces of polytopes can also be topologically nontrivial.

Stronger still, Nikolai Mnëv showed in 1986, as a corollary of his celebrated Universality Theorem for oriented matroids, that realization spaces of polytopes can be arbitrarily complicated in a precisely specified sense [25].

Theorem 1.2 (Mnëv's Universality Theorem, 1986)**.** *Every primary basic semialgebraic set is stably equivalent to the realization space of some oriented matroid.*

Corollary 1.3. *Every primary basic semialgebraic set is stably equivalent to the realization space of some polytope.*

Universality theorems could cynically be described as a form of Murphy's law, as they ensure that certain classes of objects must exhibit any arbitrarily complicated structure up to some obvious limit. In Mnëv's universality theorem, for example, this obvious limit is that realization spaces of oriented matroids are always primary basic semialgebraic sets. Simplified proofs of Mnëv's prolific result can be seen for example in [35, 27], and generalizations in [24, 18, 21, 5, 22, 38, 23]. The last five provide a scheme-theoretic view. In [14] Ruchira Datta proved another famous universality theorem for Nash equilibria. For another kind of universality, see the universality theorem for planar mechanical linkages by Denis Jordan and Marcel Steiner in [20].

The term *stable equivalence* is a central focus of this thesis. The specific definition used by Mnëv is discussed in more detail in Section 8 under the name "S1 equivalence." Morally, stable equivalence between semialgebraic sets should preserve at least homotopy type and algebraic number type; we say that stably equivalent sets differ only by *trivial fibration*.

After Mnëv's results, the primary remaining question was whether the dimension of the polytope in Corollary 1.3 could be fixed. In 1996, Jürgen Richter-Gebert published a theorem answering this in the affirmative: realization spaces become arbitrarily complicated already for 4-polytopes. His result, called the Universality Theorem for 4-polytopes, was published in [28] and announced in the research report [31].

Theorem 1.4 (Universality Theorem for 4-polytopes, 1996)**.** *Let $V \subseteq \mathbb{R}^n$ be a primary basic semialgebraic set. Then there exists a* 4*-polytope $\boldsymbol{P}$ such that the following properties hold*:

(1) *V and $\mathcal{R}(\boldsymbol{P})$ are homotopy equivalent*;

(2) *If A is a proper subfield of the real algebraic numbers and $\mathcal{R}(\boldsymbol{P}) \subseteq \mathbb{R}^m$, then*

$$V \cap A^n = \varnothing \iff \mathcal{R}(\boldsymbol{P}) \cap A^m = \varnothing;$$

(3) *The face lattice of $\boldsymbol{P}$ can be computed in polynomial time from the equations and inequalities defining V.*

However, as Tobias Boege noted in his blog post [9], Richter-Gebert's proof contains a critical error. It relies on a new notion of stable equivalence between semialgebraic sets to prove the first two claims, but this new definition allows for homotopy equivalence to fail in some specific cases, invalidating Theorem 1.4 (1). The same flawed definition is reused at least in Karim Adiprasito's universality theorem for neighborly polytopes [2], and in Richter-Gebert's new proof of Mnëv's universality theorem in [27]. Harald Günzel later published a completely novel proof of the universality theorem for 4-polytopes in [19], which inadvertently corrects the issue. Günzel's version of stable equivalence is similar to Richter-Gebert's, but sufficiently strict to ensure homotopy equivalence.

The goal of this thesis is to provide a correction to Richter-Gebert's proof by improving his definition of stable equivalence. Our new version of the definition is strong enough to preserve homotopy type, yet weak enough for the technical details of Richter-Gebert's proof to work. We also wish to review and consolidate the somewhat scattered literature available on stable equivalence.

The thesis is structured as follows. In Sections 2, 3 and 4 we cover the necessary preliminaries in polyhedral geometry, topology and real algebraic geometry, respectively. Section 4 also includes more detailed discussion on why Richter-Gebert's stable equivalence fails, along with our correction. Furthermore, we provide a complete proof of the existence of a Shor normal form for any primary basic semialgebraic set, see [35]. This striking result is essential to Richter-Gebert's proof of the universality theorem, and also has applications elsewhere. Prior to this thesis, the details behind the arguments for preservation of stable equivalence in the construction of the Shor normal form have not been published.

Sections 5, 6 and 7 focus on the proof of the universality theorem. In Section 5 we cover certain ways to combine polytopes and extend them to higher dimensions. Some of the proofs are given in more detail than in [28]. Section 6 includes important building blocks constructed with the methods from the previous section. Section 7 finally combines everything to produce the universality theorem.

Section 8 is a survey of all existing notions of stable equivalence and related concepts. In addition to listing every definition we could find, we prove some relationships between them and describe their hierarchy.

We conclude this introduction with some words on visualization of polytopes. Figures are not strictly necessary to the proofs, but they are helpful in aiding intuition. Polyhedra are easy to represent in the plane through, *e.g.*, orthogonal projections, and we will include some shading to make features easier to distinguish. To visualize 4-polytopes we use *Schlegel diagrams*: the idea is to represent a d-polytope $\boldsymbol{P}$ in $(d-1)$-dimensional space by projecting the boundary of $\boldsymbol{P}$ onto one of its facets F using a point $\boldsymbol{p}$, which lies outside $\boldsymbol{P}$ but still sufficiently close. See Figure 1 for examples in dimension 3.

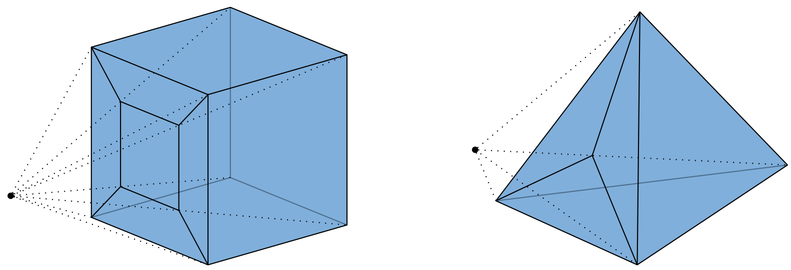


Figure 1: Constructing Schlegel diagrams of a cube and a tetrahedron.

Schlegel diagrams are useful, since the projection induces a polytopal subdivision of the facet F. The parts of the subdivision correspond with the other facets of $\boldsymbol{P}$, and their adjacency correctly represents the combinatorial structure of $\boldsymbol{P}$ even if geometric information is clearly lost. Since we are primarily interested in combinatorial structure over specific geometric detail, this is a suitable compromise. The three-dimensional Schlegel diagrams of 4-polytopes can easily be represented in the plane, such as the two examples in Figure 2. Victor Schlegel first introduced the idea in [33], and the books [17, 41] by Branko Grünbaum and Günter Ziegler provide a more thorough treatment of the subject.

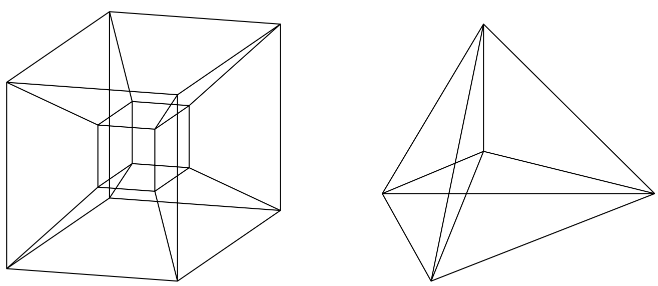


Figure 2: Schlegel diagrams of a 4-cube and a 4-simplex.

2 Polyhedral geometry

In this section we will introduce all of the basic tools within polyhedral geometry that are relevant to the present work. The first step is to recall the definition and properties of linear transformations on vector spaces, and we finish by covering polytopes, cones and their realization spaces. The theory is heavily reliant on projective geometry, see *e.g.* [30].

2.1 Transformations

Definition 2.1. Let V be a vector space over a field F. A *linear form* on V is a linear function $f : V \to F$. We denote the set of linear forms on V, *i.e.*, the *dual space* of V, by V^*.

In euclidean space, linear forms are simply real-valued linear functions. For example, recall the inner product $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{y}_i.$$

Fixing some $\boldsymbol{a} \in \mathbb{R}^n$ we get a linear form $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{a} \rangle$ since, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$f(\lambda \boldsymbol{x} + \boldsymbol{y}) = \sum_{i=1}^{n} (\lambda x_i + y_i) a_i = \lambda \sum_{i=1}^{n} x_i a_i + \sum_{i=1}^{n} y_i a_i = \lambda f(\boldsymbol{x}) + f(\boldsymbol{y}).$$

In an inner product space V, it turns out that every linear form can be described in terms of the inner product. Concretely, if $f \in V^*$ then there exists some $\boldsymbol{v} \in V$ with $f(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ for all $\boldsymbol{u} \in V$. If V is infinite-dimensional we need to further assume that f is bounded, but for the scope of this thesis all vector spaces may be assumed to be of finite dimension.

In light of this fact, it is natural to identify the linear form f directly with the corresponding vector $\boldsymbol{v}$. This allows us to write, *e.g.*, $f(\boldsymbol{u}) = \langle \boldsymbol{u}, f \rangle$ with a slight abuse of notation. Furthermore, we note that the kernel of f forms a linear hyperplane of V, and any linear hyperplane can be obtained in this manner.

Recall that linear transformations are defined as linear functions between vector spaces. In affine geometry we also need to consider *affine transformations*, and the special case of *affine forms*, which are counterparts to linear transformations and linear forms respectively. An affine transformation A is a composition of a linear transformation and a translation; formally

$$A : \begin{array}{ccc} V & \longrightarrow & W \\ \boldsymbol{v} & \longmapsto & M\boldsymbol{v} + \boldsymbol{t} \end{array}$$

for some vector spaces V and W over F, a linear transformation $M : V \to W$ and a vector $\boldsymbol{t} \in W$. Hence affine forms $V \to F$ are functions $\boldsymbol{x} \mapsto f(\boldsymbol{x}) + t$ for some $f \in V^*$ and $t \in F$, and kernels of affine forms on inner product spaces are translations of linear hyperplanes, called affine hyperplanes.

We will furthermore utilize *projective transformations* $\mathbb{R}^d \to \mathbb{R}^d$, which for our purposes are comprised of three distinct parts. For a point $\boldsymbol{x} \in \mathbb{R}^d$, a projective transformation is defined by the following steps:

(1) Homogenization $\boldsymbol{x} = (x_1, \dots, x_d) \mapsto \boldsymbol{x}^{\text{hom}} = (x_1, \dots, x_d, 1)$, with the result interpreted in homogeneous coordinates;

(2) Transformation $\boldsymbol{x}^{\text{hom}} \mapsto M\boldsymbol{x}^{\text{hom}}$ for some $M \in \mathbb{R}^{(d+1)\times(d+1)}$;

(3) Dehomogenization by intersecting $M\boldsymbol{x}^{\text{hom}}$ with some affine hyperplane $H \subseteq \mathbb{R}^{d+1}$. The result can be interpreted as a point of $\mathbb{R}^d$ by identifying H with $\mathbb{R}^d$.

We call a projective transformation τ of a bounded set $S \subseteq \mathbb{R}^d$ *admissible*, if M in step (2) is invertible and $\tau(S)$ is bounded. Note that projective transformations preserve convexity, since each of the three steps do so.

Example 2.2. We can use projective transformations to describe all affine transformations: for translations $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{t}$ we use the map

$$\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} + \boldsymbol{t} \\ 1 \end{bmatrix},$$

and similarly for linear transformations $\boldsymbol{x} \mapsto M\boldsymbol{x}$ we have

$$\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} = \begin{bmatrix} M\boldsymbol{x} \\ 1 \end{bmatrix}.$$

This hints at projective transformations being a significantly wider class than affine transformations. The matrix can be chosen freely, and the dehomogenizing hyperplane need not be $x_{d+1} = 1$.

2.2 Polytopes and cones

In an intuitive sense a *polyhedral set* is a region of euclidean spaces bounded by affine hyperplanes. Familiar examples include, *e.g.*, the orthants of $\mathbb{R}^d$, polygons in $\mathbb{R}^2$ and polyhedra in $\mathbb{R}^3$; the latter two examples are, of course, also examples of polytopes.

Definition 2.3. A subset $S \subseteq \mathbb{R}^n$ is called *polyhedral* if it is obtained as an intersection of finitely many closed affine halfspaces of $\mathbb{R}^n$. In other words, we have

$$S = \{\boldsymbol{x} \in \mathbb{R}^n \mid f_i(\boldsymbol{x}) \geqslant 0 \text{ for } 1 \leqslant i \leqslant k\},$$

for some affine forms $f_1, \ldots, f_k$.

All polyhedral sets are convex. This follows from the fact that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are elements of a polyhedral set S defined by affine forms $f_1, \ldots, f_k$, then

$$\begin{aligned} f_i((1-t)\boldsymbol{x} + t\boldsymbol{y}) &= \langle (1-t)\boldsymbol{x} + t\boldsymbol{y}, \boldsymbol{a}_i\rangle + b_i \\ &= (1-t)\langle \boldsymbol{x}, \boldsymbol{a}_i\rangle + t\langle \boldsymbol{y}, \boldsymbol{a}_i\rangle + b_i \\ &\geqslant \min(\langle \boldsymbol{x}, \boldsymbol{a}_i\rangle, \langle \boldsymbol{y}, \boldsymbol{a}_i\rangle) + b_i \\ &\geqslant 0, \end{aligned}$$

where $f_i = \langle -, \boldsymbol{a}_i\rangle + b_i$, for all $t \in [0,1]$ and $1 \leqslant i \leqslant k$.

A *convex polytope* is essentially a bounded polyhedral set, and their counterparts in homogeneous coordinates are called *polyhedral cones*. Since we only handle polytopes that are convex in this thesis, we will drop the specification from here on. We consider polytopes as objects embedded in affine spaces and cones as objects in linear spaces. The following terminology from linear algebra is important.

Definition 2.4. The *affine*, *convex*, *linear* and *positive hulls* of a subset $S \subseteq \mathbb{R}^d$ are, respectively,

$$\begin{aligned} \operatorname{aff}(S) &= \left\{ \sum_{i=1}^n \lambda_i \boldsymbol{x}_i \mid n \in \mathbb{N},\ \lambda_i \in \mathbb{R},\ \boldsymbol{x}_i \in S \text{ for } 1 \leqslant i \leqslant n \text{ and } \sum_{i=1}^n \lambda_i = 1 \right\}, \\ \operatorname{conv}(S) &= \left\{ \sum_{i=1}^n \lambda_i \boldsymbol{x}_i \mid n \in \mathbb{N},\ \lambda_i \geqslant 0,\ \boldsymbol{x}_i \in S \text{ for } 1 \leqslant i \leqslant n \text{ and } \sum_{i=1}^n \lambda_i = 1 \right\}, \\ \operatorname{lin}(S) &= \left\{ \sum_{i=1}^n \lambda_i \boldsymbol{x}_i \mid n \in \mathbb{N},\ \lambda_i \in \mathbb{R} \text{ and } \boldsymbol{x}_i \in S \text{ for } 1 \leqslant i \leqslant n \right\} \text{ and} \\ \operatorname{pos}(S) &= \left\{ \sum_{i=1}^n \lambda_i \boldsymbol{x}_i \mid n \in \mathbb{N},\ \lambda_i \geqslant 0 \text{ and } \boldsymbol{x}_i \in S \text{ for } 1 \leqslant i \leqslant n \right\}. \end{aligned}$$

The affine hull can, as the name suggests, be interpreted as the affine counterpart of the linear hull. Concretely, $\operatorname{aff}(S)$ forms the smallest affine subspace of $\mathbb{R}^d$ containing S. Convex hulls and positive hulls have a similar symmetry: the convex hull of a set $S \subseteq \mathbb{R}^d$ is the smallest convex subset of $\mathbb{R}^d$ containing S, whereas the positive hull of S is the smallest cone from the origin containing S. In this sense, the positive and linear hulls are the homogeneous counterparts to the affine and linear hulls, respectively.

The distinction between the affine and linear hulls of S gives us two natural interpretations for the dimension of S. We say that S has *affine dimension d* if $\operatorname{aff}(S)$ is d-dimensional, and similarly that S has *linear dimension d* if $\operatorname{lin}(S)$ is d-dimensional. Intuitively, affine dimension is useful when discussing concepts in affine geometry, such as polytopes. Linear dimension applies better to homogeneous objects like cones.

Definition 2.5. Let $\boldsymbol{P} = (\boldsymbol{p}_i)_{i \in X} \in \mathbb{R}^{d \times |X|}$ be a finite point configuration in $\mathbb{R}^d$.

- If $\boldsymbol{P}$ has affine dimension d and $\operatorname{conv}\bigl(\boldsymbol{P}|_{X \setminus \{i\}}\bigr) \neq \operatorname{conv}(\boldsymbol{P})$ for all $i \in X$, then $\boldsymbol{P}$ is called a *d-polytope*.
- If $\boldsymbol{P}$ has linear dimension d and $\operatorname{pos}\bigl(\boldsymbol{P}|_{X \setminus \{i\}}\bigr) \neq \operatorname{pos}(\boldsymbol{P})$ for all $i \in X$, then $\boldsymbol{P}$ is called a *d-cone*.
- The *associated cone* of a d-polytope $\boldsymbol{P} \in \mathbb{R}^{d \times n}$ is the $(d+1)$-cone

$$\boldsymbol{P}^{\mathrm{hom}} = \left(\boldsymbol{p}_i^{\mathrm{hom}}\right)_{i \in X} \in \mathbb{R}^{(d+1) \times n}.$$

- The *faces* of a d-cone $\boldsymbol{P}$ are the sets $\{i \in X \mid h(\boldsymbol{p}_i) = 0\}$, where h is a linear form nonnegative on all points $\boldsymbol{p}_i$ of $\boldsymbol{P}$.
- The *faces* of a polytope are the faces of its associated cone.

We treat polytopes as point configurations instead of as convex hulls, which is more traditional. In our case, this distinction avoids cumbersome notation in many of the proofs.

Equivalently, polytopes can be described as polyhedral sets that are bounded in the sense that they contain no ray of the form $\{\boldsymbol{x} + t\boldsymbol{y} \mid t \geqslant 0\}$ for any $\boldsymbol{y} \neq 0$. This characterization can be very convenient in some cases. Proving that these two versions are equivalent is nontrivial, we refer to the first section in [41].

Some faces have special names: the 0-dimensional, 1-dimensional and $(d-1)$-dimensional faces of a d-polytope $\boldsymbol{P}$ are called vertices, edges and facets, respectively. We will denote the set of faces, vertices and facets of $\boldsymbol{P}$ by $\operatorname{faces}(\boldsymbol{P})$, $\operatorname{vert}(\boldsymbol{P})$ and $\operatorname{facets}(\boldsymbol{P})$.

Definition 2.6. The *face lattice* of a d-polytope $\boldsymbol{P}$ is the lattice

$$\operatorname{FL}(\boldsymbol{P}) = (\operatorname{faces}(\boldsymbol{P}), \subseteq),$$

the set of faces ordered by inclusion. A lattice isomorphic to the face lattice of a polytope is called *polytopal*.

The face lattice describes the combinatorial type of a polytope, and is completely determined by its facets. Combinatorial type is often of central importance, since it in some sense encapsulates the essence of the structure of the polytope and forgets unneeded geometric detail.

Face lattices have nice order-theoretic properties. They are ranked by dimension of the faces, and atomistic as every face is spanned by vertices. They are furthermore coatomistic, since every face is uniquely determined as an intersection of facets, as discussed above. This topic is discussed in more detail in, *e.g.*, [41, Section 2.2] and [17].

Due to our focus on combinatorial type, it is often convenient to be freed from any geometric restrictions by working fully within the realm of combinatorics. By the coatomicity of the face lattice, the set $P = \mathrm{facets}(\boldsymbol{P})$ of a d-polytope $\boldsymbol{P}$ fully describes the combinatorial type of $\boldsymbol{P}$. Such an object P is called a *combinatorial d-polytope*, and the definition will be extended further to even include certain nonpolytopal combinatorial types.

Definition 2.7. A set $P \subseteq 2^X$ for a finite index set X is a *combinatorial d-polytope* if it can be obtained by the following recursive definition:

(1) The set $\mathrm{facets}(\boldsymbol{P})$ of a d-polytope $\boldsymbol{P}$ is a combinatorial d-polytope;

(2) Let P and Q be combinatorial d-polytopes on X and Y respectively. If

$$P \cap Q = \{F\} = \{X \cap Y\} \quad \text{and} \quad P|_F = Q|_F,$$

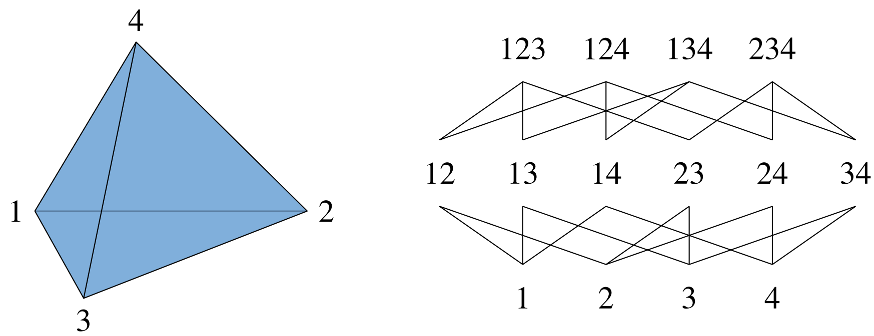
then $(P \cup Q) \setminus \{F\}$ is a combinatorial d-polytope. Here $P|_F$ denotes the restriction

$$P|_F = \{F' \cap F \mid F' \in P\}.$$

If it is the case that the face lattice resulting from the second part of the above definition is polytopal, it models gluing together two d-polytopes $\boldsymbol{P}$ and $\boldsymbol{Q}$ along combinatorially isomorphic facets $F_{\boldsymbol{P}} \in \mathrm{facets}(\boldsymbol{P})$ and $F_{\boldsymbol{Q}} \in \mathrm{facets}(\boldsymbol{Q})$. We will later introduce this concept more rigorously as the *connected sum* operation.

Example 2.8. Figure 3 shows the face lattice of a tetrahedron $\boldsymbol{P}$ with X and $\emptyset$ excluded for clarity. It is clear that $\mathrm{FL}(\boldsymbol{P})$ remains invariant under small translations of the vertices, so it is not unique to $\boldsymbol{P}$. Furthermore, we can see how the intersections of the facets of $\boldsymbol{P}$ determine the rest of its faces.

A *realization* of a combinatorial d-polytope P is a d-polytope $\boldsymbol{Q}$ satisfying $P = \mathrm{facets}(\boldsymbol{Q})$; we say that $\boldsymbol{Q}$ *realizes* P. Again, it is possible that no such polytope $\boldsymbol{Q}$ exists, but usually there are many. If P was given by the set of facets of some specific d-polytope $\boldsymbol{P}$, we say directly that $\boldsymbol{Q}$ is a realization of $\boldsymbol{P}$. Note that this is equivalent to requiring $\mathrm{FL}(\boldsymbol{P}) = \mathrm{FL}(\boldsymbol{Q})$.



$\boldsymbol{P}$ FL($\boldsymbol{P}$)

Figure 3: The face lattice of a tetrahedron.

With these notions in mind, we want the *realization space* of a (combinatorial) polytope to include all of its nontrivial realizations and factor out the rest. In particular, realizations obtained by nondegenerate affine transformations of other realizations should be left out. A simple way of achieving this is by fixing a set of $d + 1$ affinely independent points on the polytope.

Definition 2.9. An *affine basis* of a d-polytope $\boldsymbol{P} = (\boldsymbol{p}_i)_{i \in X}$ is a set $B = \{b_1, \ldots, b_{d+1}\} \subseteq X$ such that the vertices corresponding to B are affinely independent in any realization of $\boldsymbol{P}$.

The following recursive process always returns an affine basis of a d-polytope $\boldsymbol{P}$ indexed over X:

(1) Choose any $b_{d+1} \in X$;

(2) Choose a facet F of $\boldsymbol{P}$ such that $b_{d+1} \notin F$, find an affine basis $\{b_1, \ldots, b_d\}$ of $\boldsymbol{P}|_F$.

This works, since by convexity the points $b_1, \ldots, b_d$ must span an affine hyperplane in any realization of $\boldsymbol{P}$, and b_{d+1} is chosen outside that hyperplane.

Definition 2.10. Let $\boldsymbol{P}$ be a d-polytope with an affine basis $B = \{b_1, \ldots, b_{d+1}\}$. The *realization space* $\mathcal{R}(\boldsymbol{P}, B)$ is the set of all realizations $\boldsymbol{Q}$ of $\boldsymbol{P}$ such that $\boldsymbol{p}_i = \boldsymbol{q}_i$ for all $i \in B$.

In other words, all elements of the realization space fix a certain affine basis. We will see later that the choice of basis does not change the structure of the realization space up to a very strict form of equivalence.

Our definition of realization spaces represents but one of several related objects. The most obvious way Definition 2.10 could have differed is by directly factoring the set of all realizations of $\boldsymbol{P}$ by the action of the group of all affine transformations $\mathbb{R}^d \to \mathbb{R}^d$.

This really achieves the same goal, considering that in our definition the $d + 1$ affinely independent fixed points imply that no nontrivial affine transformation of $\boldsymbol{P}$ can be included in $\mathcal{R}(\boldsymbol{P}, B)$. Our definition will also be easier to work with especially in Section 4, where we prove that realization spaces are primary basic semialgebraic sets.

Other related models of realization spaces, including the so-called Grassmannian model, the Gale transformation model and the slack variety, along with their connections, are discussed in detail in [16]. *Centered* realization spaces are covered in [26], with results regarding dimension. The overview article [42] provides a concise explanation of the connection between realization spaces of polytopes, point configurations and polyhedral surfaces. The standard model we use is the one discussed most often, such as in [28, 41].

Example 2.11. Let $\boldsymbol{P} = (\boldsymbol{p}_i)_{i=1}^5 \in \mathbb{R}^{3\times 5}$ be a tetrahedron with $\boldsymbol{p}_1 = 0$, $\boldsymbol{p}_2 = (1, 0, 0)$, $\boldsymbol{p}_3 = (0, 1, 0)$ and $\boldsymbol{p}_4 = (0, 0, 1)$. Figure 4 shows three polytopes from the realization space $\mathcal{R}(\boldsymbol{P}, B)$ with $B = \{1, 2, 3, 4\}$.

