Beyond Single Factors: Multifactor Volatility-Timing in Norway

Master Thesis

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ABSTRACT

We study whether volatility-timing strategies applied to well-known asset pricing factors improve risk-adjusted returns in the Norwegian stock market. We do it by constructing individual-factors, individual-factor portfolios and mean-variance multifactor portfolios, applying volatility management techniques based on factor and market volatility, and evaluating their out-of-sample performance. We find that individual volatility-managed factors and individual volatility-managed factor portfolios generally fail to deliver statistically significant improvements, whereas a volatility-managed multifactor portfolio based on market volatility achieves a modestly significant Sharpe ratio increase. We conclude that while volatility-timing may enhance performance under specific conditions, the results lack robustness when accounting for time periods, factor dependencies, and real-world frictions, questioning their practical applicability.

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[1]Special thanks to Raman Uppal at EDHEC for his insightful teaching on the topic and for patiently addressing all our questions related to his paper, which serves as the main reference for this thesis.

Disclaimer

This thesis does not include text generated or suggested directly by AI. We used Grammarly and Writefull to check for grammar and spelling mistakes, deciding ourselves whether to accept or reject the suggestions. ChatGPT was used to improve the clarity and structure of the text. For the coding part of the research, we used GitHub Copilot to make the code more efficient and ChatGPT to suggest or improve parts of the code. Additionally, we used Perplexity to help conduct research online, ensuring access to a broad range of resources and information.

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1 Introduction

The pursuit of higher returns has always been a central goal for investors, yet it comes attached to the inescapable reality of risk. Conventional financial theory, embodied in models like the Capital Asset Pricing Model (CAPM), posits a stable, linear relationship between the two: greater risk should, in theory, yield greater reward. But what if this relationship falters under scrutiny? What if the tools we use to balance risk and return could be sharpened to better navigate turbulent markets? This thesis investigates these questions in the context of the Norwegian stock market, exploring whether dynamic strategies can enhance portfolio performance where static assumptions fall short.

Specifically, we examine the potential of volatility-timing strategies, methods that adjust investment exposure based on past volatility, to improve risk-adjusted returns. Our focus is twofold. First, we assess whether volatility-timing can enhance the performance of well-known asset pricing factors–such as market, size, value, momentum, and betting-against-beta–in Norway's equity market. We specifically test their ability to deliver superior out-of-sample returns. Second, we construct multifactor portfolios that are dynamically adjusted based on past volatility and test whether they outperform a traditional static benchmark. These questions strike at the heart of modern portfolio management and challenge the predictability of risk-return dynamics.

This study builds on a growing body of research on volatility-timing strategies. The foundational work by Moreira and Muir (2017) introduced the concept of scaling investment exposure inversely to past variance to enhance Sharpe ratios, an idea that has shown promise in the context of the Norwegian stock market. However, subsequent critiques by Cederburg et al. (2020) and Barroso and Detzel (2021) have highlighted significant limitations, including inconsistent out-of-sample performance and the detrimental impact of transaction costs. Building on these critiques, DeMiguel et al. (2024) advocated for the use of multi-factor portfolios that incorporate real-world frictions in out-of-sample testing. In this thesis, we replicate their methodological approach, though we do not account for transaction costs in our empirical analysis. Nonetheless, we critically examine the implications of real-world frictions, such as transaction costs, on the efficacy of volatility-timing strategies. By applying these frameworks to Oslo Børs, our aim is to ascertain whether volatility-timing offers tangible benefits for investors in the Norwegian stock market.

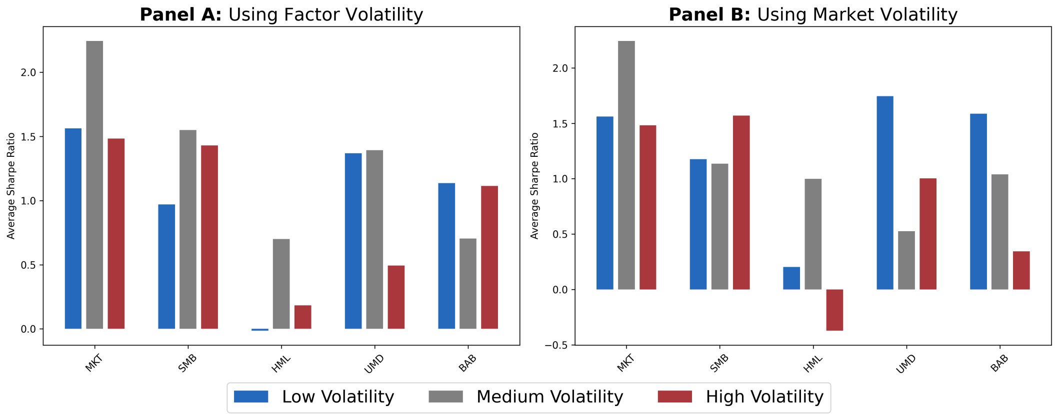
Figure 1 illustrates the relationship between realized volatility in period t and the average risk-adjusted return in period $t + 1$. Previous studies in Norway have found this relationship to be statistically significant, particularly for specific factors when using their own volatility measures. However, as shown in the figure, our analysis suggests that this significance is not consistent across all factors. Moreover, results vary significantly depending on whether market volatility or factor-specific volatility is used, with only a few factors showing notably higher Sharpe ratios during low volatility periods. In light of these findings, this thesis aims to re-evaluate and expand on these observations using improved methodologies and to investigate the potential benefits of a multi-factor portfolio approach.

Our analysis addresses two core research questions: Can volatility-managed individual asset pricing factors deliver superior out-of-sample returns in the Norwegian market? And does a multifactor portfolio, with weights adjusted inversely to volatility, outperform its unconditional counterpart? The answers promise to illuminate the efficacy of volatility-timing in a smaller, unique market setting.

The thesis proceeds as follows. Section 2 reviews the theoretical and empirical foundations of volatility-timing and multi-factor models. Section 3 outlines our methodology and hypotheses. Section 4 describes the data and preprocessing steps, while Section 5 presents our empirical findings. Section 6 concludes

Figure 1: Factor Risk-Return Trade-Off Across Volatility Regimes

This plot illustrates how the risk-return trade-off of the five factors varies across different levels of realized volatility. Specifically, we sort the months in our sample into terciles based on the monthly time series of realized volatility in time t. For each category, we compute the risk-return trade-off from the factor using the return and volatility in $t+1$ and report the average value in each category. Panel A uses the factors own volatility and Panel B uses only the markets volatility. The time period is 1993-2023.



with implications and directions for future research. Through this structure, we seek to contribute to the nuanced relationship of risk and return in the Norwegian stock market.

2 Literature Review

This literature review examines the existing research on volatility-managed investment strategies, with a primary focus on individual factor-based approaches and multifactor portfolio applications. A complementary review is provided in Appendix A, including foundational literature on modern portfolio theory, asset pricing factors, performance evaluation techniques, and additional studies related to volatility management.

2.1 Volatility-Timing Factor Strategies

Volatility-timing strategies have received substantial attention in academia, the financial press, and among practitioners. BlackRock describes their Managed Volatility V.I. Fund as follows: "In periods of heightened volatility, the portfolio will de-risk into less volatile assets like fixed income and cash and re-risk when market turbulence subsides." This statement captures the essence of volatility-timed strategies. However, academic opinion remains divided regarding their efficacy. Although Asness (2016) of AQR expresses skepticism about the feasibility and benefits of factor timing, Moreira and Muir (2017) argue that volatility-timing is both achievable and advantageous.

Moreira and Muir (2017) demonstrate that investors can improve the Sharpe ratio by reducing exposure to risk factors during periods of elevated volatility, which can also produce larger alphas. Their approach involves scaling down factor exposures following market crashes–such as those experienced in 1929, 1987, and 2008–based on the observation that returns do not rise proportionally with risk during volatile periods. By lowering the exposure when volatility is high, investors can reduce risk to a greater extent than returns.

To evaluate the effectiveness of their strategy, they run a regression on the volatility-managed factor and its counterpart. They argue, under theoretical assumptions, that the intercept in the regression should be zero, implying a proportional relationship between risk and return. However, significant deviations from zero suggest that increased risk is not compensated by proportional returns[2]. Their empirical results indicate statistically significant coefficients and nonzero intercept values, thereby supporting the argument that volatility-timing can enhance portfolio performance.

Several attempts have been made to replicate and extend this methodology using Norwegian equity data. Johansen and Eckhoff (2016) report that volatility-timing generally leads to higher alphas and improved Sharpe ratios, further showing that volatility forecasts derived from GARCH models are effective. Similarly, Bakken and Horvei (2022) document significant alphas for the factors HML, PR1YR, and UMD when applying the approach of Moreira and Muir (2017). However, they contend that these results are not feasible in real-world settings, mainly due to trading frictions, citing the findings of Barroso and Detzel (2021). Furthermore, Alme and Aarsland (2022) observe that volatility-timing improves returns, particularly during financial crises, although the benefits appear to be reduced in the post-COVID-19 period.

2.1.1 Limitations and Critiques of Volatility-Timing Strategies

The conclusions drawn by Moreira and Muir (2017) have been met with considerable criticism. Cederburg et al. (2020) argue that their reported gains are not replicable in out-of-sample tests, attributing the observed results to a look-ahead bias originating from the way they define the alpha values. Their replication and extension of the strategy reveal no consistent evidence that volatility-managed portfolios consistently outperform. Specifically, they find that only 8 of 103 volatility-managed factors exhibit Sharpe ratios signifi-

[2]Under the assumptions of Markowitz and Black-Litterman, they argue that it holds that $\mu_{t,mkt} = \gamma_{mkt}\sigma^2_{t,mkt}$, supporting the expectation of $\alpha = 0$. We will revisit this view in Section 2.1.1 as it is critized by Cederburg et al. (2020).

cantly different from their unmanaged counterparts, with these instances concentrated primarily in momentum-based strategies.

Additionally, Barroso and Detzel (2021) demonstrate that once transaction costs are accounted for, the advantages of volatility-timing dissipate largely. They show that such strategies often entail prohibitively high turnover, particularly due to large positions in small-cap stocks, leading to elevated trading costs. The sole exception appears to be the market factor, which retains its importance owing to lower associated transaction costs. However, even for the market factor, improved performance is restricted to periods characterized by elevated investor sentiment, in line with theories suggesting a sentiment-driven underreaction to volatility fluctuations.

2.1.2 Multifactor Frameworks for Volatility-Timing

In response to these critiques, DeMiguel et al. (2024) propose an improved framework for volatility-timing that addresses the identified limitations. They begin by constructing individual-factor portfolios managed by variance using the methodology of Moreira and Muir (2017), subsequently forming a mean-variance portfolio that combines the managed and the unmanaged factors. To counteract the concerns raised by Cederburg et al. (2020), they employ a bootstrapping approach to assess statistical significance robustly. They also explicitly model transaction costs and account for trading diversification in the optimization problem to counteract the critics raised by Barroso and Detzel (2021).

Their findings indicate that, when transaction costs and trading diversification are properly incorporated, only the UMD, IA, and BAB, 3 of 9, factors exhibit statistically significant improvements in Sharpe ratios, with IA significant at the level 5% and the others at 10%.

Building on these insights, they introduce a new volatility-timing framework characterized by four key features:

1. The use of multi-factor portfolios instead of isolated factor portfolios, extending the parametric portfolio policy framework.
2. Dynamic adjustment of relative factor weights, as opposed to static weights, allowing weights to vary with volatility.
3. Incorporation of trading diversification into transaction cost calculations, following the methodology of DeMiguel et al. (2020).
4. Optimization of factor weights while explicitly integrating transaction costs into the objective function.

Using this refined approach, they demonstrate that volatility-managed multi-factor portfolios consistently outperform both their unconditional counterparts and the earlier strategies proposed by Moreira and Muir (2017) even out-of-sample. They find that the drivers for their portfolio, are split into three parts. Firstly, the conditional multifactor portfolio has greater trading diversification then its unmanaged counterpart. Secondly, their approach takes transaction costs into the mean-variance optimization, helping to penalizing factors with high costs. Lastly, allowing the weights to adjust with volatility for each factor. Section 3 will explain their methodology in detail.

3 Methodology and Testable Hypotheses

This section outlines the methodology for constructing volatility-managed *individual*-factors, volatility-managed *individual*-factor portfolios, unconditional volatility-managed *multifactor* portfolios, and conditional volatility-managed *multifactor* portfolios by replicating the methodology of DeMiguel et al. (2024).

3.1 Hypothesis for Research Question 1 (RQ-1)

> *Can well-known asset pricing factors be volatility-managed to achieve superior returns, on average, out-of-sample in the Norwegian stock market?*

To address RQ-1, we begin by defining volatility-managed *individual*-factors following the methodology of Moreira and Muir (2017). Specifically, the return of the kth volatility-managed individual-factor is given by:

$$r_{k,t+1}^{\sigma_k} = \frac{c}{\sigma_{k,t}^2} r_{k,t+1}, \tag{1}$$

where $r_{k,t+1}$ represents the unmanaged return of the kth factor in month $t+1$, $\sigma_{k,t}^2$ is the realized variance of the kth factor for month t used as a proxy for the variance in $t+1$, estimated using the sample variance of daily factor returns, and c is a scaling parameter ensuring that the volatility of the managed factor matches that of the unmanaged factor[3].

Furthermore, we construct an alternative set of returns, $r_{k,t+1}^{\sigma_m}$, where factor returns are scaled by market variance instead of their own variance. This follows the same formulation as Equation 1, but replaces $\sigma_{k,t}^2$ with the market's realized variance, $\sigma_{m,t}^2$[4].

[3] The variable c is merely a scaling parameter to make the factors comparable in terms of standard deviations and does not influence the significance or create a look-a-head bias.

[4] The scaling parameter c is adjusted to ensure that the volatility of the managed and unmanaged factors remains identical in both cases.

Finally, the volatility-managed individual-factor portfolio is constructed as the mean-variance combination of the unmanaged factor and its volatility-managed counterpart, whether scaled by its own volatility or by market volatility[5]. We employ an expanding window approach, using the first 120 months as the starting window, to obtain the portfolio weights using the mean-variance framework. A risk-aversion parameter, γ, of 5 is applied across all mean-variance approaches. Furthermore, we introduce a short-selling constraint to prevent extreme weights and reduce estimation error. The choice of an expanding window of 120 months, a γ of 5, and a short-selling constraints are all in line with the methodology used by DeMiguel et al. (2024)[6]. To assess the effectiveness of volatility-management on *individual*-factors, we analyze the statistical significance of differences in Sharpe ratios, thereby addressing RQ-1.

3.2 Hypothesis for Research Question 2 (RQ-2)

> *Can a volatility-managed multifactor portfolio, adjusting weights inversely to volatility, on average out-of-sample, outperform its unconditional counterpart in the Norwegian stock market?*

We define a conditional mean-variance multifactor portfolio that allows the relative weights of different factors to vary with either market volatility or factor volatility[7]. Additionally, we use volatility rather than variance to scale the returns, as Moreira and Muir (2017) and Barroso and Detzel (2021) point out that using volatility reduces transaction costs.

[5]When doing mean-variance to get the weights, we use the realized volatility in the same period as the returns. That is, for month $t+1$, we get the covariance matrix and mean return from $r_{k,t}^{\sigma_k} = \frac{c}{\sigma_{k,t}^2} r_{k,t}$ from past dates as this is the relationship we want to exploit.

[6]We play with all of these choices in the robustness checks.

[7]DeMiguel et al. (2024) consider only market volatility, arguing that this is a conservative choice, as the effect is even stronger when using factor volatility. For completeness, we include both factor and market volatility.

We define a conditional multifactor portfolio at time t as:

$$w_t(\theta_t) = \sum_{k=1}^{K} x_{k,t}\theta_{k,t}, \tag{2}$$

where K is the number of factors, $x_{k,t} \in \mathbb{R}^{N_t}$ is the stock portfolio associated with the kth factor at time t, with N_t denoting the number of stocks available at time t. The term $\theta_{k,t}$ represents the portfolio weight on the kth factor at time t, and $\theta_t = (\theta_{1,t}, \theta_{2,t}, \dots, \theta_{K,t})$ is the factor-weight vector at time t.

Each factor weight, $\theta_{k,t}$, is modeled as an affine function of the inverse of volatility, defined as

$$\theta_{k,t} = a_k + b_k \frac{1}{\sigma_t}, \tag{3}$$

where σ_t represents realized volatility, either of the market or the factor, estimated as the sample volatility of daily market returns in month t. A positive value of b_k implies that the model reduces exposure to the kth factor when realized volatility is high. Additionally, the model allows each factor's weight to vary independently, as in general, $b_i \neq b_j$ for $i \neq j$.

If $r_{t+1} \in \mathbb{R}^{N_t}$ is the vector of stock returns for month $t+1$, and $r_{k,t+1} \equiv x_{k,t}^\top (r_{t+1} - r_{f,t+1} 1_{N_t}) \in \mathbb{R}$ is the kth factor return for month $t+1$, where $r_{f,t+1}$ is the return of the risk-free asset at time $t+1$ and 1_{N_t} is the N_t-dimensional vector of ones, then substituting (3) into (2) gives the return of our conditional multifactor portfolio as:

$$r_{p,t+1}(\theta_t) = \sum_{k=1}^{K} r_{k,t+1} \left(a_k + b_k \frac{1}{\sigma_t} \right) = \sum_{k=1}^{K} r_{k,t+1}\theta_{k,t}. \tag{4}$$

To simplify implementation, we define an "extended" factor portfolio-weight matrix $X_{\text{ext},t}$, factor-return vector $r_{\text{ext},t+1}$, and factor-weight vector η as:

$$
X_{\text{ext},t} = \begin{bmatrix} x_{1t}^\top \\ x_{2t}^\top \\ \vdots \\ x_{Kt}^\top \\ x_{1t}^\top \times \frac{1}{\sigma_t} \\ x_{2t}^\top \times \frac{1}{\sigma_t} \\ \vdots \\ x_{Kt}^\top \times \frac{1}{\sigma_t} \end{bmatrix}, \quad r_{\text{ext},t+1} = \begin{bmatrix} r_{1,t+1} \\ r_{2,t+1} \\ \vdots \\ r_{K,t+1} \\ r_{1,t+1} \times \frac{1}{\sigma_t} \\ r_{2,t+1} \times \frac{1}{\sigma_t} \\ \vdots \\ r_{K,t+1} \times \frac{1}{\sigma_t} \end{bmatrix}, \quad \text{and} \quad \eta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_K \\ b_1 \\ b_2 \\ \vdots \\ b_K \end{bmatrix}. \tag{5}
$$

The conditional mean-variance multifactor portfolio is given by the extended factor-weight vector, η, which optimizes the mean-variance utility of an investor with risk-aversion parameter γ:

$$
\max_{\eta \geq 0} \ \widehat{\mu}_{\text{ext}}^\top \eta - \frac{\gamma}{2} \eta^\top \widehat{\Sigma}_{\text{ext}} \eta, \tag{6}
$$

where $\widehat{\mu}_{\text{ext}}$ and $\widehat{\Sigma}_{\text{ext}}$ represent the sample mean and covariance matrix of the extended factor-return vector, respectively. Here, $\widehat{\mu}_{\text{ext}}^\top \eta$ and $\eta^\top \widehat{\Sigma}_{\text{ext}} \eta$ denote the sample mean and variance of the conditional multifactor portfolio return, both constructed using realized volatility in the same periods as returns, as with the individual-factor portfolios[8].

To minimize estimation errors and prevent extreme weights, we impose non-negativity constraints on the factor weight parameters, such that $a_k \geq 0$ and $b_k \geq 0$, ensuring $\eta \geq 0$. We also investigate the use of shrinkage according to Ledoit and Wolf (2004a,b). In this same framework, we define the unconditional multifactor portfolio in an identical manner, by adding the constraint $b_k = 0$ to the problem (6).

[8]Since our portfolios are zero-cost, we do not impose constraints requiring portfolio weights to sum to one.

We then assess the effectiveness of volatility-management by comparing differences in Sharpe ratios and look for significant alphas to answer RQ-2. Specifically, to evaluate the statistical significance of our results, without a look-ahead bias, for RQ-1 and RQ-2, we compute p-values using a bootstrap approach. Specifically, we generate 100,000 bootstrap samples of the volatility-managed and corresponding unmanaged return series, employing the stationary block-bootstrap method with an average block length of five. The return series are bootstrapped jointly to preserve the inherent correlation between them, as the performance of the unmanaged portfolio directly influences the managed portfolio's returns. From these bootstrap samples, we construct the empirical distribution of the differences in Sharpe ratios. The p-value is then calculated as the proportion of instances in which the Sharpe ratio difference is less than zero.

4 Data and Preprocessing

This study focuses on equities listed on the Norwegian stock market. Stock return data were collected from the Oslo Børs Information (OBI) database. Additionally, the value-weighted indices of all stocks listed on the Oslo Børs are used as the market factor (MKT) and were obtained from Bernt A. Ødegaard's website[9]. The value factor (HML), size factor (SMB), and Carhart momentum factor (UMD) were likewise sourced from Ødegaard's dataset. Risk-free rates were gathered from Norges Bank, Oslo Børs, and NoRe, and accessed through the same source. Our dataset, summarized in Table 1, includes both daily and monthly observations covering the period from January 1980 to December 2023, yielding a sample of 44 years. As the BAB factor requires three years of historical data for construction, it commences in 1983[10].

