Fuzzy Logic Report

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Introduction

This report is created as part of the Computational Intelligence & Deep Reinforcement Learning course in Electrical and Computer Engineering department of Aristotle University of Thessaloniki. It analyzes the Fuzzy Logic field through applications and graphic visualizations in the form of exercises. The exercises are organized in the `fuzzy` module of the repository with respective names. Developed code in Python, along with detailed comments, can be found in the emily-palaska/GeneticFuzzyExercises GitHub Repository.

Exercise 2.8

Given the fuzzy relation $R =$ 'x is close to the origin with y', where:

$$\mu_R(x, y) = e^{-(x^2+y^2)}$$

And:

$$R = \int_{X\times Y} \frac{\mu_R(x, y)}{(x, y)} = \int_{X\times Y} \frac{e^{-(x^2+y^2)}}{(x, y)}$$

(a) Plot R graphically.

(b) Repeat the same for the fuzzy relation R = 'x is close to the perimeter of a circle of radius 1 with y'

Results

We compute and visualize the fuzzy relations over a grid of points $(x, y) \in [-2, 2] \times [-2, 2]$. The graphic visualizations are depicted in Figure 1.

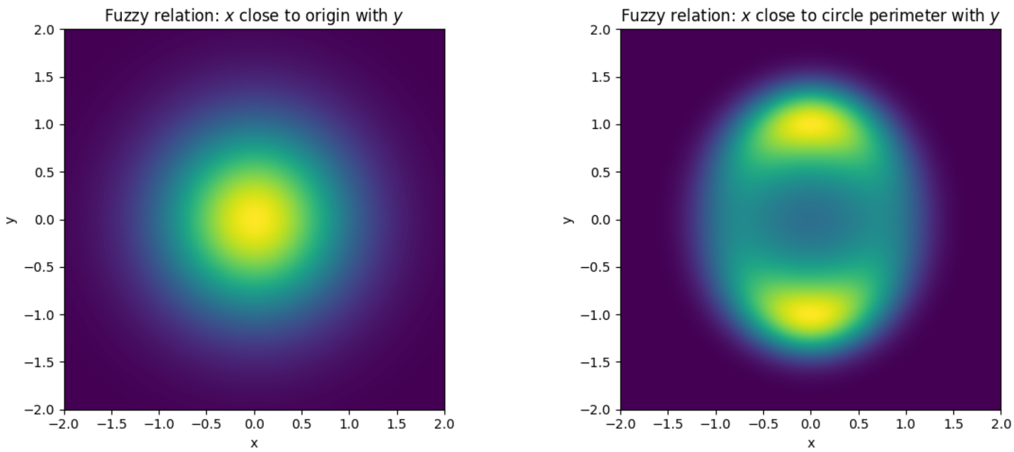


Figure 1: Visualization of fuzzy relations for exercise 2.8

(a) Fuzzy relation: x close to the origin with y

The membership function is:

$$\mu_R(x, y) = e^{-(x^2+y^2)}$$

This results in a 2D Gaussian centered at the origin, where the membership grade decreases radially outward from the origin.

(b) Fuzzy relation: x close to the perimeter of a circle of radius 1 with y

We define the membership as:

$$\mu_R(x, y) = e^{-((x^2+y^2-1)^2+y^2)}$$

This models the case where x is close to the perimeter of a unit circle and simultaneously affected by y, with highest membership around the circular boundary and centerline $y = 0$.

Exercise 2.12

Two independent but similar probability distributions are given, described by the equations:

$$dP(x_1) = e^{x_1}\, dx_1$$

$$dP(x_2) = x_2 e^{-x_2}\, dx_2$$

where $x_1, x_2 \geq 0$.

We are asked to model the "similarity" between x_1 and x_2 using a fuzzy set (relation), and to find the probability of this fuzzy event occurring.

