

Numerisk Analys

FMNF05

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Kapitel 0: Fundamentals

0.1 Evaluating a Polynomial

0.1.1 a)

$$P(x) = 6x^4 + x^3 + 5x^2 + x + 1 = 1 + x(1 + x(5 + x(1 + 6x)))$$

With nested:

$$6 \cdot 1/3 + 1 = 3$$

$$3 \cdot 1/3 + 5 = 6$$

$$6 \cdot 1/3 + 1 = 3$$

$$3 \cdot 1/3 + 1 = 2$$

Without nested:

$$6 \cdot (1/3)^4 + (1/3)^3 + 5 \cdot (1/3)^2 + 1/3 + 1$$

$$6/81 + 1/27 + 5/9 + 1/3 + 1$$

$$6/81 + 3/81 + 45/81 + 27/81 + 81/81$$

$$162/81 = 2$$

b)

$$P(x) = -3x^4 + 4x^3 + 5x^2 - 5x + 1 = 1 + x(-5 + x(5 + x(4 - 3x)))$$

With nested:

$$-3 \cdot 1/3 + 4 = 3$$

$$3 \cdot 1/3 + 5 = 6$$

$$6 \cdot 1/3 - 5 = -3$$

$$-3 \cdot 1/3 + 1 = 0$$

Without nested:

$$-3 \cdot (1/3)^4 + 4 \cdot (1/3)^3 + 5 \cdot (1/3)^2 - 5 \cdot (1/3) + 1$$

$$-3/81 + 4/27 + 5/9 - 5/3 + 1$$

$$-3/81 + 12/81 + 45/81 - 135/81 + 81/81$$

$$0/81 = 0$$

c)

$$P(x) = 2x^4 + x^3 - x^2 + 1 = 1 + x(0 + x(-1 + x(1 + 2x)))$$

With nested:

$$2 \cdot 1/3 + 1 = 5/3$$

$$5/3 \cdot 1/3 - 1 = -4/9$$

$$-4/9 \cdot 1/3 = -4/27$$

$$-4/27 \cdot 1/3 + 1 = 77/81$$

Without nested:

$$2 \cdot (1/3)^4 + (1/3)^3 - (1/3)^2 + 1$$

$$2/81 + 1/27 - 1/9 + 1$$

$$2/81 + 3/81 - 9/81 + 81/81$$

$$77/81$$

0.1.2 a)

$$P(x) = 6x^3 - 2x^2 - 3x + 7 = 7 + x(-3 + x(-2 + 6x))$$

With nested:

$$6 \cdot (-1/2) - 2 = -5$$

$$-5 \cdot (-1/2) - 3 = -1/2$$

$$-1/2 \cdot (-1/2) + 7 = 29/4$$

b)

$$P(x) = 8x^5 - x^4 - 3x^3 + x^2 - 3x + 1 = 1 + x(-3 + x(1 + x(-3 + x(-1 + 8x))))$$

With nested:

$$8 \cdot (-1/2) - 1 = -5$$

$$-5 \cdot (-1/2) - 3 = -1/2$$

$$-1/2 \cdot (-1/2) + 1 = 5/4$$

$$5/4 \cdot (-1/2) - 3 = -29/8$$

$$-29/8 \cdot (-1/2) + 1 = 45/16$$

c)

$$P(x) = 4x^6 - 2x^4 - 2x + 4 = 4 + x(-2 + x(0 + x(0 + x(-2 + x(0 + 4x))))$$

With nested:

$$4 \cdot (-1/2) = -2$$

$$-2 \cdot (-1/2) - 2 = -1$$

$$-1 \cdot (-1/2) = 1/2$$

$$1/2 \cdot (-1/2) = -1/4$$

$$-1/4 \cdot (-1/2) - 2 = -15/8$$

$$-15/8 \cdot (-1/2) + 4 = 79/16$$

0.1.3

$$P(x) = x^6 - 4x^4 + 2x^2 + 1 = 1 + x^2(2 + x^2(-4 + x^2))$$

With nested:

$$(1/2)^2 - 4 = -15/4$$

$$-15/4 \cdot (1/2)^2 + 2 = 17/16$$

$$17/16 \cdot (1/2)^2 + 1 = 81/64$$

0.1.4 a)