4.1 Filters

Previous studies, including Novy-Marx and Velikov (2021), have found that a significant share of the returns produced by the BAB factor stem from an excessive focus on small and illiquid stocks. In a similar vein, Barroso and Detzel (2021) suggest that this emphasis on small-cap stocks results in higher transaction costs, particularly when deploying volatility-timing strategies. Moreover, Ødegaard (2021) highlight the necessity to exclude these stocks from analyses involving OBI data, as their reported returns are frequently inflated. To address these issues, we implement data screening methods in line with those employed by Ødegaard (2021). Specifically, we omit stocks priced under 10 NOK if their market capitalization is either below 10 million NOK or not available. Additionally, stocks with market capitalizations under 1 million NOK

[9] https://ba-odegaard.no/

[10] Since our estimation window starts with 120 months, our volatility-timed portfolio returns will then start in 1993 for all in and out-of sample computations.

are also excluded. Table 1 demonstrates a more consistent stock universe after applying these filters.

4.2 Data Cleaning and Manipulation

Regarding the risk-free rate, we encountered 86 missing daily observations, which were imputed using simple linear interpolation. The construction of the BAB factor follows the methodology detailed in Appendix B.4. In the OBI daily and monthly datasets, several observations for the year 2020–particularly between June and December–were missing. To address this, we supplemented the data using Yahoo Finance, Bloomberg, and S&P CIQ to the extent possible; however, a noticeably lower number of stock return observations remains during this period.

Table 1: Data Summary

This table summarizes key variables used in the analysis. "Stocks" refers to unfiltered data, while "Stocks (F)" is the filtered subset. For "Stocks" and "Stocks (F)," N indicates the number of stocks, for the other variables it indicates the number of observations. Periods are based on monthly data. Means and standard deviations (SD) are based on monthly returns, annualized. Max and Min reflect the highest and lowest monthly returns. All returns, except risk-free (RF) and stocks, are excess returns.

Variable	**Frequency**	**Period**	**N**	**Mean**	**SD**	**Max**	**Min**
Stocks	D & M	1980:01-2023:12	1074	1473.47%	13305.53%	249900%	-100%
Stocks (F)	D & M	1980:01-2023:12	995	14.41%	30.02%	11567%	-100%
RF	D & M	1980:01-2023:12	10290 & 492	5.71%	1.33%	2.07%	0.01%
MKT	D & M	1980:01-2023:12	10290 & 492	17.51%	20.80%	19.95%	-25.95%
SMB	D & M	1981:02-2023:12	10290 & 492	15.01%	18.81%	62.52%	-11.95%
HML	D & M	1981:07-2023:12	10290 & 492	8.01%	24.58%	91.54%	-29.09%
UMD	D & M	1981:07-2023:12	10290 & 492	13.40%	21.36%	22.41%	-25.95%
BAB	D & M	1983:01-2023:12	10290 & 492	16.70%	20.78%	37.97%	-22.28%

For the empirical analysis, we employ heteroskedasticity-consistent standard errors, specifically those introduced by Newey and West (1987). Unless otherwise specified, a 5% significance level is applied throughout the study. To ensure comparability across all analyses, the results presented, except Table 1, are based on the period from February 1993 to December 2023.

5 Empirical Analysis of volatility-timing Performance

In this section, we examine the economic benefits and statistical significance of volatility-management in-sample and out-of-sample. Section 5.1 evaluates the performance of volatility-managed *individual*-factors and volatility-managed *individual*-factor portfolios, which have been a focal point in the existing literature. Furthermore, Section 5.2 evaluates the performance of the conditional *multifactor* portfolio proposed by DeMiguel et al. (2024) which compares them to the unconditional *multifactor* portfolio. Lastly, Section 5.3 seeks to understand what drives the portfolio by examining the movements of the weights and testing its robustness.

5.1 Volatility-Management of Individual-Factors and Portfolios

To begin, we establish the foundation for our analysis by first examining individual volatility-timed factors, following the approach proposed by Moreira and Muir (2017). Next, we construct a mean-variance portfolio that combines the original factor with its volatility-timed counterpart, both in-sample and out-of-sample, as suggested by DeMiguel et al. (2024). Through this approach, we aim to validate the criticisms raised by Cederburg et al. (2020), namely that volatility-timing does not work out-of-sample despite prior research finding it significant in the Norwegian stock market.

For each factor analyzed, Table 2 presents the annualized Sharpe ratio for the unmanaged factor, $\mathrm{SR}(r_k)$, the volatility-managed individual-factor, $\mathrm{SR}(r_k^\sigma)$, and the volatility-managed individual-factor portfolio, $\mathrm{SR}(r_k, r_k^\sigma)$. Additionally, we report the p-value for the difference in Sharpe ratios relative to the original factor. More specifically, Panel A reports the performance of each individual-factor using factor volatility. Panel B presents the in-sample volatility-managed individual-factor portfolios based on factor volatility, while Panel C shows the out-of-sample results for the same approach. Similarly, Panel D examines the performance of individual-factors using market volatility, Panel E displays the in-sample volatility-managed portfolios based on market volatility, and Panel F presents the corresponding out-of-sample results. The in-sample portfolios in Panels B and E utilize the entire sample period to determine the weights that maximize the Sharpe ratio, thereby introducing look-ahead bias, which is argued to be consistent with prior literature.

Panel A compares each volatility-managed factor to its unmanaged counterpart. Among the managed factors, only UMD display a higher Sharpe ratio, approaching statistical significance at the 10% level. This finding is particularly noteworthy, as previous academic studies in the US suggest that BAB are one of the factors that benefit the most from volatility-management. However, the table reports a similar relationship to that shown in Figure 1, with BAB patterns appearing more consistent when using market volatility.

Panel B reports results somewhat in line with the methodology of Moreira and Muir (2017), which has been criticized for its in-sample nature. The Sharpe ratio of the volatility-managed individual-factor portfolio, denoted as $\mathrm{SR}(r_k, r_k^{\sigma_k})$, is higher for three of the five factors analyzed, and the remaining factors show identical value per construction. However, statistical significance is observed only for the UMD factor and at the 10% level.

Panel C presents the out-of-sample performance, where Sharpe ratios are predictably lower compared to their in-sample counterparts. We still observe UMD and MKT with higher Sharpe ratios, where only UMD is significant at the 10% level. Comparing Panels A, B, and C, we observe that volatility-timing using the factor's own volatility, as proposed by Moreira and Muir (2017), generally does not produce statistically significant improvements, except for,

Table 2: Performance of Volatility-Managed individual-factor Portfolios

This table presents the annualized Sharpe ratios for each factor considered in our analysis, including the Sharpe ratios of the unmanaged factor, SR(r_k), the volatility-managed individual-factor, SR(r_k^{σ}), and the Sharpe ratio of the mean-variance portfolio that combines the unmanaged factor with its managed counterpart, SR(r_k, r_k^{σ}), as constructed using Equation 1. Panels A, B, and C focus on scaling using factor volatility, while Panels D, E, and F focus on scaling using market volatility. Additionally, the table reports p-values to assess the statistical significance of differences in Sharpe ratios. To ensure consistency with the mean-variance portfolio evaluations, the sample period spans from February 1993 to December 2023.

	MKT	SMB	HML	UMD	BAB
***Panel A**: Factor and volatility-managed counterpart using σ_k*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR($r_k^{\sigma_k}$)	0.948	0.906	0.160	0.753	0.426
p-value(SR($r_k^{\sigma_k}$) - SR(r_k))	0.593	0.679	0.836	0.113	0.906
***Panel B**: In-sample volatility-timed individual-factor portfolio using σ_k*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR(r_k, $r_k^{\sigma_k}$)	1.026	0.963	0.244	0.757	0.924
p-value(SR(r_k, $r_k^{\sigma_k}$) - SR(r_k))	0.430	0.549	0.584	0.095	0.592
***Panel C**: Out-of-sample volatility-timed individual-factor portfolio using σ_k*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR(r_k, $r_k^{\sigma_k}$)	1.001	0.926	0.217	0.710	0.737
p-value(SR(r_k, $r_k^{\sigma_k}$) - SR(r_k))	0.411	0.596	0.607	0.093	0.636
***Panel D**: Factor and volatility-managed counterpart using σ_m*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR($r_k^{\sigma_m}$)	0.948	0.892	0.262	0.696	1.113
p-value(SR($r_k^{\sigma_m}$) - SR(r_k))	0.457	0.627	0.558	0.130	0.488
***Panel E**: In-sample volatility-timed individual-factor portfolio using σ_m*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR(r_k, $r_k^{\sigma_m}$)	1.026	0.987	0.269	0.697	1.114
p-value(SR(r_k, $r_k^{\sigma_m}$) - SR(r_k))	0.419	0.567	0.510	0.123	0.397
***Panel F**: Out-of-sample volatility-timed individual-factor portfolio using σ_m*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR(r_k, $r_k^{\sigma_m}$)	1.001	0.949	0.284	0.668	1.034
p-value(SR(r_k, $r_k^{\sigma_m}$) - SR(r_k))	0.411	0.587	0.533	0.117	0.451

momentum, which aligns with the findings of Cederburg et al. (2020). Notably, compared to Panels A and C, all p-values decrease when using a mean-variance approach, indicating more significant results.

Panels D, E, and F replicate the analysis using market volatility instead of factor volatility. In these cases, Sharpe ratios are generally higher for portfolios compared to those obtained using factor volatility. However, no statistically significant outputs are observed at the 10% level, even in the in-sample analysis. As expected, we observe an improvement in the BAB factor, although it remains insignificant. We again obtain lower p-values in Panel F compared to Panel D indicating that a mean-variance approach seem to help. In general in the table, we see the pattern we observed in Figure 1, which shows that UMD, using factor volatility, showed the most significant difference in Sharpe ratio.

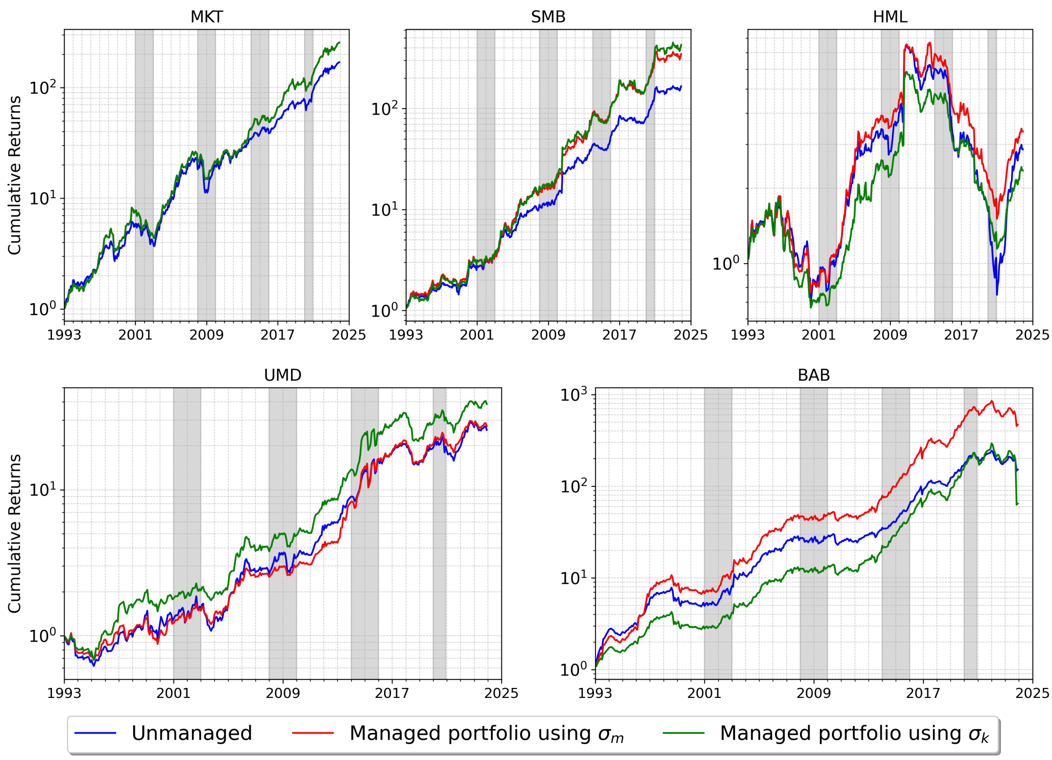
Our findings are consistent with previous academic papers that identify UMD as the factor most responsive to volatility-management in the Norwegian stock market when using factor volatility. We also find that UMD performs best in terms of Sharpe ratio, and is the only significant results we obtain at the 10% level, when using factor volatility. However, unlike earlier studies that reported significant results for HML, SMB, and occasionally other factors, our results do not support these findings. This is likely due to differences in methodology, particularly in the way significance is measured, as previous studies followed the approach of Moreira and Muir (2017). Overall, our results suggest that volatility-timing on individual-factors or individual-factor portfolios does not produce significant outcomes[11]. Incorporating mean-variance optimization looks to yield better results then just investing in the volatility-managed factor alone, suggesting that a multifactor approach could be more effective.

[11] In Appendix C.1, we apply shrinkage during the estimation of volatility-timed individual-factor portfolios. This method leads to a marginal improvement in results, although the enhancements are not substantial. This outcome is somewhat anticipated, as the estimation error when selecting between only two factors is typically minimal.

To visually illustrate these findings, Figure 2 shows the cumulative returns of each unmanaged factor (blue) alongside its corresponding portfolios of individual-factors managed by factor volatility (green) and market volatility (red). Note that the MKT factor remains identical across both individual-factor portfolios per construction. The cumulative returns are presented as total returns, representing the growth of an investment in each factor beginning in February 1993. The volatility-managed individual-factors are scaled to match the volatility of the unmanaged factor[12].

Figure 2: Cumulative Returns: Volatility-Managed vs. Unmanaged individual-factor Portfolios (Out-of-Sample)

The five graphs display the cumulative return of the unmanaged factor (blue line) and its corresponding volatility-managed individual-factor portfolios using factor volatility (green line) and using market volatility (red line) over the period from February 1993 to December 2023. The volatility-managed individual-factor portfolio is scaled to match the variance of the unmanaged portfolio, ensuring comparability in risk-adjusted performance.



[12]First, scaling the factor returns does not affect the Sharpe ratios. Second, since these portfolios are self-financing, they generate payoffs rather than traditional returns; however, for simplicity, we refer to these payoffs as returns

Upon looking at the cumulative returns illustrated in Figure 2, it becomes clear that portfolios adjusted for volatility tend to outperform their unmanaged versions in terms of cumulative returns. For the SMB portfolio, the results are consistent whether considering market or factor volatility, whereas HML benefits more when market volatility is applied. With regard to UMD, utilizing the factor's own volatility gives better outcomes, while the BAB factor displays higher cumulative returns when market volatility is employed. This is mostly due to the extreme event in November 2023, during which the BAB factor experienced a single-month return of -20%. In the preceding month, however, the annualized volatility of the factor was relatively low at only 6%, resulting in a disproportionately high allocation to the factor under volatility timing. This mismatch between low estimated volatility and realized negative returns led to a negative return of over 300%, and underscores a potential vulnerability of volatility-based scaling approaches without funding constraints. Concerning the MKT factor, employing volatility-timing, especially during crises, can potentially mitigate losses[13]. The trends highlighted in the figure are consistent with those in Table 2. In general, significant differences are observed when switching between factor and market volatilities, with a tendency for market volatility to boost overall factor performance. However, it should be noted that higher cumulative returns do not necessarily equate to significantly better returns or Sharpe ratios; for instance, while volatility-timed SMB portfolios outperformed the unmanaged version in the mid-2000s, they essentially mirrored the unmanaged performance thereafter.

Based on the evidence presented, we conclude that, consistent with the findings of Cederburg et al. (2020) and DeMiguel et al. (2024), a volatility-managed portfolio constructed from individual-factors can outperform its unmanaged counterpart, but not significantly when evaluated out-of-sample. Although

[13] Appendix D shows that the drawdowns of the volatility-managed factors can be smaller during crises like 2008, although in other crises, they may lead to greater drawdowns.

this analysis does not explicitly account for transaction costs, it is reasonable to argue that including such costs would further diminish the Sharpe ratios. We observe that incorporating a mean-variance approach, without accounting for transaction costs, yields better results than simply scaling the factor. Therefore, we further conclude that the positive findings reported in previous studies, which employed the approach of Moreira and Muir (2017), are not valid when evaluated out-of-sample.

5.2 Performance of Mean-Variance Multifactor Portfolios

In the previous section, we analyzed the performance of volatility-managed individual-factors and portfolios, an area that has been explored in the context of the Norwegian stock market. In this section, we shift focus to a multifactor perspective by considering an investor who has the ability to invest in, and apply volatility-management to, all five factors simultaneously. Specifically, we compare three portfolios: the unconditional mean-variance multifactor portfolio (UMV) and two conditional mean-variance multifactor portfolios–one using market-volatility scaled factors (CMV_m) and another using factor-volatility scaled factors (CMV_k)[14]. We also estimate each portfolio with and without shrinkage, to observe the effect of estimation error. Each portfolio is obtained by solving the optimization problem described in Equation (6). For the UMV portfolio, we impose an additional constraint by setting $b_k = 0$ for all $k = 1, 2, \ldots, K$. In essence, the UMV portfolio solves the same optimization problem, but without access to returns scaled by past volatility.

For each of the portfolios, Table 3 reports the out-of-sample annualized mean return, standard deviation, and Sharpe ratio, along with the p-values for the

[14] DeMiguel et al. (2024) focus exclusively on market volatility; however, given the lack of significant results in the previous section, we include both market and factor volatility for completeness.

difference in Sharpe ratios between the unconditional and conditional portfolios. Additionally, the table presents the annualized alpha from a time-series regression of the conditional portfolios' out-of-sample returns on those of the unconditional portfolio, including the Newey-West t-statistic for the alpha[15].

Table 3: Performance of Mean-Variance Multifactor Portfolios

This table reports the out-of-sample performance, with and without shrinkage, of the three multifactor portfolios: the two conditional mean-variance multifactor portfolios (CMV), obtained by solving Equation 6, and the unconditional mean-variance multifactor portfolio (UMV), also derived from Equation 6 but with the additional constraint that $b_k = 0$ for $k = 1, 2, \ldots, K$. This constraint ensures that the weights on the K factors are set to zero for the volatility-managed factor by construction. For each portfolio, the table presents the annualized mean, standard deviation, Sharpe ratio, and p-values for the difference between the Sharpe ratios of the CMV portfolios and the UMV portfolio. Additionally, the table reports the annualized alpha from the time-series regression of the CMV's out-of-sample returns on those of the UMV, along with the Newey-West t-statistic for alpha. The evaluation period spans from February 1993 to December 2023.

	Without Shrinkage			**With Shrinkage**		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
Mean	0.499	0.601	0.696	0.447	0.620	0.722
Standard deviation	0.305	0.333	0.429	0.275	0.335	0.444
Sharpe ratio	1.639	1.805	1.623	1.624	1.852	1.627
p-value(SR_{CMV} - SR_{UMV})		0.127	0.565		0.029	0.466
α (%)		9.966	4.812		13.421	7.565
$t(\alpha)$		2.741	1.147		3.820	1.665

Table 3 shows that all the conditional multifactor portfolios return higher mean returns and standard deviations compared to the unconditional portfolios. It is important to highlight that the high mean returns across all portfolios stem from the fact that we do not impose leverage constraints, as all portfolios are constructed to be zero-cost. We also observe higher Sharpe ratios for three out of four conditional portfolios, with CMV_k without shrinkage being the only portfolio that underperforms its counterpart. In terms of Sharpe ratio, CMV_m without shrinkage comes close to being significant at a 10% level but when including shrinkage,we observe a significantly higher Sharpe ratio at the 3% level, and a 14% increase in Sharpe ratio. Unlike DeMiguel et al. (2024),

[15]Regression equation: $CMV_t = \alpha + \beta \times UMV_t + \epsilon_t$

our results are not significant with factor volatility. While the use of factor volatility results in a higher mean return, it also leads to an increased standard deviation, rendering the additional mean insufficient in a risk-return setting. Referring back to Table 2, we observed that only the momentum factor was statistically significant when factor volatility was employed as a scaler. This finding might initially suggest that the multifactor portfolio would perform better under a factor volatility scaling approach. However, as shown in Figure 1, when applying factor volatility, only one instance of a higher Sharpe ratio was recorded. In contrast, when utilizing market volatility as the scaling variable, three Sharpe ratios were higher, suggesting that market volatility scaling benefits multiple factors simultaneously, while factor volatility primarily enhances the performance of a single factor. The table above further illustrates the impact of estimation error in our methodology, underscoring a considerable difference from the usage of shrinkage. In both conditional portfolios we see an increase in Sharpe ratio due to shrinkage, which is likely due to the larger number of parameters to estimate when including more returns in the model, however, for the UMV portfolio, we observe a lower Sharpe ratio when using shrinkage which raises uncertainty to the results. In terms of alpha, all conditional portfolios show positive values, where both CMV_m show significant values[16]. We see that shrinkage makes the alphas bigger as well as more significant.