Results

We model the similarity between $x_1 \sim \text{Exponential}(1)$ and $x_2 \sim \text{Gamma}(2, 1)$ with the fuzzy membership function:

$$\mu(x_1, x_2) = \exp\left(-\alpha(x_1 - x_2)^2\right), \quad \alpha = 1$$

Then the probability of the fuzzy event is:

$$P = \iint_0^\infty \mu(x_1, x_2) \cdot p_1(x_1) \cdot p_2(x_2)\, dx_1\, dx_2$$

Using numerical integration over the domain $[0, 10] \times [0, 10]$, we find:

$$P \approx 0.3901$$

This result quantifies the overall similarity between x_1 and x_2 based on their distributions. The model is visualized graphically in Figure 2.

Exercise 2.17

(a) Write a function that implements the following right-open S-shaped membership function:

$$S(x; l, r) = \begin{cases} 0, & x \leq l \\ 2\left(\frac{x-l}{r-l}\right)^2, & l < x < \frac{l+r}{2} \\ 1 - 2\left(\frac{r-x}{r-l}\right)^2, & \frac{l+r}{2} < x \leq r \\ 1, & r < x \end{cases}$$

(b) Plot the function for various values of the parameters l and r.

(c) Find the intersection point of $S(x; l, r)$.

(d) Show that the derivative of $S(x; l, r)$ with respect to x is continuous.

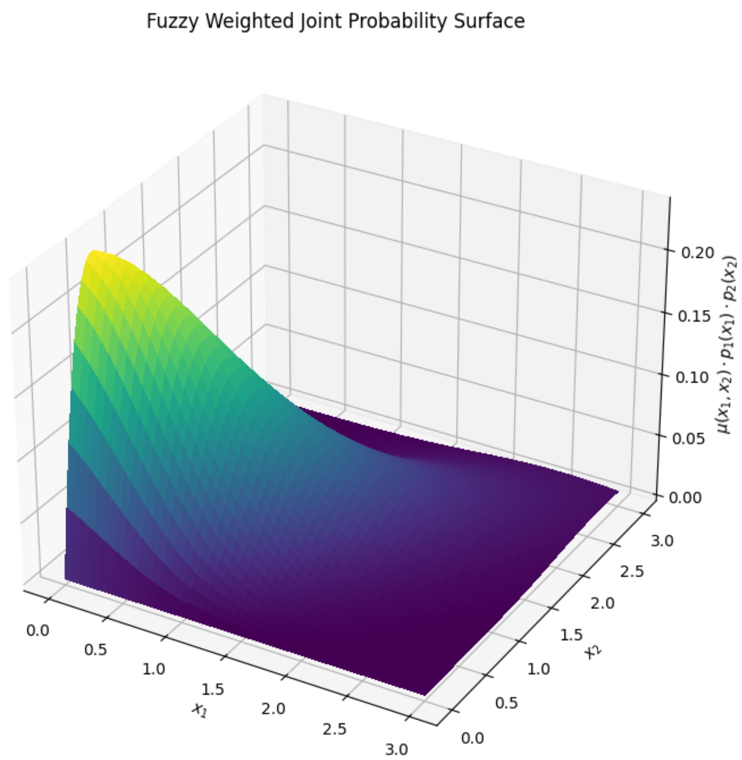


Figure 2: Visualization of fuzzy relations for exercise 2.12

Results

(a) Function Implementation:

The function $S\,(x; l, r)$ was implemented in Python using the PyTorch library in the file `fuzzy_2_17.py`. It inputs the x data points as a `torch.Tensor` and the l, r parameters as float numbers, returning the respective values y for every given point as a `torch.Tensor` as well.

(b) Graphic Visualization:

The function is plotted in Figure 3 for the l, r pairs $(2, 6), (2, 7), (3, 6), (3, 7)$, confirming the S-shaped trajectory and visualizing how these parameters affect its slope.