$$P(x) = 1 + x(1/2 + (x - 2)(1/2 + (x - 3)(-1/2)))$$

With nested:

$$-1/2 \cdot (5 - 3) + 1/2 = -1/2$$

$$-1/2 \cdot (5 - 2) + 1/2 = -1$$

$$-1 \cdot 5 + 1 = -4$$

b)

$$P(x) = 1 + x(1/2 + (x - 2)(1/2 + (x - 3)(-1/2)))$$

With nested:

$$-1/2 \cdot (-1 - 3) + 1/2 = 5/2$$

$$5/2 \cdot (-1 - 2) + 1/2 = -7$$

$$-7 \cdot (-1) + 1 = 8$$

0.1.5 a)

$$P(x) = 4 + x(4 + (x - 1)(1 + (x - 2)(3 + 2(x - 3))))$$

With nested:

$$2 \cdot (1/2 - 3) + 3 = -2$$

$$-2 \cdot (1/2 - 2) + 1 = 4$$

$$4 \cdot (1/2 - 1) + 4 = 2$$

$$2 \cdot (1/2) + 4 = 5$$

b)

$$P(x) = 4 + x(4 + (x - 1)(1 + (x - 2)(3 + 2(x - 3))))$$

With nested:

$$2 \cdot (-1/2 - 3) + 3 = -4$$

$$-4 \cdot (-1/2 - 2) + 1 = 11$$

$$11 \cdot (-1/2 - 1) + 4 = -25/2$$

$$-25/2 \cdot (-1/2) + 4 = 41/4$$

0.1.6 a)

$$P(x) = a_0 + a_5x^5 + a_{10}x^{10} + a_{15}x^{15} = a_0 + x^5(a_5 + x^5(a_{10} + x^5(a_{15})))$$

$$a_{15}x^5 + a_{10} = b_1 \quad 5 \text{ multiplications and 1 addition}$$

$$b_1x^5 + a_5 = b_2 \quad (\text{since } x^5 \text{ is calculated}) \text{ 1 multiplications and 1 addition}$$

$$b_2x^5 + a_0 = b_3 \quad 1 \text{ multiplications and 1 addition}$$

$5 + 1 + 1 = 7$ multiplications, $1 + 1 + 1 = 3$ addition.

b)

$$P(x) = a_7x^7 + a_{12}x^{12} + a_{17}x^{17} + a_{22}x^{22} + a_{27}x^{27} = x^7(a_7 + x^5(a_{12} + x^5(a_{17} + x^5(a_{22} + x^5(a_{27}))))))$$

$$a_{27}x^5 + a_{22} = b_1 \quad 5 \text{ multiplications and 1 addition}$$

$$b_1x^5 + a_{17} = b_2 \quad (\text{since } x^5 \text{ is calculated}) \text{ 1 multiplications and 1 addition}$$

$$b_2x^5 + a_{12} = b_3 \quad 1 \text{ multiplications and 1 addition}$$

$$b_3x^5 + a_7 = b_4 \quad 1 \text{ multiplications and 1 addition}$$

$$b_4x^7 = b_4 \quad 2 \text{ multiplications}$$

$5 + 1 + 1 + 1 + 1 + 2 = 10$ multiplications, $1 + 1 + 1 + 1 = 4$ addition.

0.1.7 n multiplications, $2n$ addition.

(c) 0.1.1

```
format long
x = 1.00001;
p = nest(50, ones(1,51), x)
q = (x^51-1)/(x-1)
estError = abs(p-q)
```

Output:

```
p=51.012752082749991
q=51.012752082745230
estError=0.000000000004761
```

(c) 0.1.2

$$P(x) = 1 - x + x^2 - x^3 + \dots + x^{98} - x^{99} = 1 - x + x^2(1 - x) + \dots + x^{98}(1 - x) = \sum_{k=0}^{49} x^{2k}(1 - x) = (1 - x) \sum_{k=0}^{49} (x^2)^k = (1 - x) \frac{1 - (x^2)^{50}}{1 - x^2} = 1 - x^{100}$$

```
format long
x = 1.00001;
p = nest(99, (-1).^(0:99), x)
q = (1-x^100)
estError = abs(p-q)
```