However, DeMiguel et al. (2024) also reports that the transaction costs associated with the CMV portfolio are higher than those of the UMV portfolio. This finding aligns with the argument put forth by Barroso and Detzel (2021), who suggested that such costs may offset the observed differences in performance as well as their statistical significance. Furthermore, we observe that the difference in Sharpe ratios and p-values in our results is smaller than that reported

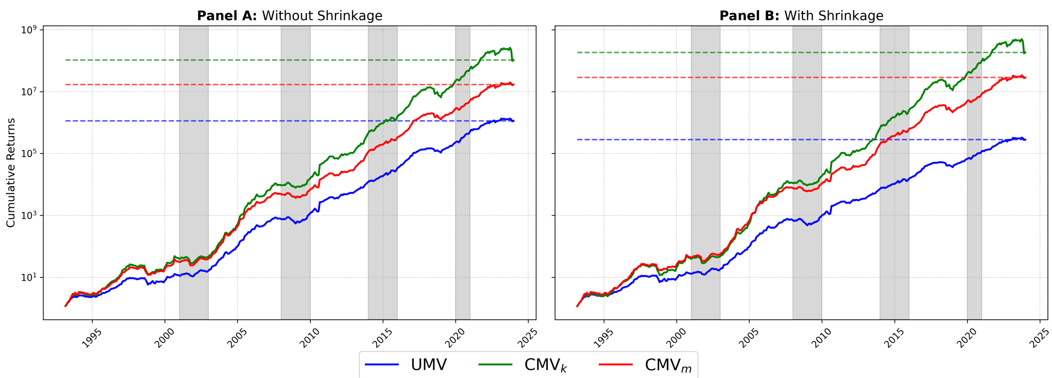
[16] For the t-statistic, a one-month lag is employed. We have tested multiple lag structures and find that $t(\alpha)$ remains statistically significant throughout all of them.

by DeMiguel et al. (2024) before accounting for transaction costs, who obtain p-values of zero across the board. Although we lack transaction cost data, when compared to the existing literature, it is important to remember that a smaller market, like Norway, may not have the same transaction costs as a larger market like the US, but the conditional portfolios will, in any case, incur higher transaction costs than their unconditional counterpart[17].

Moving on, Figure 3 presents the out-of-sample cumulative returns of the unconditional mean-variance multifactor portfolio (UMV) alongside the two conditional mean-variance multifactor portfolios (CMV_m and CMV_k) with and without shrinkage. To easier compare their end values, we plotted a dashed line that is easier to compare cross panels.

Figure 3: Cumulative Returns: Conditional vs. Unconditional Mean-Variance Multifactor Portfolios (Out-of-Sample)

This figures illustrates the out-of-sample cumulative returns of the unconditional mean-variance multifactor portfolio (UMV) alongside the two conditional mean-variance multifactor portfolios (CMV_m and CMV_k) over the period from February 1993 to December 2023. Panel A displays results without shrinkage, while Panel B displays results with shrinkage.



The figure demonstrates that both conditional portfolios consistently outperform the unconditional portfolio over time, where using factor volatility clearly comes out on top given its higher mean in Table 3. Due to the negative return

[17]Determining whether transaction costs in Norway are larger or smaller compared to the US is challenging. On one hand, Norway's smaller market may result in higher transaction costs, yet on the other hand, employing factors with fewer stocks might enhance trading diversification. Overall, one can generally assumed that these costs are higher in Norway than in the US.

of the BAB factor, CMV_k sees a negative return of 56% in November when not using shrinkage, and a 59% negative return when using shrinkage. When applying shrinkage, we see a fall in the CMV_k and UMV and a rise in CMV_m in terms of ending value. The improvement we observed in the past table when applying shrinkage, could therefore primarily result from the decline of UMV, rather than an improvement in CMV. We do also see a higher Sharpe ratio for UMV without shrinkage pointing at this, which raises the concern if shrinkage makes the strategy better or just the benchmark worse[18].

When comparing the results presented in Section 5.1 with the findings of this section, it becomes evident that constructing a multifactor volatility-timed portfolio can, enhance the Sharpe ratio and produce significant alphas out-of-sample, prior to accounting for transaction costs. However, it is important to recognize that these results may lose significance once transaction costs are taken into account, and only holds true when using market volatility and shrinkage that can worsen the benchmark. Multifactor portfolios, will achieve higher transaction costs, compared to individual-factor portfolios. Furthermore, when comparing the multifactor portfolios to the individual-factor portfolios, it is not surprising to observe that the multifactor approach outperforms the individual-factor portfolios across all performance metrics without accounting for transaction costs.

5.3 Understanding the Portfolio

The findings presented in the previous section demonstrate that our conditional multifactor portfolio–incorporating market volatility and shrinkage–substantially outperforms the unconditional benchmark in terms of both Sharpe ratio and cumulative returns. In the following section, we will examine the differences in portfolio behavior and then adjust our model

[18]We find that p-value($\text{SR}_{CMV} - \text{SR}_{UMV}$), when UMV is not using shrinkage and CMV_m is using shrinkage, to be 0.11.

assumptions to evaluate robustness. Specifically, this section focuses on the conditional multifactor portfolio that achieved the strongest performance, namely the versions integrating shrinkage.

5.3.1 Time Variation in Weights

To begin analyzing our model, we first examine whether it assigns weights to the original factor or its volatility-scaled counterpart. It is important to note that a positive value of b_k indicates a reduction in exposure to the kth factor when realized volatility is high. Figure 4 presents the estimates of a_k (solid lines) and b_k (dashed lines), from the affine model specified in Equation 3, for both conditional multifactor portfolios incorporating market volatility (CMV_m, in red) and factor-specific volatility (CMV_k, in green), both under the application of shrinkage.

When looking at the figure, we observe that the market factor exclusively loads on its volatility-managed returns, assigning a weight of zero to the original factor. This pattern similarly shows for SMB, HML, and UMD, where the original factor return almost consistently have a weight of zero. Differences between the use of factor volatility and market volatility shows firstly for SMB and UMD; specifically, the market volatility based portfolio tends to allocate relatively higher weights to the original factor compared to the factor volatility based portfolio, thereby reducing the weight on the volatility-managed returns. In the case of BAB, a distinct pattern emerges: neither portfolio assigns a weight of zero to either the original factor or its volatility-managed counterpart, suggesting that both contribute meaningfully to portfolio performance. Furthermore, the market volatility based portfolio maintains stable weights favoring the volatility-managed BAB return, whereas the factor volatility based portfolio exhibits similar weighting across both return series until after the 2008 financial crisis, after which a pronounced shift towards a higher weight on the

Figure 4: a_k and b_k Out-of-Sample and Over Time

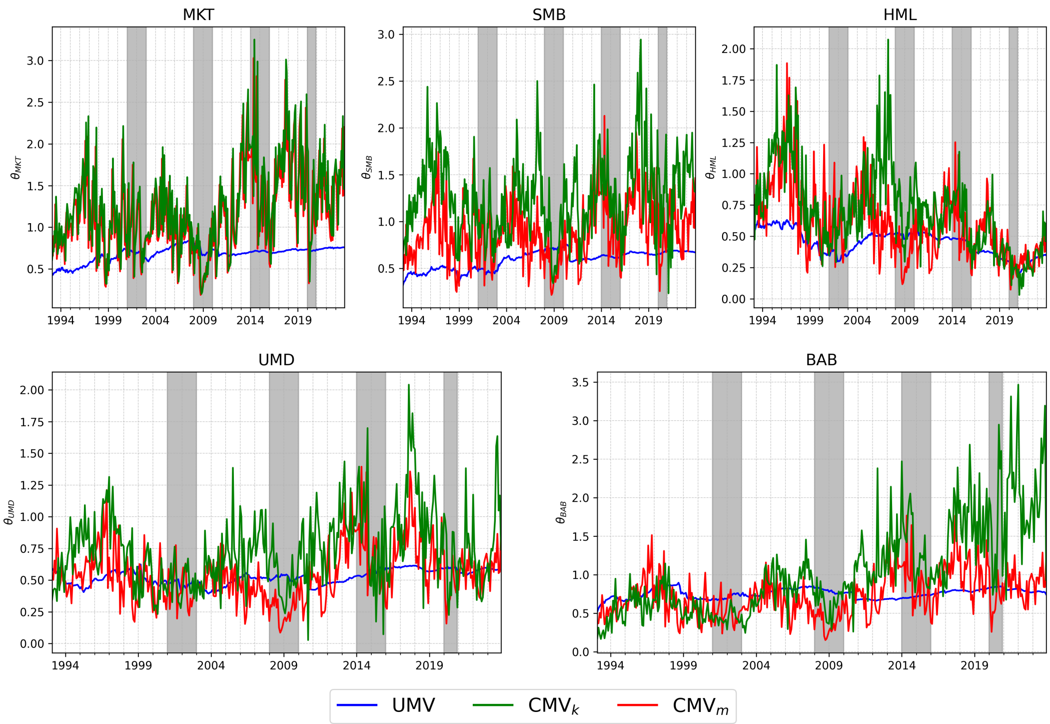
This figure presents the estimates of a_k (solid lines) and b_k (dashed lines), ifrom the affine model specified in Equation 3, for both conditional multifactor portfolios incorporating market volatility (CMV$_m$, depicted in red) and factor-specific volatility (CMV$_k$, shown in green), under the application of shrinkage. The time period spans from 1993 to 2023.

volatility-managed return and a lower weight on the original return becomes apparent. This issue was previously observed in November 2023, when the volatility-managed BAB factor crashed. In Appendix C.2, we find that the CMV$_m$ significantly outperforms the UMV during this period, indicating that differences in the allocation to the volatility-timed BAB factor may have led to a lower Sharpe ratio. In general, we can conclude from the figure that the model favors the volatility-timed returns over the original factor in general.

Next, we examine the evolution of weights, θ_k, from Equation 3, through time. Figure 5 displays the weights for the unconditional multifactor portfolio (UMV, blue line), the conditional multifactor portfolios incorporating market volatility (CMV$_m$, red line), and those using factor volatility (CMV$_k$, green line), all under the application of shrinkage.

Figure 5: Portfolios Weights Out-of-Sample and Over Time

This figure displays the weights for the unconditional multifactor portfolio (UMV, blue line), the conditional multifactor portfolios incorporating market volatility (CMV_m, red line), and those using factor volatility (CMV_k, green line), all under the application of shrinkage. The time period spans from 1993 to 2023.



Upon examining the figure, it is evident that the UMV portfolio weights, depicted in blue, remain relatively stable over time compared to the CMV portfolios. The UMV portfolio consistently assigns approximately a weight of 0.5 to each factor, with some deviations, particularly during periods of financial crises. In contrast, the CMV portfolios display considerably greater variation in their weights. As indicated in the previous table, the MKT factor receives approximately equal weighting in both conditional portfolios, resulting in nearly identical θ_k values. For the SMB and UMD factors, the CMV_k portfolio generally allocate higher weights, reflecting the greater weight on volatility-managed returns relative to the original returns. In the case of the HML factor, the weight distributions are relatively similar, with differences primarily attributable to variations in volatility. Finally, for the BAB factor in the CMV_k portfolio, we observe the highest weight, reaching approximately

3.5, which leads to a very negative return in November 2023. A notable shift is also apparent post-2008, where CMV_k assigns a greater weight to the volatility-managed return, thereby increasing the overall factor weight. The superior performance of CMV_m may thus be attributed to its more stable weight on the BAB factor in the post 2008 crisis. Another reason, may be the ability to time all factors using market volatility which could be a better predictor for future volatility then the factor's own volatility. It is also important to note that the factors, in a smaller market like Norway, include less stocks then in the US, making the factor and its volatility more unstable.

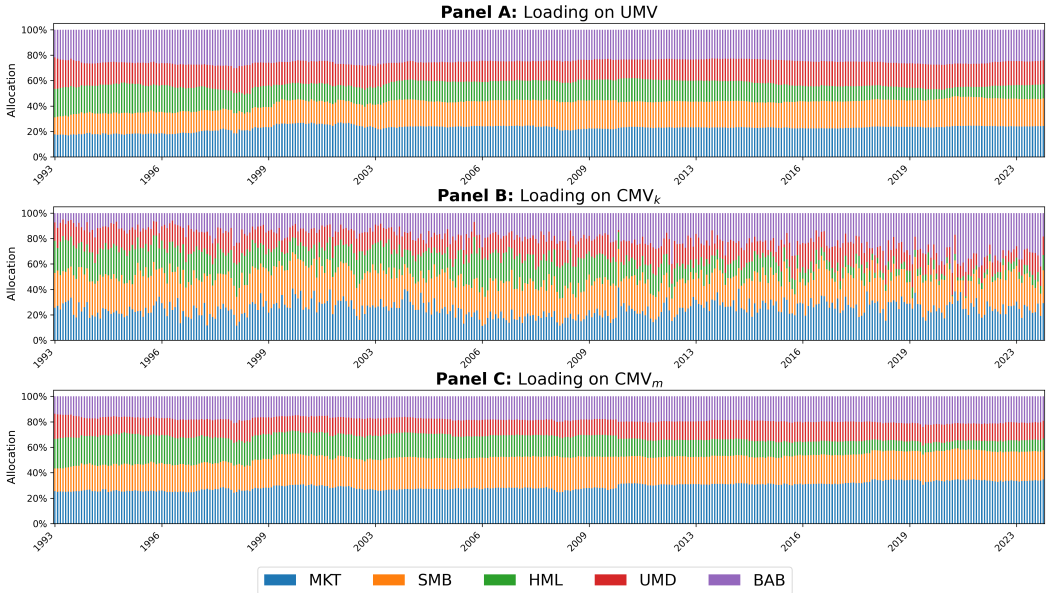
Lastly, we examine Figure 6. This figure illustrates the distribution of weights in our portfolios with shrinkage applied. Panel A displays the weight distribution for the unconditional multifactor portfolio; Panel B depicts the distribution for the conditional multifactor portfolio based on factor volatility; and Panel C shows the distribution for the conditional multifactor portfolio based on market volatility.

Panel A presents the UMV portfolio, which consistently maintains a balanced approach across the five factors without one factor predominating over time. For example, during the early 1990s, weights are allocated roughly equally (around 20-30% per factor), and even during turbulent times such as 2008-2009, the portfolio avoids significant biases. Similarly, the CMV_m portfolio in Panel C exhibits a comparable pattern to UMV. This is largely due to the adjustment by $\frac{1}{\sigma_m}$ across all factors, effectively behaving as if the portfolio itself is scaled by this term[19]. Panel B, on the other hand, showcases significant variations in the allocation arising from the unique volatility adjustments of each one. The CMV_k portfolio efficiently tries identifies the ideal factor weights and modifies these weights based on the individual-factor's own volatility, in-

[19]Although this isn't entirely precise mathematically, since only b_k is adjusted by $\frac{1}{\sigma_m}$, while a_k remains unaffected. But as we saw in Figure 4, the values of a_k and b_k, are relative stable over time.

Figure 6: Portfolios Weight Allocation Out-of-Sample and Over Time

This figure illustrates the distribution of weights in our portfolios with shrinkage applied. Panel A displays the weight distribution for the unconditional multifactor portfolio; Panel B depicts the distribution for the conditional multifactor portfolio based on factor volatility; and Panel C shows the distribution for the conditional multifactor portfolio based on market volatility. The period covered extends from 1993 to 2023.



creasing investment when the volatility is minimal. We see clearly that the portfolio jumps a lot back and forth on allocation, which in this environment, could be one of the reasons for its lack in significance.

In summary, this section concludes that while UMV portfolios adjust weights, these adjustments are insufficient in capturing all the evolving risk-return relationships. The CMV_m portfolio gives a higher weight to the original factor return compared to CMV_k, making it able to behave more flexible to balance the risk-return relationship. Although scaling factors by factor volatility may moderate timing of factor downturns and upturns, the weights seem too unstable to produce significant results. The consistently changing weights of the portfolio using factor volatility, could also lead to even higher transaction costs, diminishing the risk adjusted return even further.

5.3.2 Evaluating Portfolio Robustness

This section focuses on evaluating the stability of our portfolio. We will specifically alter parameters to understand their effect on the importance of our results. To avoid excessively lengthy tables, all tables concerning robustness are included in Appendix C, with references made here. When concluding robustness analyses, we shall primarily concentrate on the most stable portfolio, CMV_m with shrinkage, unless indicated otherwise. The appendix shows both portfolios, CMV_m and CMV_k, with and without shrinkage. Additionally, Appendix D also presents the histograms, drawdowns and correlations of all of our return series. For the multifactor portfolios, we note a favorable return distribution, characterized by positive skewness and significant upsides. Generally, there are reduced drawdowns, notably during crises like 2008, although differences are less pronounced during the dot-com bubble and the COVID-19 pandemic.

Adding leverage constraints, as shown in Appendix C.3, lowers p-values, with the highest being 3%, indicating that the portfolio remains stable even under restrictive conditions that is closer to real-life constrains. This adjustment can also reduce the risk of over-scaling into a factor like BAB at the end of 2023. Appendix C.4 reveals that altering the risk aversion parameter, γ, has no impact on the results, suggesting that the portfolio's performance is consistent across different levels of investor risk tolerance. Furthermore, Appendix C.5 indicates that permitting short-selling slightly reduces significance, likely due to increased estimation errors, but yet the results retain statistical significance, reinforcing robustness. Finally, Appendix C.6 demonstrates that the portfolio maintains significance across estimation windows ranging from 1 to 9 years, while non-shrinkage portfolios perform better with shorter windows, validating the conservative choice of a 10-year estimation window.

However, there are aspects where the portfolio's robustness is less certain, particularly regarding factor dependencies and scaling parameters. Appendix C.7 shows that substituting volatility with variance as the scaler leads to insignificant results, likely due to the absence of a scaling parameter, c, in multifactor portfolios, which complicates the mean-variance optimization. Additionally, Appendix C.8 highlights the portfolio's sensitivity to specific factors: removing the MKT and SMB factors enhances performance, removing BAB and UMD reduces the significance a little, and excluding the HML factor renders the portfolio insignificant. When excluding the MKT factor, we see a sharp decline in the Sharpe ratio of UMV leading to more significant differences in Sharpe ratios. For exclusion of HML we find that the Sharpe ratio of the UMV portfolio gets a steep bump, leading to a nonsignificant difference in Sharpe ratios. These findings suggest that the portfolio's success relies heavily on inclusion of certain factors, either to boost the conditional portfolios, or to crush the unconditional portfolio.

The portfolio's robustness also varies across different time periods, as explored in Appendix C.9. When the data is divided into 10-year segments, significance is only observed in the 2003-2012 period, with a p-value of 6%. Similarly, for 5-year intervals, significance is limited to the 1993-1997 period. This inconsistency indicates that the portfolio's outperformance may be tied to specific market conditions or economic cycles, raising questions about its stability over time and in the future.

Regarding volatility proxies, we have until now used the volatility in time t as a proxy for the volatility in $t + 1$. In Appendix C.10, we look into the use of a GARCH(1,1) model to predict next month's volatility instead of relying on the previous month's data. More specifically, we use a daily data and a window of 24 months to estimate the volatility for the next 21 days. This approach results in a p-value of 2%, lower than the original findings, suggesting that the portfolio not only remains robust to alternative volatility estimation methods but may also benefit from more sophisticated forecasting techniques, potentially enhancing its statistical significance.

6 Conclusion

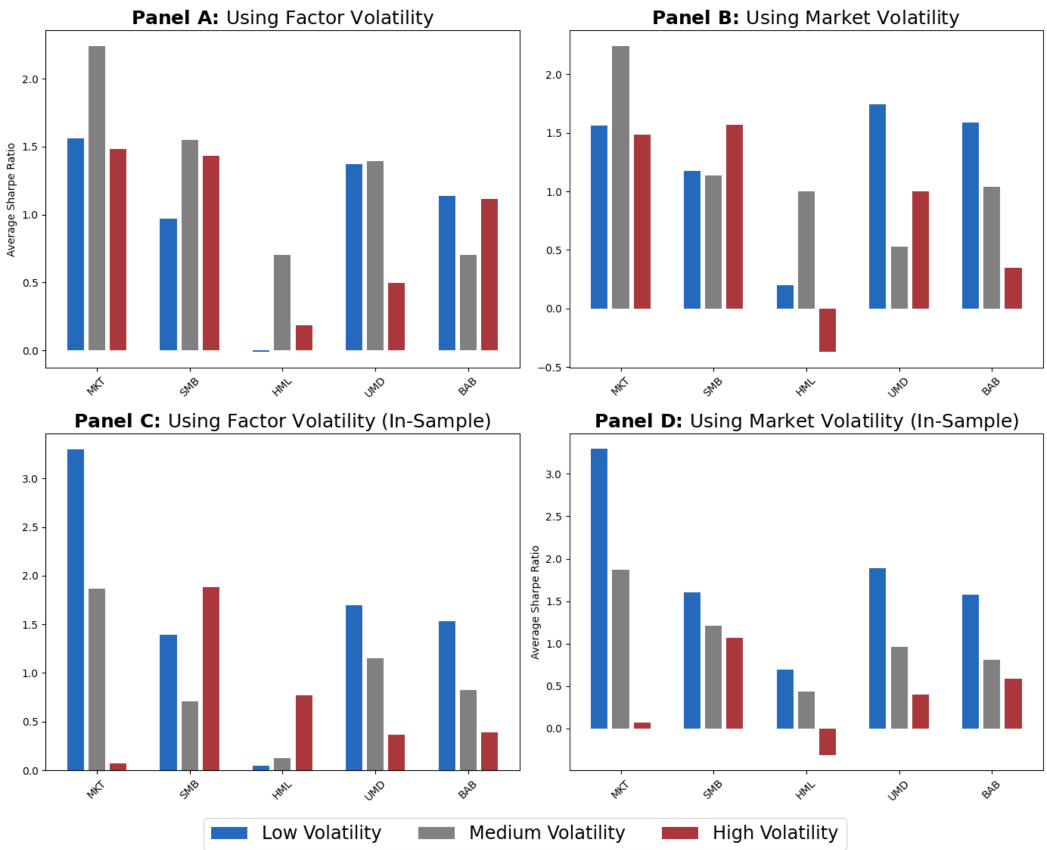
This thesis set out to investigate whether volatility-managed asset pricing factors could deliver superior risk-adjusted returns in the Norwegian stock market. Addressing RQ-1, our results indicate that volatility-timing applied to individual-factors and individual-factor portfolios does not yield statistically significant improvements when evaluated out-of-sample. Although when using factor volatility we observe a statistically significant value for UMD at the 10% level, the results could turn nonsignificant when incorporating transaction costs.