Notice that the four points corresponding to B are fixed in all pictured realizations, whereas the fifth point moves around. In all cases care needs to be taken to preserve convexity of the resulting object: the region in which $\boldsymbol{p}_5$ can be placed is defined by strict inequalities of the linear forms defining the hyperplanes supporting facets $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 3, 4\}$.

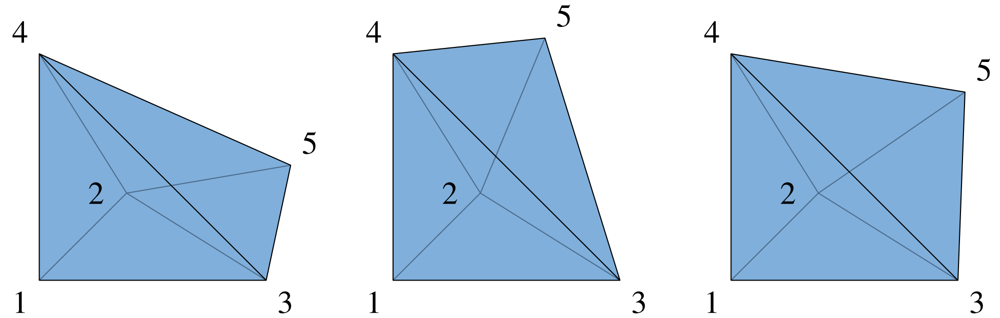


Figure 4: Elements from a realization space.

3 Topology

Topology refers to the study of certain properties of geometric objects that remain invariant under continuous deformations of the object. This type of property is called a topological invariant. We are particularly interested in *homotopy type* due to its role in the universality theorem. The area of research can roughly be divided into point-set topology and algebraic topology. The former, which we will focus on, is more set-theoretic, whereas the latter involves tools from abstract algebra. Our treatment will follow along the lines of the first chapter in Glen Bredon's book [13].

3.1 Basic notions

Definition 3.1. A pair (X, τ) with $\tau \subseteq 2^X$ is a *topological space* if τ satisfies the following conditions:

(1) If $U, V \in \tau$ then $U \cap V \in \tau$;

(2) If $\boldsymbol{U} \subseteq \tau$ then $\bigcup \boldsymbol{U} \in \tau$;

(3) $X \in \tau$ and $\varnothing \in \tau$.

The set τ is called a *topology* on X and the elements of τ are called *open sets*. Any $C \subseteq X$ such that $X \setminus C \in \tau$ is called a *closed set*. A set which is both open and closed is called *clopen*.

A topology τ on a set X restricts naturally to subsets $S \subseteq X$ by intersecting the open sets of X with S; concretely, we define the *induced topology* on S by $\tau_S = \{U \cap S \mid U \in \tau\}$. It is easy to see that (S, τ_S) indeed satisfies the conditions of a topological space.

Topological spaces are very general. We will exclusively be working with subsets of euclidean space, which implies more structure than the basic axioms imply. The central notion introduced in these nicer spaces is that of distance between points.

Definition 3.2. A *metric space* is a set X equipped with a *metric* $\mathrm{dist} : X \times X \to \mathbb{R}$ satisfying the following conditions for all $x, y, z \in X$:

(1) (Positivity) $\mathrm{dist}(x, y) \geqslant 0$, with equality if and only if $x = y$;

(2) (Symmetry) $\mathrm{dist}(x, y) = \mathrm{dist}(y, x)$;

(3) (Triangle inequality) $\mathrm{dist}(x, z) \leqslant \mathrm{dist}(x, y) + \mathrm{dist}(y, z)$.

As a familiar example, Euclidean space $\mathbb{R}^n$ is a metric space equipped with its canonical norm metric

$$\text{dist} : \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (\boldsymbol{x}, \boldsymbol{y}) & \longmapsto & \|\boldsymbol{x} - \boldsymbol{y}\|_2, \end{array}$$

where $\|-\|_2 : \mathbb{R}^n \to \mathbb{R}$ is the euclidean norm defined for $\boldsymbol{x} \in \mathbb{R}^n$ by

$$\|\boldsymbol{x}\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}.$$

It is easy to check that this metric satisfies positivity, symmetry and the triangle inequality. Metric spaces are also topological spaces when equipped with a natural topology induced by the distance function.

Lemma 3.3. *A metric space X is topological, with $U \subseteq X$ open if for each $x \in U$ there is an $\varepsilon > 0$ such that the ε-ball*

$$B_\varepsilon(x) = \{y \in X \mid \text{dist}(x, y) < \varepsilon\} \subseteq U.$$

Notice that all ε-balls of a metric space X are open: let $B_\varepsilon(x)$ be an ε-ball and $y \in B_\varepsilon(x)$. Defining $\delta = \varepsilon - \text{dist}(x, y)$, we then have for $z \in B_\delta(y)$ by the triangle inequality that

$$\text{dist}(x, z) \leqslant \text{dist}(x, y) + \text{dist}(y, z) < \text{dist}(x, y) + \varepsilon - \text{dist}(x, y) = \varepsilon,$$

so $B_\delta(y) \subseteq B_\varepsilon(x)$. To further categorize topological spaces, we use the so-called *separation axioms*. These are a set of increasingly strict conditions which describe, in a sense, how precisely a given topology discerns points in the space.

Definition 3.4. Let X be a topological space.

(1) X is a *T_0-space*, if for any two distinct $x, y \in X$ there is an open set U such that $|U \cap \{x, y\}| = 1$;

(2) X is a *T_1-space*, if for any two distinct $x, y \in X$ there are open sets U and V such that $x \in U$, $y \notin U$, $y \in V$ and $x \notin V$;

(3) X is a *T_2-space* or *Hausdorff*, if for any two distinct $x, y \in X$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$;

(4) X is a *T_3-space* or *regular*, if it is a T_1-space such that for any $x \in X$ and closed set C not containing x, there are disjoint open sets U and V with $x \in U$ and $C \subseteq V$;

(5) X is a *T_4-space* or *normal Hausdorff*, if it is a T_1-space such that for any two disjoint closed sets C and D there are disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$.

Notice that the separation axioms indeed increase in strictness: any T_n space is also a T_{n-1} space for $2 \leqslant n \leqslant 4$. In particular, all metric space are normal Hausdorff and thus also regular, Hausdorff, T_1 and T_0.

Lemma 3.5. *Metric spaces are normal Hausdorff.*

Continuity of functions is central in topology. In metric spaces it is captured by the classic ε-δ definition, but we can generalize it to the following.

Definition 3.6. A map $f : X \to Y$ between two topological spaces X, Y is *continuous*, if $f^{-1}(U) \subseteq X$ is open for each open $U \subseteq Y$.

Definition 3.7. A map $f : X \to Y$ between two topological spaces is a *homeomorphism*, if it is bijective and both f and f^{-1} are continuous. X and Y are then called *homeomorphic*.

Homeomorphisms generate an equivalence relation, with transitivity taken care of by function composition. Homeomorphic spaces are topologically identical up to relabaling of the open sets: a function $f : X \to Y$ between topological spaces (X, τ) and (Y, τ') is a homeomorphism if and only if f is bijective and $f(\tau) = \tau'$.

3.2 Paracompact spaces

Compactness essentially generalizes the notion of a closed and bounded subset of euclidean space to general topological spaces. We are interested in paracompactness, which in some sense is a localization of compactness to possibly unbounded spaces. The main fact we will use later is that *semialgebraic sets*, and in particular realization spaces of polytopes, are paracompact.

Definition 3.8. If X is a topological space, $x \in X$, $N \subseteq X$ and there exists an open set $U \subseteq N$ with $x \in U$, then N is a *neighborhood* of x.

Definition 3.9. A *covering* of a topological space X is a collection $\boldsymbol{U} \subseteq 2^X$ such that $\bigcup \boldsymbol{U} = X$. If each $U \in \boldsymbol{U}$ is open, $\boldsymbol{U}$ is an *open covering*. A covering $\boldsymbol{V} \subseteq \boldsymbol{U}$ of X is a *subcover* of $\boldsymbol{U}$.

Definition 3.10. A collection $\boldsymbol{U} \subseteq 2^X$ of subsets of a topological space X is *locally finite*, if each $x \in X$ has a neighborhood N which nontrivially intersects only a finite number of the elements of $\boldsymbol{U}$.

Definition 3.11. If $\boldsymbol{U}$ and $\boldsymbol{V}$ cover a topological space X and each $U \in \boldsymbol{U}$ is a subset of some $V \in \boldsymbol{V}$, then $\boldsymbol{U}$ is a *refinement* of $\boldsymbol{V}$.

Definition 3.12. A Hausdorff space X is *paracompact*, if every open covering of X has an open and locally finite refinement.

This still seems very abstract, but paracompact spaces have special properties related to their open coverings that we will need. Recall that the *support* of a real valued map f is defined as

$$\operatorname{support}(f) = \operatorname{closure}\{x \mid f(x) \neq 0\},$$

where the closure of a set is the smallest closed set containing it.

Definition 3.13. Let X be a topological space with an open covering $\boldsymbol{U} = \{U_i \mid i \in I\}$. A *partition of unity subordinate to* $\boldsymbol{U}$ is a collection of continuous maps

$$\left\{f_j : X \to [0,1] \mid j \in J\right\}$$

with the following properties:

(1) There exists an open and locally finite refinement $\boldsymbol{V} = \left\{V_j \mid j \in J\right\}$ of $\boldsymbol{U}$ such that $\mathrm{support}(f_j) \subseteq V_j$ for $j \in J$;

(2) For all $x \in X$, the sum $\sum_{j \in J} f_j(x) = 1$.

Theorem 3.14. *If $\boldsymbol{U}$ is an open covering of a paracompact space X, then there exists a partition of unity subordinate to $\boldsymbol{U}$.*

The proof of the preceding result is somewhat technical and we will omit it here; see *e.g.* [13, Theorem 12.11] for details. We will furthermore use the following well-known theorem due to Arthur Stone [37]. For a simpler proof, see [32].

Theorem 3.15. *Metric spaces are paracompact.*

3.3 Homotopy

Homotopy equivalence is the second kind of topological equivalence we will use. Morally, if two spaces are homotopy equivalent then one can be transformed to the other by bending, shrinking or expanding parts, without creating or removing any holes in the process.

Definition 3.16. A *homotopy* of continuous maps between topological spaces X and Y is a continuous map $F : X \times [0,1] \to Y$. Two continuous maps $f, g : X \to Y$ are *homotopic*, if there exists a homotopy $F : X \times [0,1] \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for $x \in X$. We then write $f \simeq g$.

In the above definition, the topology on a product space $X \times Y$, *i.e.*, the *product topology*, is defined by the basis sets $U \times V$ for open $U \subseteq X$ and $V \subseteq Y$. This means that the open sets of $X \times Y$ are obtained as arbitrary unions of sets of this form.

Definition 3.17. A continuous map $f : X \to Y$ is a *homotopy equivalence* with *homotopy inverse* g, if there exists a continuous map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We say that X and Y are *homotopy equivalent*, or that they have the same *homotopy type*.

Both homotopy of functions and of topological spaces are equivalence relations. To show transitivity of homotopy of maps, suppose that $f, g, h : X \to Y$ are continuous maps between topological spaces X and Y such that f and g (resp. g and h) are homotopic by $F : X \times [0,1] \to Y$ (resp. $G : X \times [0,1] \to Y$) with $F(x,0) = f(x)$, $F(x,1) = G(x,0) = g(x)$ and $G(x,1) = h(x)$. We can define a new function $H : X \times [0,1] \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t) & \text{if} \quad t \leqslant \frac{1}{2}, \\ G(x,2t-1) & \text{if} \quad t > \frac{1}{2}, \end{cases}$$

so that H traverses F and G consecutively at twice the speed. Clearly H is also a homotopy, so f and h are homotopic.

On the other hand, suppose X, Y and Z are topological spaces such that X and Y (resp. Y and Z) are homotopy equivalent by $f : X \to Y$ and $g : Y \to X$ (resp. $h : Y \to Z$ and $k : Z \to Y$). Note that if $m : Y \to Y$ and $n : Y \to Y$ are continuous maps homotopic by $F : Y \times [0,1] \to Y$, then defining $G : X \times [0,1] \to Y$ by $G(x,t) = F(f(x),t)$ gives

$$G(x,0) = F(f(x),0) = (m \circ f)(x) \quad \text{and} \quad G(x,1) = F(f(x),1) = (n \circ f)(x).$$

Since compositions of continuous functions are continuous G is a homotopy, implying that $m \circ f \simeq n \circ f$. Similarly one can verify that $g \circ m \simeq g \circ n$.

Thus associativity of function composition gives

$$(g \circ k) \circ (h \circ f) = g \circ \underbrace{(k \circ h)}_{\simeq \mathrm{id}_Y} \circ f \simeq g \circ f \simeq \mathrm{id}_X,$$

and with an analogous proof

$$(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \simeq h \circ k \simeq \mathrm{id}_Z\,.$$

The question of whether two topological spaces are homotopy equivalent is, in general, difficult to answer. Typically one has to employ tools from algebraic topology, but for us the following will suffice.

Definition 3.18. A topological space X is *connected*, if $X = U \cup V$ for two disjoint open sets $U, V \subseteq X$ implies that $U = \varnothing$ or $V = \varnothing$.

Proposition 3.19. *If X and Y are homotopy equivalent topological spaces and X is connected, then also Y is connected.*

Our proof will make use of the following terminology:

- A topological space D is called *discrete*, if every subset of D is open;

- A continuous map is called *discrete valued*, if its codomain is a discrete topological space.

Proof of Proposition 3.19. We will first show that a topological space X is connected if and only if every discrete valued continuous map on X is constant.

By definition, X is connected if it is not the disjoint union of two nonempty open subsets. Hence the only clopen subsets of X are $\varnothing$ and X itself. In the discrete topology, on the other hand, every subset is clopen. Therefore, if $d : X \longrightarrow D$ is a discrete valued continuous map and $y \in d(X)$, the preimage $d^{-1}(y)$ is clopen and must equal X.

Conversely, suppose $X = U \cup V$ for two disjoint open subsets U and V of X. We can then define a continuous nonconstant map $d : X \longrightarrow \{0, 1\}$, with $\{0, 1\}$ having the discrete topology, by

$$d(x) = \begin{cases} 0 & \text{if} \quad x \in U, \\ 1 & \text{if} \quad x \in V. \end{cases}$$

Now suppose that X and Y are homotopy equivalent by the continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$, and that X is connected. Let $d : Y \longrightarrow D$ be a discrete valued continuous map on Y. Since X is connected the composition $d \circ f : X \longrightarrow D$ is constant, so the composition $d' = d \circ f \circ g : Y \longrightarrow D$ must also be constant. Since

$$d' = d \circ f \circ g \simeq d \circ \mathrm{id}_Y = d,$$

there is a homotopy $H : Y \times [0, 1] \longrightarrow D$ with $H(y, 0) = d(y)$ and $H(y, 1) = d'(y)$. Since $\{y\} \times [0, 1]$ is connected for all $y \in Y$, the image $H(\{y\} \times [0, 1])$ is connected and therefore a singleton. Thus $d(y) = H(y, 0) = H(y, 1) = d'(y)$ for any $y \in Y$, so $d = d'$. Hence any discrete valued continuous map d on Y must be constant, implying that Y is connected. $\square$

We call a topological space *homotopically trivial* or *contractible* if it is homotopy equivalent to the singleton $\{0\}$ equipped with its unique topology. Intuitively, two topological spaces are homotopy equivalent if one can be deformed to the other without creating or removing any "holes." Figure 5 contains a few examples.

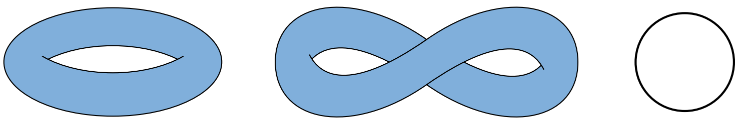


Figure 5: A few homotopy equivalent spaces.

4 Semialgebraic sets

An *algebraic set* or *algebraic variety* is the set of solutions to a collection of polynomial equations in affine space over some field F, *i.e.*, a set $V \subseteq F^n$ of the form

$$V = \{\boldsymbol{v} \in F^n \mid f_i(\boldsymbol{v}) = 0 \text{ for } i \in X\}$$

for some finite collection of polynomials $(f_i)_{i\in X} \subseteq F[x_1, \dots, x_n]$. Classically we have either $F = \mathbb{C}$, which tends to behave nicely due to the complex numbers being algebraically closed, or $F = \mathbb{R}$. Real algebraic geometry studies the latter, more restrictive case; the main focus is on a wider class of sets which also allow polynomial inequalities in their definitions. Many interesting subsets of euclidean space, such as realization spaces of polytopes, are included in this class. This section will mainly be based on the treatment in [7].

Definition 4.1. A subset $V \subseteq \mathbb{R}^n$ is *semialgebraic* if it is of the form

$$V = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \big\{\boldsymbol{v} \in \mathbb{R}^n \mid f_{i,j}(\boldsymbol{v}) \sim_{i,j} 0\big\},$$

where $f_{i,j} \in \mathbb{Z}[x_1, \dots, x_n]$ and $\sim_{i,j}$ is either $<$ or $=$ for $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant r_i$.

The set is *basic semialgebraic* if there exist collections of polynomials $(f_i)_{i\in X}$, $(g_j)_{j\in Y}$, $(h_k)_{k\in Z} \subseteq \mathbb{Z}[x_1, \dots, x_n]$ for finite index sets X, Y and Z, such that

$$V = \big\{\boldsymbol{v} \in \mathbb{R}^n \mid f_i(\boldsymbol{v}) = 0,\ g_j(\boldsymbol{v}) > 0,\ h_k(\boldsymbol{v}) \geqslant 0 \text{ for } i \in X,\ j \in Y,\ k \in Z\big\}.$$

If furthermore $Z = \varnothing$, V is called *primary basic semialgebraic*.

Proposition 4.2. *Realization spaces of polytopes are primary basic semialgebraic sets.*

Proof. Let $\boldsymbol{P} = (\boldsymbol{p}_i)_{i=1}^n \in \mathbb{R}^{d\times n}$ be a d-polytope with an affine basis B. Choose an affine basis B_F of each facet F of $\boldsymbol{P}$, and denote by $f_{\boldsymbol{Q},F} \in \big(\mathbb{R}^{d+1}\big)^*$ the linear form

$$\boldsymbol{x} \longmapsto \det\Big(\boldsymbol{Q}^{\text{hom}}|_{B_F}, \boldsymbol{x}\Big),$$

where $\boldsymbol{Q} = (\boldsymbol{q}_i)_{i=1}^n \in \mathbb{R}^{d\times n}$. The determinant is a polynomial with integer coefficients in the entries of the matrix, so $f_{\boldsymbol{Q},F} \in \mathbb{Z}[x_1, \dots, x_{n+1}]$. We will show that $\boldsymbol{Q} \in \mathcal{R}(\boldsymbol{P}, B)$ if and only if the following conditions hold:

(1) $\boldsymbol{p}_i = \boldsymbol{q}_i$ for all $i \in B$;

(2) If $F \in \operatorname{facets}(\boldsymbol{P})$ and $i \in F$, then $f_{\boldsymbol{Q},F}\big(\boldsymbol{q}_i^{\mathrm{hom}}\big) = 0$;

(3) If $F \in \operatorname{facets}(\boldsymbol{P})$ and $i, j \in \{1, \ldots, n\} \setminus F$ are two (not necessarily distinct) labels, then
$$f_{\boldsymbol{Q},F}\Big(\boldsymbol{q}_i^{\mathrm{hom}}\Big) \cdot f_{\boldsymbol{Q},F}\Big(\boldsymbol{q}_j^{\mathrm{hom}}\Big) > 0.$$

First, let $\boldsymbol{Q} \in \mathcal{R}(\boldsymbol{P}, B)$. Condition (1) holds by definition. Condition (2) holds since the vertices $\boldsymbol{Q}|_F$ lie on a hyperplane for any facet F, and the determinant maps matrices with linearly dependent column vectors to zero. Condition (3) holds since all points $\boldsymbol{q}_i^{\mathrm{hom}}$ of $\boldsymbol{Q}^{\mathrm{hom}}$ with $i \notin F$ lie in the same halfspace of the hyperplane spanned by $\boldsymbol{Q}^{\mathrm{hom}}|_F$.

Conversely, suppose $\boldsymbol{Q} = (\boldsymbol{q}_i)_{i=1}^n \in \mathbb{R}^{d \times n}$ satisfies conditions (1), (2) and (3) and fix a facet F of $\boldsymbol{P}$. By condition (2) each point $\boldsymbol{q}_i^{\mathrm{hom}}$ with $i \in F$ lies in the kernel of $f_{\boldsymbol{Q},F}$. On the other hand, there exists some $i \in \{1, \ldots, n\} \setminus F$ and condition (3) implies that $f_{\boldsymbol{Q}}^F\big(\boldsymbol{q}_i^{\mathrm{hom}}\big) \neq 0$. Combining these facts gives
$$\ker\big(f_{\boldsymbol{Q},F}\big) = \operatorname{lin}\Big(\boldsymbol{Q}^{\mathrm{hom}}|_{F_B}\Big) = \operatorname{lin}\Big(\boldsymbol{Q}^{\mathrm{hom}}|_F\Big).$$

Now, let $i, j \in \{1, \ldots, n\} \setminus F$ be two distinct labels. Condition (3) implies that $f_{\boldsymbol{Q},F}\big(\boldsymbol{q}_i^{\mathrm{hom}}\big)$ and $f_{\boldsymbol{Q},F}\Big(\boldsymbol{q}_j^{\mathrm{hom}}\Big)$ are both nonzero and have the same sign, so $\boldsymbol{q}_i^{\mathrm{hom}}$ and $\boldsymbol{q}_j^{\mathrm{hom}}$ must lie in the same halfspace of the hyperplane spanned by $\operatorname{lin}\big(\boldsymbol{Q}^{\mathrm{hom}}|_F\big)$. In other words, for each $F \in \operatorname{facets}(\boldsymbol{P})$ the hyperplane $\operatorname{lin}\big(\boldsymbol{Q}^{\mathrm{hom}}|_F\big)$ is external to $\boldsymbol{Q}^{\mathrm{hom}}$ in the sense that all points of $\boldsymbol{Q}^{\mathrm{hom}}$ lie on the same side of $\operatorname{lin}\big(\boldsymbol{Q}^{\mathrm{hom}}|_F\big)$.

It follows that $\boldsymbol{Q}$ is a d-polytope with the same facets as $\boldsymbol{P}$. The facets of a polytope uniquely determine its combinatorial type, implying that $\mathrm{FL}(\boldsymbol{P}) = \mathrm{FL}(\boldsymbol{Q})$. Together with condition (1) this shows that $\boldsymbol{Q} \in \mathcal{R}(\boldsymbol{P}, B)$. Hence $\mathcal{R}(\boldsymbol{P}, B)$ is a primary basic semialgebrac set. $\square$

Perhaps the most central property of semialgebraic sets is given by the Tarski-Seidenberg Theorem, which states that if $V \subseteq \mathbb{R}^{n+1}$ is semialgebraic then also $\pi(V) \subseteq \mathbb{R}^n$ is semialgebraic for any coordinate-deleting projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$. In general, however, we cannot conclude whether $\pi(V)$ is, *e.g.*, algebraic or primary basic semialgebraic based only on V; take for example the algebraic set $W = \big\{(v, u) \in \mathbb{R}^2 \mid v^2 + u^2 = 1\big\}$, which projects onto the basic semialgebraic set $V = \big\{v \in \mathbb{R} \mid v^2 \leqslant 1\big\}$. V is not algebraic, since a univariate polynomial $f \in \mathbb{R}[x]$ has infinitely many roots only if $f = 0$.

4.1 Stable equivalence

We use a notion of *stable equivalence* to compare primary basic semialgebraic sets. The idea is that sets in the same equivalence class should differ only by trivial fibration, and thereby at the very least have the same homotopy type. Often we also want to preserve algebraic complexity in some sense. In [28, Section 2.5], Richter-Gebert defines stable equivalence in two parts.

Definition 4.3. The primary basic semialgebraic sets V and W are *rationally equivalent* if there is a homeomorphism $f : V \to W$ such that f and f^{-1} are rational functions with rational coefficients.

Preliminary definition 4.4. Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^{n+m}$ be primary basic semialgebraic sets such that $V = \pi(W)$, where π is the projection which deletes the last m coordinates. If we can obtain W as

$$W = \left\{(\boldsymbol{v}, \boldsymbol{u}) \in \mathbb{R}^{n+m} \mid \boldsymbol{v} \in V,\ \varphi_i^{\boldsymbol{v}}(\boldsymbol{u}) > 0,\ \psi_j^{\boldsymbol{v}}(\boldsymbol{u}) = 0 \text{ for } i \in X,\ j \in Y\right\},$$

where $\varphi_i^{\boldsymbol{v}}$ and $\psi_j^{\boldsymbol{v}}$ are affine forms whose parameters depend polynomially on $\boldsymbol{v}$ and X, Y are finite index sets, then π is called a *stable projection*.

Definition 4.5. The equivalence relation generated by rational equivalence and stable projections is called *stable equivalence*. Two primary basic semialgebraic sets V and W that are in the same equivalence class with respect to stable equivalence are called *stably equivalent*, and we denote $V \approx W$.

The two parts of the definition allow us to move between dimensions as well as within the same dimension, hopefully whilst preserving important properties along the way. If two sets are rationally equivalent, preservation of homotopy type and algebraic complexity is clear. However, contradicting [28, Lemma 2.5.2], stable projections do not necessarily give homotopy equivalence. The following counterexample was found by Tobias Boege and Andreas Kretschmer [9].

Example 4.6. Let $V = \mathbb{R}$ and

$$W = \left\{(v, u) \in \mathbb{R}^2 \mid v \in V,\ v(vu - 1) = 0\right\}.$$

We have $V = \pi(W)$ and W is obtained from V by a single affine form with parameters v^2 and $-v$, so π is a stable projection. Since V is path-connected but W is not, they are not homotopy equivalent as per Lemma 3.19. This invalidates Richter-Gebert's proof of the universality theorem, as we cannot guarantee homotopy equivalence of a primary basic semialgebraic set and the realization space of its corresponding polytope. Figure 6 shows a plot of V and W.