Output:

```
p=-0.000500245079648
q=-0.001000495161746
estError=0.000500250082098
```

0.2 Binary Numbers

0.2.1 a)

$$\begin{aligned} 64/2 &= 32 \text{ R } 0 \\ 32/2 &= 16 \text{ R } 0 \\ 16/2 &= 8 \text{ R } 0 \\ 8/2 &= 4 \text{ R } 0 \\ 4/2 &= 2 \text{ R } 0 \\ 2/2 &= 1 \text{ R } 0 \\ 1/2 &= 0 \text{ R } 1 \\ (64)_{10} &= (1000000)_2 \end{aligned}$$

b)

$$\begin{aligned} 17/2 &= 8 \text{ R } 1 \\ 8/2 &= 4 \text{ R } 0 \\ 4/2 &= 2 \text{ R } 0 \\ 2/2 &= 1 \text{ R } 0 \\ 1/2 &= 0 \text{ R } 1 \\ (17)_{10} &= (10001)_2 \end{aligned}$$

c)

$$\begin{aligned}79/2 &= 32 \text{ R } 1 \\39/2 &= 19 \text{ R } 1 \\19/2 &= 9 \text{ R } 1 \\9/2 &= 4 \text{ R } 1 \\4/2 &= 2 \text{ R } 0 \\2/2 &= 1 \text{ R } 0 \\1/2 &= 0 \text{ R } 1 \\(79)_{10} &= (1001111)_2\end{aligned}$$

d)

$$\begin{aligned}227/2 &= 113 \text{ R } 1 \\113/2 &= 56 \text{ R } 1 \\56/2 &= 28 \text{ R } 0 \\28/2 &= 14 \text{ R } 0 \\14/2 &= 7 \text{ R } 0 \\7/2 &= 3 \text{ R } 1 \\3/2 &= 1 \text{ R } 1 \\1/2 &= 0 \text{ R } 1 \\(227)_{10} &= (11100011)_2\end{aligned}$$

0.2.2 a)

$$\begin{aligned}1/8 \cdot 2 &= 1/4 \text{ R } 0 \\1/4 \cdot 2 &= 1/2 \text{ R } 0 \\1/2 \cdot 2 &= 0 \text{ R } 1 \\(1/8)_{10} &= (.001)_2\end{aligned}$$

b)

$$\begin{aligned}7/8 \cdot 2 &= 3/4 \text{ R } 1 \\3/4 \cdot 2 &= 1/2 \text{ R } 1 \\1/2 \cdot 2 &= 0 \text{ R } 1 \\(7/8)_{10} &= (.111)_2\end{aligned}$$

c)

It's larger than 2, factor it out.

Integer part:

$$\begin{aligned}2/2 &= 1 \text{ R } 0 \\1/2 &= 56 \text{ R } 1\end{aligned}$$

Fractional part:

$$\begin{aligned}3/16 \cdot 2 &= 3/8 \text{ R } 0 \\3/8 \cdot 2 &= 3/4 \text{ R } 0 \\3/4 \cdot 2 &= 1/2 \text{ R } 1 \\1/2 \cdot 2 &= 0 \text{ R } 1 \\(35/16)_{10} &= (10.0011)_2\end{aligned}$$

d)

$$31/64 \cdot 2 = 31/32 \text{ R } 0$$

$$31/32 \cdot 2 = 15/16 \text{ R } 1$$

$$15/16 \cdot 2 = 7/8 \text{ R } 1$$

$$7/8 \cdot 2 = 3/4 \text{ R } 1$$

$$3/4 \cdot 2 = 1/2 \text{ R } 1$$

$$1/2 \cdot 2 = 0 \text{ R } 1$$

$$(31/64)_{10} = (.011111)_2$$

0.2.3 a)

Solve the integer and fractional part separately.