For RQ-2, we find that a volatility-managed multifactor portfolio scaled by market volatility achieves a statistically significant improvement in the Sharpe ratio compared to its unconditional counterpart. However, the lack of robustness across different time periods, factor dependencies, and real-world frictions such as transaction costs raises questions about their practical applicability. We also find that the application of shrinkage enhances the statistical significance of the results, primarily due to a lower Sharpe ratio in the benchmark portfolio, rather than a substantial increase in the Sharpe ratios of the conditional portfolios.

In examining alternative proxies of volatility, like a GARCH model, we observe noteworthy outcomes, even more pronounced than our earlier results. This leads us to Figure 7. Panels A and B in this figure is identical to the first figure in this thesis, while Panels C and D demonstrate risk-adjusted returns ranked using in-sample volatility. The data in Panels C and D clearly highlight a trend where reduced volatility corresponds with an increased Sharpe ratio. This pointing to a potential new research avenue in examining advanced volatility forecasting models for effective volatility-timing within a multifactor portfolio.

Figure 7: Factor Risk-Return Trade-Off Across Volatility Regimes Out-of-Sample and In-Sample

This plot illustrates how the risk-return trade-off of the five factors varies across different levels of realized volatility. Panels A and B are identical to Figure 1. Panels C and D are created by sorting the months into terciles based on the monthly time series of realized volatility in time t. For each category, we compute the risk-return trade-off from the factor using the return and volatility in t and report the average value in each category. Panel C uses the factor's own volatility and Panel D uses only the market's volatility. The time period is 1993-2023.



We conclude that while volatility-management can offer performance enhancements under certain conditions, these benefits are unlikely to be consistently achievable in practice. The Norwegian market's smaller size and sector composition may contribute to the observed lack of robustness, differing from findings in larger markets. Future research should focus on incorporating predictive models for volatility or assessing the role of downside volatility in enhancing portfolio performance.

APPENDIX

This section provides the technical details and supporting work that underlie the analyses conducted in this thesis. Although this material may not be of interest to the general reader, it offers a deeper understanding of the methodologies and processes used. Readers focused solely on the conclusions may choose to skip this section, but those seeking a comprehensive exploration of our approach are encouraged to delve into the details provided here.

The complete replication code and an Internet appendix can be found here[20].

A Additional Literature Review

A.1 Theoretical Foundations of Portfolio Choice

Markowitz (1952, 1959) fundamentally transformed the portfolio theory by incorporating not only the expected returns of individual assets but also their covariances, recognizing that interactions among assets play a crucial role in portfolio construction. This groundbreaking framework, which earned Markowitz the Nobel Prize in 1990, established the cornerstone of modern portfolio theory, emphasizing the trade-off between risk and return. The essence of his contribution is captured in the following equation, which identifies the optimal portfolio weights that maximize risk-adjusted returns:

$$w = \frac{1}{\gamma}(\mathbb{V}[R])^{-1}(\mathbb{E}[R] - R_f 1_N),$$

where w denotes portfolio weights, $\mathbb{V}[R]$ is the $N \times N$ covariance matrix of asset returns, $\mathbb{E}[R]$ represents the $N \times 1$ vector of expected returns, R_f is the risk-free rate, 1_N is a $N \times 1$ vector of ones, and γ is the degree of risk aversion of the investor.

[20] Our p-values are calculated through bootstrapping, which means they may vary slightly with each execution of the code.

Despite its theoretical elegance, mean-variance optimization has been subject to considerable criticism, primarily due to the sensitivity of its inputs to the estimation error. Empirical studies consistently show that sample-based estimates of expected returns, $\mathbb{E}[R]$, are highly imprecise, while estimates of the covariance matrix, $\mathbb{V}[R]$, often suffer from being ill-conditioned. Merton (1980) underscores the inherent difficulty in accurately estimating expected returns, pointing out that these estimates remain noisy and do not improve with more data points. Furthermore, the covariance matrix involves estimating a large number of parameters, which is particularly problematic when the available data is limited, leading to instability in the optimization process.

In response, several methodological advancements have been proposed to mitigate these issues:

- **Global Minimum Variance (GMV) Portfolio**: Jorion (1985, 1986) advocate for bypassing expected returns altogether by focusing exclusively on minimizing portfolio variance, effectively removing the need to estimate expected returns.
- **Constraints on Portfolio Weights**: Studies such as Frost and Savarino (1988), Chopra (1993), and Jagannathan and Ma (2003) recommend imposing constraints–such as prohibiting short-selling–to stabilize optimization and reduce sensitivity to estimation noise.
- **Shrinkage Estimators**: Ledoit and Wolf (2004a,b) propose shrinking the sample covariance matrix towards a more structured, stable target, thereby reducing the estimation error and improving robustness.

Further critique is offered by DeMiguel et al. (2009), who demonstrate that, even with shrinkage techniques applied, the mean-variance optimized portfolios tend to underperform the simple equally weighted portfolio, $1/N$, in out-of-sample tests based on the Sharpe ratio. In addition, they highlight that such portfolios typically experience significantly higher turnover, resulting in elevated transaction costs. Interestingly, their findings indicate that the GMV portfolio–particularly when the covariance matrix is shrunk through the Ledoit-Wolf shrinkage methodology and short-sale constraints is applied–exhibits relatively improved performance, though, this is the only portfolio that beats the $1/N$ benchmark.

Addressing these limitations from a different angle, Brandt et al. (2009) introduce a parametric portfolio policy, built on critical insight. Rather than first modeling asset returns and subsequently deriving optimal portfolio weights, they propose specifying a parametric factor model directly for the portfolio weights. This approach markedly reduces the dimensionality of the optimization problem, particularly in a factor investment context, thereby simplifying the portfolio construction process while maintaining flexibility and tractability.

A.2 Theoretical Foundations of Asset Pricing Models

The CAPM was first introduced by Sharpe (1964) as a foundational model establishing a positive linear relationship between systematic risk (beta) and expected returns. Later, Lintner (1965) and Mossin (1966) formalized and extended the framework. The model is expressed as:

$$\mathbb{E}[R_i] = R_f + \beta_i(\mathbb{E}[R_m] - R_f), \quad \text{where } \beta_i = \frac{Cov(R_i, R_m)}{Var(R_m)}.$$

In this equation, $\mathbb{E}[R_i]$ represents the expected return on the asset i, R_f is the risk-free rate, β_i measures the systematic risk of the asset, and $\mathbb{E}[R_m] - R_f$ is the market risk premium.

Since its introduction, the CAPM has been one of the most widely used models in finance, but has also faced significant criticism. For example, Black (1972) shows that the security market line is flatter than predicted by the CAPM.

This means that low-beta stocks earn higher returns than the model predicts, while high-beta stocks earn lower returns than expected.

Roll (1977) raised another critique, emphasizing that CAPM tests rely on proxies for the market portfolio, which may not fully represent all investable assets, making the model's validity difficult to confirm. Similarly, Banz (1981) identified the "size effect", where smaller firms exhibit higher risk-adjusted returns than predicted by the CAPM, introducing the concept of a size premium that the model fails to incorporate.

Based on these critiques, Fama and French (1992) demonstrated that variables such as size and value have significant explanatory power for asset returns, leading to the development of the Fama-French three-factor model. This model introduced the Small Minus Big (SMB) factor, which captures the size effect, and the High Minus Low (HML) factor, which represents value.

Furthermore, Jegadeesh and Titman (1993) documented the momentum effect, showing that stocks with strong past performance tend to continue to perform well in the short term, another anomaly not captured by the CAPM. This insight led to the introduction of the Up Minus Down (UMD) factor by Carhart (1997), capturing momentum as an additional explanatory variable.

The literature is divided into two perspectives. The first perspective attributes it to structural constraints, such as funding limitations. This view underpins the Betting Against Beta (BAB) factor introduced by Frazzini and Pedersen (2014), which has been further supported by empirical findings from Oliver and Mikhail (2018) and Pelster (2024). This perspective suggests that leverage constraints force investors to overweight high-beta stocks, distorting the relationship between beta and returns.

The second perspective focuses on investor behavior. Brunnermeier and Parker (2007) argue that investors are drawn to lottery-like stocks, which exhibit a high probability of large short-term moves. This behavior is idiosyncratic rather than systematic and has been explored further in studies like Ang et al. (2006, 2009) and Bali et al. (2011), which identify significant factors linked to these preferences.

Building on this, Bali et al. (2017) sort stocks by beta and demonstrate that, after accounting for lottery demand, the abnormal returns associated with low-beta stocks disappear. They also show that lottery demand increases buying pressure on stocks with a high probability of extreme short-term moves, flattening the SML and generating alpha for portfolios that are long low-beta stocks and short high-beta stocks.

Based on these findings, Asness et al. (2013) explore the interplay between value and momentum factors, demonstrating their effectiveness in various asset classes. Their research highlights that value and momentum are negatively correlated, making them highly complementary when combined in a single portfolio. By doing so, investors can achieve better diversification and risk-adjusted returns, further challenging the simplicity of the CAPM and reinforcing the need for multi-factor models.

Betting against beta (BAB) is then introduced by Frazzini and Pedersen (2014), which longs low-beta stocks while shorting high-beta stocks building on the findings of Black (1972). It has also been supported by empirical findings from Oliver and Mikhail (2018) and Pelster (2024). They suggested that leverage constraints force investors to overweight high-beta stocks, distorting the relationship between beta and returns.

The literature is divided into two perspectives. The first perspective attributes it to structural constraints, such as funding limitations. This view underpins the Betting Against Beta (BAB) factor introduced by Frazzini and Pedersen (2014), which has been further supported by empirical findings from Oliver and Mikhail (2018) and Pelster (2024). This perspective suggests that lever-

age constraints force investors to overweight high-beta stocks, distorting the relationship between beta and returns.

Studies focusing on the Norwegian market have highlighted key differences in the applicability of asset pricing models. For example, Fylling and Jacobsen (2023) argues that these factors exhibit lower stability in the Norwegian market compared to their performance in the United States. Similarly, Lønø and Svendsen (2019) shows that the inclusion of additional factors, such as momentum and robust minimum weakness, beyond the traditional Fama-French three-factor model, does not significantly enhance the explanatory power of returns in the Norwegian stock market.

A.3 Performance Evaluation Metrics

The Sharpe ratio, introduced by Sharpe (1966), is a widely used metric to evaluate the performance of investment funds relative to the level of risk taken. It is defined as:

$$SR_p = \frac{\mathbb{E}[R_p - R_f]}{\sqrt{\mathbb{V}[R_p - R_f]}} = \frac{\text{excess mean return}}{\text{total portfolio risk}},$$

where $\mathbb{E}[R_p - R_f]$ represents the excess return of the portfolio over the risk-free rate and $\sqrt{\mathbb{V}[R_p - R_f]}$ is the standard deviation of the excess return of the portfolio.

Despite its widespread use, it has been subject to several criticisms. For example, Leland (1999) highlights that it is unsuitable when return distributions deviate from normality, particularly when skewness or kurtosis is present. Additionally, (Lo, 2002, eq. (20) & Table 2) demonstrates that scaling the SR by $\sqrt{t}$ is valid only under the assumption of Independent and Identically Distributed (IID) returns with no serial correlation. If a negative serial correlation exists, the scaling factor can be significantly larger, revealing the limitations of this adjustment in non-IID contexts. Given these criticisms, the Sharpe ratio is still one of the most widely used ratios to account for portfolio returns.

A.4 Additional Volatility-Timing Literature

In addition to the previously discussed literature, recent studies have further explored the concept of factor timing, providing new insights and methodologies for dynamically adjusting exposure to risk factors. Ehsani and Linnainmaa (2022) demonstrate that stock momentum is largely driven by factor momentum, suggesting that the momentum effect is a manifestation of timing other factors rather than an independent risk factor. Gupta and Kelly (2019) document robust momentum behavior across 65 characteristic-based equity factors globally, showing that a time-series factor momentum portfolio achieves an annual Sharpe ratio of 0.84, highlighting its potential to enhance portfolio performance. Haddad et al. (2020) explore the predictability of market-neutral equity factors, finding them strongly and predictably robust, which allows investors to outperform static factor investing by exploiting this predictability. Methodological advancements are also significant: Miller et al. (2015) propose a risk-oriented model using nonparametric methods to predict factor failures based on broad risk measures, outperforming static and momentum-based approaches, while Hodges et al. (2017) advocate for a multi-predictor approach–combining business cycle indicators, valuation metrics, relative strength, and dispersion measures–to effectively time smart beta factors like value, size, momentum, quality, and minimum-volatility. Additionally, De Franco et al. (2017) compare linear and nonlinear factor models for capturing volatility, finding nonlinear models more robust, with implications for factor timing strategies reliant on accurate volatility modeling.

Complementing the existing body of work, several papers have delved deeper into the relationship between market risk and return, particularly through volatility timing strategies. Fleming et al. (2001) show that volatility timing strategies outperform unconditionally efficient static portfolios with the same target expected return and volatility, with results robust to estimation risk and transaction costs. In Fleming et al. (2003), they follow-up leverages intradaily returns for precise volatility estimates, revealing that risk-averse investors would pay 50 to 200 basis points annually for these benefits, emphasizing the economic value of accurate volatility forecasting. Marquering and Verbeek (2004) find strong evidence for market timing by predicting stock index returns and volatility, noting that mean-variance investors can achieve profitability despite constraints like no short sales and high transaction costs, especially during high-volatility periods. Gomez-Cram (2021) examines stock return behavior across business cycles, showing returns are predictably negative post-recession onset before rising, a pattern tied to slow investor reactions, their market timing approach boosts the buy-and-hold Sharpe ratio by 60%.

B Data Handling

B.1 Stock Returns

For the stock returns, we begin by identifying the missing tickers. Since we have the company names, we can manually fill in these missing tickers. The replacements are as follows:

- Kongsberg Automotive → KOA,
- Northern Offshore and Northern Ocean Ltd. → NOL,
- EVRY → EVRY,
- BW Energy Limited → BWE,
- Atlantic Sapphire → ASA,

- Pexip Holding → PEXIP.

Some tickers remain missing, but these correspond to companies that do not have a full data trading month and are therefore excluded from the dataset[21]. Going forward, we need to take into account the companies with different tickers but the same ISIN. Here, we simply replace the old tickers with the new ones.

Furthermore, we observe missing returns and prices for the period from 2020-07-01 to 2020-11-30. To address this, we retrieve data from Yahoo Finance, Bloomberg, and S&P CIQ, filling the gaps to the best of our ability. For the missing period, we also require values for outstanding shares to compute the market capitalization. We handle missing values by interpolating between known values if there are 2020-06 and 2020-12 data points. If only one of these values is available, we use it to populate the missing data.

Regarding data filtering, we restrict the dataset to the period 1980-2023 due to limited availability for 2024. Additionally, we apply the following data filters:

- Exclude all returns where the market capitalization is less than 10 million NOK.

- Remove returns where the stock price is less than 10 NOK and the market capitalization is less than 10 million NOK or missing. The rationale behind this is that filtering purely based on price may not be appropriate, as market capitalization better reflects the company's size rather than price alone.

B.2 Factors

We include the market factor (MKT) as the value-weighted stock market. We also subtract the risk-free rate accordingly.

[21] This does not make a difference for our analysis, as we use thresholds over a month for our factor replication.

B.3 Risk-Free Rate

For the risk-free rate, we address missing daily values by interpolation to ensure consistency with our stock return data.

B.4 Betting Against Beta (BAB) Replication

We express our gratitude to Masood and Guttulsrød (2024) for providing access to their replication code and robustness tests.

B.4.1 Beta Estimation

We estimate betas using excess stock returns and excess market returns. Daily data is used for beta estimation, since higher frequency data improve covariance estimates, as noted by Merton (1980). Consequently, volatilities also benefit from higher-frequency data.

Once the daily betas are estimated, we convert them into monthly betas by selecting the last beta available at the end of each month. This ensures continuity by carrying over the most recent estimate from the previous period into the subsequent month's analysis. This approach aligns with conventional financial data resampling techniques, maintaining consistency while preserving the latest market information at each period's transition.

The beta estimation follows:

$$\hat{\beta}_i^{ts} = \hat{\rho}\frac{\hat{\sigma}_i}{\hat{\sigma}_m}, \tag{7}$$

where $\hat{\sigma}_i$ and $\hat{\sigma}_m$ represent the estimated volatilities of the stock and the stock market, respectively, and $\hat{\rho}$ denotes their correlation.

A rolling window approach is applied, using a one-year period for volatility estimation and a five-year period for correlation estimation, as correlations tend to evolve more gradually than volatilities. Daily log returns are used to estimate volatilities, whereas correlations are estimated using overlapping three-day log returns:

$$r_{i,t}^{3d} = \sum_{k=0}^{2} \ln(1 + r_{t+k}^{i}). \tag{8}$$

This approach mitigates the impact of non-synchronous trading. Furthermore, we require a minimum of 120 non-missing data points for volatility estimation and 750 non-missing data points for correlation estimation to ensure robustness.

Finally, we shrink the time series estimate of beta towards the cross-sectional mean following the methodology of Vasicek (1973) and Elton et al. (2003). Mathematically, this is expressed as:

$$\hat{\beta}_i = w_i \hat{\beta}_i^{TS} + (1 - w_i)\hat{\beta}^{XS}, \tag{9}$$

where the cross-sectional mean beta is set as $\hat{\beta}^{XS} = 1$ and the shrinkage weight is defined as $w_i = 0.6$ for simplicity.

B.4.2 Portfolio Weights

Our objective is to construct a long-short portfolio by taking long positions in low-beta stocks and short positions in high-beta stocks. To achieve this, we design portfolio weights that appropriately adjust to the data.

Each year, we compute an $n \times 1$ vector of beta ranks, defined as:

$$z_i = \text{rank}(\beta_{it}), \tag{10}$$

where β_{it} represents the estimated beta for stock i at time t. We then compute the average rank:

$$\bar{z} = \frac{1_n^\top z}{n}, \tag{11}$$

where n is the number of betas and 1_n is an $n \times 1$ vector of ones. The portfolio weights for the high-beta (w_H) and low-beta (w_L) portfolios are then computed as:

$$w_H = k(z - \bar{z})^+, \quad \text{and} \quad w_L = k(z - \bar{z})^-, \tag{12}$$

where k is a normalizing constant given by:

$$k = \frac{2}{1_n^\top |z - \bar{z}|}. \tag{13}$$

The operators x^+ and x^- indicate the positive and negative elements of a vector x, respectively, and return zero for non-positive/non-negative values. This ensures that stocks with beta estimates above the average receive positive weights in the high-beta portfolio, while those below the average receive positive weights in the low-beta portfolio.

To maintain consistency and ensure that portfolio weights sum to one by construction, we apply this methodology separately to both monthly and daily betas. This separation prevents distortions arising from frequency differences in beta estimation while preserving the intended risk exposures in the long and short positions.

B.4.3 Hedging Procedure

Finally, we compute the BAB factor return using the following formulation:

$$r_{t+1}^{BAB} = \frac{1}{\beta_t^L}(r_{t+1}^L - r_{t+1}^f) - \frac{1}{\beta_t^H}(r_{t+1}^H - r_{t+1}^f), \tag{14}$$

where:

$$
\begin{aligned}
r_{t+1}^{L} &= w_L^\top r_{t+1}, & \beta_t^L &= \beta_t^\top w_L, \\
r_{t+1}^{H} &= w_H^\top r_{t+1}, & \beta_t^H &= \beta_t^\top w_H.
\end{aligned}
$$

This formulation ensures that the strategy is market-neutral by scaling the portfolio to achieve a zero market beta. The long and short positions are adjusted to neutralize exposure to systematic risk, allowing the factor returns to reflect the performance of the beta anomaly rather than overall market movements.