Richter-Gebert's definition of stable projections can relatively easily be fixed. Notice that the counterexample in Example 4.6 only fails at a point v where the dimensions of the fibers $\pi^{-1}(v')$ decrease in every neighborhood of v. This turns out to be the only mode of failure.

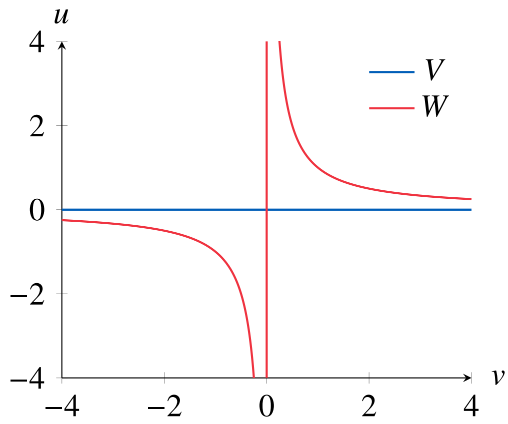


Figure 6: Stable equivalence does not imply homotopy equivalence.

Definition 4.7. Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^{n+m}$ be primary basic semialgebraic sets such that $V = \pi(W)$, where π is the projection which deletes the last m coordinates. If we obtain W as

$$W = \left\{ (\boldsymbol{v}, \boldsymbol{u}) \in \mathbb{R}^{n+m} \mid \boldsymbol{v} \in V,\ \varphi_i^{\boldsymbol{v}}(\boldsymbol{u}) > 0,\ \psi_j^{\boldsymbol{v}}(\boldsymbol{u}) = 0 \text{ for } i \in X,\ j \in Y \right\},$$

where each $\varphi_i^{\boldsymbol{v}}$ and $\psi_j^{\boldsymbol{v}}$ are affine forms whose parameters depend polynomially on $\boldsymbol{v}$, X and Y are finite index sets, and furthermore the fibers $\pi^{-1}(\boldsymbol{v})$ have constant dimension locally around every point $\boldsymbol{v} \in V$, then π is called a *stable projection*.

We remark that this is likely not the weakest correction. Consider for example the semialgebraic set

$$W = \left\{ (u, v) \in \mathbb{R}^2 \mid uv = 0 \right\}$$

consisting of the x- and y-axis of $\mathbb{R}^2$. The set W projects stably onto $\mathbb{R}$ as per the preliminary Definition 4.4, as it arguably should considering that W and $\mathbb{R}$ are homotopy equivalent, but since $\pi^{-1}(0)$ is one-dimensional our new definition disallows it.

From now on, stable projections will exclusively refer to Definition 4.7. If we want to discuss the old definition, we will specify it. The new definition was proposed by Tobias Boege in [9], along with the proof of the following result.

Definition 4.8. The *algebraic complexity* or *algebraic number type* of a semialgebraic set $V \subseteq \mathbb{R}^n$ is the smallest possible degree of an algebraic field extension F of $\mathbb{Q}$ such that $V \cap F^n \neq \emptyset$.

Lemma 4.9. *Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^{n+m}$ be stably equivalent primary basic semialgebraic sets. Then the following hold:*

(1) *V and W are homotopy equivalent;*

(2) *For a proper subfield A of the real algebraic numbers,*

$$V \cap A^n = \varnothing \iff W \cap A^{n+m} = \varnothing.$$

In particular, V and W have the same algebraic number type.

Proof. It is easy to verify that both claims hold for rational equivalence, so suppose instead that $V = \pi(W)$ is a stable projection such that

$$W = \left\{(\boldsymbol{v}, \boldsymbol{u}) \in \mathbb{R}^{n+m} \mid \boldsymbol{v} \in V,\ \varphi_i^{\boldsymbol{v}}(\boldsymbol{u}) > 0,\ \psi_j^{\boldsymbol{v}}(\boldsymbol{u}) = 0 \text{ for } i \in X,\ j \in Y\right\},$$

as per the definition.

We fix a point $(\boldsymbol{v}, \boldsymbol{u}) \in W$. The goal is to extend the map $\boldsymbol{v} \mapsto (\boldsymbol{v}, \boldsymbol{u})$ to some neighborhood of $\boldsymbol{v}$ in V, so that the extension is continuous.

and aim to extend it to a neighborhood of $\boldsymbol{v} \in V$. In small enough neighborhoods of $\boldsymbol{v}$ the strict inequalities $\varphi_j^{\boldsymbol{v}}(\boldsymbol{u}) > 0$ are satisfied, so it suffices to analyze the system $A_{\boldsymbol{v}}\boldsymbol{u} = b_{\boldsymbol{v}}$ given by the polynomial equations $\psi_j^{\boldsymbol{v}}(\boldsymbol{u}) = 0$. Here $A_{\boldsymbol{v}}$ is a $|Y| \times m$ matrix and $b_{\boldsymbol{v}}$ a vector, both having entries consisting of polynomials in $\boldsymbol{v}$.

We assumed that the solution space to $A_{\boldsymbol{v}}\boldsymbol{u} = b_{\boldsymbol{v}}$ is constant-dimensional in an open neighborhood $U_{\boldsymbol{v}}$ of $\boldsymbol{v}$, which implies that the rank r of $A_{\boldsymbol{v}}$ is also constant in $U_{\boldsymbol{v}}$. Hence $A_{\boldsymbol{v}}$ has an $r \times r$ submatrix with nonzero determinant, to which we can apply Cramer's rule and continuously extend $\boldsymbol{u}$ to a unique set of solutions for all $\boldsymbol{v}' \in U_{\boldsymbol{v}}$. This gives us a set of local sections $\{\sigma_{\boldsymbol{v}} \mid \boldsymbol{v} \in V\}$.

V is a metric space equipped with the euclidean metric inherited from $\mathbb{R}^n$, and hence paracompact. Therefore there exists a partition of unity $\{\rho_{\boldsymbol{v}} : V \to [0,1] \mid \boldsymbol{v} \in V'\}$ for some $V' \subseteq V$, subordinate to the open covering $\{U_{\boldsymbol{v}} \mid \boldsymbol{v} \in V\}$. We define

$$\sigma(\boldsymbol{v}') = \sum_{\boldsymbol{v} \in V'} \rho_{\boldsymbol{v}}(\boldsymbol{v}')\sigma_{\boldsymbol{v}}(\boldsymbol{v}').$$

The summation is finite and well-defined, since there is a locally finite refinement $\left\{U'_{\boldsymbol{v}} \mid \boldsymbol{v} \in V'\right\}$ of $\{U_{\boldsymbol{v}} \mid \boldsymbol{v} \in V\}$, and $\mathrm{support}(\rho_{\boldsymbol{v}}) \subseteq U'_{\boldsymbol{v}}$. The function is continuous since every $\rho_{\boldsymbol{v}}$ and $\sigma_{\boldsymbol{v}}$ is continuous, and convexity of $\pi^{-1}(\boldsymbol{v}')$ shows that $\sigma(\boldsymbol{v}') \in \pi^{-1}(\boldsymbol{v}')$.

The convexity of the fibers furthermore shows that $\sigma(V)$ is a strong deformation retract of W. Since V and $\sigma(V)$ are homeomorphic, we have that V and W are homotopy equivalent. This concludes the proof of (1).

We now prove (2) for stable projections. If $V \cap A^n = \varnothing$ then also $W \cap A^{n+m} = \varnothing$, since π just deletes coordinates. Conversely, suppose that there exists a point $\boldsymbol{v} \in V \cap A^n$. As in the proof of (1), we can extend $\boldsymbol{v}$ by Cramer's rule to a point $(\boldsymbol{v}, \boldsymbol{u}) \in W$. Since Cramer's rule is a rational function with rational coefficients we have $\boldsymbol{u} \in A^m$ and hence $(\boldsymbol{v}, \boldsymbol{u}) \in A^{n+m}$. $\square$

We can see that affine-linearity in the polynomials used to recover the higher dimensional set is not necessary for the proof of homotopy equivalence to go through; it would be enough for the fibers to be convex. On the other hand, the additional constraint ensures directly that the algebraic number type is preserved. We will also see that the proof of the universality theorem essentially gives the condition for free, so this does not cause any extra complications.

It remains to verify that our definition of stable equivalence satisfies a few necessary results. We call a semialgebraic set *trivial* if it is stably equivalent to $\{0\}$.

Lemma 4.10. *Stable equivalence has the following properties*:

(1) *A primary basic semialgebraic set $V \subseteq \mathbb{R}^n$ is stably equivalent to the set*

$$W = \left\{\lambda \boldsymbol{v} \in \mathbb{R}^{n+1} \mid \boldsymbol{v} \in V^{\text{hom}},\ \lambda > 0\right\};$$

(2) *Nonempty sets defined by strict affine inequalities and equations are trivial*;

(3) *The interior of a polytope is trivial.*

Proof. For (1), let $V' = \{(\boldsymbol{v}, \lambda) \mid \boldsymbol{v} \in V,\ \lambda > 0\}$ and define $f : V' \to W$ by $f(\boldsymbol{v}, \lambda) = (\lambda \boldsymbol{v}, \lambda)$. The function f has a well defined rational inverse $f^{-1}(\boldsymbol{w}, \lambda) = (\boldsymbol{w}/\lambda, \lambda)$, so V' and W are rationally equivalent. Since V is a stable projection of V', we have $V \approx W$.

For (2), let $V \subseteq \mathbb{R}^n$ be a semialgebraic set defined by strict affine inequalities and equations, *i.e.*, the relative interior of a polyhedral set and hence globally constant dimensional. Let

$$V' = \left\{\lambda \boldsymbol{v} \in \mathbb{R}^{n+1} \mid \boldsymbol{v} \in V^{\text{hom}},\ \lambda > 0\right\}.$$

Since V' is defined by strict linear inequalities and equations, $\{0\}$ is a stable projection of $V' \times \{0\}$. Since also V' is a stable projection of $V' \times \{0\}$ and $V' \approx V$ by (1), we have $V \approx \{0\}$.

Claim (3) is a special case of (2). □

Finally, we shall see that realization spaces of polytopes are invariant of the choice of affine basis up to rational equivalence. Since we will specifically work with realization spaces up to stable equivalence, this result allows us to refer simply to the realization space $\mathcal{R}(\boldsymbol{P})$ of a polytope $\boldsymbol{P}$ without ambiguity.

Lemma 4.11. *Let B_1 and B_2 be affine bases of a d-polytope $\boldsymbol{P}$. Then $\mathcal{R}(\boldsymbol{P}, B_1)$ and $\mathcal{R}(\boldsymbol{P}, B_2)$ are rationally equivalent.*

Proof. Denote $\boldsymbol{P} = (\boldsymbol{p}_i)_{i\in X}$ and let $\boldsymbol{Q} = (\boldsymbol{q}_i)_{i\in X} \in \mathcal{R}(\boldsymbol{P}, B_1)$. The vertices $(\boldsymbol{q}_i)_{i\in B_2}$ are affinely independent since B_2 is an affine basis of $\boldsymbol{P}$. Hence there exists a unique nondegenerate affine transformation $\tau_{\boldsymbol{Q}} : \mathbb{R}^d \to \mathbb{R}^d$ with

$$\tau_{\boldsymbol{Q}}(\boldsymbol{q}_i) = \boldsymbol{p}_i \quad \text{for} \quad i \in B_2,$$

and by Cramer's rule the coefficients of $\tau_{\boldsymbol{Q}}$ are rational functions in the coordinates of $\boldsymbol{P}$ and $\boldsymbol{Q}$. Thus the map

$$f : \begin{array}{ccc} \mathcal{R}(\boldsymbol{P}, B_1) & \longrightarrow & \mathcal{R}(\boldsymbol{P}, B_2) \\ \boldsymbol{Q} & \longmapsto & \tau_{\boldsymbol{Q}}(\boldsymbol{Q}) \end{array}$$

is continuous, and has an inverse constructed in the same way. In conclusion, f is a rational homeomorphism with rational coefficients and a rational inverse, showing that $\mathcal{R}(\boldsymbol{P}, B_1)$ and $\mathcal{R}(\boldsymbol{P}, B_2)$ are rationally equivalent. □

4.2 Shor's normal form

Up to stable equivalence, we want to be able to describe primary basic semialgebraic sets with polynomials that are as simple as possible. This pre-processing step is essential for the proof of the universality theorem, because it reduces the number and complexity of the necessary constructions. The following result due to Peter Shor, [35], accomplishes this goal to a surprising degree: we only need elementary addition and multiplication, and the variables can be assumed to be totally ordered.

Definition 4.12. A triple $\mathcal{S} = (n, \boldsymbol{A}, \boldsymbol{M})$ where $n \in \mathbb{N}$ and $\boldsymbol{A}, \boldsymbol{M} \subseteq \{1, \ldots, n\}^3$ are such that $(i, j, k) \in \boldsymbol{A} \cup \boldsymbol{M}$ implies $i \leqslant j < k$, is called a *Shor normal form*. The Shor normal form defines a semialgebraic set $V(\mathcal{S}) \subseteq \mathbb{R}^n$ by

$$\begin{aligned} 1 < x_1 < \ldots < x_n&, \\ x_i + x_j = x_k \quad &\text{for} \quad (i, j, k) \in \boldsymbol{A}, \\ x_i \cdot x_j = x_k \quad &\text{for} \quad (i, j, k) \in \boldsymbol{M}. \end{aligned}$$

Theorem 4.13. *For every primary basic semialgebraic set $W \subseteq \mathbb{R}^m$ there exists a Shor normal form $\mathcal{S} = (n, \boldsymbol{A}, \boldsymbol{M})$ such that $W \approx V(S)$.*

Proof. The proof of this theorem will be purely constructive, following the ideas in Shor's original paper. We will, however, need to be more precise regarding the arguments on stable equivalence. The proof is comprised of three distinct steps, each of which brings the construction closer to Shor's normal form.

Step 1. We first show that W is rationally equivalent to a primary basic semialgebraic set defined by equations and inequalities of the form

$$x_i < x_j, \quad x_i + x_j = x_k, \quad x_i \cdot x_j = x_k.$$

This can be achieved by iteratively replacing each addition and multiplication in the defining polynomials of W by an intermediate variable. Concretely, *e.g.* $2x_1^3 + x_2^2 - 2$ becomes $v_{2x_1^3+x_2^2-2}$ where

$$\begin{aligned} v_{2x_1^3+x_2^2-2} &= v_{2x_1^3} + v_{x_2^2}, & v_{2x_1^3} &= v_2 \cdot v_{x_1^3}, & v_{x_1^3} &= v_{x_1^2} \cdot x_1, \\ v_{x_1^2} &= x_1 \cdot x_1, & v_{x_2^2} &= x_2 \cdot x_2, & v_2 &= 1 + 1. \end{aligned}$$

Defining polynomials of primary basic semialgebraic sets have integer coefficients, so this method allows us to construct every such polynomial using only the variables $x_1, \ldots, x_m$ and the constants 0 and 1. Polynomial inequalities defining W become simply comparisons of these variables. The constructed primary basic semialgebraic V is rationally equivalent to W since for each point $(x_1, \ldots, x_m, v_1, \ldots, v_k) \in V$, the new coordinates $v_1, \ldots, v_k$ are determined from $(x_1, \ldots, x_m) \in W$ by rational functions.

Step 2. Suppose that W is defined by equations and inequalities of the form

$$x_i < x_j, \quad x_i + x_j = x_k, \quad x_i \cdot x_j = x_k.$$

Our next goal is to construct a new primary basic semialgebraic set $V \approx W$ of the same form, but with all variables strictly greater than one.

We begin by introducing variables called v_{x_i+a} in V for $1 \leqslant i \leqslant m$; these will end up corresponding with values $x_i + a$ for variables x_i of W and some sufficiently large a. The idea is to create equivalent defining equations and inequalities for V using these new variables, so that any new variables we define also are strictly greater than one. We introduce variables v_a, v_{2a}, v_{a^2} and v_{a+a^2}, along with equations

$$v_{2a} = v_a + v_a, \quad v_{a^2} = v_a \cdot v_a, \quad v_{a+a^2} = v_a + v_{a^2}.$$

To enforce the necessary inequalities we add the inequalties $v_a > 1$ and $v_{x_i+a} > 1$ for all $1 \leqslant i \leqslant m$. For the rest of the variables this is implied automatically. For reasons that will later become apparent we also need variables v_{x_i+a+1} and the constraints

$$v_{2a} > v_{x_i+a+1}, \quad v_{x_i+a+1} = v_{x_i+a} + 1$$

for $1 \leqslant i \leqslant m$. The next step is to mirror the defining equations and inequalities of W in V.

- If $x_i < x_j$ in W, we enforce $v_{x_i+a} < v_{x_j+a}$ in V.
- If $x_i + x_j = x_k$ in W, we define in V the variables and equations

$$v_{x_i+a} + v_{x_j+a} = v_{x_i+x_j+2a}, \quad v_{x_i+x_j+2a} = v_{x_k+a} + v_a.$$

- If $x_i \cdot x_j = x_k$ in W, we define in V the variables and equations

$$\begin{aligned} v_{x_i+a} \cdot v_{x_j+a} &= v_{x_ix_j+ax_i+ax_j+a^2}, \\ v_a \cdot v_{x_i+a} &= v_{ax_i+a^2}, \\ v_a \cdot v_{x_j+a} &= v_{ax_j+a^2}, \\ v_{ax_i+a^2} + v_{ax_j+a^2} &= v_{ax_i+ax_j+2a^2}, \\ v_{x_ix_j+ax_i+ax_j+a^2} + v_{a+a^2} &= v_{x_ix_j+ax_i+ax_j+a+2a^2}, \\ v_{ax_i+ax_j+2a^2} + v_{x_k+a} &= v_{x_ix_j+ax_i+ax_j+a+2a^2}. \end{aligned}$$

This concludes the construction of V. It remains to show that V and W are stably equivalent. To this end, consider the sets

$$\begin{aligned} V' &= \left\{(x_1, \ldots, x_m, a) \in \mathbb{R}^{m+1} \mid (x_1, \ldots, x_m) \in W \text{ and } a > \max_i |x_i| + 1\right\}, \\ V'' &= \left\{(x_1 + a, \ldots, x_m + a, a) \in \mathbb{R}^{m+1} \mid (x_1, \ldots, x_m, a) \in V'\right\}. \end{aligned}$$

The constraint $a > \max_i |x_i| + 1$ can be reformulated as $a > x_i + 1$ and $a > -x_i + 1$ for all $1 \leqslant i \leqslant m$, so V' is primary basic semialgebraic and projects stably onto W. Since V' and V'' are rationally equivalent, $W \approx V''$. We want to show that V'' is rationally equivalent to V. The following diagram clarifies the situation:

$$\begin{tikzcd} V' \arrow[r, "\cong"] \arrow[d, "\pi"'] & V'' \arrow[r, dotted, "\cong"] & V \\ W \arrow[urr, dotted, dash, "\approx"'] & & \end{tikzcd} \tag{4.1}$$

Let $\boldsymbol{x} = (x_1 + a, \ldots, x_m + a, a) \in V''$ and consider the natural mapping f defined by $x_i + a \mapsto v_{x_i+a}$ and $a \mapsto v_a$. We have $v_a > 1$, $v_{x_i+a} = x_i + a > x_i + |x_i| + 1 \geqslant 1$ and

$$v_{2a} = v_a + v_a = a + a > a + x_i + 1 = v_{x_1+a+1}.$$

By construction of V, the point $f(\boldsymbol{x})$ automatically satisfies the constraints in V reflecting comparison, addition and multiplication in W. Thus $f(\boldsymbol{x})$ defines a point in V.

Conversely, let $\boldsymbol{v} = (v_{x_1+a}, \ldots, v_{x_m+a}, v_a)$ be a point in V; for the moment we disregard the rest of the variables since they are uniquely determined by $\boldsymbol{v}$. We will show that $(v_{x_1+a} - v_a, \ldots, v_{x_m+a} - v_a) \in W$.

- If $x_i < x_j$ is a constraint in W we have $v_{x_i+a} - v_a < v_{x_j+a} - v_a$.
- If $x_i + x_j = x_k$ is a constraint in W we get

$$(v_{x_i+a} - v_a) + (v_{x_j+a} - v_a) = v_{x_i+x_j+2_a} - 2v_a = v_{x_k+a} - v_a.$$

- If $x_i \cdot x_j = x_k$ is a constraint in W we get

$$\begin{aligned}(v_{x_i+a} - v_a)(v_{x_j+a} - v_a) &= v_{x_i+a} \cdot v_{x_j+a} - v_a \cdot v_{x_i+a} - v_a \cdot v_{x_j+a} + v_a \cdot v_a \\ &= v_{x_i x_j + a x_i + a x_j + a^2} - v_{a x_i + a^2} - v_{a x_j + a^2} + v_{a^2} \\ &= v_{x_i x_j + a x_i + a x_j + a + 2a^2} - v_{a+a^2} - v_{a x_i + a x_j + 2a^2} + v_{a^2} \\ &= v_{x_k+a} - v_a.\end{aligned}$$

We furthermore have that

$$\underbrace{v_{x_i+a}}_{<v_{2a}-1} - v_a + 1 < 2v_a - 1 - v_a + 1 = v_a \quad \text{and} \quad v_a - \underbrace{v_{x_i+a}}_{>1} + 1 < v_a,$$

so $\boldsymbol{v} \in V''$ by definition. In hindsight, this shows that $V'' = \pi'(V)$ where π' is the rational bijection removing all variables except v_a and v_{x_i+a}, $1 \leqslant i \leqslant m$. Thus $W \approx V$.

Step 3. Suppose that W is defined by equations and inequalities of the form

$$x_i < x_j, \quad x_i + x_j = x_k, \quad x_i \cdot x_j = x_k,$$

and $x_i > 1$ for all $1 \leqslant i \leqslant m$. The final step is to construct a primary basic semialgebraic set $V \approx W$ of the same form, but with totally ordered variables.

In the spirit of the previous step, we first introduce variables $v_{x_i+b^i}$ in V for $1 \leqslant i \leqslant m$. These will end up corresponding with values $x_i + b^i$ where b is sufficiently large, giving a total order. We also define variables v_{b^i} for powers $1 \leqslant i \leqslant r$ up to some later determined $r \in \mathbb{N}$, and add equations $v_{b^i} = v_{b^{i-1}} \cdot v_b$. We define $v_{b^i+1} = v_{b^i} + 1$, $v_{b^i+b} = v_{b^i} + v_b$, and enforce

$$v_{b^i} < v_{b^i+1} < v_{x_i+b^i} < v_{b^i+b} < v_{b^{i+1}} \tag{4.2}$$

for all $1 \leqslant i \leqslant m$; the first and the last inequality follow from the definition of the variables, we included them for clarity. Thus all variables defined so far are totally ordered in a way reflecting their subscripts. To ensure that the gaps are large enough, requiring $v_b > 6$ suffices.

In a manner similar to step 2, the defining inequalities and equations of W should be given equivalent counterparts in V. This is now more complicated, since we need to make sure that all intermediate variables we introduce respect the natural total order.

- Suppose $x_i < x_j$ is a constraint in W. We choose an unused large-enough power $\alpha > n$ of b and introduce the variables, equations and inequalities

$$v_{b^\alpha} = v_{b^i} + v_{b^\alpha - b^i}, \qquad v_{b^\alpha} = b_{b^j} + v_{b^\alpha - b^j},$$
$$v_{x_i + b^\alpha} = v_{x^i + b^i} + v_{b^\alpha - b^i}, \quad v_{x_j + b^\alpha} = v_{x^j + b^j} + v_{b^\alpha - b^j},$$
$$v_{x_i + b^\alpha} < v_{x_j + b^\alpha}.$$

  To get the desired total order, if $i < j$ we also enforce

$$v_{b^\alpha - b^j} < v_{b^\alpha - b^i} < v_{b^\alpha} < v_{x_i + b^\alpha},$$

  which again corresponds with the subscripts. If the $i > j$, the order of $v_{b^\alpha - b^j}$ and $v_{b^\alpha - b^i}$ is swapped. If α was chosen to be large enough, we also have a natural ordering for these new variables in relation to the others.

- Suppose $x_i + x_j = x_k$ is a constraint in W. We choose three unused large-enough powers α, β, γ of b such that $n < \alpha < \beta < \gamma$, and introduce the variables and equations

$$\begin{aligned} v_{b^\alpha} &= v_{b^i} + v_{b^\alpha - b^i}, \\ v_{b^\beta} &= v_{b^j} + v_{b^\beta - b^j}, \\ v_{b^\gamma} &= v_{b^k} + v_{b^\gamma - b^k}, \end{aligned}$$

$$\begin{aligned} v_{x_i + b^\alpha} &= v_{x_i + b^i} + v_{b^\alpha - b^i}, \\ v_{x_j + b^\beta} &= v_{x_j + b^j} + v_{b^\beta - b^j}, \\ v_{x_k + b^\gamma} &= v_{x_k + b^k} + v_{b^\gamma - b^k}, \end{aligned}$$

$$\begin{aligned} v_{b^\gamma} &= v_{b^\alpha} + v_{b^\gamma - b^\alpha}, \\ v_{b^\gamma - b^\alpha} &= v_{b^\beta} + v_{b^\gamma - b^\alpha - b^\beta}, \\ v_{x_i + x_j + b^\alpha + b^\beta} &= v_{x_i + b^\alpha} + v_{x_j + b^\beta}, \\ v_{x_k + b^\gamma} &= v_{x_i + x_j + b^\alpha + b^\beta} + v_{b^\gamma - b^\alpha - b^\beta}. \end{aligned}$$

  Here the first six equations do preliminary work, and the last four model the addition. We again enforce the natural orders

$$v_{b^\alpha - b^i} < v_{b^\alpha} < v_{x_i + b^\alpha},$$
$$v_{b^\beta - b^j} < v_{b^\beta} < v_{x_j + b^\beta} < v_{x_i + x_j + b^\beta + b^\alpha},$$
$$v_{b^\gamma - b^\alpha - b^\beta} < v_{b^\gamma - b^\alpha} < v_{b^\gamma - b^k} < v_{b^\gamma} < v_{x_k + b^\gamma},$$

  and with α, β, γ chosen to be large enough these variables can be ordered in relation to the other variables according to their subscripts.