Integer part:

$$10/2 = 5 \text{ R } 0$$

$$5/2 = 2 \text{ R } 1$$

$$2/2 = 1 \text{ R } 0$$

$$1/2 = 0 \text{ R } 1$$

Fractional part:

$$.5 \cdot 2 = 0 \text{ R } 1$$

Sum:

$$(10.5)_{10} = (1010.1)_2$$

b)

$$1/3 \cdot 2 = 2/3 \text{ R } 0$$

$$2/3 \cdot 2 = 1/3 \text{ R } 1$$

$$1/3 \cdot 2 = 2/3 \text{ R } 0$$

The period is two.

$$(1/3)_{10} = (.0\overline{1})_2$$

c)

$$5/7 \cdot 2 = 3/7 \text{ R } 1$$

$$3/7 \cdot 2 = 6/7 \text{ R } 0$$

$$6/7 \cdot 2 = 5/7 \text{ R } 1$$

$$5/7 \cdot 2 = 3/7 \text{ R } 1$$

The period is three.

$$(5/7)_{10} = (.1\overline{01})_2$$

- d) Solve the integer and fractional part separately.

Integer part:

$$12/2 = 6 \text{ R } 0$$

$$6/2 = 3 \text{ R } 0$$

$$3/2 = 1 \text{ R } 1$$

$$1/2 = 0 \text{ R } 1$$

Fractional part:

$$.8 \cdot 2 = .6 \text{ R } 1$$

$$.6 \cdot 2 = .2 \text{ R } 1$$

$$.2 \cdot 2 = .4 \text{ R } 0$$

$$.4 \cdot 2 = .8 \text{ R } 0$$

$$.8 \cdot 2 = .6 \text{ R } 1$$

The period is four.

Sum:

$$(12.8)_{10} = (1100.\overline{1100})_2$$

- e) Solve the integer and fractional part separately.

Integer part:

$$55/2 = 27 \text{ R } 1$$

$$27/2 = 13 \text{ R } 1$$

$$13/2 = 6 \text{ R } 1$$

$$6/2 = 3 \text{ R } 0$$

$$3/2 = 1 \text{ R } 1$$

$$1/2 = 0 \text{ R } 1$$

Fractional part:

$$.4 \cdot 2 = .8 \text{ R } 0$$

$$.8 \cdot 2 = .6 \text{ R } 1$$

$$.6 \cdot 2 = .2 \text{ R } 1$$

$$.2 \cdot 2 = .4 \text{ R } 0$$

$$.4 \cdot 2 = .8 \text{ R } 0$$

The period is four.

Sum:

$$(55.4)_{10} = (110111.\overline{0110})_2$$

f)

$$.1 \cdot 2 = .2 \text{ R } 0$$

$$.2 \cdot 2 = .4 \text{ R } 0$$

$$.4 \cdot 2 = .8 \text{ R } 0$$

$$.8 \cdot 2 = .6 \text{ R } 1$$

$$.6 \cdot 2 = .2 \text{ R } 1$$

$$.2 \cdot 2 = .4 \text{ R } 0$$

The period is four after first bit.

$$(0.1)_{10} = (0.0001\overline{1})_2$$

0.2.4 a) Solve the integer and fractional part separately.

Integer part:

$$11/2 = 5 \text{ R } 1$$

$$5/2 = 2 \text{ R } 1$$

$$2/2 = 1 \text{ R } 0$$

$$1/2 = 0 \text{ R } 1$$

Fractional part:

$$.25 \cdot 2 = .5 \text{ R } 0$$

$$.5 \cdot 2 = 0 \text{ R } 1$$

Sum:

$$(11.25)_{10} = (1101.01)_2$$

b)

$$2/3 \cdot 2 = 1/3 \text{ R } 1$$

$$1/3 \cdot 2 = 2/3 \text{ R } 0$$

$$2/3 \cdot 2 = 1/3 \text{ R } 1$$

The period is two.

$$(2/3)_{10} = (.1\overline{0})_2$$

c)

$$3/5 = 0.6$$

$$.6 \cdot 2 = .2 \text{ R } 1$$

$$.2 \cdot 2 = .4 \text{ R } 0$$

$$.4 \cdot 2 = .8 \text{ R } 0$$

$$.8 \cdot 2 = .6 \text{ R } 1$$

$$.6 \cdot 2 = .2 \text{ R } 1$$

The period is four.

$$(3/5)_{10} = (.1\overline{001})_2$$

- d) Solve the integer and fractional part separately.