Additionally, we introduce an offsetting position in the risk-free asset to make the strategy self-financing. This ensures that the returns are generated purely from the beta spread rather than any excess leverage or capital allocation considerations.

C Robustness Tests for Volatility Timing

In this section, we will take a closer look at the assumptions we take during our analysis, and what happens if we change them.

C.1 Individual Factor Portfolios with Shrinkage

Table 4 illustrates that applying shrinkage to individual factor portfolios does not lead to significant results. This is anticipated because most estimation errors occur when multiple factors are employed, offering the model a wider range of choices. Given our scenario with only two options, the estimation error is inherently reduced, but we do observe some more signficiant values.

C.2 Post Financial Crisis

As seen in Section 5.3.1, we see a clear distinction between the loadings on the BAB factor for the CMV_k and CMV_m. Table 5 shows the significance of

Table 4: Performance of Volatility-Managed Individual Factor Portfolios (With Shrinkage)

This table present the same as Table 2, only while shrinking the covariance matrix.

	MKT	SMB	HML	UMD	BAB
***Panel A**: Factor and volatility-managed counterpart using σ_k*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR($r_k^{\sigma_k}$)	0.948	0.906	0.160	0.753	0.426
p-value(SR($r_k^{\sigma_k}$) - SR(r_k))	0.594	0.680	0.835	0.111	0.905
***Panel B**: In-sample volatility-timed individual-factor portfolio using σ_k*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR(r_k, $r_k^{\sigma_k}$)	1.024	0.966	0.241	0.733	0.823
p-value(SR(r_k, $r_k^{\sigma_k}$) - SR(r_k))	0.397	0.505	0.793	0.068	0.777
***Panel C**: Out-of-sample volatility-timed individual-factor portfolio using σ_k*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR(r_k, $r_k^{\sigma_k}$)	1.002	0.933	0.276	0.707	0.744
p-value(SR(r_k, $r_k^{\sigma_k}$) - SR(r_k))	0.386	0.547	0.652	0.076	0.758
***Panel D**: Factor and volatility-managed counterpart using σ_m*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR($r_k^{\sigma_m}$)	0.948	0.892	0.262	0.696	1.113
p-value(SR($r_k^{\sigma_m}$) - SR(r_k))	0.438	0.591	0.592	0.118	0.579
***Panel E**: In-sample volatility-timed individual-factor portfolio using σ_m*					
SR(r_k)	0.974	0.945	0.244	0.614	0.924
SR(r_k, $r_k^{\sigma_m}$)	1.024	0.986	0.259	0.694	1.045
p-value(SR(r_k, $r_k^{\sigma_m}$) - SR(r_k))	0.390	0.520	0.528	0.112	0.464
***Panel F**: Out-of-sample volatility-timed individual-factor portfolio using σ_m*					
SR(r_k, $r_k^{\sigma_m}$)	1.002	0.956	0.366	0.667	1.032
p-value(SR(r_k, $r_k^{\sigma_m}$) - SR(r_k))	0.385	0.539	0.502	0.109	0.507

the portfolios after 2010, and we see that the CMV_m significantly outperforms even the CMV_k in this period.

C.3 Imposing Leverage Constraints

We are now implementing leverage constraints within the model, setting a maximum allocation limit for each factor that ranges between 20% and 500%. As depicted in Table 6, an intriguing pattern surfaces: stricter leverage constraints result in more statistically significant outcomes when market volatility

Table 5: Performance of Mean-Variance Multifactor Portfolios with Short-Selling Constraints Relaxed

This table replicates the results of Table 3, with the distinction that the time period is from 2010-2023.

	Without Shrinkage			**With Shrinkage**		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
Mean	0.548	0.621	0.753	0.444	0.627	0.791
Standard deviation	0.341	0.335	0.487	0.284	0.332	0.506
Sharpe ratio	1.608	1.857	1.545	1.567	1.892	1.564
p-value(SR_{CMV} - SR_{UMV})		0.187	0.675		0.094	0.529
α ($)		11.940	2.486		15.457	5.579
$t(\alpha)$		2.556	0.329		2.994	0.752

is used to timed. This likely occurs because leverage constraints help reduce estimation error by forcing the model to be more selective in its allocations. In the portfolio that incorporates factor volatility and shrinkage, we observe a p-value of at least 3% across all leverage constraints.

C.4 Sensitivity to Risk Aversion Parameter (γ)

Next, we adjust the value of γ in the optimization. As illustrated in Table 7 and consistent with the findings of DeMiguel et al. (2024), the results remain largely unaffected by variations in γ, indicating that the model's performance is robust to changes in the risk aversion parameter.

C.5 Relaxing Short-Selling Constraints

Table 8 illustrates the performance of mean variance multifactor portfolios with short-selling permitted. The results, as shown in the table, demonstrate a small reduction in significance. This likely arises from increased estimation errors associated with short-selling, corroborating findings from previous studies.

C.6 Sensitivity to Rolling Window Length

Subsequently, we adjust the window length for optimization in Table 9. Importantly, the CMV_m portfolio maintains its significance across all window modifications when shrinkage is used, whereas the non-shrinkage portfolio is more significant when using a smaller window.

C.7 Replacing Volatility with Variance

The performance results of multifactor portfolios using mean variance, as shown in Table 10, substitute variance for volatility, following the approach of Moreira and Muir (2017). The data reveals that employing variance leads to statistically insignificant outcomes, which calls the robustness of these results into question. A possible explanation for this observation is estimation error; since returns adjusted by variance are significantly higher than those adjusted by volatility, the model encounters greater difficulty in estimating the allocations accurately.

C.8 Factor Exclusion Analysis

We systematically remove specific factors from the optimization model to assess their influence on the portfolio's performance, as detailed in Table 14. Notably, excluding the MKT and SMB factors leads to significantly improved outcomes for both CMV_m. While removing UMD and BAB lead to less significant, but still significant when using shrinkage, results. Interestingly, the superior performance observed after removing MKT primarily arises from the lower Sharpe ratios associated with the UMV portfolios. Additionally, excluding the HML factor renders both CMV_m portfolios insignificant, due to the increase in the Sharpe ratio in the UMV. Removing BAB leads to a significant decrease in all Sharpe ratios, indicating that it one of the front runners for the Sharpe ratios.

C.9 Time-Period Subsample Analysis

Dividing the observation period into three distinct decades, as depicted in Table 12, uncovers a fascinating trend. Notably, only the decade from 2003 to 2012 shows significance at a 6% level for CMV_m when using shrinkage. For all other portfolios across the decades, none are significant. Examining five-year intervals in Table 13, the use of shrinkage yields significant results for both portfolios during 1993–1997, but for all the other periods, none of the portfolios are significant.

C.10 Using GARCH(1,1) as Proxy for Volatility

We then evaluate whether a superior proxy for the volatility in month t can be identified, other than using the volatility from $t-1$. We employ a straightforward GARCH(1,1) model similar to the approach by Johansen and Eckhoff (2016). This involves using a 24-month window, with a single lag in both returns and volatility, supplemented by daily volatility data, to predict and average the volatility for the next 21 days. These averaged forecasts are then used to adjust the returns. As indicated in the table, this approach does produce a p-value of 2% which is 1% higher then our past value, indicating that a better estimate for volatility would probably improve the results.

Table 6: Performance of Mean-Variance Multifactor Portfolios under Leverage Constraints

This table replicates the results of Table 3, but imposes leverage constraints by restricting the maximum allocation to each factor, thereby evaluating the effect of limiting factor exposures on the portfolio's performance.

	Without Shrinkage			**With Shrinkage**		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
***Panel A**: Maximum weight of 10% per factor*						
Mean	0.074	0.512	0.527	0.074	0.522	0.524
Standard deviation	0.046	0.278	0.330	0.046	0.284	0.327
Sharpe ratio	1.601	1.844	1.598	1.601	1.836	1.601
p-value(SR_{CMV} - SR_{UMV})		0.007	0.488		0.010	0.477
α (%)		10.277	3.918		10.308	4.024
$t(\alpha)$		4.116	1.448		3.960	1.497
***Panel B**: Maximum weight of 20% per factor*						
Mean	0.147	0.589	0.693	0.148	0.617	0.712
Standard deviation	0.092	0.321	0.427	0.093	0.333	0.439
Sharpe ratio	1.592	1.836	1.625	1.601	1.852	1.620
p-value(SR_{CMV} - SR_{UMV})		0.023	0.387		0.024	0.425
α (%)		13.375	9.673		14.645	9.792
$t(\alpha)$		4.133	2.124		4.080	2.055
***Panel C**: Maximum weight of 50% per factor*						
Mean	0.365	0.596	0.696	0.357	0.620	0.722
Standard deviation	0.222	0.329	0.429	0.222	0.335	0.444
Sharpe ratio	1.643	1.814	1.623	1.608	1.852	1.627
p-value(SR_{CMV} - SR_{UMV})		0.080	0.552		0.027	0.426
α (%)		10.332	5.891		14.005	8.660
$t(\alpha)$		3.230	1.425		3.760	1.949
***Panel D**: Maximum weight of 100% per factor*						
Mean	0.500	0.601	0.696	0.447	0.620	0.722
Standard deviation	0.305	0.333	0.429	0.275	0.335	0.444
Sharpe ratio	1.639	1.805	1.623	1.624	1.852	1.627
p-value(SR_{CMV} - SR_{UMV})		0.128	0.569		0.029	0.466
α (%)		9.935	4.781		13.421	7.565
$t(\alpha)$		2.733	1.139		3.820	1.642
***Panel E**: Maximum weight of 200% per factor*						
Mean	0.499	0.601	0.696	0.447	0.620	0.722
Standard deviation	0.305	0.333	0.429	0.275	0.335	0.444
Sharpe ratio	1.639	1.805	1.623	1.624	1.852	1.627
p-value(SR_{CMV} - SR_{UMV})		0.130	0.562		0.030	0.466
α (%)		9.966	4.812		13.421	7.565
$t(\alpha)$		2.741	1.147		3.820	1.642
***Panel F**: Maximum weight of 500% per factor*						
Mean	0.499	0.601	0.696	0.447	0.620	0.722
Standard deviation	0.305	0.333	0.429	0.275	0.335	0.444
Sharpe ratio	1.639	1.805	1.623	1.624	1.852	1.627
p-value(SR_{CMV} - SR_{UMV})		0.129	0.563		0.029	0.469
α (%)		9.966	4.812		13.421	7.565
$t(\alpha)$		2.741	1.147		3.820	1.642

Table 7: Sensitivity of Mean-Variance Multifactor Portfolios to Risk Aversion Parameter (γ)

This table presents the same analysis as Table 3, altering the value of γ in the optimization to investigate the sensitivity of the portfolio performance to varying levels of risk aversion.

	Without Shrinkage			**With Shrinkage**		
	UMV	CMV$_m$	CMV$_k$	UMV	CMV$_m$	CMV$_k$
***Panel A**: Using* $\gamma = 1$						
Mean	2.490	2.978	3.493	2.234	3.102	3.607
Sharpe ratio	1.642	1.801	1.635	1.624	1.853	1.626
p-value(SR$_{\text{CMV}}$ - SR$_{\text{UMV}}$)		0.148	0.542		0.028	0.470
α (%)		48.958	25.436		67.213	37.871
$t(\alpha)$		2.631	1.246		3.842	1.627
***Panel B**: Using* $\gamma = 2$						
Mean	1.247	1.490	1.739	1.117	1.550	1.804
Sharpe ratio	1.643	1.803	1.617	1.624	1.853	1.629
p-value(SR$_{\text{CMV}}$ - SR$_{\text{UMV}}$)		0.146	0.592		0.028	0.465
α (%)		24.464	11.378		33.605	19.242
$t(\alpha)$		2.629	1.047		3.836	1.667
***Panel C**: Using* $\gamma = 3$						
Mean	0.830	0.997	1.161	0.745	1.033	1.202
Sharpe ratio	1.639	1.803	1.621	1.624	1.852	1.629
p-value(SR$_{\text{CMV}}$ - SR$_{\text{UMV}}$)		0.135	0.570		0.029	0.464
α (%)		16.560	7.902		22.328	12.655
$t(\alpha)$		2.691	1.110		3.820	1.652
***Panel D**: Using* $\gamma = 4$						
Mean	0.623	0.750	0.870	0.559	0.775	0.902
Sharpe ratio	1.636	1.805	1.622	1.624	1.852	1.631
p-value(SR$_{\text{CMV}}$ - SR$_{\text{UMV}}$)		0.123	0.559		0.029	0.460
α (%)		12.586	6.084		16.765	9.586
$t(\alpha)$		2.770	1.152		3.815	1.675
***Panel E**: Using* $\gamma = 6$						
Mean	0.417	0.500	0.580	0.373	0.517	0.601
Sharpe ratio	1.642	1.812	1.627	1.624	1.850	1.628
p-value(SR$_{\text{CMV}}$ - SR$_{\text{UMV}}$)		0.129	0.564		0.029	0.466
α (%)		8.423	4.034		11.140	6.345
$t(\alpha)$		2.691	1.152		3.842	1.651
***Panel F**: Using* $\gamma = 7$						
Mean	0.357	0.429	0.497	0.319	0.443	0.516
Sharpe ratio	1.642	1.810	1.630	1.623	1.850	1.628
p-value(SR$_{\text{CMV}}$ - SR$_{\text{UMV}}$)		0.133	0.555		0.028	0.460
α (%)		7.169	3.534		9.576	5.503
$t(\alpha)$		2.660	1.187		3.846	1.670
***Panel G**: Using* $\gamma = 8$						
Mean	0.313	0.375	0.436	0.278	0.387	0.451
Sharpe ratio	1.641	1.807	1.631	1.623	1.851	1.629
p-value(SR$_{\text{CMV}}$ - SR$_{\text{UMV}}$)		0.141	0.555		0.029	0.462
α (%)		6.185	3.060		8.405	4.884
$t(\alpha)$		2.630	1.186		3.820	1.691
***Panel H**: Using* $\gamma = 9$						
Mean	0.278	0.333	0.388	0.247	0.344	0.401
Sharpe ratio	1.639	1.806	1.634	1.622	1.851	1.627
p-value(SR$_{\text{CMV}}$ - SR$_{\text{UMV}}$)		0.137	0.536		0.029	0.465
α (%)		5.517	2.791		7.517	4.316
$t(\alpha)$		2.654	1.234		3.839	1.677

Table 8: Performance of Mean-Variance Multifactor Portfolios with Short-Selling Constraints Relaxed

This table replicates the results of Table 3, with the distinction that short-selling is allowed in the optimization.

	Without Shrinkage			**With Shrinkage**		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
Mean	0.501	0.584	0.698	0.447	0.618	0.716
Standard deviation	0.307	0.368	0.491	0.275	0.335	0.447
Sharpe ratio	1.634	1.586	1.422	1.623	1.845	1.601
p-value(SR_{CMV} - SR_{UMV})		0.653	0.933		0.039	0.557
α (%)		11.436	4.979		13.447	7.130
$t(\alpha)$		2.159	0.805		3.790	1.484

Table 9: Sensitivity of Mean-Variance Multifactor Portfolios to Window Length

This table presents the same analysis as Table 3, altering the length of the estimation window in the optimization to investigate the sensitivity of portfolio performance to varying window lengths.

	Without Shrinkage			With Shrinkage		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
***Panel A**: Using a window of 1 year*						
Mean	0.463	0.555	0.641	0.412	0.574	0.674
Sharpe ratio	1.388	1.527	1.435	1.384	1.572	1.439
p-value(SR_{CMV} - SR_{UMV})		0.042	0.294		0.016	0.282
α (%)		8.668	6.829		12.450	9.951
$t(\alpha)$		3.204	1.979		4.017	2.303
***Panel B**: Using a window of 2 years*						
Mean	0.455	0.541	0.629	0.408	0.561	0.658
Sharpe ratio	1.396	1.528	1.423	1.402	1.575	1.436
p-value(SR_{CMV} - SR_{UMV})		0.061	0.385		0.028	0.361
α (%)		8.304	5.915		11.634	8.824
$t(\alpha)$		3.004	1.718		3.709	2.023
***Panel C**: Using a window of 3 years*						
Mean	0.435	0.523	0.610	0.392	0.542	0.643
Sharpe ratio	1.365	1.503	1.401	1.360	1.555	1.421
p-value(SR_{CMV} - SR_{UMV})		0.062	0.353		0.014	0.275
α (%)		8.476	6.243		11.999	9.531
$t(\alpha)$		3.007	1.787		3.933	2.204
***Panel D**: Using a window of 4 years*						
Mean	0.443	0.538	0.628	0.397	0.556	0.662
Sharpe ratio	1.398	1.559	1.454	1.375	1.603	1.476
p-value(SR_{CMV} - SR_{UMV})		0.043	0.283		0.006	0.158
α (%)		9.376	7.296		13.067	11.074
$t(\alpha)$		3.247	2.054		4.281	2.577
***Panel E**: Using a window of 5 years*						
Mean	0.439	0.533	0.626	0.397	0.552	0.657
Sharpe ratio	1.394	1.541	1.443	1.393	1.593	1.461
p-value(SR_{CMV} - SR_{UMV})		0.068	0.304		0.014	0.235
α (%)		8.826	6.909		11.932	9.392
$t(\alpha)$		2.960	1.922		3.935	2.312

Continued.

	Without Shrinkage			With Shrinkage		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
Panel F*: Using a window of 6 years*						
Mean	0.459	0.546	0.639	0.408	0.560	0.664
Sharpe ratio	1.464	1.586	1.469	1.430	1.627	1.472
p-value(SR_{CMV} - SR_{UMV})		0.131	0.483		0.017	0.328
α (%)		8.013	4.932		11.832	7.958
$t(\alpha)$		2.600	1.363		3.933	2.009
Panel G*: Using a window of 7 years*						
Mean	0.449	0.537	0.624	0.397	0.550	0.649
Sharpe ratio	1.429	1.567	1.439	1.396	1.611	1.443
p-value(SR_{CMV} - SR_{UMV})		0.098	0.467		0.011	0.303
α (%)		8.402	5.074		12.324	8.153
$t(\alpha)$		2.773	1.392		4.123	2.023
Panel H*: Using a window of 8 years*						
Mean	0.461	0.554	0.642	0.413	0.569	0.666
Sharpe ratio	1.511	1.669	1.512	1.499	1.711	1.518
p-value(SR_{CMV} - SR_{UMV})		0.099	0.511		0.022	0.413
α (%)		9.230	5.119		12.431	7.557
$t(\alpha)$		2.852	1.331		3.869	1.779
Panel I*: Using a window of 9 years*						
Mean	0.479	0.575	0.666	0.428	0.593	0.691
Sharpe ratio	1.568	1.727	1.561	1.555	1.775	1.565
p-value(SR_{CMV} - SR_{UMV})		0.115	0.538		0.025	0.443
α (%)		9.511	4.966		12.931	7.521
$t(\alpha)$		2.781	1.238		3.854	1.698

Table 10: Performance of Mean-Variance Multifactor Portfolios using Variance Instead of Volatility

This table replicates the results of Table 3, with the distinction that factors are scaled using variance and not volatility.

	Without Shrinkage			With Shrinkage		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
Mean	0.499	0.584	0.713	0.447	0.541	0.731
Standard deviation	0.305	0.351	0.482	0.275	0.333	0.542
Sharpe ratio	1.639	1.663	1.481	1.624	1.626	1.350
p-value(SR_{CMV} - SR_{UMV})		0.486	0.833		0.525	0.880
α (%)		10.376	6.645		17.319	9.438
$t(\alpha)$		2.349	1.011		3.060	1.012

Table 11: Factor Exclusion Analysis: Performance of Mean-Variance Multi-factor Portfolios

This table replicates the results of Table 3, with the distinction that factors are excluded one at a time to assess the impact of each factor's removal on the statistical significance of the results.

	Without Shrinkage			**With Shrinkage**		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
***Panel A**: Removing the MKT factor*						
Mean	0.369	0.429	0.501	0.334	0.451	0.518
Standard deviation	0.300	0.295	0.408	0.268	0.303	0.422
Sharpe ratio	1.231	1.456	1.227	1.247	1.492	1.226
p-value(SR_{CMV} - SR_{UMV})		0.046	0.536		0.007	0.544
α (%)		9.084	3.103		10.314	3.810
$t(\alpha)$		3.111	0.897		3.836	0.975
***Panel B**: Removing the SMB factor*						
Mean	0.337	0.434	0.494	0.326	0.459	0.498
Standard deviation	0.232	0.273	0.346	0.224	0.283	0.353
Sharpe ratio	1.453	1.588	1.427	1.459	1.622	1.411
p-value(SR_{CMV} - SR_{UMV})		0.039	0.535		0.023	0.582
α (%)		7.220	3.843		8.575	4.729
$t(\alpha)$		3.215	0.947		3.546	1.083
***Panel C**: Removing the HML factor*						
Mean	0.459	0.534	0.633	0.429	0.545	0.647
Standard deviation	0.250	0.290	0.360	0.236	0.290	0.370
Sharpe ratio	1.835	1.843	1.758	1.816	1.880	1.750
p-value(SR_{CMV} - SR_{UMV})		0.459	0.700		0.260	0.656
α (%)		4.153	3.521		8.363	6.015
$t(\alpha)$		1.776	0.853		2.798	1.292
***Panel D**: Removing the UMD factor*						
Mean	0.424	0.534	0.617	0.383	0.553	0.636
Standard deviation	0.278	0.308	0.405	0.256	0.314	0.416
Sharpe ratio	1.522	1.734	1.525	1.498	1.763	1.527
p-value(SR_{CMV} - SR_{UMV})		0.132	0.510		0.050	0.425
α (%)		11.528	6.110		13.788	8.817
$t(\alpha)$		2.708	1.292		3.414	1.801
***Panel E**: Removing the BAB factor*						
Mean	0.394	0.494	0.592	0.334	0.507	0.608
Standard deviation	0.308	0.341	0.424	0.275	0.338	0.434
Sharpe ratio	1.278	1.450	1.396	1.217	1.502	1.403
p-value(SR_{CMV} - SR_{UMV})		0.227	0.197		0.040	0.022
α (%)		9.675	7.868		14.065	10.772
$t(\alpha)$		2.425	2.482		3.117	3.791

Table 12: Ten-Year Time-Period Subsample Performance of Mean-Variance Multifactor Portfolios

This table replicates the results of Table 3, but splits the sample period into three distinct ten-year intervals to assess the consistency of factor performance across different market conditions.