- Suppose $x_i \cdot x_j = x_k$ is a constraint in W. We choose three unused large-enough powers α, β, γ of b such that $n < \alpha < \beta < \gamma = \alpha + \beta$. We introduce the first six equations as above, *i.e.*,

$$\begin{aligned} v_{b^\alpha} &= v_{b^i} + v_{b^\alpha - b^i}, \\ v_{b^\beta} &= v_{b^j} + v_{b^\beta - b^j}, \\ v_{b^\gamma} &= v_{b^k} + v_{b^\gamma - b^k}, \\ v_{x_i + b^\alpha} &= v_{x_i + b^i} + v_{b^\alpha - b^i}, \\ v_{x_j + b^\beta} &= v_{x_j + b^j} + v_{b^\beta - b^j}, \\ v_{x_k + b^\gamma} &= v_{x_k + b^k} + v_{b^\gamma - b^k}. \end{aligned}$$

  To model multiplication, we further require

$$\begin{aligned} v_{2b^\gamma} &= v_{b^\gamma} + v_{b^\gamma}, \\ v_{x_i x_j + b^\beta x_i + b^\alpha x_j + b^\gamma} &= v_{x_i + b^\alpha} \cdot v_{x_j + b^\beta}, \\ v_{x_i x_j + b^\beta x_i + b^\alpha x_j + 3b^\gamma} &= v_{x_i x_j + b^\beta x_i + b^\alpha x_j + b^\gamma} + v_{2b^\gamma}, \\ v_{b^\beta x_i + b^\gamma} &= v_{x_i + b^\alpha} \cdot v_{b^\beta}, \\ v_{b^\alpha x_j + b^\gamma} &= v_{x_j + b^\beta} \cdot v_{b^\alpha}, \\ v_{b^\alpha x_j + b^\beta x_i + 2b^\gamma} &= v_{b^\alpha x_j + b^\gamma} + v_{b^\beta x_i + b^\gamma}, \\ v_{x_i x_j + b^\beta x_i + b^\alpha x_j + 3b^\gamma} &= v_{b^\alpha x_j + b^\beta x_i + 2b^\gamma} + v_{x_k + b^\gamma}. \end{aligned}$$

  One can again check that it is possible to enforce an order on these new variables that is consistent with their subscripts. Thus all variables are totally ordered.

This concludes the construction of V. It remains to verify that V and W are stably equivalent, which we will do similarly as before. We define the sets

$$\begin{aligned} V' &= \big\{(x_1, \ldots, x_m, b) \in \mathbb{R}^{m+1} \mid (x_1, \ldots, x_m) \in W \text{ and } b > \max\{x_1, \ldots, x_m, 6\}\big\}, \\ V'' &= \big\{(x_1 + b, \ldots, x_m + b^m, b) \in \mathbb{R}^{m+1} \mid (x_1, \ldots, x_m, b) \in V'\big\}. \end{aligned}$$

The constraint $b > \max\{x_1, \ldots, x_m, 6\}$ can be rewritten as $b > 6$ and $b > x_i$ for all $1 \leqslant i \leqslant m$, so V' is primary basic semialgebraic and projects stably onto W. Since V' and V'' are rationally equivalent it suffices to show that $V'' \approx V$, as in the diagram (4.1). In fact, we will again see that V'' and V are essentially the same sets, disregarding the additional variables of V.

Let $\boldsymbol{x} = (x_1 + b, \ldots, x_m + b^m, b) \in V''$ and consider the natural mapping f defined by $x_i + b^i \mapsto v_{x_i + b^i}$ and $b \mapsto v_b$. Since we ordered the variables of V according to their subscripts, this mapping automatically respects the total order of the variables in V. Similarly one can check that each constraint in V reflecting comparison, addition or multiplication in W is defined to respect the subscripts. Thus $f(\boldsymbol{x})$ defines a point in V.

Conversely, let $\boldsymbol{v} = (v_{x_1+b}, \ldots, v_{x^m+b^m}, v_b)$ be a point in V; for the moment we disregard the rest of the variables since they are uniquely determined by $\boldsymbol{v}$. We will show that $(v_{x_1+b} - v_b, \ldots, v_{x_m+b^m} - v_{b^m}) \in W$.

- By the inequalities in (4.2) we have $v_{x_i+b^i} > v_{b^i+1} = v_{b^i} + 1$, implying that $v_{x_i+b^i} - v_{b^i} > 1$ for all $1 \leqslant i \leqslant m$.
- Suppose $x_i < x_j$ is a constraint in W. We have

$$v_{x_i+b^i} - v_{b^i} = v_{x_i+b^\alpha} - \underbrace{v_{b^\alpha-b^i} - v_{b^i}}_{=v_{b^\alpha}} < v_{x_j+b^\alpha} - v_{b^\alpha} = v_{x_j+b^j} - v_{b^j}.$$

- Suppose $x_i + x_j = x_k$ is a constraint in W. We get

$$\begin{aligned}(v_{x_i+b^i} - v_{b^i}) + (v_{x_j+b^j} - v_{b^j}) &= v_{x_i+b^\alpha} - v_{b^\alpha} + v_{x_j+b^\beta} - v_{b^\beta} \\ &= v_{x_i+x_j+b^\alpha+b^\beta} + v_{b^\gamma-b^\alpha-b^\beta} - v_{b^\gamma} \\ &= v_{x_k+b^\gamma} - v_{b^\gamma} \\ &= v_{x_k+b^k} - v_{b^k}.\end{aligned}$$

- Suppose $x_i \cdot x_j = x_k$ is a constraint in W. We get

$$\begin{aligned}(v_{x_i+b^i} - v_{b^i})(v_{x_j+b^j} - v_{b^j}) &= (v_{x_i+b^\alpha} - v_{b^\alpha})(v_{x_j+b^\beta} - v_{b^\beta}) \\ &= v_{x_i+b^\alpha} v_{x_j+b^\beta} - v_{b^\alpha} v_{x_j+b^\beta} - v_{b^\beta} v_{x_i+b^\alpha} + v_{b^\alpha} v_{b^\beta} \\ &= v_{x_i x_j b^\alpha x_j + b^\beta x_i + b^\gamma} - v_{b^\alpha x_j + b^\gamma} - v_{b^\beta x_i + b^\gamma} + v_{b^\gamma} \\ &= v_{x_i x_j + b^\alpha x_j + b^\beta x_i + 3b^\gamma} - v_{2b^\gamma} - v_{b^\alpha x_j + b^\beta x_i + 2b^\gamma} + v_{b^\gamma} \\ &= v_{x_k+b^\gamma} - v_{b^\gamma} \\ &= v_{x_k+b^k} - v_{b^k}.\end{aligned}$$

Thus the point $(v_{x_1+b} - v_b, \ldots, v_{x_m+b^m} - v_{b^m})$ satisfies all defining inequalities and equations of W.

Lastly, notice that by (4.2) we get $v_{x_i+b^i} < v_{b^i+b} = v_{b^i} + v_b$, and therefore $v_b > v_{x^i+b^i} - v_{b^i}$ for all $1 \leqslant i \leqslant m$. Since furthermore $v_b > 6$, we finally get $\boldsymbol{v} \in V''$. Thus V and V'' are rationally equivalent by the homeomorphism $\pi' : V \longrightarrow V''$ which removes all additional variables of V.

Conclusion. We showed that for any primary basic semialgebraic set $W \subseteq \mathbb{R}^m$ there exists a semialgebraic set $V \subseteq \mathbb{R}^n$ with variables $x_1, \ldots, x_n$, a total order

$$1 < x_1 < \ldots < x_m$$

and all defining equations of the form

$$x_i + x_j = x_k \quad \text{or} \quad x_i \cdot x_j = x_k,$$

such that $V \approx W$. The set V defines a Shor normal form $\mathcal{S} = (n, \boldsymbol{A}, \boldsymbol{M})$ by

$$\begin{aligned} (i,j,k) \in \boldsymbol{A} &\iff x_i + x_j = x_k \text{ is a constraint,} \\ (i,j,k) \in \boldsymbol{M} &\iff x_i \cdot x_j = x_k \text{ is a constraint} \end{aligned}$$

for $(i,j,k) \in \{1,\ldots,n\}^3$, since the total order of the variables automatically implies that $i \leqslant j < k$ whenever $(i,j,k) \in \boldsymbol{A} \cup \boldsymbol{M}$. Clearly $V = V(\mathcal{S})$, proving the theorem. □

5 Operations on polytopes

In order to prove the universality theorem, we need a set of constructions called *basic building blocks*. All of our constructions are based on *pyramids*, *prisms*, *tents*, *connected sums* and *Lawrence extensions*. These blocks will eventually be glued together to create complicated polytopes that model entire systems of polynomial equations and inequalities. For a set $X = \{1, \dots, n\}$ we denote by X' the copy $\{1', \dots, n'\}$ of X.

Definition 5.1. Let P be a combinatorial d-polytope on X and $y \notin X$. The *pyramid* over P is the combinatorial $(d+1)$-polytope

$$\mathrm{pyr}(P, y) = \{F \cup \{y\} \mid F \in P\} \cup \{X\}.$$

A pyramid over a d-polytope $\boldsymbol{P}$ can be realized by embedding $\boldsymbol{P}$ in a hyperplane H of $\mathbb{R}^{d+1}$, choosing a point $\boldsymbol{p} \notin H$, and taking the convex hull of $\boldsymbol{P} \cup \{\boldsymbol{p}\}$.

Definition 5.2. Let P be a combinatorial d-polytope on X. The *prism* over P is the combinatorial d-polytope

$$\mathrm{prism}(P) = \{F \cup F' \mid F \in P\} \cup \{X, X'\}.$$

A prism over a d-polytope $\boldsymbol{P}$ can be realized by embedding $\boldsymbol{P}$ in a hyperplane of $\mathbb{R}^{d+1}$, projecting it onto the same hyperplane translated in a direction orthogonal to itself, and taking the convex hull of $\boldsymbol{P}$ and its projection.

Definition 5.3. Let $P = \{\{1, 2\}, \{2, 3\}, \dots, \{n, 1\}\}$ be a combinatorial n-gon with $n \geqslant 4$, let $e = \{i, i+1\}$ and $f = \{j, j+1\}$ be two nonadjacent edges, and let a, b be new labels. We define a *tent* over P as the combinatorial 3-polytope

$$\begin{aligned}\mathrm{tent}^{e,f}(P, a, b) = {} & \{\{k, k+1, a\} \mid i+1 \leqslant k \leqslant j-1\} \\ & \cup \{\{k, k+1, b\} \mid j+1 \leqslant k \leqslant i-1\} \\ & \cup \{\{i, i+1, a, b\}, \{j, j+1, a, b\}\} \\ & \cup \{1, \dots, n\}.\end{aligned}$$

We can realize a tent by embedding an n-gon in the xy-plane with two nonadjacent edges parallel to the x-axis, adding points $(0, 0, 1)$ and $(1, 0, 1)$, and taking the convex hull. Figure 7 shows a pyramid, a prism and a tent over a pentagon.

We will occasionally refer to pyramids and tents without specifying the labels of the new points, *e.g.*, $\mathrm{pyr}(P)$ and $\mathrm{tent}^{1,5}(P)$. We do this in cases where only the basis P of the construction is of interest. Furthermore, we may talk about pyramids, prisms and tents over realizations of polytopes instead of over their combinatorial counterparts. Since $\operatorname{pyr}(P)$, $\operatorname{prism}(P)$ and $\operatorname{tent}^{i,j}(P)$ are always realizable when P is polytopal, this causes no problems.

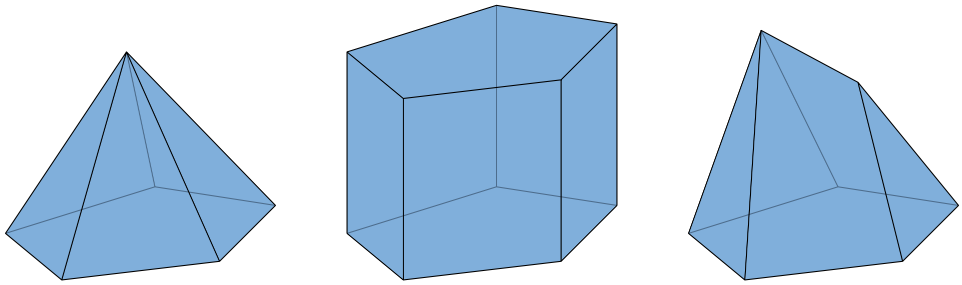


Figure 7: A pyramid, prism and tent over a pentagon.

Constructing a pyramid is the simplest way to lift a polytope to higher dimensions. Since the base of a d-dimensional pyramid spans a hyperplane of $\mathbb{R}^d$, any affine basis of the pyramid must contain its tip. This allows us to directly relate the realization space of a pyramid with the realization space of its basis. We will use this fact in the proof of the universality theorem to identify the realization space of a 4-polytope with the realization space of one of its 2-faces.

Lemma 5.4. *Let $\boldsymbol{P}$ be a d-polytope and $\boldsymbol{p} \notin \operatorname{aff}(\boldsymbol{P})$. Then $\mathcal{R}(\boldsymbol{P})$ and $\mathcal{R}(\operatorname{pyr}(\boldsymbol{P}, \boldsymbol{p}))$ are isomorphic.*

Proof. Any affine basis of $\operatorname{pyr}(\boldsymbol{P}, \boldsymbol{p})$ must contain $\boldsymbol{p}$. Hence the mapping

$$\begin{array}{ccc} \mathcal{R}(\boldsymbol{P}) & \longrightarrow & \mathcal{R}(\operatorname{pyr}(\boldsymbol{P}, \boldsymbol{p})) \\ (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) & \longmapsto & (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{p}) \end{array}$$

identifies $\mathcal{R}(\boldsymbol{P})$ with $\mathcal{R}(\operatorname{pyr}(\boldsymbol{P}, \boldsymbol{p}))$. $\square$

5.1 Connected sums

The connected sum operation describes gluing polytopes together. This models the second part of Definition 2.7 in the realizable case.

Let P and Q be combinatorial d-polytopes on X and Y, respectively. We assume that P and Q have a common facet $F \in P \cap Q$ such that, after a possible relabeling of the indices, we have $X \cap Y = F$ and $P|_F = Q|_F$. In the realizable case, this means that P and Q share a facet F of the same combinatorial type.

Definition 5.5. Let P and Q be combinatorial d-polytopes as characterized above.

- The *connected sum* $P \heartsuit_F Q$ of P and Q is the combinatorial d-polytope

$$P \heartsuit_F Q = (P \cup Q) \setminus \{F\}.$$

- If $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are d-polytopes such that $\mathrm{FL}(\boldsymbol{P}) = P$, $\mathrm{FL}(\boldsymbol{Q}) = Q$ and $\mathrm{FL}(\boldsymbol{R}) = P \heartsuit_F Q$, then $\boldsymbol{R}$ is a *connected sum* of $\boldsymbol{P}$ and $\boldsymbol{Q}$ if

$$\boldsymbol{R}|_X = \tau_P(\boldsymbol{P}) \quad \text{and} \quad \boldsymbol{R}|_Y = \tau_Q(\boldsymbol{Q})$$

  for some admissible projective transformations τ_P and τ_Q.

In general, we cannot section a realization of a connected sum into realizations of its original components. For example, there are many realizations $\boldsymbol{R}$ of $\boldsymbol{P} \heartsuit_F \boldsymbol{Q}$ from Figure 8 such that $\mathrm{aff}(\boldsymbol{R}|_F)$ is three-dimensional, since the points corresponding to F need not remain coplanar. In this thesis we are only interested in connected sums for which this sectioning into the original components is always possible.

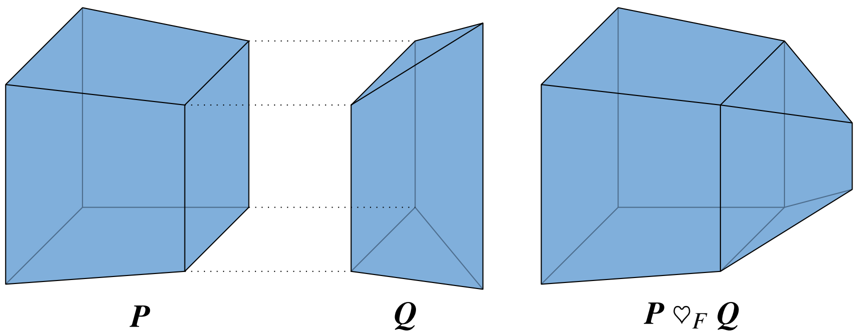


Figure 8: A connected sum of two 3-polytopes.

Definition 5.6. The *k-skeleton* of a d-polytope $\boldsymbol{P}$ is the polyhedral complex of its k-faces (cf. [17]). We call $\boldsymbol{P}$ *necessarily flat* if every polyhedral embedding of its $(d-1)$-skeleton in $\mathbb{R}^n$ for $n > d$ has at most affine dimension d.

For the reason noted above, the only necessarily flat polygon is the triangle: in all other cases the vertices of the polygon may span more than two dimensions when embedded in, *e.g.*, $\mathbb{R}^3$. In contrast, already for polyhedra there are many more possibilities.

Lemma 5.7. *Pyramids, prisms and tents over n-gons are necessarily flat.*

Proof. A pyramid $\mathrm{pyr}(P, y)$ contains P as a facet and only one additional point, so its $(d-1)$-skeleton is at most d-dimensional.

Suppose $\boldsymbol{P} \subseteq \mathbb{R}^m$ is a realization of the 2-skeleton of a prism over an n-gon $G = \{1, \dots, n\}$. By convexity of G the points $1, 2, 3$ are not collinear, so the set $\{1, 2, 3, 1'\}$ is an affine basis of $\boldsymbol{P}$. Since $\{1, 1', 2, 2'\}$ and $\{2, 2', 3, 3'\}$ are facets of $\boldsymbol{P}$,

$$\boldsymbol{p}_{2'} \in \operatorname{aff}(\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_{1'}\}) \quad \text{and} \quad \boldsymbol{p}_{3'} \in \operatorname{aff}(\{\boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_{2'}\}).$$

Furthermore,

$$\{\boldsymbol{p}_4, \dots, \boldsymbol{p}_n\} \subseteq \operatorname{aff}(\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\}) \quad \text{and} \quad \{\boldsymbol{p}_{4'}, \dots, \boldsymbol{p}_{n'}\} \subseteq \operatorname{aff}(\{\boldsymbol{p}_{1'}, \boldsymbol{p}_{2'}, \boldsymbol{p}_{3'}\}).$$

Hence $\operatorname{aff}(\boldsymbol{P}) = \operatorname{aff}(\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_{1'}\})$ which is at most three-dimensional.

Finally, suppose $\boldsymbol{P} \subseteq \mathbb{R}^m$ is a realization of the 2-skeleton of $\operatorname{tent}^{e_1,e_2}(G, a, b)$ with respect to edges e_1 and e_2. All points of $\boldsymbol{P}$ lie on the plane supporting G or on the plane supported by e_1, a and b. Both planes contain e_1, so their intersection is one-dimensional. Hence $\boldsymbol{P}$ has at most affine dimension 3. $\square$

Necessarily flat faces are useful. Assume that $\boldsymbol{R}$ is a realization of $P \heartsuit_F Q$ for combinatorial d-polytopes P and Q, and that F is necessarily flat. It is then possible to section $\boldsymbol{R}$ by the affine hyperplane spanned by F into realizations of P and Q, as we desired. If P and Q impose restrictions on the realizability of F, *i.e.*, ensure that the shape of F is not arbitrarily prescribable, this means that the same restrictions must apply on $\boldsymbol{R}$ as well.

Suppose for example that P and Q each force the collinearity of a distinct set of at least three points on the necessarily flat face F. Then both sets of points must be collinear in any realization of $P \heartsuit_F Q$, which hints at $\mathcal{R}(\boldsymbol{R})$ having more complex structure. This is the fundamental fact which allows us to construct increasingly complicated realization spaces.

The question remains whether a connected sum is realizable. We find that it indeed always is, given that the connective facets are projectively equivalent.

Lemma 5.8. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be realizations of the combinatorial d-polytopes P and Q respectively. If the facets $\boldsymbol{P}|_F$ and $\boldsymbol{Q}|_F$ are projectively equivalent, then there exists a projective transformation τ such that $\boldsymbol{P} \cup \tau(\boldsymbol{Q})$ is a realization of $\boldsymbol{P} \heartsuit_F \boldsymbol{Q}$.*

Proof. We may assume that $\boldsymbol{Q}$ satisfies

$$\operatorname{conv}(\boldsymbol{Q}) \cap \operatorname{conv}(\boldsymbol{P}) = \operatorname{conv}(\boldsymbol{Q}|_F) = \operatorname{conv}(\boldsymbol{P}|_F),$$

since $\boldsymbol{Q}|_F$ and $\boldsymbol{P}|_F$ are projectively equivalent.

Denote by $H_0, \dots, H_r$ the affine hyperplanes defining the facets of $\boldsymbol{Q}$, so that H_0 defines F. Let $h_0, \dots, h_r \in \left(\mathbb{R}^{d+1}\right)^*$ denote the corresponding linear forms, with the positive half-space of each hyperplane containing the interior of $\boldsymbol{Q}$. We choose a point $\boldsymbol{q}$ on the negative side of H_0 and the positive side of $H_1, \dots, H_r$. By convexity of $\operatorname{conv}(\boldsymbol{Q})$ this region is nonempty. Similarly we choose a point $\boldsymbol{p}$ for $\boldsymbol{P}$ using the hyperplanes defining $\operatorname{facets}(\boldsymbol{P})$ with positive half-spaces containing the interior of $\boldsymbol{P}$.

Now choose an affine basis B of the facet F and consider the projective transformation τ defined by the matrix

$$T = \left[\boldsymbol{Q}^{\mathrm{hom}}|_B \quad -\boldsymbol{p}^{\mathrm{hom}}\right] \left[\boldsymbol{Q}^{\mathrm{hom}}|_B \quad \boldsymbol{q}^{\mathrm{hom}}\right]^{-1} \tag{5.1}$$

and standard dehomogenization by the hyperplane $x_{d+1} = 1$.

We define a cone C by the affine hyperplanes $\{\operatorname{aff}(\boldsymbol{Q}|_G \cup \{\boldsymbol{q}\}) \mid G \in \operatorname{facets}(F)\}$; in other words, C is a cone centered at $\boldsymbol{q}$ with defining hyperplanes spanned by $\boldsymbol{q}$ and facets of F. Let $H = \operatorname{aff}(\boldsymbol{Q}|_G \cup \{\boldsymbol{q}\})$ be such a hyperplane, *i.e.*,

$$H = \left\{\lambda \boldsymbol{q} + \sum_{i \in G} \lambda_i \boldsymbol{q}_i \mid \lambda, \lambda_i \in \mathbb{R} \text{ for } i \in G \text{ and } \lambda + \sum_{i \in G} \lambda_i = 1\right\}.$$

Since at least two facets of $\boldsymbol{Q}$ intersect to form G, there exists some $j \in \{1, \dots, r\}$ with $h_j(\boldsymbol{q}_i^{\mathrm{hom}}) = 0$ for all $i \in G$. For any $\boldsymbol{x} = \lambda \boldsymbol{q} + \sum_{i \in G} \lambda_i \boldsymbol{q}_i \in H$ this implies

$$h_j\left(\boldsymbol{x}^{\mathrm{hom}}\right) = \lambda h_j\left(\boldsymbol{q}^{\mathrm{hom}}\right) + \sum_{i \in G} \lambda_i \underbrace{h_j\left(\boldsymbol{q}_i^{\mathrm{hom}}\right)}_{=0} = \lambda \underbrace{h_j\left(\boldsymbol{q}^{\mathrm{hom}}\right)}_{>0} \text{ and}$$

$$h_0\left(\boldsymbol{x}^{\mathrm{hom}}\right) = \lambda h_0\left(\boldsymbol{q}^{\mathrm{hom}}\right) + \sum_{i \in G} \lambda_i \underbrace{h_0\left(\boldsymbol{q}_i^{\mathrm{hom}}\right)}_{=0} = \lambda \underbrace{h_0\left(\boldsymbol{q}^{\mathrm{hom}}\right)}_{<0},$$

by definition of the linear forms $h_0, \dots, h_r$. Hence $h_0(\boldsymbol{x}^{\mathrm{hom}}) > 0$ only if $h_j(\boldsymbol{x}^{\mathrm{hom}}) < 0$ for any $\boldsymbol{x} \in H$, showing that the defining hyperplanes of C never intersect the interior of $\boldsymbol{Q}$. It is clear that C contains some point from the interior of $\boldsymbol{Q}$, which thus implies that $\operatorname{conv}(\boldsymbol{Q}) \subseteq C$.