Integer part:

$$3/2 = 1 \text{ R } 1$$

$$1/2 = 0 \text{ R } 1$$

Fractional part:

$$.2 \cdot 2 = .4 \text{ R } 0$$

$$.4 \cdot 2 = .8 \text{ R } 0$$

$$.8 \cdot 2 = .6 \text{ R } 1$$

$$.6 \cdot 2 = .2 \text{ R } 1$$

$$.2 \cdot 2 = .4 \text{ R } 0$$

The period is four.

Sum:

$$(3.2)_{10} = (11.\overline{0011})_2$$

- e) Solve the integer and fractional part separately.

Integer part:

$$30/2 = 15 \text{ R } 0$$

$$15/2 = 7 \text{ R } 1$$

$$7/2 = 3 \text{ R } 1$$

$$3/2 = 1 \text{ R } 1$$

$$1/2 = 0 \text{ R } 1$$

Fractional part:

$$.6 \cdot 2 = .2 \text{ R } 1$$

$$.2 \cdot 2 = .4 \text{ R } 0$$

$$.4 \cdot 2 = .8 \text{ R } 0$$

$$.8 \cdot 2 = .6 \text{ R } 1$$

$$.6 \cdot 2 = .2 \text{ R } 1$$

The period is four.

Sum:

$$(30.6)_{10} = (11110.\overline{1001})_2$$

- f) Solve the integer and fractional part separately.

Integer part:

$$99/2 = 49 \text{ R } 1$$

$$49/2 = 24 \text{ R } 1$$

$$24/2 = 12 \text{ R } 0$$

$$12/2 = 6 \text{ R } 0$$

$$6/2 = 3 \text{ R } 0$$

$$3/2 = 1 \text{ R } 1$$

$$1/2 = 0 \text{ R } 1$$

Fractional part:

$$.9 \cdot 2 = .8 \text{ R } 1$$

$$.8 \cdot 2 = .6 \text{ R } 1$$

$$.6 \cdot 2 = .2 \text{ R } 1$$

$$.2 \cdot 2 = .4 \text{ R } 0$$

$$.4 \cdot 2 = .8 \text{ R } 0$$

$$.8 \cdot 2 = .6 \text{ R } 1$$

The period is four after the first bit.

Sum:

$$(99.9)_{10} = (1100011.1\overline{1100})_2$$

0.2.5 Solve the integer and fractional part separately. At least 4 decimal points (3.1416) will give the correct answer.

Integer part:

$$3/2 = 1 \text{ R } 1$$

$$1/2 = 0 \text{ R } 1$$

Fractional part:

$$.14159265358979 \cdot 2 = .28318530717958 \text{ R } 0$$

$$.28318530717958 \cdot 2 = .56637061435916 \text{ R } 0$$

$$.56637061435916 \cdot 2 = .13274122871832 \text{ R } 1$$

$$.13274122871832 \cdot 2 = .26548245743664 \text{ R } 0$$

$$.26548245743664 \cdot 2 = .53096491487328 \text{ R } 0$$

$$.53096491487328 \cdot 2 = .06192982974656 \text{ R } 1$$

$$.06192982974656 \cdot 2 = .12385965949312 \text{ R } 0$$

$$.12385965949312 \cdot 2 = .24771931898624 \text{ R } 0$$

$$.24771931898624 \cdot 2 = .49543863797248 \text{ R } 0$$

$$.49543863797248 \cdot 2 = .99087727594496 \text{ R } 0$$

$$.99087727594496 \cdot 2 = .98175455188992 \text{ R } 1$$

$$.98175455188992 \cdot 2 = .96350910377984 \text{ R } 1$$

$$.96350910377984 \cdot 2 = .92701820755968 \text{ R } 1$$

Sum:

$$(\pi)_{10} \approx (11.0010010000111)_2$$

0.2.6 Do it the same way as in the last exercise. At least 4 decimal points (2.7183) will give the correct answer.

$$(e)_{10} \approx (10.1011011111100)_2$$

0.2.7 a)

$$2^6 + 2^4 + 2^2 + 1 = 64 + 16 + 4 + 1 = 85$$

b) Solve the integer and fractional part separately.