	Without Shrinkage			**With Shrinkage**		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
***Panel A**: Period: 1993–2002*						
Mean	0.305	0.418	0.452	0.321	0.458	0.457
Standard deviation	0.250	0.321	0.364	0.254	0.330	0.376
Sharpe ratio	1.221	1.304	1.242	1.264	1.388	1.214
p-value(SR_{CMV} - SR_{UMV})		0.303	0.467		0.270	0.637
α (\$)		7.074	5.883		9.675	4.725
$t(\alpha)$		1.244	0.943		1.418	0.708
***Panel B**: Period: 2003–2012*						
Mean	0.694	0.785	0.949	0.624	0.791	0.979
Standard deviation	0.420	0.411	0.529	0.367	0.395	0.541
Sharpe ratio	1.652	1.910	1.794	1.700	2.001	1.810
p-value(SR_{CMV} - SR_{UMV})		0.151	0.094		0.058	0.126
α (\$)		14.074	10.080		16.989	10.756
$t(\alpha)$		2.664	2.547		3.407	1.842
***Panel C**: Period: 2013–2023*						
Mean	0.523	0.631	0.733	0.425	0.647	0.769
Standard deviation	0.239	0.283	0.408	0.206	0.298	0.429
Sharpe ratio	2.187	2.231	1.797	2.065	2.172	1.794
p-value(SR_{CMV} - SR_{UMV})		0.377	0.913		0.234	0.794
α (\$)		7.006	-5.724		9.796	-1.156
$t(\alpha)$		1.603	-0.540		2.187	-0.106

Table 13: Five-Year Time-Period Subsample Performance of Mean-Variance Multifactor Portfolios

This table replicates the results of Table 3, but splits the sample period into six distinct five-year intervals to assess the consistency of factor performance across different market conditions.

	Without Shrinkage			With Shrinkage		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
***Panel A**: Period: 1993–1997*						
Mean	0.487	0.678	0.710	0.521	0.750	0.707
Standard deviation	0.212	0.277	0.287	0.225	0.296	0.298
Sharpe ratio	2.295	2.450	2.473	2.320	2.534	2.375
p-value(SR_{CMV} - SR_{UMV})		0.114	0.099		0.046	0.374
α (\$)		8.023	12.519		11.489	13.030
$t(\alpha)$		1.661	1.977		1.974	1.582
***Panel B**: Period: 1998–2002*						
Mean	0.091	0.127	0.147	0.078	0.145	0.156
Standard deviation	0.275	0.355	0.412	0.267	0.356	0.427
Sharpe ratio	0.331	0.357	0.357	0.292	0.408	0.365
p-value(SR_{CMV} - SR_{UMV})		0.482	0.499		0.369	0.433
α (\$)		2.789	3.010		6.139	5.008
$t(\alpha)$		0.299	0.286		0.596	0.455
***Panel C**: Period: 2003–2007*						
Mean	0.972	1.231	1.354	0.914	1.260	1.391
Standard deviation	0.293	0.374	0.420	0.282	0.386	0.429
Sharpe ratio	3.315	3.293	3.224	3.242	3.260	3.246
p-value(SR_{CMV} - SR_{UMV})		0.510	0.676		0.468	0.495
α (\$)		5.408	3.723		13.579	16.738
$t(\alpha)$		0.946	0.507		1.404	1.205
***Panel D**: Period: 2008–2012*						
Mean	0.524	0.446	0.657	0.424	0.423	0.681
Standard deviation	0.543	0.453	0.644	0.454	0.405	0.659
Sharpe ratio	0.964	0.985	1.020	0.934	1.045	1.033
p-value(SR_{CMV} - SR_{UMV})		0.470	0.228		0.182	0.163
α (\$)		1.761	4.611		5.491	8.197
$t(\alpha)$		0.374	0.773		1.134	1.059
***Panel E**: Period: 2013–2017*						
Mean	0.724	0.914	1.059	0.591	0.968	1.125
Standard deviation	0.211	0.252	0.322	0.193	0.286	0.346
Sharpe ratio	3.436	3.626	3.287	3.062	3.383	3.252
p-value(SR_{CMV} - SR_{UMV})		0.272	0.670		0.151	0.191
α (\$)		16.019	8.287		21.357	17.124
$t(\alpha)$		2.284	0.730		2.480	2.164
***Panel F**: Period: 2018–2022*						
Mean	0.383	0.418	0.482	0.310	0.404	0.497
Standard deviation	0.257	0.298	0.464	0.217	0.296	0.483
Sharpe ratio	1.487	1.402	1.039	1.425	1.363	1.028
p-value(SR_{CMV} - SR_{UMV})		0.632	0.908		0.609	0.879
α (\$)		1.043	-12.652		2.175	-10.313
$t(\alpha)$		0.197	-0.945		0.403	-0.717

Table 14: Using a GARCH(1,1) model to predict volatility

This table replicates the results of Table 3, with the distinction that factors are excluded one after one to assess the impact of each factor's removal on the statistical significance of the results.

	Without Shrinkage			**With Shrinkage**		
	UMV	CMV_m	CMV_k	UMV	CMV_m	CMV_k
Mean	0.499	0.598	0.695	0.447	0.626	0.705
Standard deviation	0.305	0.331	0.430	0.275	0.343	0.421
Sharpe ratio	1.639	1.807	1.617	1.624	1.828	1.674
p-value(SR_{CMV} - SR_{UMV})		0.130	0.590		0.020	0.275
α (%)		10.112	4.363		11.376	6.947
$t(\alpha)$		2.683	1.077		3.886	2.030

D Additional Statistics

Within our replication code, we incorporate histograms, drawdowns, correlation matrices, and cumulative returns for all the factors and factor portfolios discussed in this thesis. We report the figures here, but will not comment on them other then what has been done in the thesis itself. You can also access the supplementary appendix here.

D.1 Histograms

Figure 8: Histogram of the Original Factors

This figure illustrates a histogram and the skewness and kurtosis of the factor returns. The time period spans from 1993 to 2023.

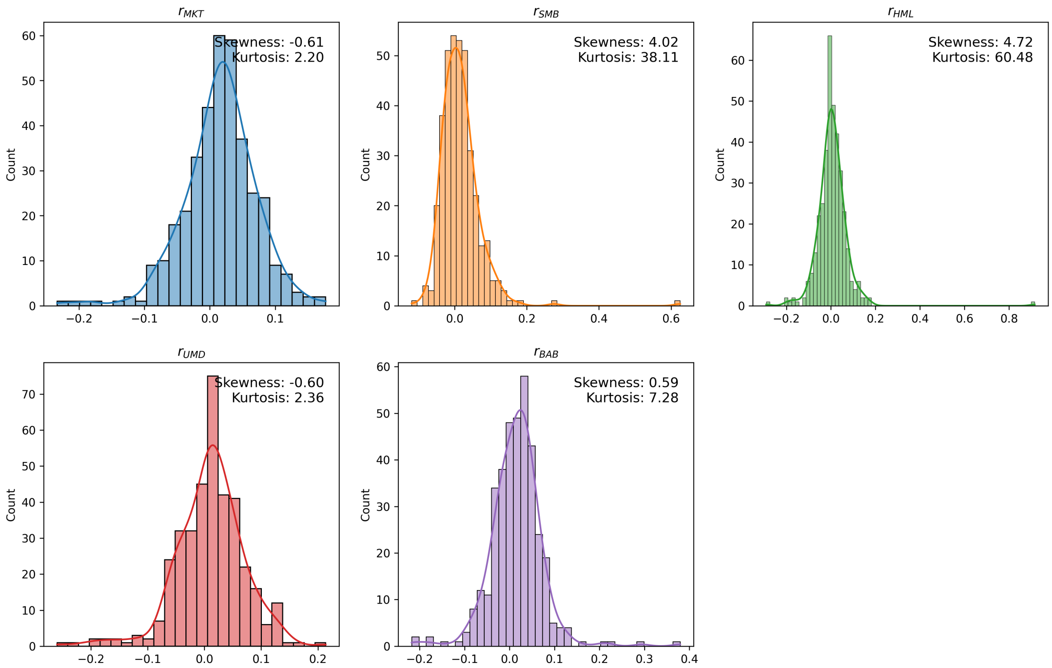


Figure 9: Histograms of Individual Volatility-Managed Factor Returns using Factor Volatility

This figure illustrates a histogram and the skewness and kurtosis of the individual volatility-managed factor returns using factor volatility. The time period spans from 1993 to 2023.

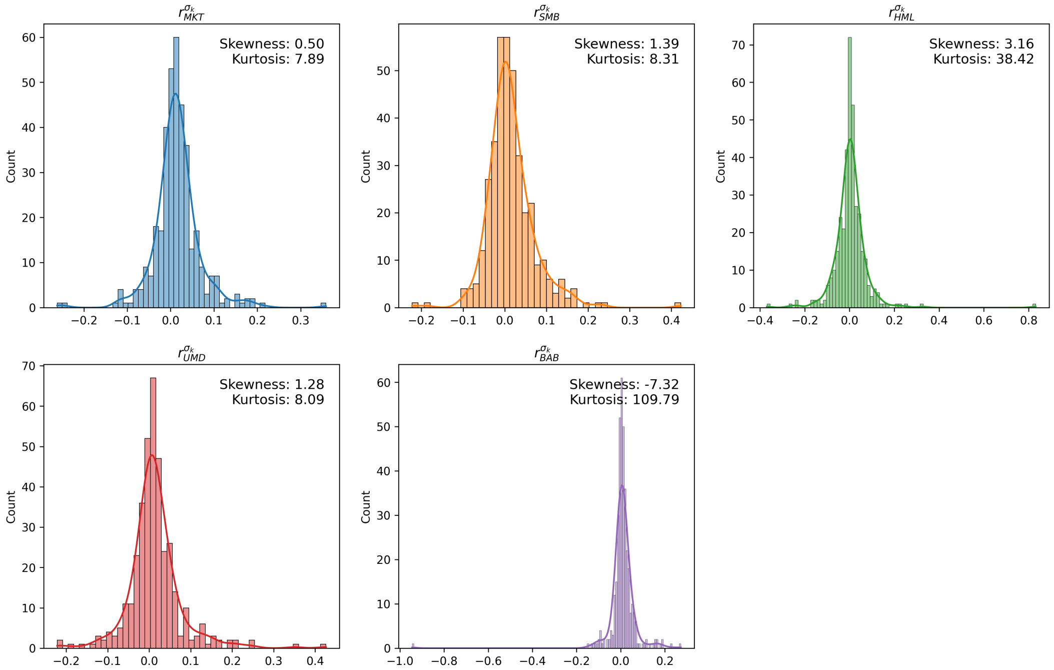


Figure 10: Histograms of Individual Volatility-Managed Factor Returns using Market Volatility

This figure illustrates a histogram and the skewness and kurtosis of the individual volatility-managed factor returns using market volatility. The time period spans from 1993 to 2023.

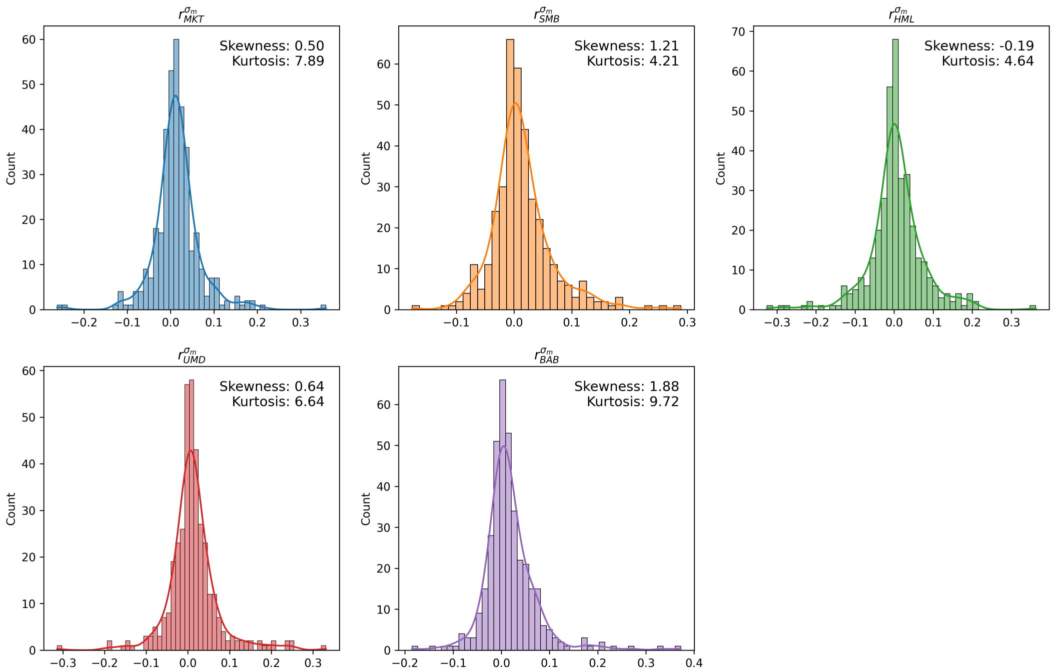


Figure 11: Histograms of Individual Volatility-Managed Factor Portfolio Returns using Factor Volatility

This figure illustrates a histogram and the skewness and kurtosis of the individual volatility-managed factor portfolio returns using factor volatility. The time period spans from 1993 to 2023.

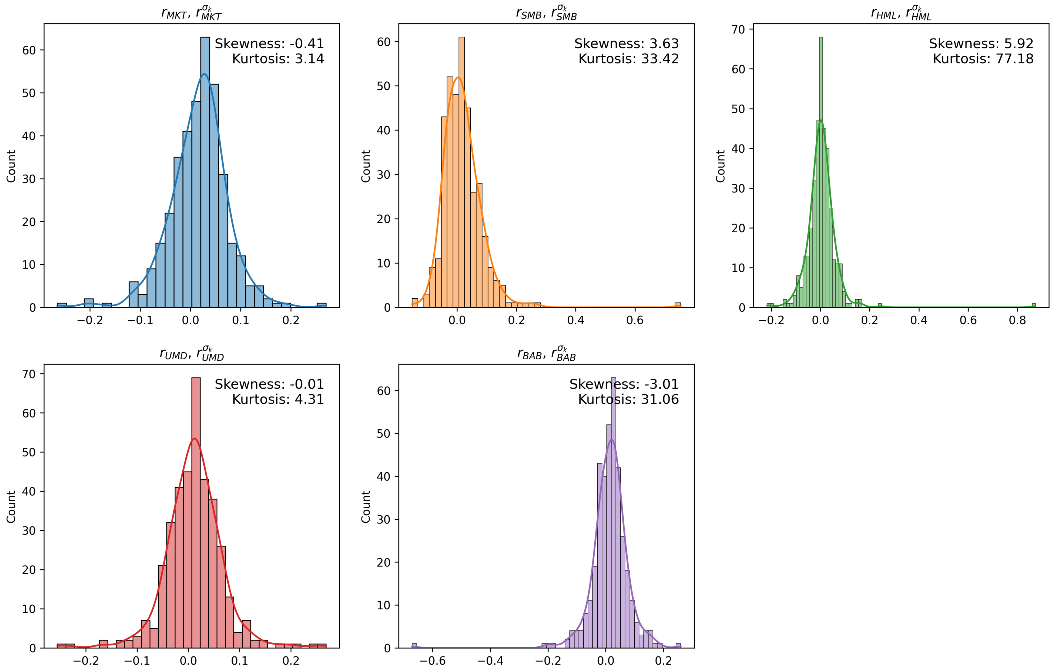


Figure 12: Histograms of Individual Volatility-Managed Factor Portfolio Returns using Market Volatility

This figure illustrates a histogram and the skewness and kurtosis of the individual volatility-managed factor portfolio returns using market volatility. The time period spans from 1993 to 2023.

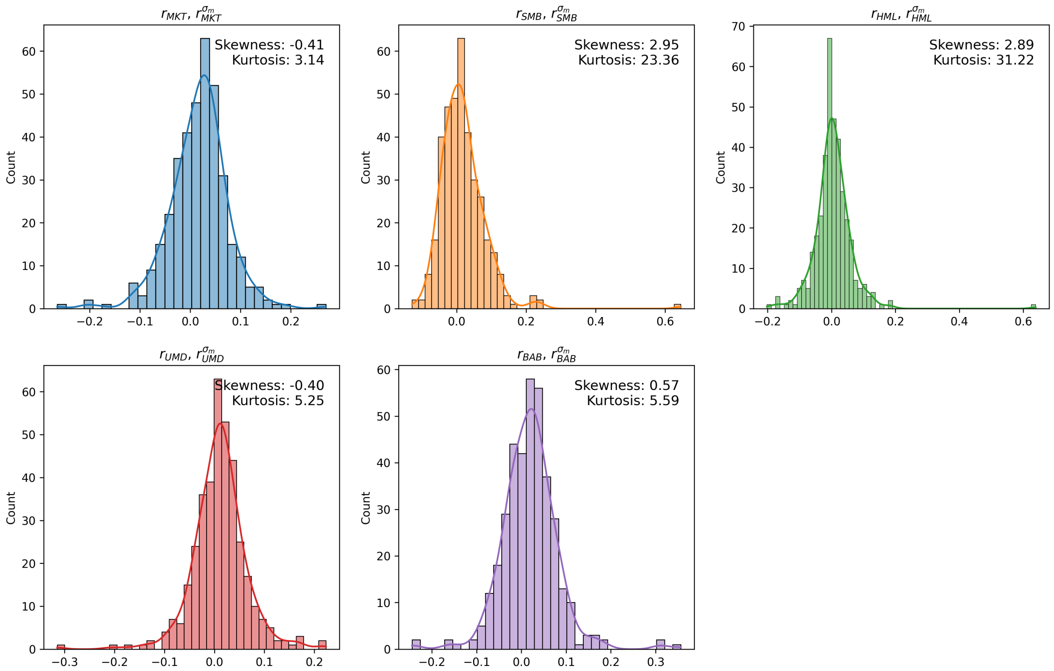
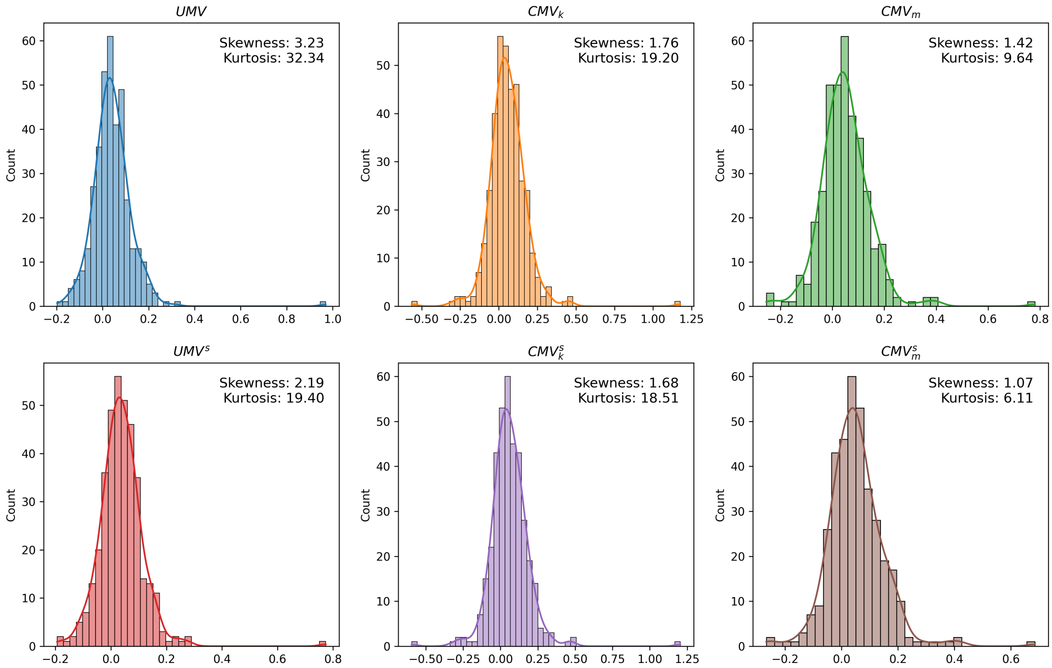


Figure 13: Histogram of Multifactor Volatility-Timed Portfolios

This figure illustrates a histogram and the skewness and kurtosis of the multifactor volatility-managed portfolios, where the superscript s indicated shrinkage. The time period spans from 1993 to 2023.