For any $\boldsymbol{x} \in C$ we can write

$$\boldsymbol{x} = \boldsymbol{q} + \sum_{i \in F} \lambda_i (\boldsymbol{q}_i - \boldsymbol{q}) = \left(1 - \sum_{i \in F} \lambda_i\right) \boldsymbol{q} + \sum_{i \in F} \lambda_i \boldsymbol{q}_i$$

for some coefficients $\lambda_i \geqslant 0$, so

$$T\boldsymbol{x}^{\mathrm{hom}} = \left(1 - \sum_{i \in F} \lambda_i\right) \underbrace{T\boldsymbol{q}^{\mathrm{hom}}}_{=-\boldsymbol{p}^{\mathrm{hom}}} + \sum_{i \in F} \lambda_i \underbrace{T\boldsymbol{q}_i^{\mathrm{hom}}}_{=\boldsymbol{q}_i^{\mathrm{hom}}}$$

$$= \left(\sum_{i \in F} \lambda_i - 1 \right) \boldsymbol{p}^{\text{hom}} + \sum_{i \in F} \lambda_i \boldsymbol{q}_i^{\text{hom}},$$

implying that

$$\tau(\boldsymbol{x}) = \left(\sum_{i \in F} \lambda_i - 1 \right) \boldsymbol{p} + \sum_{i \in F} \lambda_i \boldsymbol{q}_i. \tag{5.2}$$

By definition of the convex hull

$$\sum_{i \in F} \lambda_i \begin{cases} < 1 & \text{if} \quad \boldsymbol{x} \in \operatorname{conv}(\boldsymbol{Q}|_F \cup \{\boldsymbol{q}\}) \setminus \operatorname{conv}(\boldsymbol{Q}|_F), \\ = 1 & \text{if} \quad \boldsymbol{x} \in \operatorname{conv}(\boldsymbol{Q}|_F), \\ > 1 & \text{if} \quad \boldsymbol{x} \in C \setminus \operatorname{conv}(\boldsymbol{Q}|_F \cup \{\boldsymbol{q}\}). \end{cases}$$

Hence if $\boldsymbol{x} \in \operatorname{conv}(\boldsymbol{Q})$ we have $\sum_{i \in F} \lambda_i - 1 \geqslant 0$, so $\tau(\boldsymbol{x}) \in \operatorname{conv}(\boldsymbol{Q}|_F \cup \{\boldsymbol{p}\})$ by the formula in (5.2). Since τ furthermore fixes $\boldsymbol{Q}|_F$, the full point configuration $\boldsymbol{P} \cup \tau(\boldsymbol{Q})$ now has all points in extreme positions due to our choice of $\boldsymbol{p}$. This concludes the proof. □

Example 5.9. Consider the unit cube $\boldsymbol{Q} = (0, 1)^3$ along with points $\boldsymbol{q} = (2, 1/2, 1/2)$ and $\boldsymbol{p} = (-1, 1/2, 1/2)$. Choosing the facet F closest to $\boldsymbol{q}$, these satisfy the assumptions in the proof of Lemma 5.8. Figure 9 shows the projective transformation proposed in (5.1). To be precise, we get

$$\tau(\boldsymbol{Q}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1/3 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1 & 1/3 & 2/3 & 1/3 & 2/3 \\ 0 & 0 & 1 & 1 & 1/3 & 1/3 & 2/3 & 2/3 \end{bmatrix}.$$

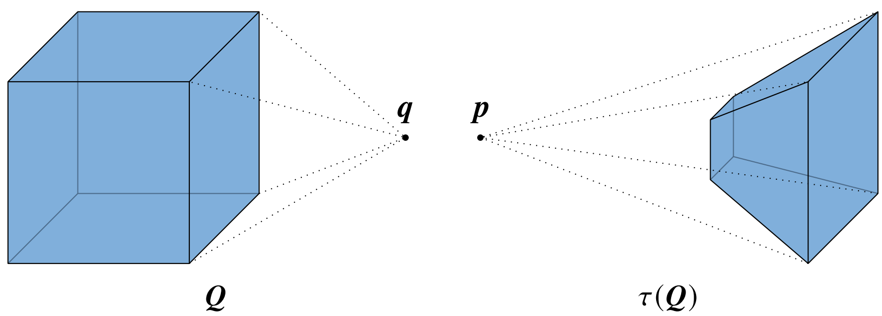


Figure 9: Mapping a cube into a pyramid.

5.2 Lawrence extension

So-called *Lawrence polytopes*, or *Lawrence extensions* of polytopes, were originally discussed in [4]. The idea was based upon a related notion for oriented matroids, introduced by Lawrence in 1980 (unpublished). The version used by Richter-Gebert is more general.

Definition 5.10. Let $\boldsymbol{P} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_n) \in \mathbb{R}^{d\times n}$ be a d-cone and $\boldsymbol{q} \in \mathbb{R}^d$ an additional point such that $\boldsymbol{q} \notin \operatorname{pos}(\boldsymbol{P})$. The *single-point Lawrence extension* $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ is the $(n + 2) \times (d + 1)$ homogeneous point configuration

$$\Lambda(\boldsymbol{P}, \boldsymbol{q}) = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{q} & -\boldsymbol{q} \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}_1 & \cdots & \boldsymbol{p}_n & \boldsymbol{q} & -\boldsymbol{q} \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}. \tag{5.3}$$

After some work it will turn out that single-point Lawrence extensions are cones. As in the definition, let $\boldsymbol{P} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_n) \in \mathbb{R}^{d\times n}$ be a d-cone and $\boldsymbol{q} \in \mathbb{R}^d$ an additional point such that $\boldsymbol{q} \notin \operatorname{pos}(\boldsymbol{P})$. We will label the points of $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ by $1, \dots, n, \overline{y}, \underline{y}$ in the order of (5.3).

It is possible to dehomogenize $\Lambda(\boldsymbol{P}, \boldsymbol{q})$, since a linear form $(h_1, \dots, h_d, N) \in \left(\mathbb{R}^{d+1}\right)^*$ is strictly positive on all points of $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ whenever $(h_1, \dots, h_d)$ is strictly positive on $\boldsymbol{P}$ and N is sufficiently large. It still remains to show that all points of $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ lie in extreme positions; in fact, we will get a full characterisation of the facets of $\Lambda(\boldsymbol{P}, \boldsymbol{q})$.

For a linear form $h \in \left(\mathbb{R}^d\right)^*$, denote by $\boldsymbol{P}_h^+$, $\boldsymbol{P}_h^-$ and $\boldsymbol{P}_h^0$ the indices of $\boldsymbol{P}$ on which h is positive, negative and zero respectively. We call h a *cocircuit* if $\boldsymbol{P}_h^0$ is maximal, *i.e.*, there is no nonzero form $g \in \left(\mathbb{R}^d\right)^*$ such that $\boldsymbol{P}_h^0 \subsetneq \boldsymbol{P}_g^0$. A linear form h is *external* to $\boldsymbol{P}$ if $\boldsymbol{P}_h^- = \varnothing$.

Lemma 5.11. *If h is a cocircuit of $\boldsymbol{P} \cup \{\boldsymbol{q}\}$ and external to $\boldsymbol{P}$, then*

$$\boldsymbol{P}_h^0 \cup \begin{cases} \{\overline{y}, \underline{y}\} & \text{if } \ h(\boldsymbol{q}) = 0, \\ \{\overline{y}\} & \text{if } \ h(\boldsymbol{q}) < 0, \\ \{\underline{y}\} & \text{if } \ h(\boldsymbol{q}) > 0 \end{cases} \tag{5.4}$$

is a facet of $\Lambda(\boldsymbol{P}, \boldsymbol{q})$. The only facet of $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ which is not of this form is $\{1, \dots, n\}$.

Proof. We denote $\Lambda = \Lambda(\boldsymbol{P}, \boldsymbol{q}) = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{x}_{\overline{y}}, \boldsymbol{x}_{\underline{y}})$. The facets of Λ are defined by cocircuits $h \in \left(\mathbb{R}^{d+1}\right)^*$ that are external to Λ, meaning that that $\Lambda_h^- = \varnothing$ and Λ_h^0 is maximal. Let $h = (h_1, \dots, h_{d+1})$ be a linear form satisfying $\Lambda_h^- = \varnothing$ and denote $h' = (h_1, \dots, h_d) \in \left(\mathbb{R}^d\right)^*$. Assuming $h' \neq 0$, we define $g = (h_1, \dots, h_d, |h'(\boldsymbol{q})|)$.

Now, if $h(\boldsymbol{x}_i) = 0$ for $1 \leqslant i \leqslant n$ then

$$g(\boldsymbol{x}_i) = h'(\boldsymbol{p}_i) + |h'(\boldsymbol{q})| \cdot 0 = h(\boldsymbol{x}_i) = 0.$$

Furthermore, if $h(\boldsymbol{x}_{\overline{y}}) = h'(\boldsymbol{q}) + h_{d+1} = 0$ then $h_{d+1} \geqslant 0$; otherwise we would have $h'(\boldsymbol{q}) > 0$ implying that $h(\boldsymbol{x}_{\underline{y}}) = -h'(\boldsymbol{q}) + h_{d+1} < 0$, contradicting our assumption $\Lambda_h^- = \varnothing$. It follows that $h'(\boldsymbol{q}) \leqslant 0$, so

$$g(\boldsymbol{x}_{\overline{y}}) = h'(\boldsymbol{q}) + |h'(\boldsymbol{q})| = 0.$$

Similarly if $h(\boldsymbol{x}_{\underline{y}}) = 0$ then also $g(\boldsymbol{x}_{\underline{y}}) = 0$, so we conclude that $\Lambda_h^0 \subseteq \Lambda_g^0$. Thus any facet-defining linear form $h = (h_1 \dots, h_{d+1})$ of Λ such that $h' = (h_1 \dots, h_d) \neq 0$, must be of the form $h_{d+1} = |h'(\boldsymbol{q})|$. Clearly Λ_h^0 is maximal when h' is a cocircuit of $\boldsymbol{P} \cup \{\boldsymbol{q}\}$, and h' must be external to $\boldsymbol{P}$ to satisfy $\Lambda_h^0 = \varnothing$. Such a linear form h with restriction h' gives exactly the facet described in (5.4).

The rest of the facets of Λ are defined by linear forms with $h' = 0$, so the only possibility is (a positive multiple of) $h = (0, \dots, 0, 1)$. This gives $h(\boldsymbol{x}_i) = 0$ for all $1 \leqslant i \leqslant n$ and $h(\boldsymbol{x}_{\overline{y}}) = h(\boldsymbol{x}_{\underline{y}}) = 1$, so h defines the facet $\{1, \dots, n\}$. □

Lemma 5.12. *The single-point Lawrence extension $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ is a $(d+1)$-cone.*

Proof. We will show that all points of $\Lambda(\boldsymbol{P}, \boldsymbol{q}) = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{x}_{\overline{y}}, \boldsymbol{x}_{\underline{y}})$ lie in extreme positions. For an index $1 \leqslant i \leqslant n$, let $h_1, \dots, h_d \in (\mathbb{R}^d)^*$ be d distinct facet-defining linear forms of $\boldsymbol{P}$ such that $\bigcap_{j=1}^d \boldsymbol{P}_{h_j}^0 = \{i\}$. By Lemma 5.11 $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ has facets

$$F_j = \boldsymbol{P}_{h_j}^0 \cup \begin{cases} \{\overline{y}, \underline{y}\} & \text{if} \quad h(\boldsymbol{q}) = 0, \\ \{\overline{y}\} & \text{if} \quad h(\boldsymbol{q}) < 0, \\ \{\underline{y}\} & \text{if} \quad h(\boldsymbol{q}) > 0 \end{cases}$$

for all $1 \leqslant j \leqslant n$ and a facet $\{1, \dots, n\}$, so we can express

$$\{1, \dots, n\} \cap \bigcap_{j=1}^{d} F_j = \bigcap_{j=1}^{d} \boldsymbol{P}_{h_j}^0 = \{i\}.$$

Therefore $\boldsymbol{x}_i$ is a vertex of $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ for each $1 \leqslant i \leqslant n$.

To show that $\boldsymbol{x}_{\overline{y}}$ and $\boldsymbol{x}_{\underline{y}}$ lie in extreme positions, we resort to the definition of cones. Since $\operatorname{pos}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \subseteq \{(x_1 \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid x_{d+1} = 0\}$ and $\boldsymbol{x}_{\underline{y}}$ lies in the hyperplane $x_{d+1} = 1$, $\boldsymbol{x}_{\underline{y}} \notin \operatorname{pos}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$. Thus

$$\operatorname{pos}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \neq \operatorname{pos}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{x}_{\underline{y}}).$$

Suppose that $\boldsymbol{x}_{\overline{y}} \in \operatorname{pos}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{x}_{\underline{y}})$. Then we can express $\boldsymbol{x}_{\overline{y}} = \lambda \boldsymbol{x}_{\underline{y}} + \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i$ for some coefficients $\lambda, \lambda_1, \dots, \lambda_n \geqslant 0$. But we must have $\lambda = 1$, so restricting to the first d coordinates this implies that

$$\boldsymbol{q} = -\boldsymbol{q} + \sum_{i=1}^{n} \lambda_i \boldsymbol{p}_i \iff \boldsymbol{q} = \sum_{i=1}^{n} \frac{\lambda_i}{2} \boldsymbol{p}_i,$$

which contradicts our assumption $\boldsymbol{q} \notin \operatorname{pos}(\boldsymbol{P})$. Hence $\boldsymbol{x}_{\overline{y}} \notin \operatorname{pos}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n, \boldsymbol{x}_{\underline{y}})$, and by definition $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ is a $(d+1)$-cone. □

Since single-point Lawrence extensions of cones are cones themselves, it makes sense to speak of single-point Lawrence extensions of a polytope $\boldsymbol{P}$ as the corresponding Lawrence extension of the associated cone of $\boldsymbol{P}$. As discussed earlier the resulting cone can then be dehomogenized, giving an interpretation for the Lawrence extension as a polytope. Note that the specific hyperplane used for dehomogenization does not affect the combinatorial type of the polytope, as any two choices are related by a projective transformation (cf. [28, Section 2]).

In some cases it is useful to consider Lawrence extensions of several points. The above results simplify this generalization.

Definition 5.13. Let $\boldsymbol{P} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_n) \in \mathbb{R}^{d \times n}$ be a d-cone and $\boldsymbol{Q} = (\boldsymbol{q}_1, \dots, \boldsymbol{q}_m) \in \mathbb{R}^{d \times m}$ be additional points such that $\boldsymbol{q}_i \notin \operatorname{pos}(\boldsymbol{P})$ for all $1 \leqslant i \leqslant m$. The *Lawrence extension* $\Lambda(\boldsymbol{P}, \boldsymbol{Q})$ is the $(d+m) \times (n+2m)$ homogeneous point configuration defined recursively by

$$\Lambda(\boldsymbol{P}, \boldsymbol{Q}) = \Lambda\Big(\Lambda\big(\boldsymbol{P}, \boldsymbol{Q}|_{\{1,\dots,m-1\}}\big), \boldsymbol{q}_m \times 0^{m-1}\Big);$$

the point $\boldsymbol{q}_m$ is padded with zeros to keep dimensions consistent. With some reordering, we explicitly get

$$\Lambda(\boldsymbol{P}, \boldsymbol{Q}) = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{Q} & -\boldsymbol{Q} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{p}_1 & \cdots & \boldsymbol{p}_n & \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_m & -\boldsymbol{q}_1 & \cdots & -\boldsymbol{q}_m \\ 0 & \cdots & 0 & 1 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 1 \end{bmatrix}. \tag{5.5}$$

We will label the points of a Lawrence extension by $1, \dots, n, \overline{1}, \dots, \overline{m}, \underline{1}, \dots, \underline{m}$ in the order of (5.5). We denote $X = \{1, \dots, n\}$, $\overline{Y} = \{\overline{1}, \dots, \overline{m}\}$ and $\underline{Y} = \{\underline{1}, \dots, \underline{m}\}$. By Lemmas 5.11 and 5.12 the following result follows easily from the recursive definition of Lawrence extensions.

Lemma 5.14. *Let $\boldsymbol{P} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) \in \mathbb{R}^{d\times n}$ be a d-cone and $\boldsymbol{Q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_m) \in \mathbb{R}^{d\times m}$ be additional points such that $\boldsymbol{q}_i \notin \operatorname{pos}(\boldsymbol{P})$ for all $1 \leqslant i \leqslant m$. If h is a cocircuit of $\boldsymbol{P} \cup \boldsymbol{Q}$ and external to $\boldsymbol{P}$, then*

$$\boldsymbol{P}_h^0 \cup \left\{\bar{i} \mid i \notin \boldsymbol{Q}_h^+\right\} \cup \left\{\underline{i} \mid i \notin \boldsymbol{Q}_h^-\right\} \tag{5.6}$$

is a facet of $\Lambda(\boldsymbol{P}, \boldsymbol{Q})$. The rest of the facets of $\Lambda(\boldsymbol{P}, \boldsymbol{Q})$ are given by

$$(X \cup \overline{Y} \cup \underline{Y}) \setminus \left\{\bar{i}, \underline{i}\right\} \quad \textit{for} \quad 1 \leqslant i \leqslant m. \tag{5.7}$$

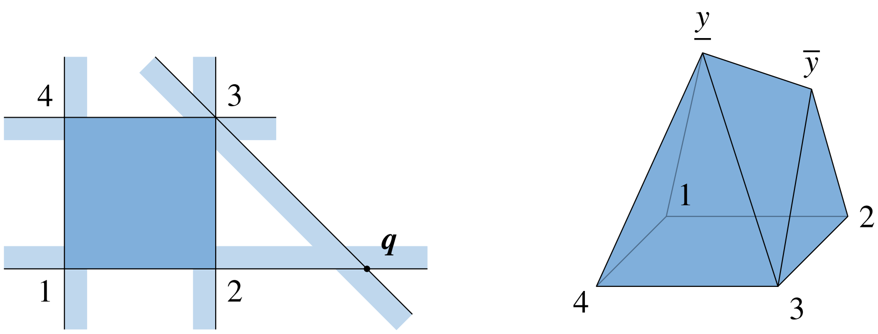
Furthermore, $\Lambda(\boldsymbol{P}, \boldsymbol{Q})$ is a $(d+m)$-cone.

Example 5.15. Consider the unit square $\boldsymbol{P} = (0,1)^2$ and the point $\boldsymbol{q} = (2,0)$. Since $\boldsymbol{q}^{\text{hom}} \notin \operatorname{pos}(\boldsymbol{P}^{\text{hom}})$, the Lawrence extension $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ is defined. In Figure 10a we can see the five affine cocircuits of $\boldsymbol{P} \cup \{\boldsymbol{q}\}$ that are external to $\boldsymbol{P}$. The positive half-spaces of the cocircuits are indicated by a lighter shade of blue.

By Lemma 5.11 it follows that $\Lambda(\boldsymbol{P}, \boldsymbol{q})$ has facets

$$\underbrace{\{1,2,\bar{y},\underline{y}\}, \quad \{1,4,\underline{y}\}, \quad \{2,3,\bar{y}\}, \quad \{3,4,\underline{y}\}, \quad \{3,\bar{y},\underline{y}\},}_{\text{type (5.6)}} \quad \underbrace{\{1,2,3,4\}}_{\text{type (5.7)}}.$$

Figure 10b shows a polytope with these facets, *i.e.*, a realization of $\Lambda(\boldsymbol{P}, \boldsymbol{q})$.



(a) Cocircuits. (b) $\Lambda(\boldsymbol{P}, \boldsymbol{q})$.

Figure 10: A single-point Lawrence extension of the unit square.

In Figure 11 we have another example of a realization of a single-point Lawrence extension of a polygon. Here $\boldsymbol{q}$ is chosen as the intersection of the supporting lines of two nonadjacent edges. Similar ideas can be used to force certain nontrivial collinearities in higher dimensions, as we will see later.

In this thesis we will not explore properties of Lawrence extensions further. More detailed discussion can be found in, *e.g.*, [28, 4]; Richter-Gebert specifically lists interesting examples in [28, Section 3.4]. On the other hand, Lawrence extensions will be central in the construction of several polytopes used to prove the universality theorem.

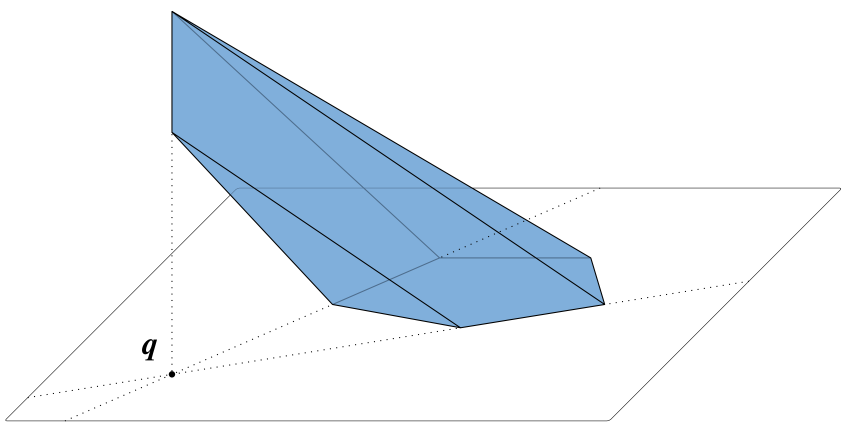


Figure 11: Lawrence extension of a pentagon.

6 Building blocks

The proof of the universality theorem for realization spaces of 4-polytopes relies on a set of basic constructions—the so called *basic building blocks*—using which we construct polytopes modeling addition and multiplication, and finally a polytopes representing any given primary basic semialgebraic set. In this expository section we introduce the building blocks briefly, for the sake of completeness and ease of reading. The specific results do not make use of stable equivalence, so we refer to [28, Part 2] for all proofs.

Outline of the proof

The idea is to encode variables of a Shor normal form $\mathcal{S} = (n, \boldsymbol{A}, \boldsymbol{M})$ as line slopes of a $2(n+3)$-gon $\boldsymbol{G}$ on a 4-polytope. The convexity of $\boldsymbol{G}$ gives us the strict total order required by $\mathcal{S}$, and the equations corresponding to $\boldsymbol{A}$ and $\boldsymbol{M}$ will be forced by additional constructions.

For a set $X = (a_1, \dots, a_n)$ of labels, let $\boldsymbol{G}(X)$ denote an n-gon with edges labeled by X in the same order. By $\boldsymbol{G}[X]$ we denote a $2n$-gon with edges labeled by, in this order,

$$a_1, \dots, a_n, a_1', \dots, a_n'.$$

The unique line joining two distinct points a and b is denoted by $a \vee b$, and the unique intersection point of two distinct lines ℓ and k by $\ell \wedge k$. We call $a \vee b$ the *join* of a and b, and $\ell \wedge k$ the *meet* of ℓ and k. The meet of two line segments is defined as the meet of their supporting lines. In particular, a vertex v of $\boldsymbol{G}(X)$ is labeled $a_i \wedge a_j$, where a_i and a_j are labels of the edges incident to v.

Definition 6.1. A *computation frame* is a $2(n+3)$-gon $\boldsymbol{G}[X]$ with edge labels of the form

$$X = (\mathbf{0}, \mathbf{1}, x_1, \dots, x_n, \infty).$$

If the intersections of the edge supporting lines of all edge pairs (i, i') for $i \in X$ are collinear, then $\boldsymbol{G}[X]$ is *normal*. We denote $i \wedge i' = \boldsymbol{p}_i$. If $\boldsymbol{G}[X]$ is normal, the line containing each $\boldsymbol{p}_i$ is called the *computation line* of $\boldsymbol{G}[X]$.

Figure 12 shows an example of a normal computation frame $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x_1, x_2, \infty]$. We will from here on often switch implicitly between the affine setting and projective space; this happens naturally whenever we homogenize a polytope. Thus it makes to allow the computation line of a normal computation frame to be the line at infinity, like ℓ in the figure.

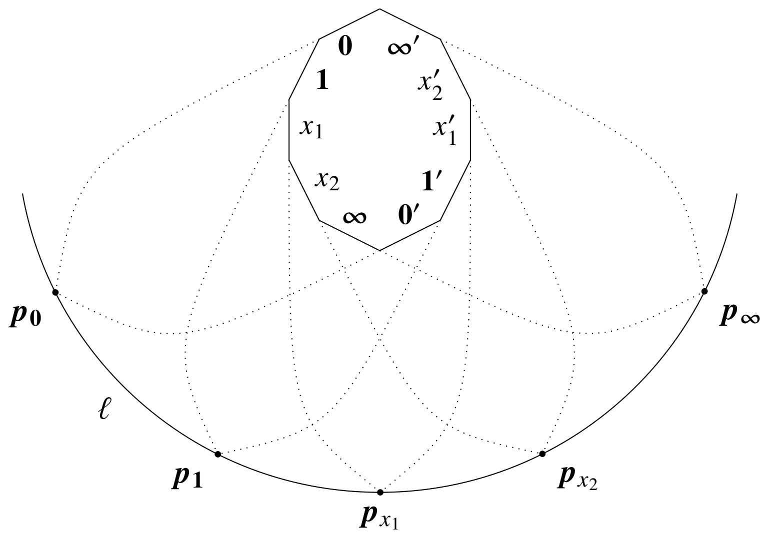


Figure 12: A normal computation frame $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x_1, x_2, \infty]$.

Definition 6.2. The *cross ratio* $(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{c}, \boldsymbol{d})$ of four points on a line is

$$(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{c}, \boldsymbol{d}) = \frac{|\boldsymbol{a}, \boldsymbol{c}| \cdot |\boldsymbol{b}, \boldsymbol{d}|}{|\boldsymbol{a}, \boldsymbol{d}| \cdot |\boldsymbol{b}, \boldsymbol{c}|},$$

where $|-,-|$ denotes the oriented euclidean distance between two points.

Crucially, the cross ratio is invariant under projective transformations, allowing us to extend the definition to include points at infinity. The cross ratio can be used to provide a well-defined scale, and hence a measure of distance, with respect to three chosen points $\mathbf{0}$, $\mathbf{1}$ and ∞ on a line in projective space. In particular, for a normal computation frame $\boldsymbol{G}[X]$ we can take the points $\boldsymbol{p_0}$, $\boldsymbol{p_1}$ and $\boldsymbol{p_\infty}$ and define a scale

$$\gamma(\boldsymbol{p}) = (\boldsymbol{p}, \boldsymbol{p_1} \mid \boldsymbol{p_0}, \boldsymbol{p_\infty}),$$

for points $\boldsymbol{p}$ on the computation line ℓ of $\boldsymbol{G}[X]$.

We are now ready to outline our goals regarding the rest of the proof of the universality theorem. Again, let $\mathcal{S} = (n, \boldsymbol{A}, \boldsymbol{M})$ be a Shor normal form. We will construct a 4-polytope $\boldsymbol{P}(\mathcal{S})$ with a face $\boldsymbol{G} = \boldsymbol{G}[\mathbf{0}, \mathbf{1}, x_1, \ldots, x_n, \infty]$, so that $\boldsymbol{G}$ is a normal computation frame for every realization of $\boldsymbol{P}(\mathcal{S})$. Moreover, $\boldsymbol{P}(\mathcal{S})$ will satisfy the following conditions:

(1) We have $\gamma(\boldsymbol{p}_{x_i}) < \gamma(\boldsymbol{p}_{x_j})$ for $i < j$ in every realization of $\boldsymbol{P}(\mathcal{S})$. This is an immediate consequence of the ordering of the edge labels;

(2) $\gamma(\boldsymbol{p}_{x_i}) + \gamma(\boldsymbol{p}_{x_j}) = \gamma(\boldsymbol{p}_{x_k})$ for $(i, j, k) \in \boldsymbol{A}$ in every realization of $\boldsymbol{P}(\mathcal{S})$;

(3) $\gamma(\boldsymbol{p}_{x_i}) \cdot \gamma(\boldsymbol{p}_{x_j}) = \gamma(\boldsymbol{p}_{x_k})$ for $(i, j, k) \in \boldsymbol{M}$ in every realization of $\boldsymbol{P}(\mathcal{S})$;

(4) For every point $(v_1, \dots, v_n) \in V(\mathcal{S})$ there exists a realization of $\boldsymbol{P}(\mathcal{S})$ such that $\gamma(\boldsymbol{p}_{x_i}) = v_i$ for $1 \leqslant i \leqslant n$.