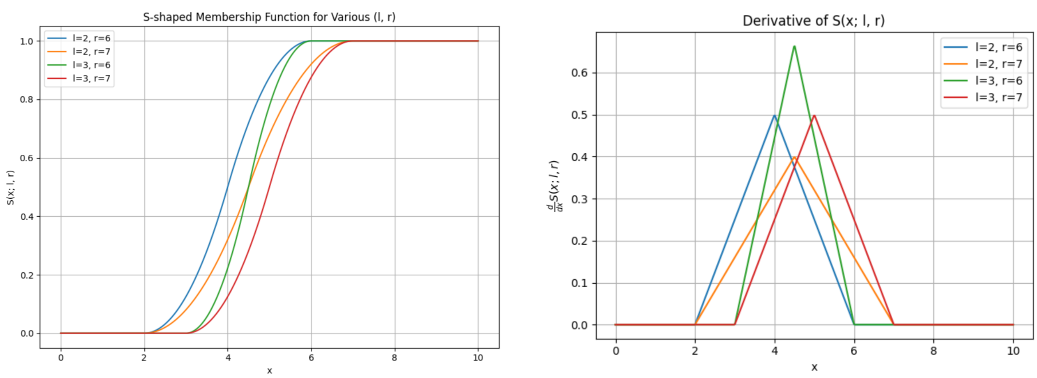


Figure 3: Plots with l, r parametric sweeps for Exercise 2.17

(c) Intersection Point:

We calculate the intersection point as

$$S\left(x; l, r\right) = 0.5 \Rightarrow x = \frac{l + r}{2}$$

This was proven experimentally by testing with the l, r pairs from the previous question and getting the results in Table 1

l	r	x	S
2	6	4.00	0.50
2	7	4.50	0.50
3	6	4.50	0.50
3	7	5.00	0.50

Table 1: Intersection values for various l and r

(d) Continuity of the Derivative:

The derivative of $S(x; l, r)$ with respect to x is given by:

$$\frac{d}{dx} S(x; l, r) = \begin{cases} 0, & x \leq l \text{ or } x > r \\ \frac{4(x-l)}{(r-l)^2}, & l < x < \dfrac{l+r}{2} \\ \frac{4(r-x)}{(r-l)^2}, & \dfrac{l+r}{2} < x < r \end{cases}$$

At $x = \frac{l+r}{2}$, the left and right derivatives both equal:

$$\frac{2}{r - l}$$

Therefore, the derivative is continuous at $x = \frac{l+r}{2}$ and everywhere else. The graphic proof of this conclusion is visualized in Figure 3.

Exercise 3.1

Given the fuzzy numbers A and B:

$$A = \frac{0.33}{6} + \frac{0.67}{7} + \frac{1.00}{8} + \frac{0.67}{9} + \frac{0.33}{10}$$

$$B = \frac{0.33}{1} + \frac{0.67}{2} + \frac{1.00}{3} + \frac{0.67}{4} + \frac{0.33}{5}$$

(i) Plot and provide a linguistic interpretation of these fuzzy numbers.

(ii) Compute and plot the fuzzy product $C = A \cdot B$.

(iii) Compute and plot the fuzzy difference $D = A - B$ and the fuzzy quotient $E = A + B$.