D.2 Drawdowns

Figure 14: Drawdowns of the Original Factors

This figure shows the drawdowns of the original factor. The time period spans from 1993 to 2023.

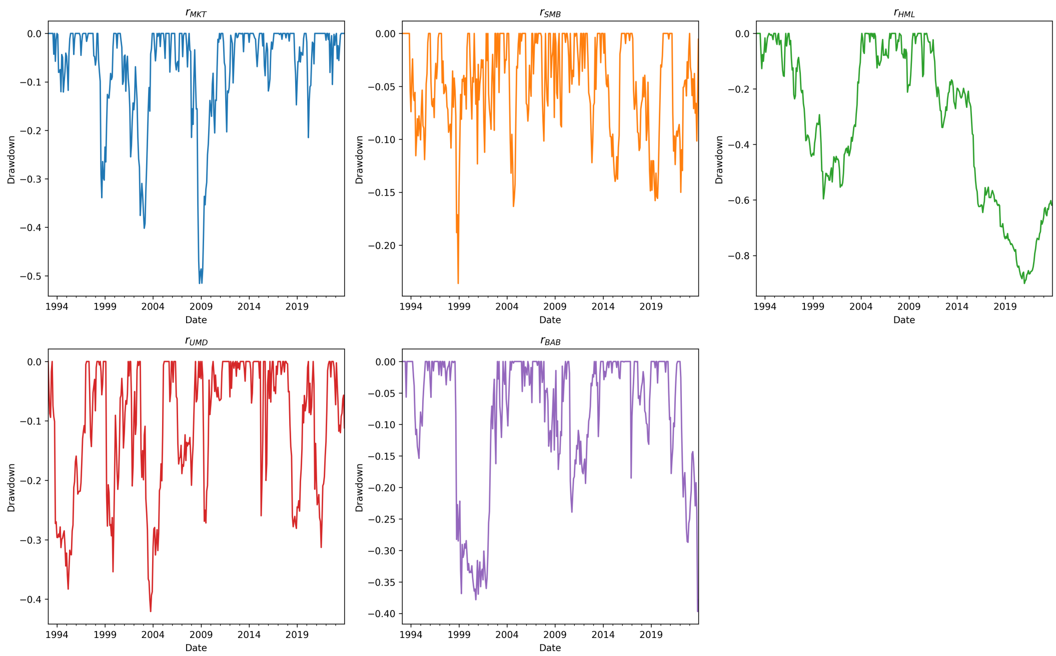


Figure 15: Drawdowns of Individual Volatility-Managed Factor Returns using Factor Volatility

This figure illustrates drawdowns of the individual volatility-managed factor returns using factor volatility. The time period spans from 1993 to 2023.

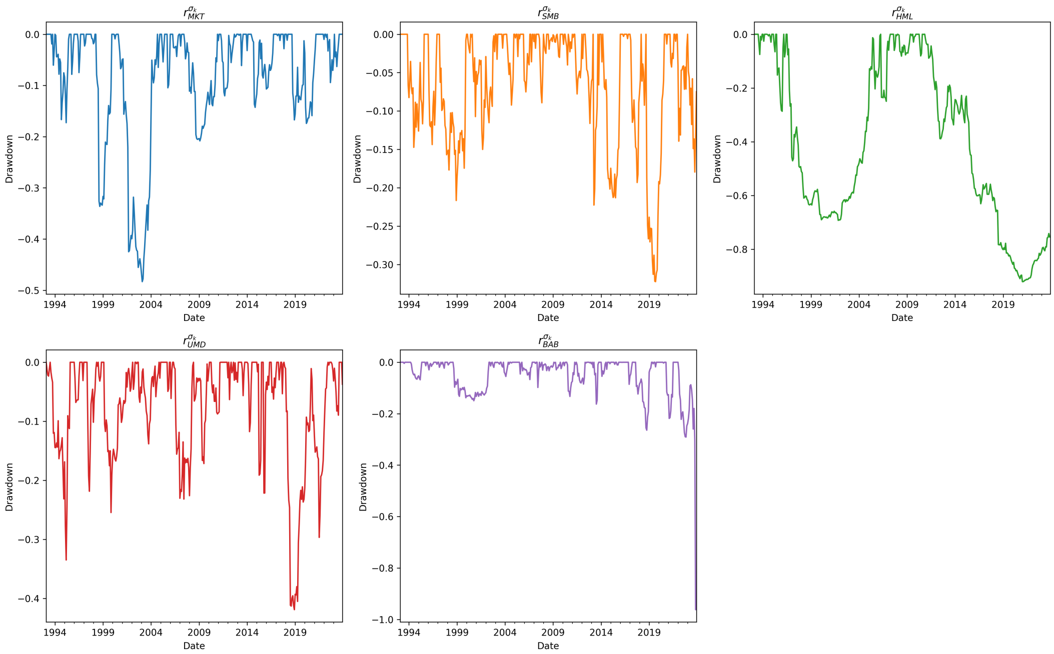


Figure 16: Histograms of Individual Volatility-Managed Factor Returns using Market Volatility

This figure illustrates drawdowns of the individual volatility-managed factor returns using market volatility. The time period spans from 1993 to 2023.

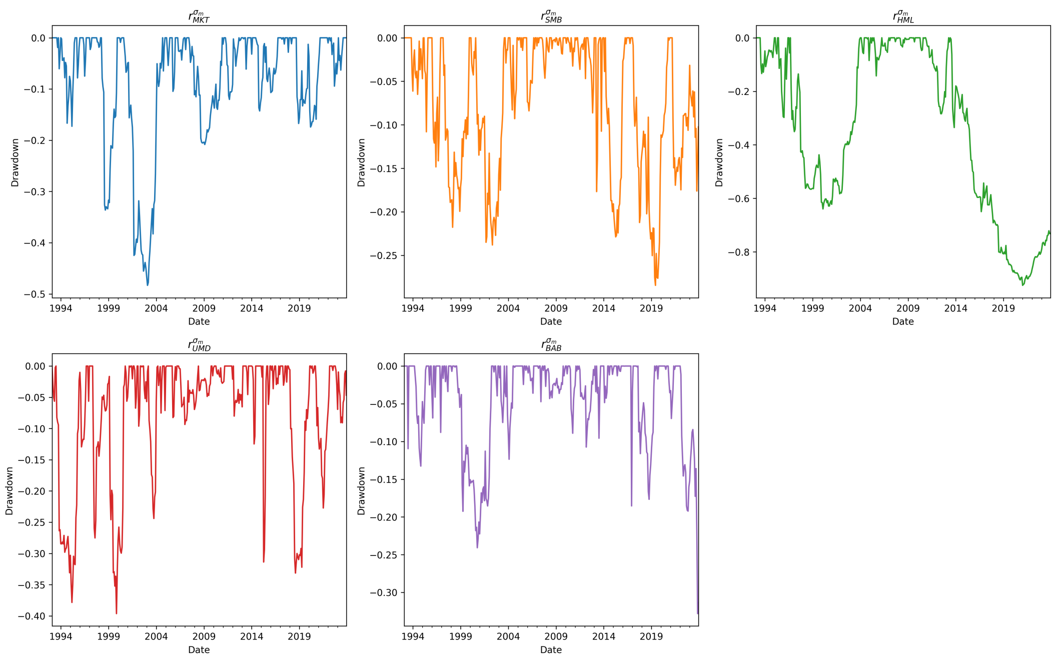


Figure 17: Drawdowns of Individual Volatility-Managed Factor Portfolio Returns using Factor Volatility

This figure illustrates drawdowns of the individual volatility-managed factor portfolio returns using factor volatility. The time period spans from 1993 to 2023.

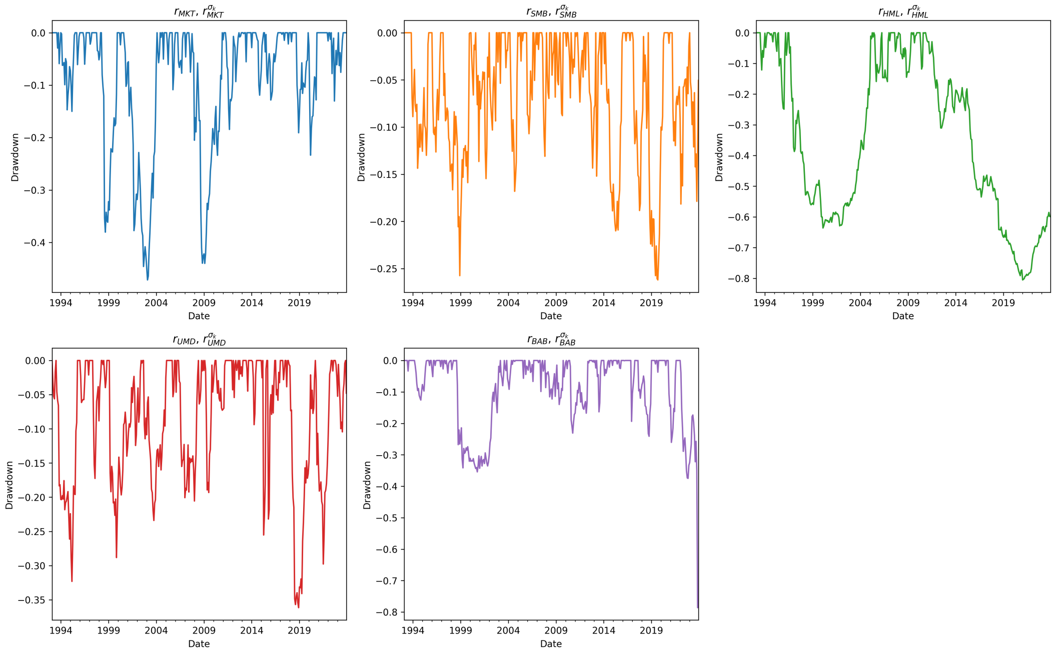


Figure 18: Drawdowns of Individual Volatility-Managed Factor Portfolio Returns using Market Volatility

This figure illustrates drawdowns of the individual volatility-managed factor portfolio returns using market volatility. The time period spans from 1993 to 2023.

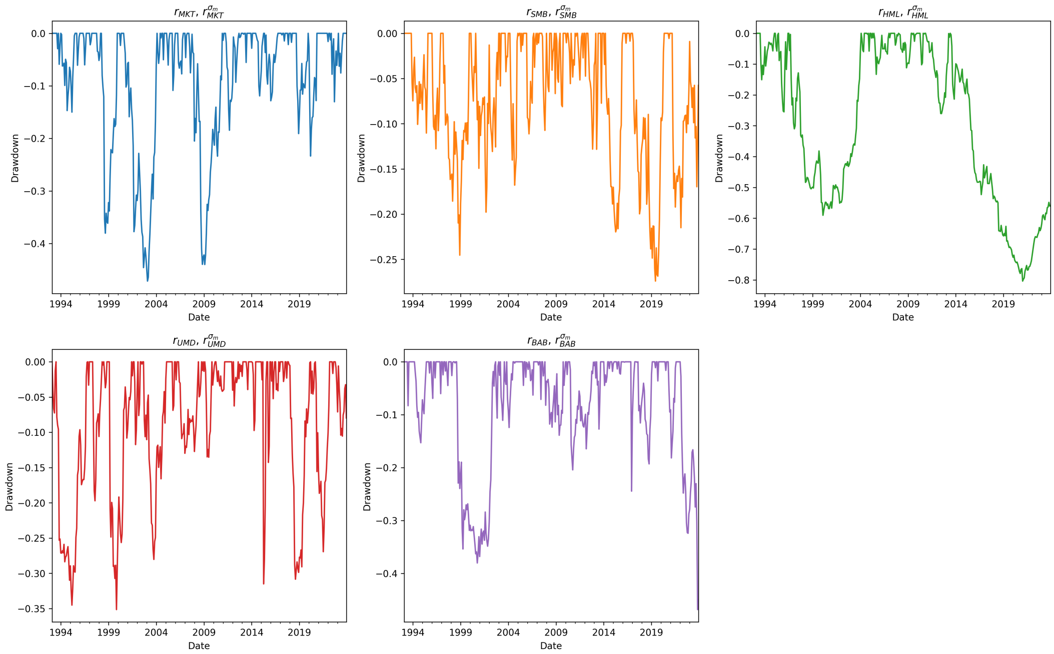
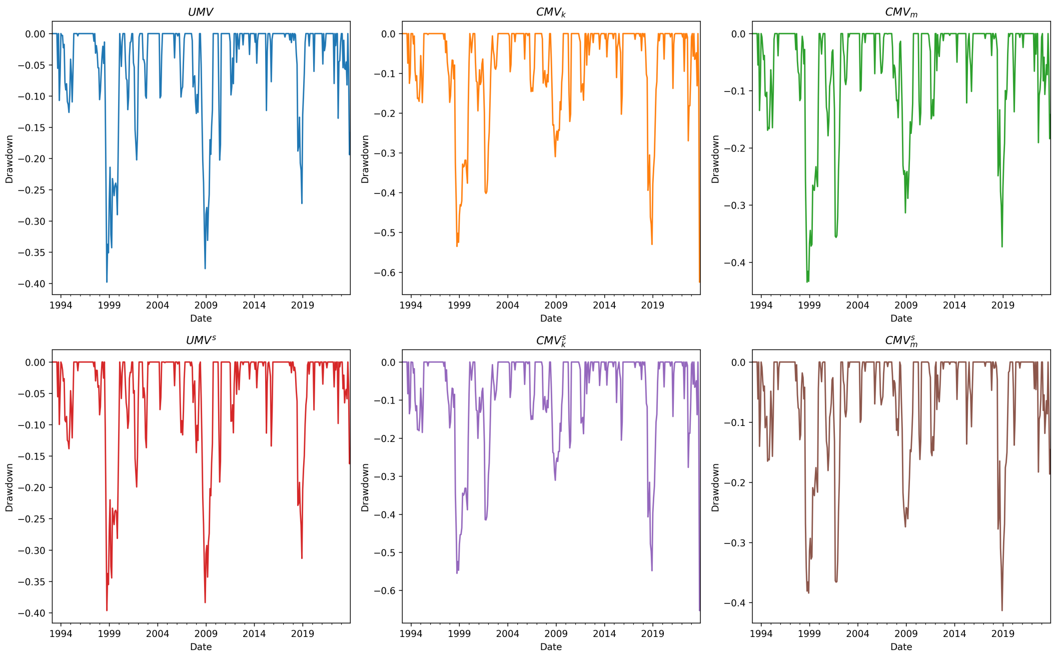


Figure 19: Drawdowns of Multifactor Volatility-Managed Portfolios

This figure shows the drawdowns of the multifactor volatility-timed factor portfolios. The time period spans from 1993 to 2023.



D.3 Correlations

Figure 20: Correlation of the Original Factors

This figure shows the correlation of the original factor. The time period spans from 1993 to 2023.

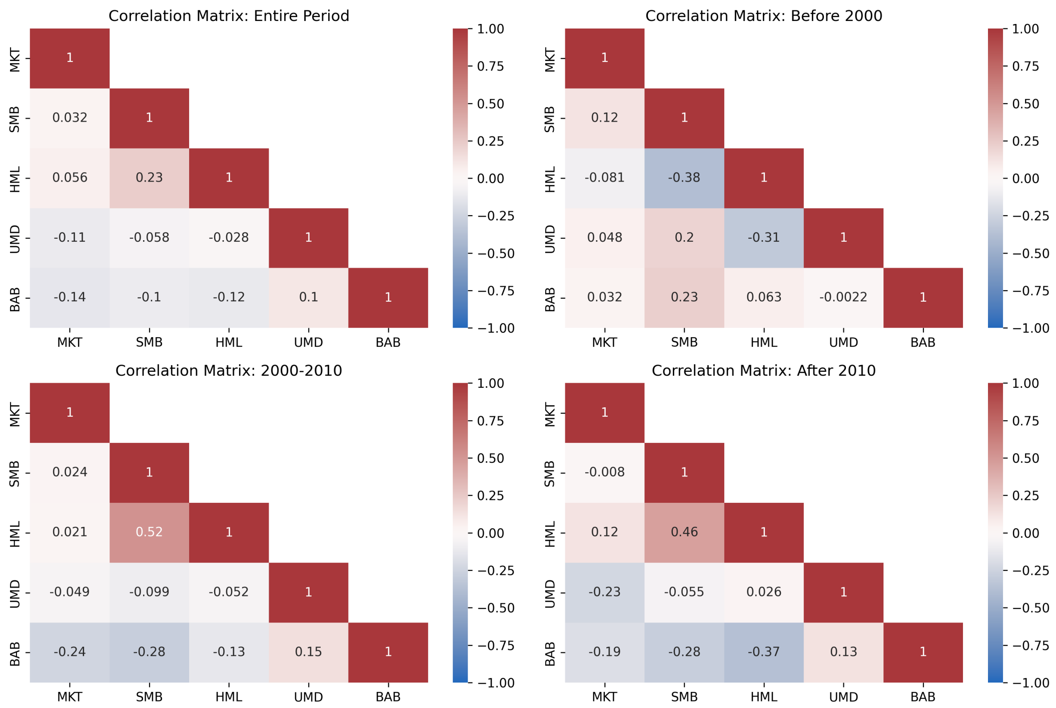


Figure 21: Correlation of Individual Volatility-Managed Factor Returns using Factor Volatility

This figure shows the correlation of the individual volatility-timed factor using factor volatility. The time period spans from 1993 to 2023.

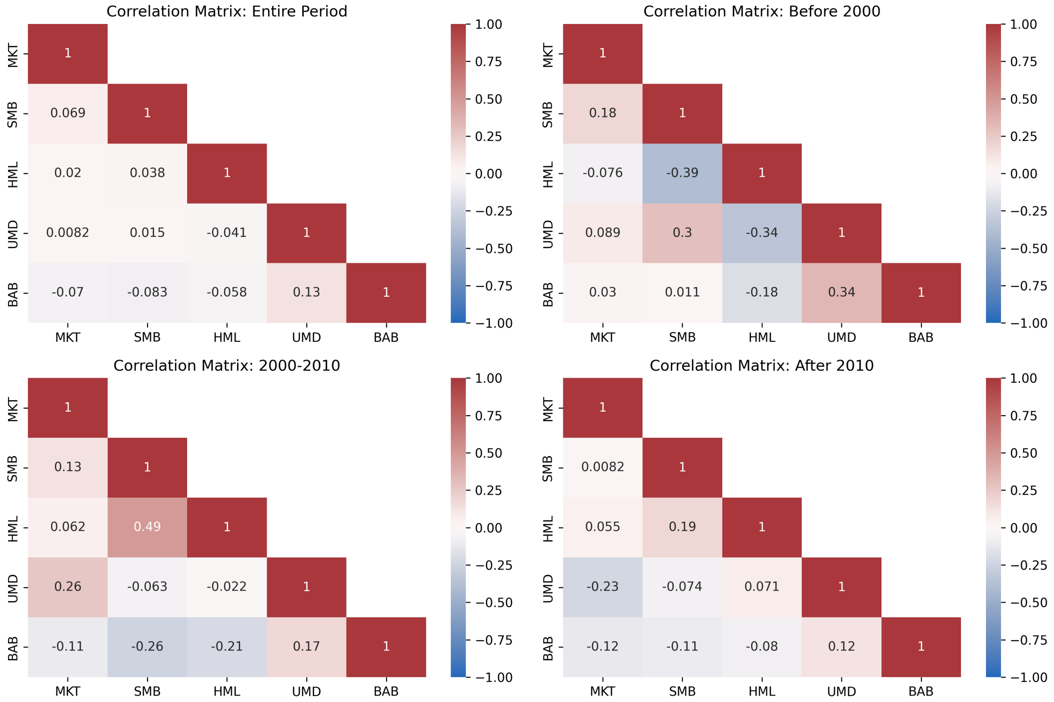


Figure 22: Correlation of Individual Volatility-Managed Factor Returns using Market Volatility

This figure shows the correlation of the individual volatility-timed factor using market volatility. The time period spans from 1993 to 2023.