Definition 6.3. Let $\boldsymbol{G}[X]$ be a normal n-gon. The process of mapping the computation line ℓ of $\boldsymbol{G}[X]$ to the line at infinity through a projective transformation is called *pre-standardization*. Embedded in $\mathbb{R}^2$, the slope of each edge $i \in X$ is denoted by s_i. If

$$s_{\mathbf{0}} = 0, \quad s_{\mathbf{1}} = 1 \quad \text{and} \quad s_{\infty} = \infty,$$

then $\boldsymbol{G}[X]$ is *standardized*.

For each slope $s_a \in [0, \infty]$, $a \in X$, of a computation frame $\boldsymbol{G}[X]$, let $\hat{a}$ denote the direction

$$\hat{a} = \begin{cases} [1 : s_a] & \text{if} \quad s_a \neq \infty \\ [0 : s_a] & \text{if} \quad s_a = \infty \end{cases}$$

in homogeneous coordinates. For four points $\boldsymbol{p}_a$, $\boldsymbol{p}_b$, $\boldsymbol{p}_c$ and $\boldsymbol{p}_d$ on a computation line ℓ we can now also write their cross ratio as

$$(\boldsymbol{p}_a, \boldsymbol{p}_b \mid \boldsymbol{p}_c, \boldsymbol{p}_d) = \frac{\det(\hat{a}, \hat{c}) \cdot \det(\hat{b}, \hat{d})}{\det(\hat{a}, \hat{d}) \cdot \det(\hat{b}, \hat{c})}.$$

On the computation line of a standardized computation frame, this gives in particular that

$$\gamma(\boldsymbol{p}_a) = \frac{\det(\hat{a}, \hat{0}) \cdot \det(\hat{1}, \hat{\infty})}{\det(\hat{a}, \hat{\infty}) \cdot \det(\hat{1}, \hat{0})} = \frac{\det \begin{bmatrix} 1 & 1 \\ s_a & 0 \end{bmatrix} \cdot \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 0 \\ s_a & 1 \end{bmatrix} \cdot \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} = s_a.$$

In other words, in the construction outlined above the points of our semialgebraic set will be in a one-to-one correspondence with the edge slopes of a standardized computation frame. It is still far from clear that these conditions give stable equivalence as we desired, since so much of the polytope's structure seems to be out of control. Miraculously, Richter-Gebert's building blocks are just rigid enough.

6.1 Basic building blocks

Our first constructions, called the *basic building blocks*, will serve as a basis for the more complicated polytopes. Roughly speaking, the aim is to create 4-polytopes for transmission of various amounts of geometric information, such as the line slopes of a certain 2-face. Combining this with a 4-polytope with restricted realizability of one 2-face, we can build polytopes with increasingly complicated realization spaces using the connected sum operation.

The transmitter

The *transmitter* $\boldsymbol{T}_X$ sends information from one 2-face to another, so that the two faces are projectively equivalent in any realization. It is constructed in the following way:

(1) Start with an n-gon $\boldsymbol{G}(X)$ and let $\boldsymbol{P} = \mathrm{prism}(\boldsymbol{G}(X))$;

(2) The edges $\{i, i'\}$ with $i \in X$ are parallel, so they meet at a point $\boldsymbol{q}_y$ at infinity;

(3) Let $\boldsymbol{T}_X = \Lambda(\boldsymbol{P}, \{\boldsymbol{q}_y\})$.

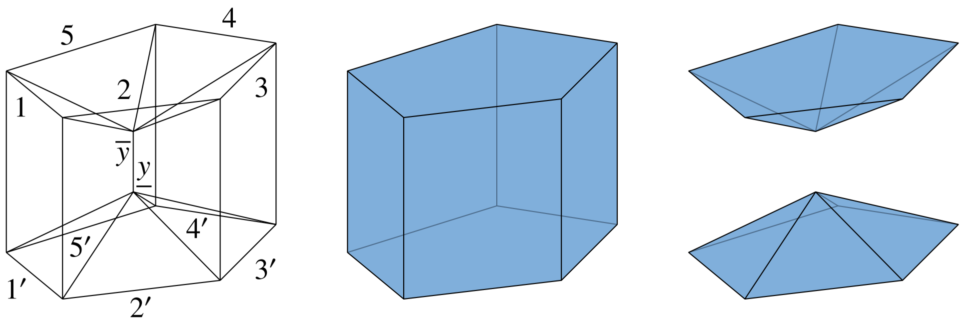
Theorem 6.4.

(1) *The combinatorial type of* $\boldsymbol{T}_X$ *is independent of the special choice of points in the constructions*;

(2) *The n-gons* $\boldsymbol{G}(X)$ *and* $\boldsymbol{G}(X')$ *are projectively equivalent in every realization of* $\boldsymbol{T}_X$;

(3) $\boldsymbol{T}_X$ *has facets* $\mathrm{pyr}(X)$, $\mathrm{pyr}(X')$ *and* $\mathrm{prism}(X)$.

Figure 13 shows a realization of a transmitter over a pentagon.



$\boldsymbol{T}_X$ prism(X) pyr(X), pyr(X')

Figure 13: A transmitter $\boldsymbol{T}_{\{1,\ldots,5\}}$ and its relevant facets.

The connector

The *connector* $\boldsymbol{C}_X$ is defined by

$$\boldsymbol{C}_X = \boldsymbol{T}_X \heartsuit_{\mathrm{prism}(X)} \boldsymbol{T}_X.$$

It has four projectively equivalent pyramids over X and X' as facets.

The forgetful transmitter

The *forgetful transmitter* $\boldsymbol{T}_X^Y$ with $X = (1, \dots, n)$ and $Y = (1', \dots, (n+1)')$ is constructed in the following way:

(1) Start with an n-gon $\boldsymbol{G}(X)$ and let $\boldsymbol{P} = \operatorname{prism}(\boldsymbol{G}(X))$;

(2) The edges joining corresponding vertices of $\boldsymbol{G}(X)$ and $\boldsymbol{G}(X')$ are parallel, so they meet at a point $\boldsymbol{q}_y$ at infinity;

(3) Truncate the vertex $1' \wedge n'$ and label the new edge on $\boldsymbol{G}(X')$ by $(n+1)'$. Call the resulting polytope $\boldsymbol{P}'$;

(4) Let $\boldsymbol{T}_X^Y = \Lambda(\boldsymbol{P}', \{\boldsymbol{q}_y\})$.

Theorem 6.5.

(1) *The combinatorial structure of* $\boldsymbol{T}_X^Y$ *is independent of the special choice of points in the construction*;

(2) *The sets of lines supporting the edges* $1, \dots, n$ *and* $1', \dots, n'$ *respectively are projectively equivalent in every realization of* $\boldsymbol{T}_X^Y$;

(3) $\boldsymbol{T}_X^Y$ *has facets* $\operatorname{pyr}(X)$ *and* $\operatorname{pyr}(Y)$.

We can chain together multiple forgetful transmitters in order to forget more than one edge. Specifically, if $X = (1, \dots, n)$ and $Y = (1', \dots, (n+k)')$ and we denote $Y_i = (1', \dots, (n+i)')$ for $1 \leqslant i \leqslant k-1$, then we may define

$$\boldsymbol{T}_X^Y = \boldsymbol{T}_X^{Y_1} \heartsuit_{\operatorname{pyr}(Y_1)} \boldsymbol{T}_{Y_1}^{Y_2} \heartsuit_{\operatorname{pyr}(Y_2)} \cdots \heartsuit_{\operatorname{pyr}(Y_{k-1})} \boldsymbol{T}_{Y_{k-1}}^{Y}.$$

The properties of Theorem 6.5 translate to $\boldsymbol{T}_X^Y$ in an obvious way.

Figure 14 shows a realization of the forgetful transmitter.

A polytope with a nonprescribable face

The entire proof of the universality theorem relies on there being 4-polytopes whose realization spaces have “controllable” nontrivialities. In this sense the next polytope we will construct, called the $\boldsymbol{X}$-polytope, is our most important building block: we are able to force a collinearity of three points that has to hold in any realization of $\boldsymbol{X}$. The polytope is constructed in the following way:

(1) Start with a hexagon $\boldsymbol{G} = \boldsymbol{G}(1, \dots, 6)$ such that the points $1 \wedge 4$, $2 \wedge 3$ and $5 \wedge 6$ are collinear;

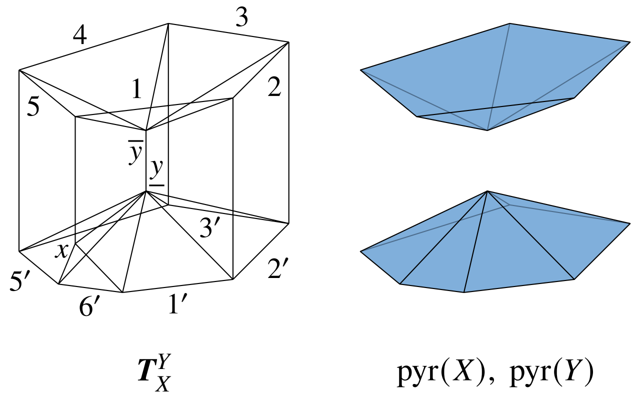


Figure 14: A forgetful transmitter $\boldsymbol{T}_{\{1,\ldots,5\}}^{\{1',\ldots,6'\}}$ and its relevant facets.

(2) Let $\boldsymbol{P} = \text{tent}^{1,4}(\boldsymbol{G})$. If a and b are the points as in Figure 15, then $(2 \wedge 3) \vee a$ and $(5 \wedge 6) \vee b$ meet at a point $\boldsymbol{q}_y$;

(3) Let $\boldsymbol{X} = \Lambda(\boldsymbol{P}, \{\boldsymbol{q}_y\})$.

Theorem 6.6.

(1) *The combinatorial structure of* $\boldsymbol{X}$ *is independent of the special choice of points in the construction*;

(2) *The points* $1 \wedge 4$, $2 \wedge 3$ *and* $5 \wedge 6$ *are collinear in every realization of* $\boldsymbol{X}$;

(3) $\boldsymbol{X}$ *has a facet* $\text{pyr}(1, \ldots, 6)$.

We write $\boldsymbol{X}(X)$ to specify the edge labels of the computation frame of $\boldsymbol{X}$ for an ordered set X of labels. The edges are labeled in the same order as in the figure.

The adapter

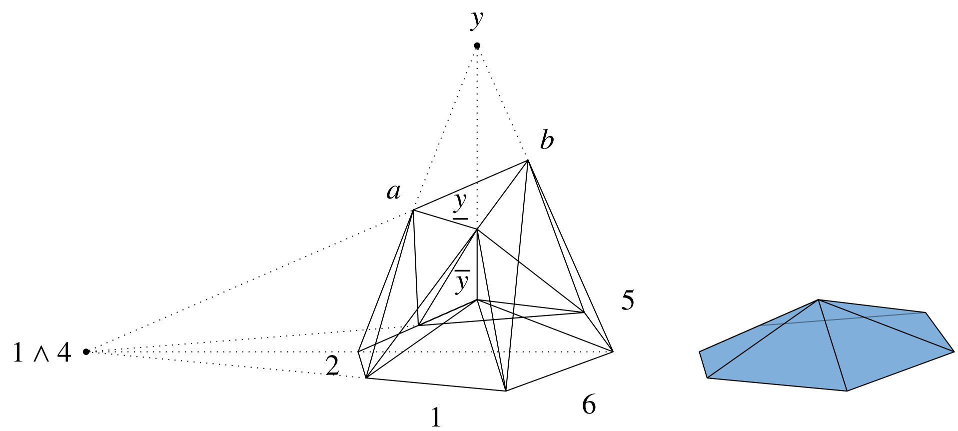
The *adapter* $\boldsymbol{A}_8$ contains a pyramid and a tent over a common 8-gon $\boldsymbol{G}(1, \ldots, 8)$ as facets, and is used to connect a building block with a tent to a building block with a pyramid. It is defined by

$$\boldsymbol{A}_8^{a,b} = \text{pyr}(\text{tent}^{a,b}(\boldsymbol{G}(1, \ldots, 8))).$$

The partial slope transmitter

The *partial slope transmitter* $\boldsymbol{Y}_8^{1,5} = \boldsymbol{Y}(1, \ldots, 8)$ is constructed in the following way:

(1) Start with an octagon $\boldsymbol{G} = \boldsymbol{G}(1, \ldots, 8)$ in which edges 1 and 5 are parallel;



$\boldsymbol{X}$ pyr$(1, \ldots, 6)$

Figure 15: A polytope $\boldsymbol{X}$ and its relevant facet.

(2) In a plane parallel to $\boldsymbol{G}$, choose an octagon $\boldsymbol{G}'$ so that all edge pairs (i, i') for $1 \leqslant i \leqslant 8$ are parallel, and the supporting lines of the edges 1, 1′, 5 and 5′ meet at a point $\boldsymbol{q}_y$;

(3) Let $\boldsymbol{Y}_8^{1,5} = \Lambda(\boldsymbol{P}, \{\boldsymbol{q}_y\})$.

Theorem 6.7.

(1) *The combinatorial structure of* $\boldsymbol{Y}_8^{1,5}$ *does not depend on the special choice of points in the construction*;

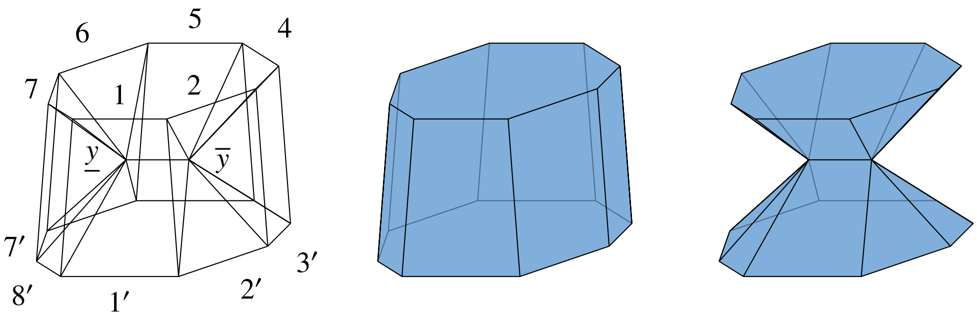
(2) *The supporting lines of the edges* 1, 1′, 5 *and* 5′ *meet at a point in every realization of* $\boldsymbol{Y}_8^{1,5}$;

(3) $\boldsymbol{Y}_8^{1,5}$ *has facets* $\mathrm{tent}^{1,5}(\boldsymbol{G})$, $\mathrm{tent}^{1',5'}(\boldsymbol{G}')$ *and* $\mathrm{prism}(\boldsymbol{G})$.

Figure 16 shows a realization of the partial slope transmitter.

The slope transmitter

The *slope transmitter* $\boldsymbol{O}_8$ is used to transmit projective conditions from an 8-gon $\boldsymbol{G} = \boldsymbol{G}(1, \ldots, 8)$ to another 8-gon $\boldsymbol{G}' = \boldsymbol{G}(1', \ldots, 8')$. It is constructed by connecting $\boldsymbol{Y}_8^{1,5}$ and $\boldsymbol{Y}_8^{2,6}$ by their prism, and then gluing adapters to the tents of $\boldsymbol{Y}_8^{1,5}$. It is described



$\boldsymbol{Y}_8^{1,5}$ prism$(\boldsymbol{G})$ tent$^{1,5}(\boldsymbol{G})$, tent$^{1',5'}(\boldsymbol{G}')$

Figure 16: A partial slope transmitter $\boldsymbol{Y}_8^{1,5}$ and its relevant facets.

more precisely by the following diagram, in which the lines represent connected sums.

$$\begin{tikzcd}
\boldsymbol{A}_8^{1,5} \arrow[r, dash, "\mathrm{tent}^{1,5}(\boldsymbol{G})"] & \boldsymbol{Y}_8^{1,5} \arrow[r, dash, "\mathrm{tent}^{1',5'}(\boldsymbol{G}')"] \arrow[d, dash, "\mathrm{prism}(\boldsymbol{G})"] & \boldsymbol{A}_8^{1',5'} \\
& \boldsymbol{Y}_8^{2,6} &
\end{tikzcd}$$

We can see that $\boldsymbol{O}_8$ has pyr$(\boldsymbol{G})$ and pyr$(\boldsymbol{G}')$ as facets.

Theorem 6.8.

(1) *If* $\boldsymbol{G} = \boldsymbol{G}(1, \dots, 8)$ *and* $\boldsymbol{G}' = \boldsymbol{G}(1', \dots, 8')$ *are two parallel* 8*-gons with*

$$1 \wedge 5 = 1' \wedge 5', \quad 2 \wedge 6 = 2' \wedge 6', \quad 3 \wedge 7 = 3' \wedge 7' \quad \text{and} \quad 4 \wedge 8 = 4' \wedge 8',$$

then $\boldsymbol{G}$ *and* $\boldsymbol{G}'$ *appear as the corresponding* 8*-gons of a realization of* $\boldsymbol{O}_8$;

(2) *If the points* $1 \wedge 5$, $2 \wedge 6$, $3 \wedge 7$ *and* $4 \wedge 8$ *of a realization of* $\boldsymbol{O}_8$ *are collinear, then*

$$1 \wedge 5 = 1' \wedge 5', \quad 2 \wedge 6 = 2' \wedge 6', \quad 3 \wedge 7 = 3' \wedge 7' \quad \text{and} \quad 4 \wedge 8 = 4' \wedge 8'.$$

The harmonic polytope

Our goal is to construct a polytope $\boldsymbol{H}(1, \dots, 8)$ with a 2-face $\boldsymbol{G} = \boldsymbol{G}(1, \dots, 8)$ such that the intersections of the edges supporting opposite sides are collinear. To accomplish this, let $\boldsymbol{G}$ satisfy the collinearities

$$\begin{gathered}
(1 \wedge 5, 8 \wedge 7, 2 \wedge 3), \quad (1 \wedge 5, 7 \wedge 6, 3 \wedge 4), \\
(3 \wedge 7, 2 \wedge 1, 4 \wedge 5), \quad (2 \wedge 5, 3 \wedge 4, 8 \wedge 7), \\
(4 \wedge 1, 2 \wedge 3, 7 \wedge 6), \quad (6 \wedge 1, 3 \wedge 4, 8 \wedge 7), \\
\text{and} \quad (8 \wedge 5, 2 \wedge 3, 7 \wedge 6).
\end{gathered} \tag{6.1}$$

Lemma 6.9. *An* 8*-gon* $\boldsymbol{G}$ *which satisfies* (6.1) *is normal, i.e., the intersections of the lines supporting opposite edges are collinear. The slopes* s_1*,* s_2*,* s_3 *and* s_4 *of edges* 1, 2, 3 *and* 4 *respectively satisfy* $(s_1, s_2 \mid s_3, s_4) = -1$ *after pre-standardization.*

The *harmonic polytope* $\boldsymbol{H}(1, \dots, 8)$ is constructed by connecting a large number of our basic building blocks. It will contain a normal 2-face $O = \boldsymbol{G}(1, \dots, 8)$. We use seven $\boldsymbol{X}$-polytopes to impose the collinearities in (6.1), forgetful transmitters to add the remaining two edges of the octagon for each $\boldsymbol{X}$-polytope, connectors to connect all the components, and finally a slope transmitter so that only the slope information remains.

Lemma 6.10.

(1) *The octagon* O *is normal in every realization of* $\boldsymbol{H}(1, \dots, 8)$;

(2) *If* O *is pre-standardized by a projective transformation, then* $(s_1, s_2 \mid s_3, s_4) = -1$;

(3) *Every normal and pre-standardized octagon* O *with* $(s_1, s_2 \mid s_3, s_4) = -1$ *can be completed to a realization of* $\boldsymbol{H}$.

The following diagram shows the exact construction of $\boldsymbol{H}(1, \dots, 8)$. All connections are made along the appropriate pyramids, implying by Lemma 5.8 that the harmonic polytope is realizable. Here the octagon O appears in the unused connective facet of $\boldsymbol{O}_8$.

$$
\begin{tikzcd}
\boldsymbol{X}(\mathbf{1},2,3,\mathbf{5},7,8) \arrow[r, dash] & \boldsymbol{T}^{(1,\dots,8)}_{(1,2,3,5,7,8)} \arrow[dr, dash, bend left] & \\
\boldsymbol{X}(\mathbf{1},3,4,\mathbf{5},6,7) \arrow[r, dash] & \boldsymbol{T}^{(1,\dots,8)}_{(1,3,4,5,6,7)} \arrow[r, dash] & \boldsymbol{C}_8 \arrow[d, dash] \\
\boldsymbol{X}(\mathbf{3},4,5,\mathbf{7},1,2) \arrow[r, dash] & \boldsymbol{T}^{(1,\dots,8)}_{(1,2,3,4,5,7)} \arrow[r, dash] & \boldsymbol{C}_8 \arrow[d, dash] \\
\boldsymbol{X}(\mathbf{2},3,4,\mathbf{5},7,8) \arrow[r, dash] & \boldsymbol{T}^{(1,\dots,8)}_{(2,3,4,5,7,8)} \arrow[r, dash] & \boldsymbol{C}_8 \arrow[d, dash] \\
\boldsymbol{X}(\mathbf{1},2,3,\mathbf{4},6,7) \arrow[r, dash] & \boldsymbol{T}^{(1,\dots,8)}_{(1,2,3,4,6,7)} \arrow[r, dash] & \boldsymbol{C}_8 \arrow[d, dash] \\
\boldsymbol{X}(\mathbf{1},3,4,\mathbf{6},7,8) \arrow[r, dash] & \boldsymbol{T}^{(1,\dots,8)}_{(1,3,4,6,7,8)} \arrow[r, dash] & \boldsymbol{C}_8 \arrow[d, dash] \\
\boldsymbol{X}(\mathbf{5},6,7,\mathbf{8},2,3) \arrow[r, dash] & \boldsymbol{T}^{(1,\dots,8)}_{(2,3,5,6,7,8)} \arrow[r, dash] & \boldsymbol{C}_8 \arrow[d, dash] \\
 & & \boldsymbol{O}_8
\end{tikzcd}
$$

6.2 Addition and multiplication

The *addition* and *multiplication* polytopes will model addition and multiplication of real numbers using nonprescribability of line slopes on a normal computation frame. In accordance with Shor's normal form, we specifically need to address addition of the form $x + y = z$ with $x < y$, and $x + x = z$. We do this with separate polytopes, since the same pair of edges describes both summands in the latter case. Multiplication is handled similarly, but with more complications.

Addition

The case $x + x = z$ is simple, because the harmonic polytope $\boldsymbol{H}[\mathbf{0}, x, 2x, \boldsymbol{\infty}]$ satisfies

$$(0, 2x \mid x, \infty) = \frac{0 - x}{2x - x} = -1$$

for all $x > 0$. We define

$$\boldsymbol{P}^{2x}[\mathbf{0}, x, 2x, \boldsymbol{\infty}] = \boldsymbol{H}[\mathbf{0}, x, 2x, \boldsymbol{\infty}].$$

The polytyope $\boldsymbol{P}^{x+y}[\mathbf{0}, x, y, x + y, \boldsymbol{\infty}]$ is more complicated. We join the harmonic polytopes

$$\boldsymbol{H}_1 = \boldsymbol{H}\left[x, \frac{x+y}{2}, y, \boldsymbol{\infty}\right] \quad \text{and} \quad \boldsymbol{H}_2 = \boldsymbol{H}\left[\mathbf{0}, \frac{x+y}{2}, x + y, \boldsymbol{\infty}\right]$$

to the 12-gon

$$\boldsymbol{G} = \boldsymbol{G}\left[\mathbf{0}, x, \frac{x+y}{2}, y, x + y, \boldsymbol{\infty}\right]$$

with forgetful transmitters and a connector. Since $x < y$ the edges are correctly ordered. Lastly, we delete the unnecessary edge pair with a forgetful transmitter so that we have a 10-gon $\boldsymbol{G}[\mathbf{0}, x, y, x + y, \boldsymbol{\infty}]$.

Theorem 6.11.

(1) *The* 10*-gon* $\boldsymbol{G}[\mathbf{0}, x, y, x+y, \boldsymbol{\infty}]$ *is a normal computation frame in every realization of* $\boldsymbol{P}^{x+y}[\mathbf{0}, x, y, x + y, \boldsymbol{\infty}]$. *After standardization the slopes satisfy* $s_x + s_y = s_{x+y}$;

(2) *Every standardized* 10*-gon* $\boldsymbol{G}[\mathbf{0}, x, y, x+y, \boldsymbol{\infty}]$ *with* $s_x+s_y = s_{x+y}$ *can be completed to a realization of* $\boldsymbol{P}^{x+y}[\mathbf{0}, x, y, x + y, \boldsymbol{\infty}]$.

The following diagram shows $\boldsymbol{P}^{x+y}$ more precisely. All the connections are along the appropriate pyramids.

$$\begin{tikzcd}
\boldsymbol{H}\big[x, \tfrac{x+y}{2}, y, \infty\big] \arrow[r, dash] & \boldsymbol{T}^{\left[\mathbf{0},x,\frac{x+y}{2},y,x+y,\infty\right]}_{\left[x,\frac{x+y}{2},y,\infty\right]} \arrow[dr, dash, bend left] & \\
\boldsymbol{H}\big[\mathbf{0}, \tfrac{x+y}{2}, x+y, \infty\big] \arrow[r, dash] & \boldsymbol{T}^{\left[\mathbf{0},x,\frac{x+y}{2},y,x+y,\infty\right]}_{\left[\mathbf{0},\frac{x+y}{2},x+y,\infty\right]} \arrow[r, dash] & \boldsymbol{C}_{12} \arrow[d, dash] \\
& & \boldsymbol{T}^{\left[\mathbf{0},x,\frac{x+y}{2},y,x+y,\infty\right]}_{\left[\mathbf{0},x,y,x+y,\infty\right]}
\end{tikzcd}$$

Multiplication

We construct the *multiplication polytopes* in a similar way, separately for $x \cdot y = z$ with $x < y$ and for $x \cdot x = z$. Especially the first case is complicated, since we need to make sure algebraic complexity never rises at any intermediate step. For the case $x \cdot x = z$, we construct $\boldsymbol{P}^{x^2}$ by connecting the harmonic polytopes

$$\boldsymbol{H}_1 = \boldsymbol{H}[-x, \mathbf{0}, x, \infty] \quad \text{and} \quad \boldsymbol{H}_2 = \boldsymbol{H}[-x, \mathbf{1}, x, x^2]$$

so that we get a computation frame $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x, x^2, \infty]$. The polytope $\boldsymbol{H}_1$ determines the slope of the edge pair labeled by $-x$, allowing $\boldsymbol{H}_2$ to determine x^2.