Integer part:

$$2^3 + 2^1 + 1 = 8 + 2 + 1 = 11$$

Fractional part:

$$1/2 + 1/8 = 5/8 = .625$$

Sum:

$$(1011.101)_2 = (11.625)_{10}$$

c) Solve the integer and fractional part separately.

Integer part:

$$2^4 + 2^2 + 2^1 + 1 = 16 + 4 + 2 + 1 = 23$$

Fractional part:

$$x = (.0\overline{1})_2$$

$$2^2 x = (01.0\overline{1})_2$$

$$(2^2 - 1)x = (01.0\overline{1})_2 - (.0\overline{1})_2 = (01)_2 = 1 \Leftrightarrow x = \frac{1}{4-1} = 1/3$$

Sum:

$$(10111.0\overline{1})_2 = (23 + 1/3)_{10} = (70/3)_{10}$$

d) Solve the integer and fractional part separately.

Integer part:

$$2^2 + 2^1 = 4 + 2 = 6$$

Fractional part:

$$x = (.1\overline{0})_2$$

$$2^2 x = (10.1\overline{0})_2$$

$$(2^2 - 1)x = (10.1\overline{0})_2 - (.1\overline{0})_2 = (10)_2 = 2 \Leftrightarrow x = \frac{2}{4-1} = 2/3$$

Sum:

$$(110.1\overline{0})_2 = (6 + 2/3)_{10} = (20/3)_{10}$$

e) Solve the integer and fractional part separately.

Integer part:

$$2^1 = 2$$

Fractional part:

$$x = (.1\overline{10})_2$$

$$2^3 x = (110.1\overline{10})_2$$

$$(2^3 - 1)x = (110.1\overline{10})_2 - (.1\overline{10})_2 = (110)_2 = 6 \Leftrightarrow x = \frac{6}{8-1} = 6/7$$

Sum:

$$(10.1\overline{10})_2 = (2 + 6/7)_{10} = (20/7)_{10}$$

f) Solve the integer and fractional part separately.

Integer part:

$$2^2 + 2^1 = 4 + 2 = 6$$

Fractional part:

$$x = (.1\overline{101})_2$$

$$y = 2x = (1.\overline{101})_2$$

$$z = (.1\overline{01})_2$$

$$2^3 z = (101.\overline{101})_2$$

$$(2^3 - 1)z = (101.\overline{101})_2 - (.1\overline{01})_2 = (101)_2 = 5 \Leftrightarrow z = \frac{5}{8-1} = 5/7$$

$$y = 1 + 5/7 = 12/7 \Leftrightarrow x = y/2 = 6/7$$

Sum:

$$(110.1\overline{101})_2 = (6 + 6/7)_{10} = (48/7)_{10}$$

f) Solve the integer and fractional part separately.

Integer part:

$$2^1 = 2$$

Fractional part:

$$x = (.010\overline{1101})_2$$

$$y = 2^3 x = (010.\overline{1101})_2$$

$$z = (.1\overline{101})_2$$

$$2^4 z = (1101.\overline{1101})_2$$

$$(2^4 - 1)z = (1101.\overline{1101})_2 - (.1\overline{101})_2 = (1101)_2 = 13 \Leftrightarrow z = \frac{13}{16-1} = 13/15$$

$$y = 2 + 13/15 = 43/15 \Leftrightarrow x = y/8 = 43/120$$

Sum:

$$(10.010\overline{1101})_2 = (2 + 43/120)_{10} = (283/120)_{10}$$

e) Solve the integer and fractional part separately.

Integer part:

$$2^2 + 2^1 + 1 = 4 + 2 + 1 = 7$$

Fractional part:

$$x = (.1)_{2^2}$$

$$2x = (1.\overline{1})_2$$

$$(2 - 1)x = (1.\overline{1})_2 - (.1)_{2^2} = (1)_2 = 1 \Leftrightarrow x = \frac{1}{2-1} = 1$$

Sum:

$$(111.\overline{1})_2 = (7 + 1)_{10} = (8)_{10}$$