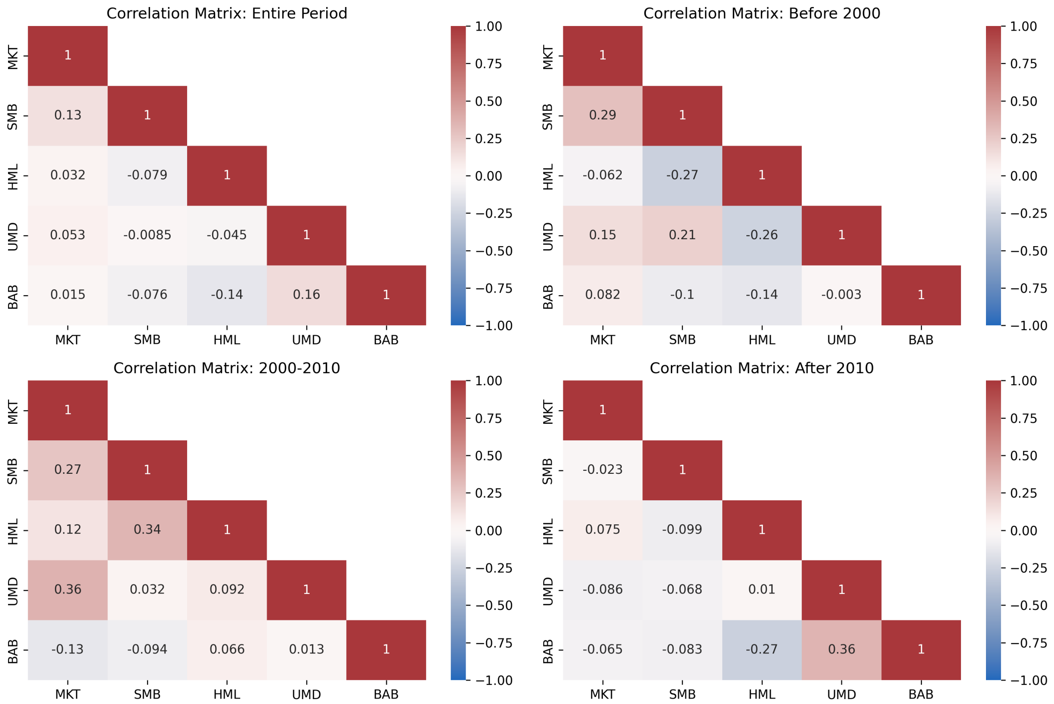


Figure 23: Correlation of Individual Volatility-Managed Factor Portfolios using Factor Volatility

This figure shows the correlation of the individual volatility-timed factor portfolio using factor volatility. The time period spans from 1993 to 2023.

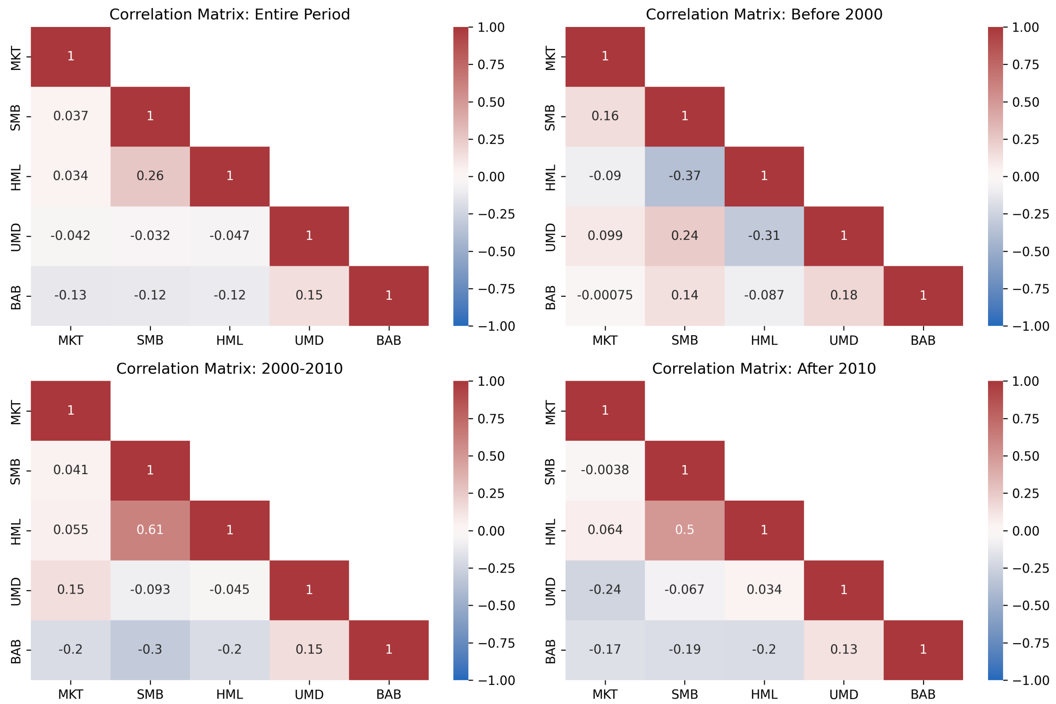


Figure 24: Correlation of Individual Volatility-Managed Factor Portfolios using Market Volatility

This figure shows the correlation of the individual volatility-timed factor portfolio using market volatility. The time period spans from 1993 to 2023.

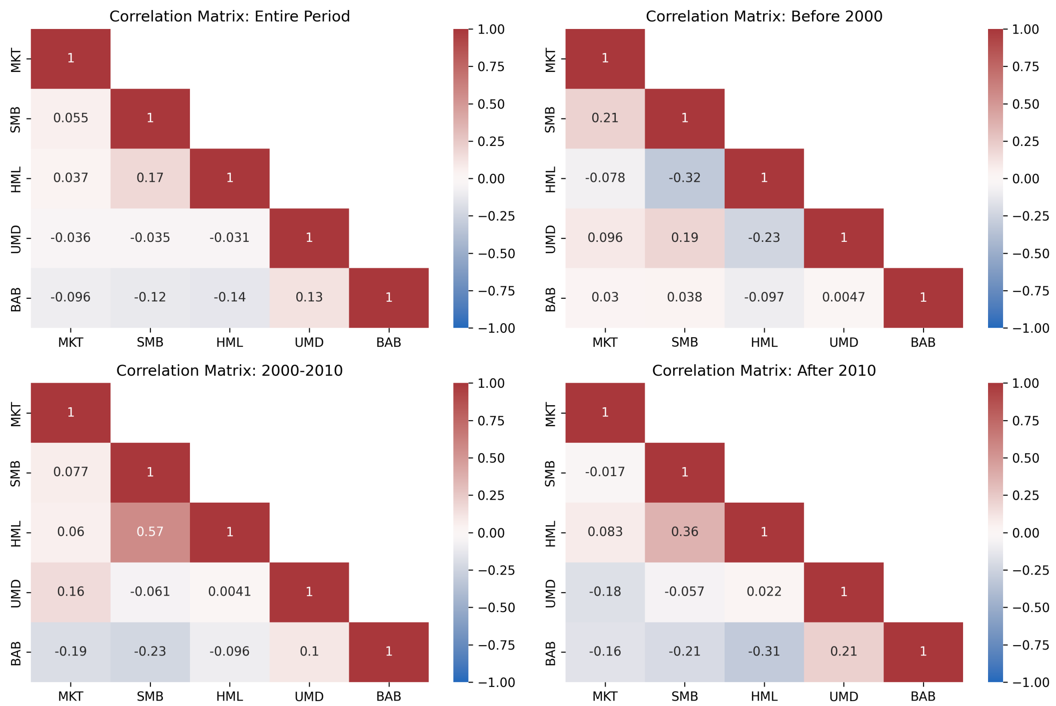
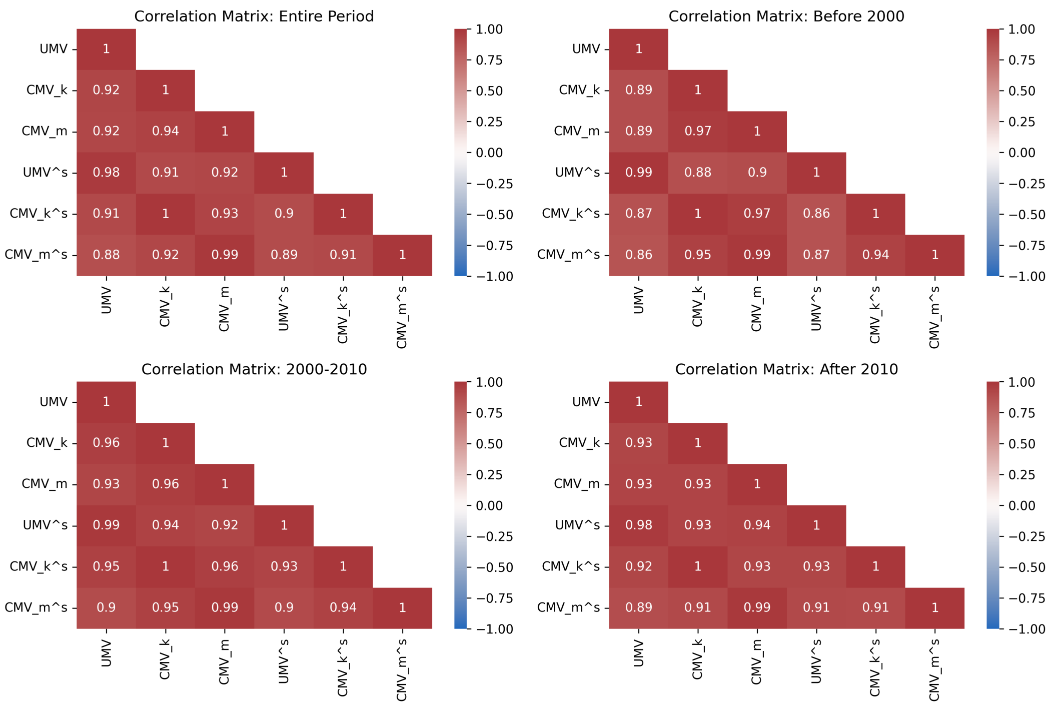


Figure 25: Correlation of Multifactor Volatility-Managed Portfolios

This figure shows the correlation of the multifactor volatility-timed factor portfolios. The time period spans from 1993 to 2023.



D.4 Cumulative returns

Figure 26: Cumulative Returns of the Original Factors

This figure shows the Cumulative Returns of the original factor. The time period spans from 1993 to 2023.

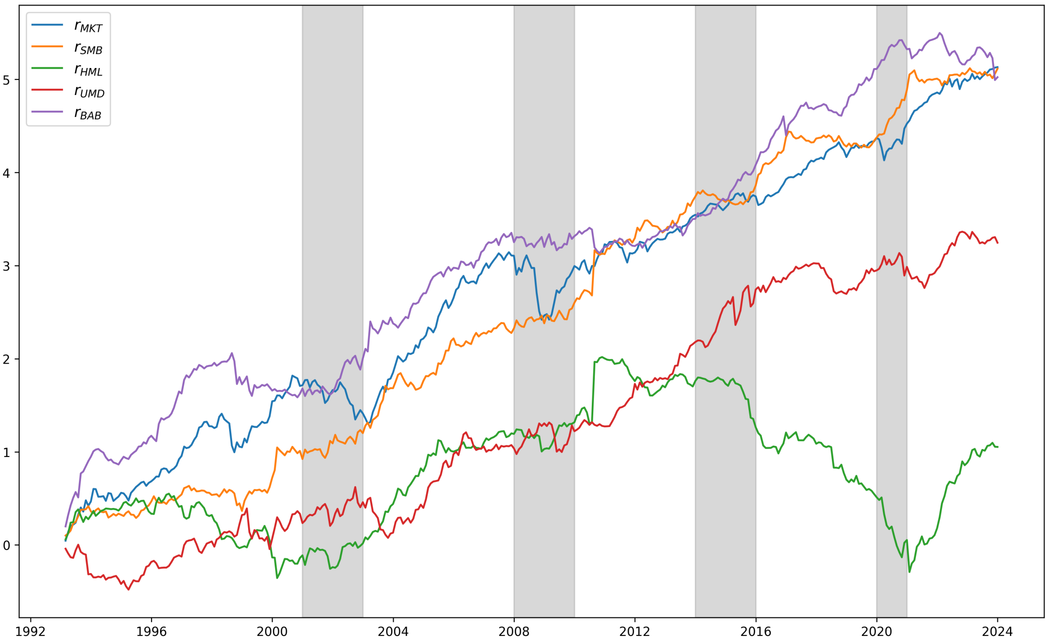


Figure 27: Cumulative Returns of Individual Volatility-Managed Factor Returns using Factor Volatility

This figure shows the Cumulative Returns of the individual volatility-timed factor using factor volatility. The time period spans from 1993 to 2023.

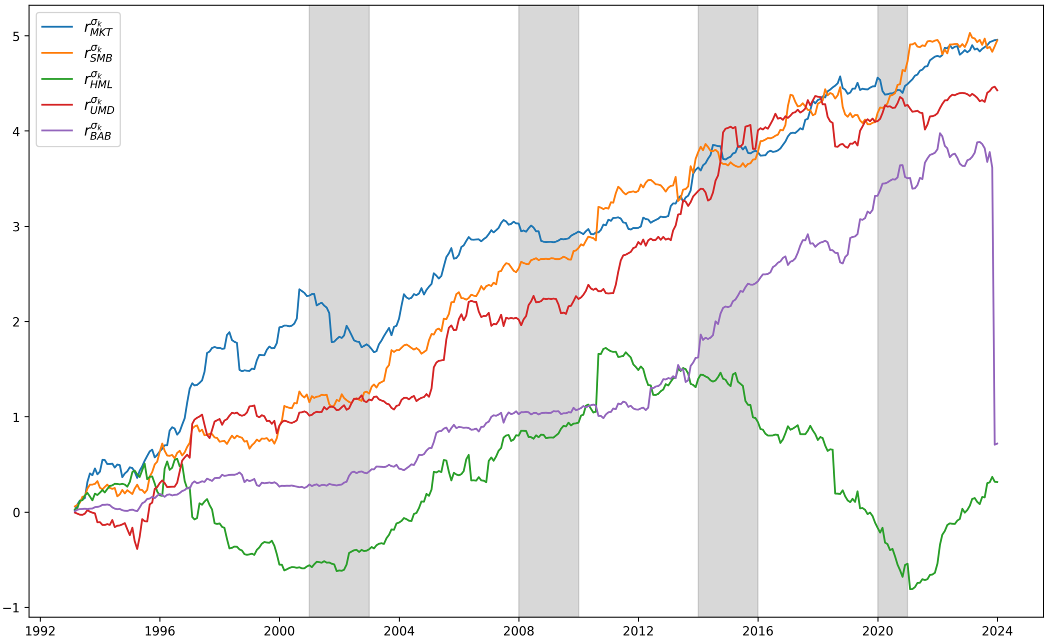


Figure 28: Cumulative Returns of Individual Volatility-Managed Factor Returns using Market Volatility

This figure shows the Cumulative Returns of the individual volatility-timed factor using market volatility. The time period spans from 1993 to 2023.

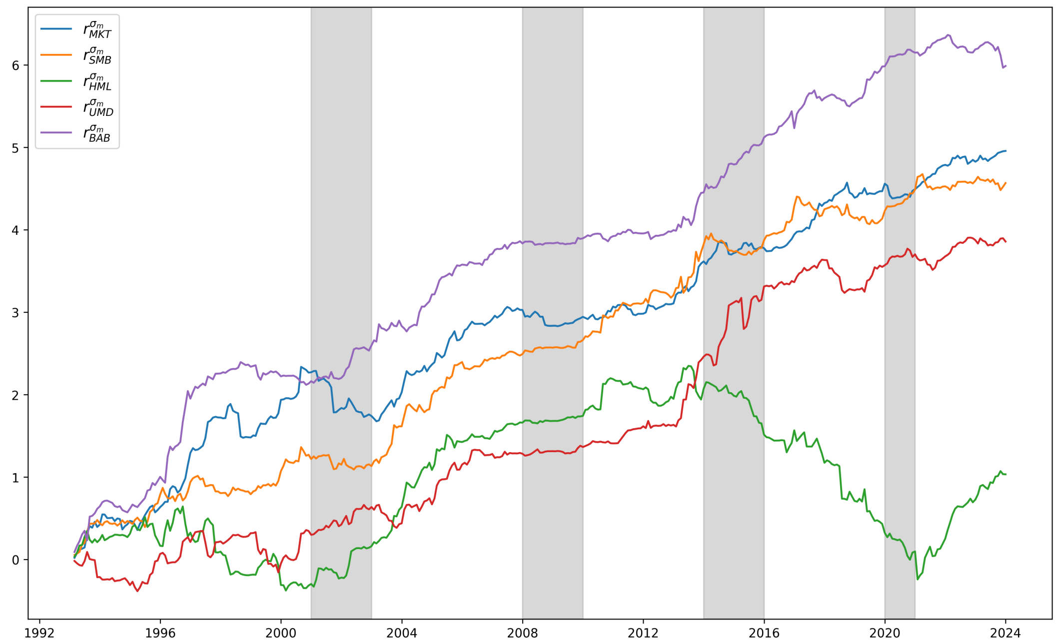


Figure 29: Cumulative Returns of Individual Volatility-Managed Factor Portfolios using Factor Volatility

This figure shows the Cumulative Returns of the individual volatility-timed factor portfolio using factor volatility. The time period spans from 1993 to 2023.

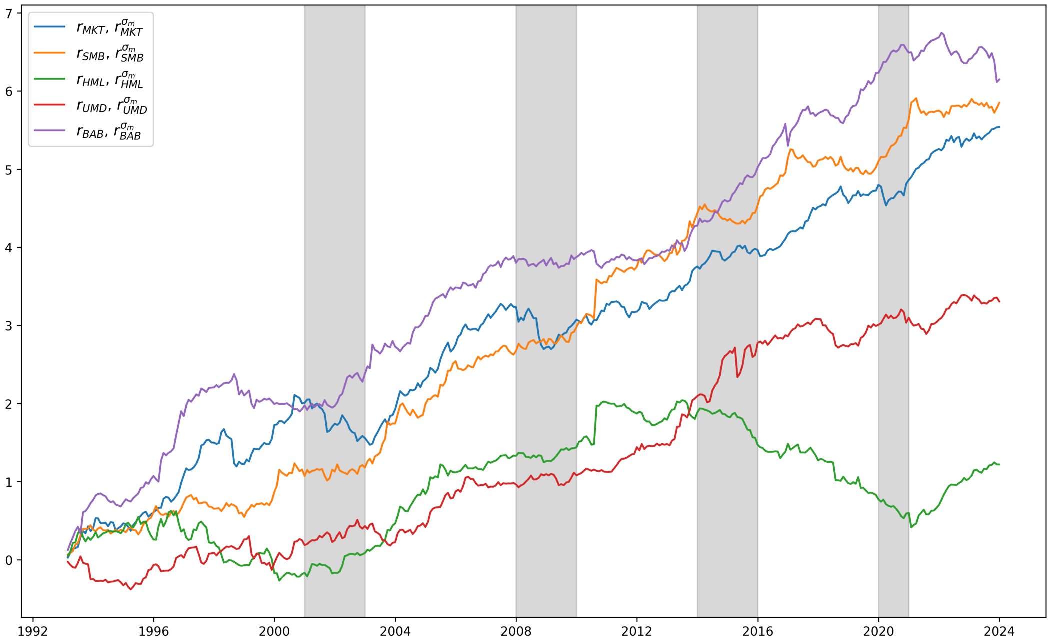
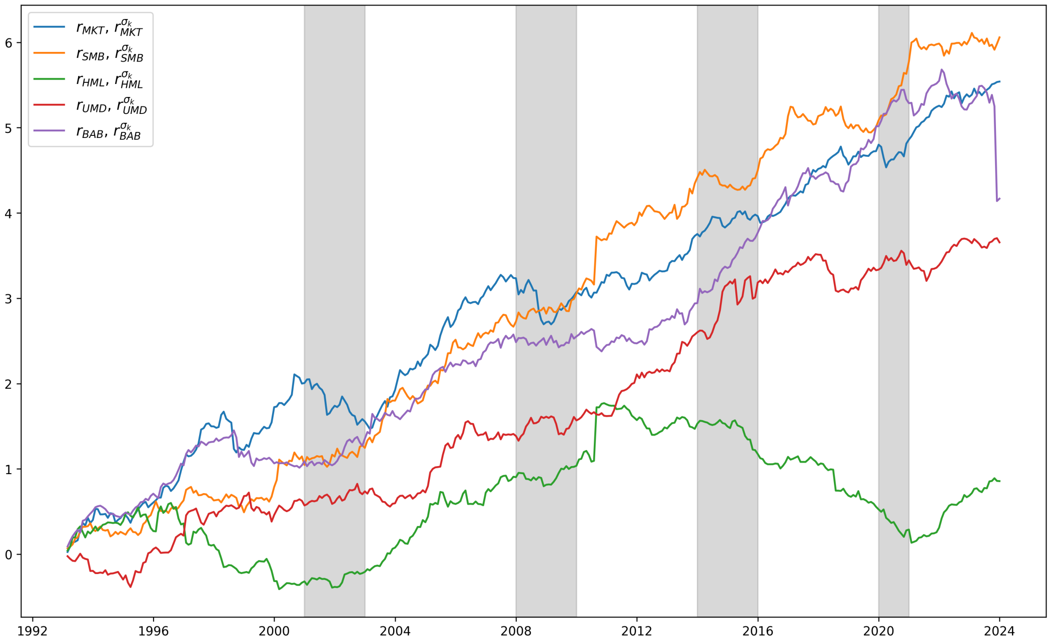


Figure 30: Cumulative Returns of Individual Volatility-Managed Factor Portfolios using Market Volatility

This figure shows the Cumulative Returns of the individual volatility-timed factor portfolio using market volatility. The time period spans from 1993 to 2023.



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