Theorem 6.12.

(1) *The* 10*-gon* $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x, x^2, \infty]$ *is a normal computation frame in every realization of* $\boldsymbol{P}^{x^2}[\mathbf{0}, \mathbf{1}, x, x^2, \infty]$. *After standardization the slopes satisfy* $s_x^2 = s_{x^2}$;

(2) *Every standardized* 10*-gon* $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x, x^2, \infty]$ *with* $s_x^2 = s_{x^2}$ *can be completed to a realization of* $\boldsymbol{P}^{x^2}[\mathbf{0}, \mathbf{1}, x, x^2, \infty]$.

Multiplications $x \cdot y = z$ could be modeled in an analogous way, but this introduces an intermediate variable $\sqrt{x \cdot y}$, which might increase algebraic complexity of the resulting polytope. Hence we instead first compute x^2 and y^2 using instances of $\boldsymbol{P}^{x^2}$, and afterwards calculate $x \cdot y = \sqrt{x^2 \cdot y^2}$.

The second step is achieved by constructing the polytope $\boldsymbol{P}^{\sqrt{x^2 \cdot y^2}}$ by connecting harmonic polytopes

$$\boldsymbol{H}_1 = \boldsymbol{H}[-xy, \mathbf{0}, xy, \infty] \quad \text{and} \quad \boldsymbol{H}_2 = \boldsymbol{H}[-xy, x^2, xy, y^2]$$

so that we get a computation frame $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x^2, xy, y^2, \infty]$. Here $\boldsymbol{H}_1$ forces $-s_{xy} = s_{-xy}$, allowing $\boldsymbol{H}_2$ to impose the desired condition. We then connect $\boldsymbol{P}^{\sqrt{x^2 \cdot y^2}}$ with instances of $\boldsymbol{P}^{x^2}$ for x^2 and y^2 using forgetful transmitters and connectors to the computation frame $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x, y, xy, \infty]$. This results in the second multiplication polytope $\boldsymbol{P}^{x \cdot y}[\mathbf{0}, \mathbf{1}, x, y, xy, \infty]$.

Theorem 6.13.

(1) *The* 12*-gon* $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x, y, xy, \infty]$ *is a normal computation frame in every realization of* $\boldsymbol{P}^{x \cdot y}[\mathbf{0}, \mathbf{1}, x, y, xy, \infty]$. *After standardization the slopes satisfy* $s_x \cdot s_y = s_{xy}$.

(2) *Every standardized* 12*-gon* $\boldsymbol{G}[\mathbf{0}, \mathbf{1}, x, y, xy, \infty]$ *with* $s_x \cdot s_y = s_{xy}$ *can be completed to a realization of* $\boldsymbol{P}^{x \cdot y}[\mathbf{0}, \mathbf{1}, x, y, xy, \infty]$.

We will now present the diagrams for all parts of the multiplication polytopes. In all cases, the connections are established by the appropriate pyramids.

The first multiplication polytope $\boldsymbol{P}^{x^2}$:

$$\begin{tikzcd}
\boldsymbol{H}[-x, \mathbf{0}, x, \infty] \arrow[r, dash] & \boldsymbol{T}^{[-x,\mathbf{0},\mathbf{1},x,x^2,\infty]}_{[-x,\mathbf{0},x,\infty]} \arrow[dr, dash, bend left] & \\
\boldsymbol{H}[-x, \mathbf{1}, x, x^2] \arrow[r, dash] & \boldsymbol{T}^{[-x,\mathbf{0},\mathbf{1},x,x^2,\infty]}_{[-x,\mathbf{1},x,x^2]} \arrow[r, dash] & \boldsymbol{C}_{12} \arrow[d, dash] \\
& & \boldsymbol{T}^{[-x,\mathbf{0},\mathbf{1},x,x^2,\infty]}_{[\mathbf{0},\mathbf{1},x,x^2,\infty]}
\end{tikzcd}$$

The intermediate step $\boldsymbol{P}^{\sqrt{x^2 \cdot y^2}}$:

$$\begin{tikzcd}
\boldsymbol{H}[-xy, \mathbf{0}, xy, \infty] \arrow[r, dash] & \boldsymbol{T}^{[-xy,\mathbf{0},\mathbf{1},x^2,xy,y^2,\infty]}_{[-xy,\mathbf{0},xy,\infty]} \arrow[dr, dash, bend left] & \\
\boldsymbol{H}[-xy, x^2, xy, y^2] \arrow[r, dash] & \boldsymbol{T}^{[-xy,\mathbf{0},\mathbf{1},x^2,xy,y^2,\infty]}_{[-xy,x^2,xy,y^2]} \arrow[r, dash] & \boldsymbol{C}_{14} \arrow[d, dash] \\
& & \boldsymbol{T}^{[-xy,\mathbf{0},\mathbf{1},x^2,xy,y^2,\infty]}_{[\mathbf{0},\mathbf{1},x^2,xy,y^2,\infty]}
\end{tikzcd}$$

The second multiplication polytope $\boldsymbol{P}^{x\cdot y}$:

$$
\begin{tikzcd}
\boldsymbol{P}^{\sqrt{x^2\cdot y^2}}[\mathbf{0},\mathbf{1},x^2,xy,y^2,\boldsymbol{\infty}] \arrow[r, dash] & \boldsymbol{T}^{[\mathbf{0},\mathbf{1},x,x^2,xy,y^2,\boldsymbol{\infty}]}_{[\mathbf{0},\mathbf{1},x^2,xy,y^2,\boldsymbol{\infty}]} \arrow[dr, dash, bend left] & \\
\boldsymbol{P}^{x^2}[\mathbf{0},\mathbf{1},x,x^2,\boldsymbol{\infty}] \arrow[r, dash] & \boldsymbol{T}^{[\mathbf{0},\mathbf{1},x,x^2,xy,y^2,\boldsymbol{\infty}]}_{[\mathbf{0},\mathbf{1},x,x^2,\boldsymbol{\infty}]} \arrow[r, dash] & \boldsymbol{C}_{14} \arrow[d, dash] \\
 & & \boldsymbol{T}^{[\mathbf{0},\mathbf{1},x,x^2,xy,y^2,\boldsymbol{\infty}]}_{[\mathbf{0},\mathbf{1},x,xy,y^2,\boldsymbol{\infty}]} \arrow[d, dash] \\
 & & \boldsymbol{T}^{[\mathbf{0},\mathbf{1},x,y,xy,y^2,\boldsymbol{\infty}]}_{[\mathbf{0},\mathbf{1},x,xy,y^2,\boldsymbol{\infty}]} \arrow[d, dash] \\
\boldsymbol{P}^{x^2}[\mathbf{0},\mathbf{1},y,y^2,\boldsymbol{\infty}] \arrow[r, dash] & \boldsymbol{T}^{[\mathbf{0},\mathbf{1},x,y,xy,y^2,\boldsymbol{\infty}]}_{[\mathbf{0},\mathbf{1},y,y^2,\boldsymbol{\infty}]} \arrow[r, dash] & \boldsymbol{C}_{14} \arrow[d, dash] \\
 & & \boldsymbol{T}^{[\mathbf{0},\mathbf{1},x,y,xy,y^2,\boldsymbol{\infty}]}_{[\mathbf{0},\mathbf{1},x,y,xy,\boldsymbol{\infty}]}
\end{tikzcd}
$$

7 The universality theorem

In this section we will prove the universality theorem for the improved notion of stable equivalence. This very closely follows [28, Section 8], with crucial differences in a few of the proofs. The first step is to construct the polytope $\boldsymbol{P}(\mathcal{S})$ corresponding to a Shor normal form $\mathcal{S} = (n, \boldsymbol{A}, \boldsymbol{M})$ as discussed earlier. We can assume that each variable is used in at least one equation, as we can otherwise delete the corresponding coordinate with a stable projection. Let

$$X = [\mathbf{0}, \mathbf{1}, x_1, \ldots, x_n, \infty]$$

and define

$$\boldsymbol{S}(\mathcal{S}) = \operatorname{pyr}(\operatorname{pyr}(\boldsymbol{G}[X])).$$

The face $\boldsymbol{G}[X]$ of $\boldsymbol{S}(\mathcal{S})$ will serve as the final computation frame of $\boldsymbol{P}(\mathcal{S})$, and $\boldsymbol{S}(\mathcal{S})$ will be connected to the rest of the polytope through a pyramid $\operatorname{pyr}(\boldsymbol{G}[X])$. Consider the collection of polytopes

$$\begin{aligned} C(\mathcal{S}) = {} & \left\{\boldsymbol{P}^{2x}(Y) \heartsuit_{\operatorname{pyr}(Y)} \boldsymbol{T}_Y^X \mid Y = [\mathbf{0}, x_i, x_k, \infty],\ (i,i,k) \in \boldsymbol{A}\right\} \\ & \cup \left\{\boldsymbol{P}^{x+y}(Y) \heartsuit_{\operatorname{pyr}(Y)} \boldsymbol{T}_Y^X \mid Y = [\mathbf{0}, x_i, x_j, x_k, \infty],\ (i,j,k) \in \boldsymbol{A},\ i \neq j\right\} \\ & \cup \left\{\boldsymbol{P}^{x^2}(Y) \heartsuit_{\operatorname{pyr}(Y)} \boldsymbol{T}_Y^X \mid Y = [\mathbf{0}, x_i, x_k, \infty],\ (i,i,k) \in \boldsymbol{M}\right\} \\ & \cup \left\{\boldsymbol{P}^{x\cdot y}(Y) \heartsuit_{\operatorname{pyr}(Y)} \boldsymbol{T}_Y^X \mid Y = [\mathbf{0}, x_i, x_j, x_k, \infty],\ (i,j,k) \in \boldsymbol{M},\ i \neq j\right\}. \end{aligned}$$

Each polytope $\boldsymbol{Q}_i \in C(\mathcal{S})$ has a facet $\operatorname{pyr}(\boldsymbol{G}[X])$ as the unused connection of its transmitter $\boldsymbol{T}_Y^X$. We call this the *join* of $\boldsymbol{Q}_i$. Using the joins of all elements in $C(\mathcal{S})$, we glue them to $\boldsymbol{S}(\mathcal{S})$ using connectors. The resulting polytope is called $\boldsymbol{P}(\mathcal{S})$. In particular, this construction forces all conditions of the defining equations of $\mathcal{S}$. The following diagram shows the construction of $\boldsymbol{P}(\mathcal{S})$, with $C(\mathcal{S}) = \{\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_k\}$. Again, all polytopes are connected by their relevant pyramids.

$$\begin{tikzcd}[arrows=dash] \boldsymbol{S}(\mathcal{S}) \arrow[r] & \boldsymbol{C}_X \arrow[r] \arrow[d] & \boldsymbol{C}_X \arrow[r, "\cdots"] \arrow[d] & \boldsymbol{C}_X \arrow[r] \arrow[d] & \boldsymbol{Q}_k \\ & \boldsymbol{Q}_1 & \boldsymbol{Q}_2 & \boldsymbol{Q}_{k-1} & \end{tikzcd}$$

Theorem 7.1. *The polytope $\boldsymbol{P}(\mathcal{S})$ of a Shor normal form $\mathcal{S} = (n, \boldsymbol{A}, \boldsymbol{M})$ has the following properties*:

(1) *$\boldsymbol{P}(\mathcal{S})$ contains a normal computation frame $\boldsymbol{G} = \boldsymbol{G}[\mathbf{0}, \mathbf{1}, x_1, \ldots, x_n, \infty]$;*

(2) *After standardization,* $\boldsymbol{G}$ *satisfies* $(s_{x_1}, \dots, s_{x_n}) \in V(\mathcal{S})$ *for every realization of* $\boldsymbol{P}(\mathcal{S})$;

(3) *For every point* $(v_1, \dots, v_n) \in V(\mathcal{S})$ *there exists a realization of* $\boldsymbol{P}(\mathcal{S})$ *with* $s_{x_i} = v_i$ *for all* $1 \leqslant i \leqslant n$ *after standardization.*

7.1 The proof

We will now prove stable equivalence in two steps. It is easiest to consider $\boldsymbol{P}(\mathcal{S})$ in terms of the basic building blocks it contains, so we inductively construct $\boldsymbol{P}(\mathcal{S})$ by

$$\begin{aligned} \boldsymbol{P}_0 &= \boldsymbol{S}(\mathcal{S}), \\ \boldsymbol{P}_i &= \boldsymbol{P}_{i-1} \heartsuit_{F_i} \boldsymbol{B}_i \quad \text{for} \quad 1 \leqslant i \leqslant m, \\ \boldsymbol{P}(\mathcal{S}) &= \boldsymbol{P}_m, \end{aligned}$$

where each $\boldsymbol{B}_i$ is one of the basic building blocks, and F_i is the facet along which it is connected to $\boldsymbol{P}_{i-1}$. The collection $\{\boldsymbol{B}_i \mid 1 \leqslant i \leqslant m\}$ represents the full set of basic building blocks used in $\boldsymbol{P}(\mathcal{S})$. Table 1 shows all combinations of $\boldsymbol{B}_i$ and F_i that can arise in the construction of $\boldsymbol{P}(\mathcal{S})$, as can be verified from the definiton of the building blocks in Section 6.

For a polytope $\boldsymbol{P}$ and a subset $Y \subseteq \operatorname{vert}(\boldsymbol{P})$ let

$$\mathcal{R}_Y(\boldsymbol{P}, B) = \{\boldsymbol{Q}|_Y \mid \boldsymbol{Q} \in \mathcal{R}(\boldsymbol{P}, B)\},$$

where $B \subseteq \operatorname{vert}(\boldsymbol{P}) \setminus Y$ is an affine basis of $\boldsymbol{P}$. We denote $Y_i = \operatorname{vert}(\boldsymbol{P}_i)$ for $1 \leqslant i \leqslant m$ and choose a basis $B_0 \subseteq Y_0$. Since each F_i is either a pyramid, a prism or a tent and therefore necessarily flat by Lemma 5.7, each realization of $\boldsymbol{P}_i$ can be constructed from a realization of $\boldsymbol{P}_{i-1}$ by adding new vertices.

Lemma 7.2. $\mathcal{R}_{Y_{i-1}}(\boldsymbol{P}(\mathcal{S}), B_0) \approx \mathcal{R}_{Y_i}(\boldsymbol{P}(\mathcal{S}), B_0)$ *for every* $1 \leqslant i \leqslant m$.

Proof. We fix a realization $\boldsymbol{P}_{i-1} \in \mathcal{R}_{Y_{i-1}}(\boldsymbol{P}(\mathcal{S}), B_0)$ and denote by $\boldsymbol{P}_{i-1}^{\text{hom}}$ its representation in homogeneous coordinates. Let $h_0, h_1, \dots, h_r \in \left(\mathbb{R}^{d+1}\right)^*$ be the linear forms which define its facets, so that h_0 defines F_i. Define

$$\begin{aligned} \mathcal{A} &= \left\{\boldsymbol{p} \in \mathbb{R}^{d+1} \mid h_0(\boldsymbol{p}) < 0,\ h_j(\boldsymbol{p}) > 0 \text{ for } 1 \leqslant j \leqslant r\right\}, \\ \mathcal{B} &= \{\boldsymbol{p} \in \mathcal{A} \mid h_\infty(\boldsymbol{p}) > 0\}, \end{aligned}$$

where $h_\infty = (0, \dots, 0, 1)$ represents the plane at infinity. The region $\mathcal{A}$ contains exactly the points $\boldsymbol{p}$ such that $\boldsymbol{P}_{i-1}^{\text{hom}} \cup \{\boldsymbol{p}\}$ is a polyhedral cone containing all facets of $\boldsymbol{P}_{i-1}^{\text{hom}}$ except for F_i, and $\mathcal{B}$ is the region in which $\boldsymbol{P}_{i-1} \cup \{\boldsymbol{p}\}$, after dehomogenization with hyperplane $x_{d+1} = 1$, is a polytope containing all facets of $\boldsymbol{P}_{i-1}$ except for F_i. Notice that both $\mathcal{A}$ and $\mathcal{B}$ are nonempty interiors of polyhedral sets, so they are open and convex, and their bounding hyperplanes depend polynomially on $\operatorname{vert}(\boldsymbol{P}_{i-1})$.

Case	Polytope $\boldsymbol{B}_i$	Facet F_i
1	$\boldsymbol{X}$	$\mathrm{pyr}(1,\dots,6)$
2	$\boldsymbol{T}_X$	$\mathrm{pyr}(X)$
3	$\boldsymbol{T}_X$	$\mathrm{prism}(X)$
4	$\boldsymbol{T}_X^Y$	$\mathrm{pyr}(Y)$
5	$\boldsymbol{T}_X^Y$	$\mathrm{pyr}(X)$
6	$\boldsymbol{Y}_8$	$\mathrm{tent}^{a,b}(1,\dots,8)$
7	$\boldsymbol{Y}_8$	$\mathrm{prism}(1,\dots,8)$
8	$\boldsymbol{A}_8$	$\mathrm{pyr}(X)$
9	$\boldsymbol{A}_8$	$\mathrm{tent}^{a,b}(X)$

Table 1: The possible joins in $\boldsymbol{P}(\mathcal{S})$.

Case 1. Let $\boldsymbol{B}_i$ be an $\boldsymbol{X}$-polytope with $F_i = \mathrm{pyr}(\boldsymbol{G}(1,\dots,6),\overline{y})$. Using notation consistent with Figure 15, we can choose coordinates for the remaining points a, b and $\underline{y}$ in the following way:

(1) Choose $\boldsymbol{p}_a$ in $\mathcal{B}$;

(2) Choose $\lambda_1, \tau_1 > 0$ such that $\boldsymbol{p}_b = \lambda_1 \boldsymbol{p}_a + \tau_1\sigma_1(1 \wedge 4) \in \mathcal{B}$;

(3) Lines $(2 \wedge 3) \vee \boldsymbol{p}_a$ and $(5 \wedge 6) \vee \boldsymbol{p}_b$ meet in a point $\boldsymbol{q}_y$. Choose $\lambda_2, \tau_2 > 0$ such that $\boldsymbol{p}_{\underline{y}} = \lambda_2\sigma_2\boldsymbol{p}_{\overline{y}} + \tau_2\boldsymbol{q}_y \in \mathcal{B}$.

Here the signs $\sigma_1, \sigma_2 \in \{+1,-1\}$ are constants assumed to have been chosen correctly so that $\boldsymbol{p}_b$ and $\boldsymbol{p}_{\underline{y}}$ lie on the correct side of the points e $\boldsymbol{p}_a$ and $\boldsymbol{p}_{\overline{y}}$, respectively. After a moment's reflection it is clear that any realization of $\boldsymbol{P}_i$ can be constructed in this way.

Since $\mathcal{B}$ is open it is always $(d+1)$-dimensional, meaning that the choice of $\boldsymbol{p}_a$ is governed by strict linear inequalities in a constant-dimensional fiber. Therefore the projection from the realization space after the choice of $\boldsymbol{p}_a$ to $\mathcal{R}_{Y_{i-1}}(\boldsymbol{P}, B_0)$ is stable.

Similarly $\mathcal{B}$ must contain a segment about $\boldsymbol{p}_a$ of the (homogeneous) line $(1 \wedge 4) \vee \boldsymbol{p}_a$, on which we pick $\boldsymbol{p}_b$. This is always a two-dimensional region, so also $\boldsymbol{p}_b$ is chosen from a constant-dimensional fiber subject to strict linear inequalities and a linear equation. Hence the projection from the realization space after the choice of $\boldsymbol{p}_b$ to the first step is stable.

The third step is similar to the second, so we conclude that

$$\mathcal{R}_{Y_{i-1}}(\boldsymbol{P}(\mathcal{S}), B_0) \approx \mathcal{R}_{Y_i}(\boldsymbol{P}(\mathcal{S}), B_0)$$

by a sequence of stable projections.

Case 2. Let $\boldsymbol{B}_i$ be a transmitter $\boldsymbol{T}_X$, $X = (1, \dots, r)$, and suppose that F_i is $\operatorname{pyr}(\boldsymbol{G}(X), \overline{y})$. We use the labeling from Figure 13 and define $\boldsymbol{p}_j = j \wedge (j+1)$, $\boldsymbol{p}_{j'} = j' \wedge (j+1)'$ with indices modulo r. In any realization of $\boldsymbol{P}_i$, the lines $\boldsymbol{p}_1 \vee \boldsymbol{p}_{1'}, \dots, \boldsymbol{p}_r \vee \boldsymbol{p}_{r'}$ meet in a point $\boldsymbol{q}_y$, and $\boldsymbol{q}_y$ must lie in $\mathcal{A}$ by convexity of $\boldsymbol{P}_i$. This implies that a realization of $\boldsymbol{P}_i$ can be constructed from $\boldsymbol{P}_{i-1}$ in the following way:

(1) Choose $\boldsymbol{q}_y$ in $\mathcal{A}$;

(2) Choose a linear form h such that $h(\boldsymbol{p}) > 0$ for all $\boldsymbol{p} \in \operatorname{vert}(\boldsymbol{P}_{i-1})$, $h(\boldsymbol{q}_y) < 0$, and the intersections $\ker(h) \cap (\boldsymbol{p}_j \vee \boldsymbol{q}_y)$ lie in $\mathcal{B}$ for each $j \in \{1, \dots, r, \overline{y}\}$;

(3) For each $j \in \{1, \dots, r, \overline{y}\}$ define $\boldsymbol{p}_{j'} = H \cap (\boldsymbol{p}_j \vee \boldsymbol{q}_y)$.

These steps can be used to construct any realization of $\boldsymbol{P}_i$. In the first step $\boldsymbol{q}_y$ is chosen from a constant-dimensional region governed by strict linear inequalities, so the projection deleting $\boldsymbol{q}_y$ is stable.

In the second step the choice of the hyperplane $\ker(h)$, and thus of the coefficients of h, is subject to strict linear inequalities that ensure $\ker(h) \cap (\boldsymbol{p}_j \vee \boldsymbol{q}_y)$ lies in $\mathcal{A}$ and that $h_\infty(\ker(h) \cap (\boldsymbol{p}_j \vee \boldsymbol{q}_y)) > 0$ for all indices j. This region is nonempty since a hyperplane very close to $\ker(h_0)$ works. Thus the projection deleting h (identified with a point in $\mathbb{R}^{d+1}$) is also stable.

Step (3) describes a rational equivalence which maps h to the new points, so

$$\mathcal{R}_{Y_{i-1}}(\boldsymbol{P}(\mathcal{S}), B_0) \approx \mathcal{R}_{Y_i}(\boldsymbol{P}(\mathcal{S}), B_0)$$

by a rational homeomorphism followed by a sequence of stable projections.

Case 3. Let $\boldsymbol{B}_i$ be a transmitter $\boldsymbol{T}_X$, $X = (1, \dots, r)$, and suppose that F_i is $\operatorname{prism}(\boldsymbol{G}(X))$. We define $\boldsymbol{q}_y$ as the point where $\boldsymbol{p}_1 \vee \boldsymbol{p}_{1'}, \dots, \boldsymbol{p}_r \vee \boldsymbol{p}_{r'}$ meet. A realization of $\boldsymbol{P}_i$ can be constructed as follows:

(1) Choose $\boldsymbol{p}_{\overline{y}}$ in $\mathcal{B}$;

(2) Choose $\lambda, \tau > 0$ such that $\boldsymbol{p}_{\underline{y}} = \lambda \boldsymbol{p}_{\overline{y}} + \tau \boldsymbol{q}_y$.

The regions in which both points are chosen are constant-dimensional nonempty interiors of polyhedral sets, so

$$\mathcal{R}_{Y_{i-1}}(\boldsymbol{P}(\mathcal{S}), B_0) \approx \mathcal{R}_{Y_i}(\boldsymbol{P}(\mathcal{S}), B_0)$$

by similar reasoning as before.

Case 4. Let $\boldsymbol{B}_i$ be a forgetful transmitter $\boldsymbol{T}_X^Y$, where $X = (1, \dots, r), Y = (1', \dots, (r+1)')$, and suppose $F_i = \operatorname{pyr}(\boldsymbol{G}(Y), \underline{y})$. Compare with Figure 14 for labeling. This case is identical to Case 2, except that we choose an additional point x on an open line segment of $(1 \wedge 6) \vee \boldsymbol{q}_y$. By similar reasoning as before, this shows that $\mathcal{R}_{Y_{i-1}}(\boldsymbol{P}(\mathcal{S}), B_0)$ and $\mathcal{R}_{Y_i}(\boldsymbol{P}(\mathcal{S}), B_0)$ are stably equivalent.

Case 5. Let $\boldsymbol{B}_i$ be a forgetful transmitter $\boldsymbol{T}_X^Y$, where $X = (1, \dots, r), Y = (1', \dots, (r+1)')$, and suppose $F_i = \operatorname{pyr}(\boldsymbol{G}(X), \overline{y})$. This is again identical to Case 2, except that a new edge $(r+1)'$ must be added. The slope of $(r+1)'$ is determined in $\boldsymbol{P}(\mathcal{S})$, since this case only arises in specific places of the addition and multiplication polytopes. By construction of those polytopes, the new direction can be computed as a rational function of the other slopes. We can construct a realization of $\boldsymbol{P}_i$ as follows:

(1) Choose $\boldsymbol{q}_y$ in $\mathcal{A}$;

(2) Choose a linear form h such that $h(\boldsymbol{p}) > 0$ for all $\boldsymbol{p} \in \operatorname{vert}(\boldsymbol{P}_{i-1})$, $h(\boldsymbol{q}_y) < 0$, and the intersections $H \cap (\boldsymbol{p}_j \vee \boldsymbol{q}_y)$, where H is the hyperplane defined by h, lie in $\mathcal{B}$ for each $j \in \{1, \dots, r-1, \overline{y}\}$;

(3) Define the vertices as in Case 2, but truncate $1' \wedge r'$ with a new edge $(r+1)'$, and call the new points $\boldsymbol{p}_{r'}$, $\boldsymbol{p}_{(r+1)'}$. We require that both points lie in $\mathcal{B}$;

(4) Choose $x \in \mathcal{B}$ on the open line segment connecting $1 \wedge r$ and $1' \wedge r'$.