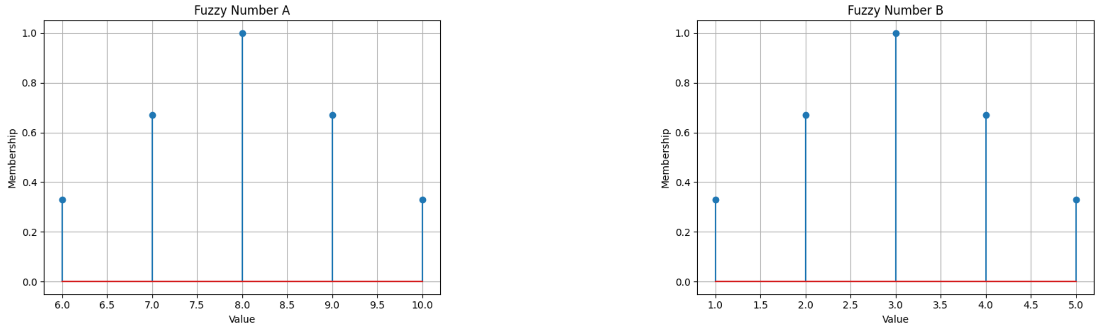
Results

The two Fuzzy Numbers are depicted in Figure 4.A is centered around $x = 8$ with the highest membership of 1.00, indicating that 8 is the most representative crisp value. B is centered around $x = 3$ with the highest membership of 1.00. A linguistic interpretation would be:

$$A : \text{approximately } 8$$

$$B : \text{approximately } 5$$

The result of the operations are visible in Figure 5.



(a) Fuzzy Number A

(b) Fuzzy Number B

Figure 4: Stem graph of Fuzzy Numbers A, B for Exercise 3.1.

Fuzzy Product $C = A \cdot B$

The fuzzy product is computed using the extension principle. For each pair (x_i, y_j) from A and B, we compute:

$$z_{ij} = x_i \cdot y_j, \quad \mu_C(z_{ij}) = \min(\mu_A(x_i), \mu_B(y_j))$$

We aggregate these using the max operator:

$$\mu_C(z) = \max_{(i,j)\,:\,x_i \cdot y_j = z} \min(\mu_A(x_i), \mu_B(y_j))$$

Fuzzy Difference $D = A - B$

The fuzzy difference is computed using the extension principle. For each pair (x_i, y_j) from A and B, we compute:

$$z_{ij} = x_i - y_j, \quad \mu_D(z_{ij}) = \min(\mu_A(x_i), \mu_B(y_j))$$

We aggregate using the max-min operation:

$$\mu_D(z) = \max_{(i,j)\,:\,x_i - y_j = z} \min(\mu_A(x_i), \mu_B(y_j))$$

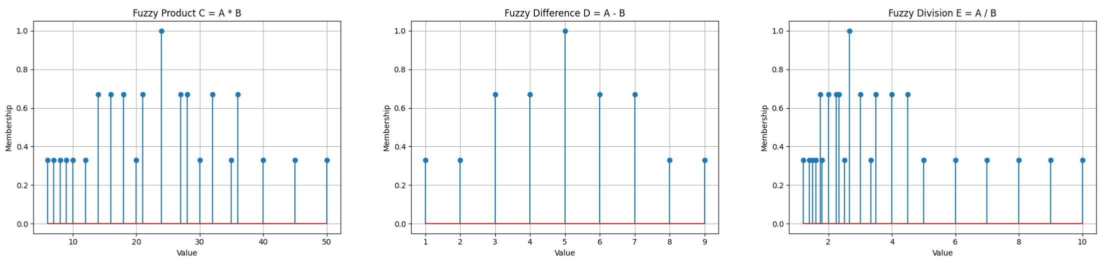
Fuzzy Sum $E = A + B$

The fuzzy sum is also computed using the extension principle. For each pair (x_i, y_j) from A and B:

$$z_{ij} = x_i + y_j, \quad \mu_E(z_{ij}) = \min(\mu_A(x_i), \mu_B(y_j))$$

Aggregation is again via the max-min rule:

$$\mu_E(z) = \max_{(i,j)\,:\,x_i + y_j = z} \min(\mu_A(x_i), \mu_B(y_j))$$

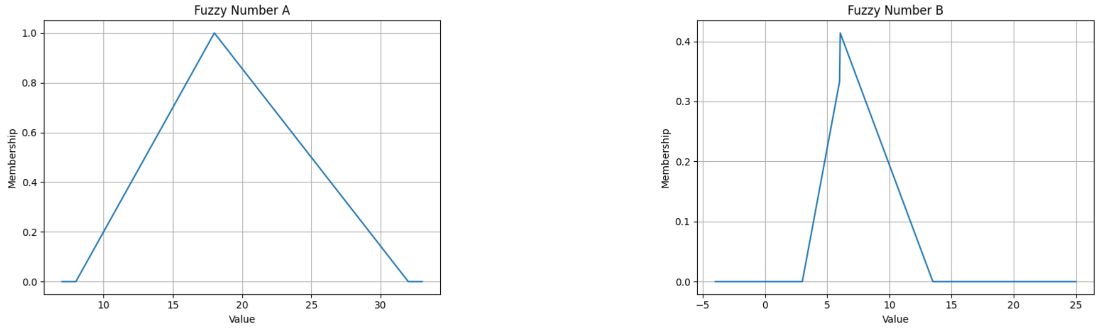


(a) Multiplication

(b) Difference

(c) Division

Figure 5: Stem graphs of fuzzy operations on A, B for Exercise 3.1



(a) Fuzzy Number A

(b) Fuzzy Number B

Figure 6: Graph of two Fuzzy Numbers A, B.

Exercise 3.2

Repeat the Exercise 3.1 for the fuzzy numbers A and B, defined via their membership functions:

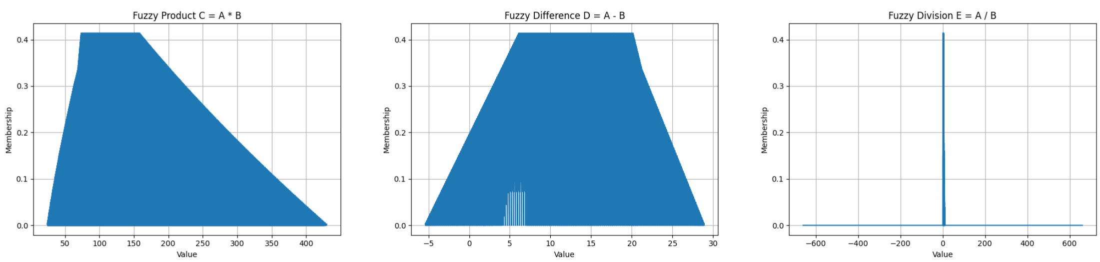
$$\mu_A(x) = \begin{cases} 0, & x \leq 8 \\ \frac{1}{10}x - \frac{8}{10}, & 8 < x < 18 \\ -\frac{1}{14}x + \frac{32}{14}, & 18 \leq x \leq 32 \\ 0, & x > 32 \end{cases}$$

$$\mu_B(x) = \begin{cases} 0, & x \leq -3 \\ \frac{1}{9}x - \frac{1}{3}, & -3 < x \leq 6 \\ -\frac{1}{18}x + \frac{3}{4}, & 6 < x \leq 24 \\ 0, & x > 24 \end{cases}$$

Results

The Fuzzy numbers A, B produced by the given membership functions $\mu_A(x), \mu_B(x)$ are visualized in Figure 6 for a linear space with `step=0.05`.

The calculation of $C = A - b$, $D = A * B$ and $E = A/B$ remain as defined above. Results are presented in Figure 7, following the expected pattern. The gap in the Subtraction plot can be explained by the membership clipping $\mu = max(0,\ \mu)$ as an area where $B > A$. The Division plot shows very low values for most memberships, except around $x = 0$.



(a) Multiplication

(b) Difference

(c) Division

Figure 7: Stem graphs of fuzzy operations on A, B for Exercise 3.2

Acknowledgments

Results presented in this work have been produced using the Aristotle University of Thessaloniki (AUTh) High Performance Computing Infrastructure and Resources. Additionally, the open-source language model ChatGPT was utilized in parts of these experiments. It generated portions of the code which were then tested and analyzed, as well as enhanced the overall readability and clarity of this report.