Since $\mathcal{B}$ is convex and open, the regions in which $(r+1)'$ and x can be chosen are nonempty. Both regions are constant-dimensional and defined by strict linear inequalities, so

$$\mathcal{R}_{Y_{i-1}}(\boldsymbol{P}(\mathcal{S}), B_0) \approx \mathcal{R}_{Y_i}(\boldsymbol{P}(\mathcal{S}), B_0)$$

by a sequence of stable projections and a rational equivalence.

Case 6. Let $\boldsymbol{B}_i$ be a $\boldsymbol{Y}_8$-polytope with $F_i = \operatorname{tent}^{1,5}(\boldsymbol{G}(1, \dots, 8))$, compare with Figure 16 for notation. We will work in the affine case, so we interpret $\mathcal{B}$ here as the dehomogoneized counterpart to the $\mathcal{B}$ defined earlier.

By construction of $\boldsymbol{P}(\mathcal{S})$, the points $1 \wedge 5$, $2 \wedge 6$, $3 \wedge 7$ and $4 \wedge 8$ lie on a line ℓ, which cannot intersect $\mathcal{B}$. Let H be a hyperplane intersecting ℓ and $\mathcal{B}$. With a suitable projective transformation, we can map ℓ to infinity so that $\boldsymbol{G}(1, \dots, 8)$ and H become parallel. We must choose $\boldsymbol{G}' = \boldsymbol{G}(1', \dots, 8')$ on H with edges parallel to $\boldsymbol{G} = \boldsymbol{G}(1, \dots, 8)$, so that $\boldsymbol{G}'$ is normal. Normality of both $\boldsymbol{G}$ and $\boldsymbol{G}'$ is forced by $\boldsymbol{P}(\mathcal{S})$. This suffices to determine $\boldsymbol{G}'$ up to the choice of parameter $x \in (1/3, 1)$ (see [28, Remark 6.1.2]) and a 2-dimensional projective transformation. Let τ be such a projective transformation, so that it maps $\boldsymbol{G}'$ into $H \cap \mathcal{B}$. Since $H \cap \mathcal{B}$ is an open set bounded by affine hyperplanes, the parameters of

τ are restrained by strict linear inequalities. The points of $\tau(\boldsymbol{G}')$ can now be calculated with rational functions, so

$$\mathcal{R}_{Y_{i-1}}(\boldsymbol{P}(\mathcal{S}), B_0) \approx \mathcal{R}_{Y_i}(\boldsymbol{P}(\mathcal{S}), B_0).$$

Cases 7 and 8. In both cases we add two new points, similar to Case 3.

Case 9. The only point to be added can be chosen anywhere in $\mathcal{B}$. □

It remains to show that the realizations of $\boldsymbol{S}(\mathcal{S})$ corresponding to realizations of $\boldsymbol{P}(\mathcal{S})$ are stably equivalent to the primary basic semialgebraic set $V(\mathcal{S})$.

Lemma 7.3. $\mathcal{R}_{Y_0}(\boldsymbol{P}(\mathcal{S}), B_0) \approx V(\mathcal{S})$.

Proof. Recall that $\boldsymbol{S}(\mathcal{S})$ was defined as $\operatorname{pyr}(\operatorname{pyr}(\boldsymbol{G}, a), b)$, where

$$\boldsymbol{G} = \boldsymbol{G}[\mathbf{0}, \mathbf{1}, x_1, \ldots, x_n, \boldsymbol{\infty}].$$

We choose the basis $B_0 = \{a, b, \mathbf{0} \wedge \boldsymbol{\infty}', \mathbf{0} \wedge \mathbf{1}, \boldsymbol{\infty}' \wedge x_n'\}$, and set without loss of generality

$$\mathbf{0} \wedge \boldsymbol{\infty}' = (0, 0), \quad \mathbf{0} \wedge \mathbf{1} = (1, 0), \quad \boldsymbol{\infty}' \wedge x_n' = (0, 1).$$

Recall that the realization space of a pyramid is isomorphic to the realization space of its basis, so it suffices to study the realizations of $\boldsymbol{G}$. By convexity, the remaining points of $\boldsymbol{G}$ have to lie in the positive quadrant of $\mathbb{R}^2$.

The realizations of $\boldsymbol{G}$ compatible with $\boldsymbol{P}(\mathcal{S})$ are the normalized computation frames with line slopes corresponding to a point in $V(\mathcal{S})$ by Theorem 7.1. Suppose first that the opposite edges in $\boldsymbol{G}$ are parallel, fix a point $(y_1, \ldots, y_n)$ in $V(\mathcal{S})$ and choose a slope $s_{\mathbf{1}} > 0$. The rest of the slopes s_{x_i} are determined by $y_i = (0, \infty \mid s_{\mathbf{1}}, s_{x_i})$. We position $\mathbf{1}$ so that it intersects $(1, 0)$, $x_{n'}$ so that it intersects $(0, 1)$, and continue to add lines in cyclic order starting from x_1, so that all previously defined vertices remain in convex positions. This parameterization of $\boldsymbol{G}$ corresponds to a set of strict linear inequalities, and the dimension of the region in which each line is chosen always has constant dimension. Hence $V(\mathcal{S})$ is stably equivalent to the set of all realizations of $\boldsymbol{G}$ with opposite edges parallel, parameterized by slopes and distances from the origin. The parameterization determines the vertices by a rational equivalence, and we can reach any remaining realization of $\boldsymbol{G}$ by projective transformations.

A projective transformation τ that leaves the basis fixed is uniquely determined by the position of a point $\boldsymbol{p} = \tau(\mathbf{1} \wedge x_i)$. The region in which $\boldsymbol{p}$ can be chosen so that $\tau(\boldsymbol{G})$ is convex is the interior of a convex polygon, and the dimension is again constant for any choice of $(y_1, \dots, y_n) \in V(\mathcal{S})$ since the region is open. Therefore we have

$$\mathcal{R}_{Y_0}(\boldsymbol{P}(\mathcal{S}), B_0) \approx V(\mathcal{S}),$$

by a sequence of stable projections and rational equivalences. □

We are now equipped to prove the universality theorem.

Theorem 7.4. *For every primary basic semialgebraic set V defined over $\mathbb{Z}$ there is a* 4*-polytope $\boldsymbol{P}$ such that $V \approx \mathcal{R}(\boldsymbol{P})$. The face lattice of $\boldsymbol{P}$ can be generated from the defining equations of V in polynomial time.*

Proof. By Theorem 4.13 there exists a Shor normal form $\mathcal{S}$ such that $V \approx V(\mathcal{S})$. By Lemma 7.3 we have $V(\mathcal{S}) \approx \mathcal{R}_{Y_0}(\boldsymbol{P}(\mathcal{S}), B_0)$, and Lemma 7.2 shows that

$$\mathcal{R}_{Y_0}(\boldsymbol{P}(\mathcal{S}), B_0) \approx \mathcal{R}_{Y_1}(\boldsymbol{P}(\mathcal{S}), B_0) \approx \dots \approx \mathcal{R}_{Y_m}(\boldsymbol{P}(\mathcal{S}), B_0) = \mathcal{R}(\boldsymbol{P}(\mathcal{S})).$$

Therefore $V \approx \mathcal{R}(\boldsymbol{P}(\mathcal{S}))$. Finally, $\mathcal{S}$ can be computed in polynomial time from the defining equations of V as can be deduced from the proof of Theorem 4.13, and the inductive construction of $\boldsymbol{P}(\mathcal{S})$ implies that its face lattice can be computed in polynomial time from $\mathcal{S}$. □

7.2 Applications

When the universality theorem for 4-polytopes was first published, it simultaneously resolved a number of long-standing conjectures in polytope theory; the main applications are described in [28, Part III]. We will recite a few of those results here, in order to demonstrate the strength and versatility of the universality theorem. The results pertain especially to the algorithmic and algebraic complexity of realizability of 4-polytopes, and to applications in other universality theorems.

Algorithmic complexity. In mathematical logic, the *existential theory of the reals* refers to the decision problem of determining the truth or falsity of a sentence

$$(\exists x_1) \cdots (\exists x_n)\ F(x_1, \dots, x_n),$$

for real variables $x_1, \dots, x_n$ and a quantifier-free formula $F(x_1, \dots, x_n)$ involving equations and inequalities of polynomials with real coefficients; see *e.g.* [3]. Likewise, by the *realizability problem for* 4*-polyotopes* we refer to the decision problem of determining whether a given combinatorial 4-polytope is realizable or not. As one form of Mnëv's unversality theorem, it had earlier been proven that the existential theory of the reals was equivalent to certain other realizability problems, such as that of oriented matroids, or polytopes without fixed dimension [25]. Richter-Gebert strengthened the result.

Theorem 7.5.

(1) *The realizability problem for* 4*-polytopes is polynomial-time equivalent to the existential theory of the reals.*

(2) *The realizability problem for* 4*-polytopes is NP-hard.*

In essence, the above result shows that the difficulty of determining whether combinatorial 4-polytopes are realizable is (polynomially) similar to deciding if a system of polynomial equations and inequalities has a solution. This is in contrast to the three-dimensional case, in which Steinitz's theorem implies faster methods.

Algebraic complexity. The task of finding the lowest-dimensional polytope with no rational realizations had previous to the universality theorem still been ongoing: the smallest known examples were 8-polytopes, see the discussion in [40]. Specifically, Micha Perles constructed an 8-polytope with 12 vertices that can be realized with coordinates over $\mathbb{Q}(\sqrt{5})$, but not over $\mathbb{Q}$. His construction was first published in Grünbaum's book [17].

The universality theorem for 4-polytopes immediately implies that there must be non-rational polytopes already in dimension four; by Steinitz's theorem this is the lowest possible. On the other hand, the least number of vertices required for such a polytope is still unknown. Richter-Gebert provided significant progress.

Theorem 7.6.

(1) *For every proper subfield A of the real algebraic numbers, there exists a* 4*-polytope not realizable over A.*

(2) *There exists a nonrational* 4*-polytope with* 33 *vertices.*

The specific construction yielding the second claim relies on an octagon with a list of forced collinearities, see [28, Section 9.2]. After some work one finds that realizing the resulting polytope requires $\sqrt{2}$ in the field of coordinates.

Universal partition theorems. We will briefly discuss one particularly interesting strengthening applicable to Mnëv's universality theorem and to the universality theorem for 4-polytopes. The topic has been discussed in detail at least in [28, 27, 18, 19].

The universal partition theorems deal with families of primary basic semialgebraic sets that partition the ambient space, as opposed to single sets. Accordingly, stable equivalence needs to be adapted to such families. We call a finite ordered collection $(V_1, \ldots, V_m)$ of pairwise disjoint primary basic semialgebraic sets $V_1, \ldots, V_m \subseteq \mathbb{R}^n$ a *semialgebraic family*.

Definition 7.7. Let $\mathcal{V} = (V_1, \ldots, V_m)$ and $\mathcal{W} = (W_1, \ldots, W_m)$ with $V_i \subseteq \mathbb{R}^n$, $W_i \subseteq \mathbb{R}^{n+d}$, be semialgebraic families such that $V_i = \pi(W_i)$ for each $1 \leqslant i \leqslant n$. If each projection $W_i \longrightarrow V_i$ is stable and use the same polynomial equations and inequalities for recovering the original set (cf. Definition 4.7), then $\mathcal{V}$ is said to be a *stable projection* of $\mathcal{W}$.

Definition 7.8. Two semialgebraic families $\mathcal{V} = (V_1, \ldots, V_m)$ and $\mathcal{W} = (W_1, \ldots, W_m)$ are *rationally equivalent* if there exists a homeomorphism

$$f : \bigcup_{i=1}^{m} V_i \longrightarrow \bigcup_{i=1}^{m} W_i$$

such that both f and f^{-1} are rational functions, and $f(V_i) = W_i$ for all $1 \leqslant i \leqslant m$.

Definition 7.9. The equivalence relation generated by stable projections and rational equivalence of semialgebraic families is called *stable equivalence*. Two semialgebraic families $\mathcal{V}$ and $\mathcal{W}$ that are in the same equivalence class with respect to stable equivalence are called *stably equivalent*, and we denote $\mathcal{V} \approx \mathcal{W}$.

The universal partition theorem for 4-polytopes deals with partitions of $\mathbb{R}^n$ generated by finite sets of polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$. A natural way of generating the partition is by examining the sets in which the values of the polynomials have some given signs.

Definition 7.10. Let $V \subseteq \mathbb{R}^n$ be a primary basic semialgebraic set and $f_1, \ldots, f_m$ polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$. For $\sigma \in \{-1, 0, +1\}^m$ we denote

$$V_\sigma = \{\boldsymbol{v} \in V \mid \operatorname{sign} f_i(\boldsymbol{v}) = \sigma_i \text{ for } 1 \leqslant i \leqslant m\}.$$

Note that V_σ is primary basic semialgebraic, and the collection $(V_\sigma)_{\sigma\in\{-1,0,+1\}^m}$ forms a partition V. In particular, $(V_\sigma)_{\sigma\in\{-1,0,+1\}^m}$ is a semialgebraic family. We are now able to state the universal partition theorem.

Theorem 7.11. *For any partition $\mathcal{V} = (V_\sigma)_{\sigma\in\{-1,0,+1\}^m}$ of $\mathbb{R}^n$ there exists a collection $(\boldsymbol{P}_\sigma)_{\sigma\in\{-1,0,+1\}^m}$ of 4-polytopes with a common affine basis B such that*

$$\mathcal{V} \approx (\mathcal{R}(\boldsymbol{P}_\sigma, B))_{\sigma\in\{-1,0,+1\}^m}.$$

The universal partition theorem directly implies the universality theorem for 4-polytopes, since any primary basic semialgebraic set can be expressed as part of a partition $(V_\sigma)_{\sigma\in\{-1,0,+1\}^m}$ of $\mathbb{R}^n$. Richter-Gebert's proof of the theorem is extremely technical, see [28, Section 11], but it can be verified that it still holds with our stricter version of stable equivalence. Harald Günzel presents another proof in [19].

8 Stable equivalence revisited

The aim of this section is to consolidate the existing work on stable equivalence into a coherent whole. The term has been used several times before, often in proofs of similar universality theorems, and nearly always with a definition differing slightly from the rest and catering to the specific application. In particular, we are interested in how exactly the definitions differ.

8.1 The list

The following list contains all notions related to stable equivalence of semialgebraic sets that we were able to find.

S1 Two semialgebraic sets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^{n+m}$ are stably equivalent if there exists a locally biregular homeomorphism f such that $W = f(V \times \mathbb{R}^m)$. Local biregularity means that in sufficiently small neighborhoods around each point of their respective domains, f and f^{-1} can be expressed as rational functions. This definition was used originally by Nikolai Mnëv in [25] to prove the universality theorem for realization spaces of oriented matroids, and later by Jürgen Bokowski and Bernd Sturmfels in [12].

S2 Two semialgebraic sets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ are stably equivalent if there exists a smooth manifold M and a diffeomorphism f such that $W = f(V \times M)$. This definition was used by Harald Günzel in [18].

S3 Let $V \subseteq \mathbb{R}^n$ be a primary basic semialgerbaic set obtained as a projection of the primary basic semialgebraic set $W \subseteq \mathbb{R}^{n+m}$ onto the first n-coordinates by the canonical projection π. The projection is called stable if the following conditions hold:

(a) The n-parametric family $F(\boldsymbol{v}) \subseteq \mathbb{R}^m$, $\boldsymbol{v} \in V$, with

$$F(\boldsymbol{v}) = \{\boldsymbol{w} \in \mathbb{R}^m \mid (\boldsymbol{v}, \boldsymbol{w}) \in W\},$$

consists of interiors of polyhedral sets;

(b) Each $F(\boldsymbol{v})$ can be described by

$$F(\boldsymbol{v}) = \left\{\boldsymbol{v} \in \mathbb{R}^m \mid \varphi_i^{\boldsymbol{v}}(\boldsymbol{w}) > 0 \text{ for } i \in X\right\},$$

where $\left\{\varphi_i^{\boldsymbol{v}} : \mathbb{R}^m \to \mathbb{R} \mid i \in X\right\}$ is a finite set of affine forms whose coefficients depend polynomially on $\boldsymbol{v}$.

This definiton was used by Harald Günzel in [19] for his alternative proof of Richter-Gebert's universality theorem.

S4 If $V \subseteq \mathbb{R}^n$ is a projection of the semialgebraic set $W \subseteq \mathbb{R}^{n+m}$ onto the first n-coordinates by the canonical projection π and we obtain W as

$$W = \left\{(\boldsymbol{v}, \boldsymbol{u}) \in \mathbb{R}^{n+m} \mid \boldsymbol{v} \in V,\ \varphi_i^{\boldsymbol{v}}(\boldsymbol{u}) > 0,\ \psi_j^{\boldsymbol{v}}(\boldsymbol{u}) = 0 \text{ for } i \in X,\ j \in Y\right\},$$

where each $\varphi_i^{\boldsymbol{v}}$ and $\psi_j^{\boldsymbol{v}}$ is a *linear* form whose parameters depend polynomially on $\boldsymbol{v}$, then π is a stable projection. Stable equivalence is, again, the equivalence relation generated by stable projections and rational equivalence. This definition was used in Richter-Gebert's original research report on his work on realization spaces of 4-polytopes [31].

S5 Two semialgebraic sets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ are *semialgebraically isomorphic* if they are homeomorphic by a function f such that the graph

$$G(f) = \{(\boldsymbol{v}, \boldsymbol{w}) \in V \times W \mid \boldsymbol{w} = f(\boldsymbol{v})\}$$

is a semialgebraic set. *Stable isomorphism* is the equivalence relation generated by semialgebraic isomorphisms and canonical projections $V \times \mathbb{R}^k \longrightarrow V$ for any natural k. This definition was used by Ruchira Datta in [14] to prove the universality theorem for Nash equilibria, and although it bears a different name it is similar in spirit.

S6 If $V \subseteq \mathbb{R}^n$ is a projection of the semialgebraic set $W \subseteq \mathbb{R}^{n+(m+m')}$ onto the first n-coordinates by the canonical projection π and we obtain W as

$$W = \left\{(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{u}') \mid \boldsymbol{v} \in V,\ \varphi_i(\boldsymbol{v}, \boldsymbol{u}) > 0,\ \psi_j(\boldsymbol{v}) = \boldsymbol{u}' \text{ for } i \in X,\ j \in Y\right\},$$

where each $\varphi_i : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}$ and $\psi_j : \mathbb{R}^n \longrightarrow \mathbb{R}^{m'}$ is a rational function such that the fibers $\pi^{-1}(\boldsymbol{v})$ are convex for each $\boldsymbol{v} \in V$, then π is a stable projection. This definition of stable projections, along with the usual rational equivalence, was used by Tobias Boege in his work on Gaussian independence models [8].

S7 Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^{n+m}$ be primary basic semialgebraic sets obtained as the disjoint union of connected primary basic semialgebraic sets

$$V = \bigsqcup_{i=1}^{k} V_i \quad \text{and} \quad W = \bigsqcup_{i=1}^{k} W_i.$$

If each V_i for $1 \leqslant i \leqslant k$ is a stable projection of the corresponding W_i in terms of Definition 4.4 in some order such that all share the same equations and inequalitites, then V is a stable projection of W. We found this definition on the wikipedia article for Mnëv's universality theorem, but have not been able to find a source in literature [39]. It could be that this is a slight misunderstanding of the definition of stable equivalence for semialgebraic families used in [27].

To avoid confusion, we will refer to the different definitions by their label in the preceding list; for example, stable isomorphism becomes S5 equivalence. The term "stable equivalence" will be reserved for the definition we have previously used.

8.2 Analysis

An important note is that S4 equivalence, although strictly stronger than Definition 4.4, is not a suitable fix since it still allows for a counterexample related to Example 4.6, namely

$$W = \left\{(v, u, u') \in \mathbb{R}^3 \mid v \in \mathbb{R},\ u' > 0,\ v(vu - u') = 0\right\}.$$

The set W again differs in connectedness from its projection $W \to \mathbb{R}$ around $v = 0$.

S1 equivalence is similar to stable equivalence in the sense that the fibers of the canonical projection $f^{-1}(W) \to V$ have constant dimension, in this case globally. The main difference is that this definition allows fibers which are not relative interiors of polyhedral sets. Preservation of homotopy type is immediately clear, since $V \times \mathbb{R}^m$ deformation retracts to $V \times \{0\} \cong V$. Preservation of algebraic number type, on the other hand, is a direct consequence of the local biregularity of f.

It intuitively appears that stable equivalence should imply S1 equivalence, at least if we assume globally constant dimensional fibers in the definition of stable projections. For a primary basic semialgebraic set $V \subseteq \mathbb{R}^n$ obtained as a stable projection of $W \subseteq \mathbb{R}^{n+m}$ we showed the existence of a global section $\sigma : V \to W$ in the proof of Lemma 4.9. Relatively open convex sets are homeomorphic to $\mathbb{R}^k$ for some particular k, so for each fiber $F_{\mathbf{v}}$ of the projection $W \to V$, $\mathbf{v} \in V$, there exists a homeomorphism $f_{\mathbf{v}} : F_{\mathbf{v}} \to \mathbb{R}^k$. The particular homeomorphism $f_{\mathbf{v}}$ should depend in some continuous sense on the hyperplanes bounding the polyhedral set $F_{\mathbf{v}}$, implying the existence of a homeomorphism $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ such that $W = f(V \times \mathbb{R}^k)$.

The issue is that we cannot guarantee piecewise biregularity of the global section σ, which means that it also cannot be guaranteed for the homeomorphism f. We still expect stable equivalence to imply S1 equivalence, but proving it would require some more tools. We do still get a few other results of this form.

Proposition 8.1. *Stable equivalence implies S*5 *equivalence.*

Proof. We will show that rational equivalence implies semialgebraic isomorphism, which equates to showing that rational functions are semialgebraic. Since S5 equivalence has no additional constraints for the projections, the claim will follow.

Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ be semialgebraic sets that are rationally equivalent by the homeomorphism $\varphi : V \to W$. Since φ is a rational function with rational coefficients, the equation $\boldsymbol{w} = \varphi(\boldsymbol{v})$ can be reformulated as $p(\boldsymbol{v}) = q(\boldsymbol{w})$ with some polynomials $p \in \mathbb{Z}[x_1, \dots, x_n]$ and $q \in \mathbb{Z}[x_1, \dots, x_m]$. Thus the graph of φ can be rewritten as

$$\begin{aligned} G(\varphi) &= \{(\boldsymbol{v}, \boldsymbol{w}) \in V \times W \mid \boldsymbol{w} = \varphi(\boldsymbol{v})\} \\ &= (V \times \mathbb{R}^m) \cap (\mathbb{R}^n \times W) \cap \left\{(\boldsymbol{v}, \boldsymbol{w}) \in \mathbb{R}^{n+m} \mid p(\boldsymbol{v}) = q(\boldsymbol{w})\right\}, \end{aligned}$$

which is clearly a semialgebraic set. □

Furthermore, it is easy to see that S3 equivalence implies stable equivalence: since S3 projections permit only strict inequalities their fibers must be constant dimensional globally. S3 equivalence also implies S6.

Lastly, stable equivalence implies S7 equivalence since we proved that stable equivalence preserves homotopy type: by Lemma 3.19 homotopy equivalent topological spaces cannot differ in connectedness. Thus the following diagram describes the current hierarchy of versions of stable equivalence:

$$\begin{tikzcd} & & \text{S3} \arrow[dl] \arrow[dr] & \\ & \text{Stable} \arrow[dl, dotted] \arrow[d] \arrow[dr] & & \text{S6} \\ \text{S1} & \text{S5} & \text{S7} & \end{tikzcd} \tag{8.1}$$

We omitted S2 equivalence since the specific manifold used affects its properties. For example, if the manifold M is contractible then S2 equivalence preserves homotopy type by the same proof as for S1 equivalence, but this is not guaranteed. We also chose to exclude S4 equivalence since it fails to preserve desired properties.

By [19], the diagram (8.1) implies that the universality theorem for 4-polytopes holds using any of the equivalences S3, S6, S5, S7 as well as with stable equivalence; indeed, the latter three are independently verified by this thesis. Morally it also should hold for S1 equivalence, as stable equivalence $V \approx W$ gives a similar homeomorphism $V \times \mathbb{R}^k \to W$, and also preserves algebraic number type.

9 Conclusions and future work

In this thesis we repaired Richter-Gebert's proof of the universality theorem for 4-polytopes. In particular, the work consisted of finding a satisfactory notion of stable equivalence which preserves homotopy type, and of verifying that the technical details of the rest of Richter-Gebert's proof still go through. In the process we provided more detailed proofs than what has been available thus far especially of existence of Shor's normal form in Theorem 4.13, of realization spaces being primary basic semialgebraic in Propositon 4.2, and of realizability of connected sums in Lemma 5.8. Lastly, we reviewed all existing notions of stable equivalence for semialgebraic sets and discussed their hierarchy.

A possible direction for future research regards the third claim in the result corresponding to our Lemma 4.9 in Richter-Gebert's research report [31] and in [29]. It is claimed that the "singularity structures" of stably equivalent sets (referring to S4 equivalence in Section 8) are "similar" or "equivalent." This claim is left without proof and the terminology is not elaborated upon. It was also entirely omitted from the book [28]. The following questions arise:

- Is there a satisfactory definition for singularities of a semialgebraic set V? What is meant by the singularity structure of V?
- Does stable equivalence in any of its forms preserve said singularity structure?

Regarding the first question, singularities of semialgebraic sets are discussed in the context of algebraic statistics, *e.g.*, in [10, 15]. In both cases the definition involves the Zariski closure of the semialgebraic set, which can lead to a lot of the structure being lost.

There are still many open questions related to polytopes and their realizations, see *e.g.* [28, Part IV] and [34, 6]. A polytope is called *simplicial* if all of its faces are simplices, and the *Koebe realization space* of a polytope $\boldsymbol{P}$ is the set of all realizations of $\boldsymbol{P}$ with all edges tangent to the unit sphere.

- Is every open primary basic semialgebraic set stably equivalent to the realization spaces of some simplicial 4-polytope?
- Are Koebe realization spaces of 3-polytopes universal?

Without fixing the dimension to four, an analogue of the first question was resolved in [2]. In the fixed dimension case the question remains open, although the affirmative answer was announced recently in Karim Adiprasito's blog [1]. Modifying Richter-Gebert's proof to yield the result for simplicial 4-polytopes has proved challenging.

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