Cheese

Let i be the store index and j be the index of the observations of a single store. I_{ij} is the display indicator.

The model is as follows:

$$\log Q_{ij} = \alpha_{0,i} + \alpha_{1,i} I_{ij} + \beta_{0,i} \log P_{ij} + \beta_{1,i} \log P_{ij} I_{ij}$$

$$= \alpha_{0,i} + \beta_{0,i} \log P_{ij} + I_{ij}(\alpha_{1,i} + \beta_{1,i} \log P_{ij})$$

We can also write this in linear regression/matrix form.

Let y_{ij} denote $\log Q_{ij}$, and for each row in the dataset define

$$x_{ij} = [1 \quad I_{ij} \quad \log P_{ij} \quad I_{ij} * \log P_{ij}]$$

Now we also define a parameter vector for each store:

$$\beta_i = [\alpha_{0,i} \quad \alpha_{1,i} \quad \beta_{0,i} \quad \beta_{1,i}]$$

Then our model may be written like this:

$$y_{ij} = x_{ij}' \beta_i + \varepsilon_{ij} \quad , w/ \quad \varepsilon_{ij} \sim N(0, \sigma_i^2).$$

Now let's define the full model.

Likelihood: $p(y_{ij} \mid \beta_i, \sigma_i^2) \sim N(x_{ij}'\beta_i, \sigma_i^2)$

Priors : $p(\beta_i) \sim N(m, s^2 I)$

$p(\sigma_i^2) \sim IG(\frac{a}{2}, \frac{b}{2})$ * set $a=4, b=2$

Hyperpriors: $p(m) \sim 1$

$p(s^2) \sim IG(\frac{1}{2}, \frac{1}{2})$

Now (on the next page) we compute the posterior.

$$p(\beta_i \mid \ldots) \propto p(\beta_i) \cdot p(y_{ij} \mid \beta_i, \sigma_i^2)$$

$$\propto \exp\left\{\frac{-1}{2s^2}(m-\beta_i)'(s^2 I)^{-1}(m-\beta_i)'\right\} \exp\left\{(y_i - X_i'\beta_i)'(\sigma_i^2 I)^{-1}(y_i - X_i'\beta_i)\right\} \text{ where } y_i = \begin{bmatrix} y_{i1} \\ \vdots \\ y_{iN_i} \end{bmatrix}$$

(Normal-Normal conjugate model done previously)

$$\rightarrow \text{Normal}(\mu^*, \Sigma_i^*) \text{ with } \Sigma_i^* = \left(\frac{1}{\sigma_i^2} X_i'X_i + \frac{1}{s^2} I\right)^{-1} \text{ and } \mu^* = \Sigma_i^{*-1}\left(\frac{1}{\sigma_i^2} X_i'Y_i + \frac{1}{s^2} m\right)$$

$$p(\sigma_i^2) \propto p(a)\, p(b)\, p(\sigma_i^2 \mid a, b)\, p(y_i \mid \beta_i, \sigma_i^2)$$

$$\propto a \exp\{-a\}\, b \exp\{-b\} (\sigma_i^2)^{-\frac{a}{2}-1} \exp\left\{\frac{-1}{\sigma_i^2} \frac{b}{2}\right\} \cdot \frac{1}{\sqrt{\det(\sigma_i^2 I)}} \exp\left\{\frac{-1}{2}(y_i - X_i'\beta_i)(\sigma_i^2 I)^{-1}(y_i - X_i'\beta_i)\right\}$$

Normal/Gamma is also conjugate:

$$\rightarrow IG\left(\frac{a}{2} + \frac{N_i}{2}, \frac{b}{2} + \frac{1}{2}(Y_i - X_i'\beta_i)'(Y_i - X_i'\beta_i)\right)$$

$$p(m \mid \ldots) \propto p(m) \prod_i p(\beta_i \mid m) \propto \prod_i \exp\left\{\frac{-1}{2}(\beta_i - m)'(s^2 I)^{-1}(\beta_i - m)\right\}$$

$$\propto \exp\left\{\frac{-1}{s^2} \sum_i (\beta_i - m)'(\beta_i - m)\right\}$$ then complete the square

$$\rightarrow \text{Normal}\left(\frac{1}{n} \sum_i \beta_i, \frac{1}{ns^2} I\right)$$

$$p(s^2 \mid \ldots) \propto p(s^2) \cdot p(\beta_i \mid s_i)$$

$$\propto (s^2)^{-\frac{1}{2}-1} \exp\left\{\frac{-1}{s^2} \cdot \frac{1}{2}\right\} \cdot \prod_{i=1} \frac{1}{\sqrt{\det(s^2 I)}} \exp\left\{-\frac{1}{2}(\beta_i - m)'(s^2 I)^{-1}(\beta_i - m)\right\}$$

$$\propto (s^2)^{-\frac{1}{2}-\frac{n}{2}-1} \exp\left\{\frac{-1}{s^2} \cdot \frac{1}{2} - \frac{1}{2s^2} \sum_i (\beta_i - m)'(\beta_i - m)\right\}$$

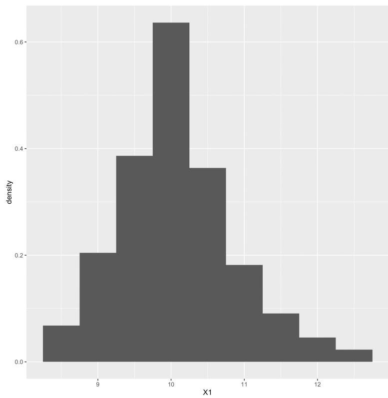
$$= (s^2)^{-(n+1)/2-1} \exp\left\{\frac{-1}{s^2} \cdot \frac{1}{2}\left(1 + \sum_i (\beta_i - m)'(\beta_i - m)\right)\right\}$$

$$\rightarrow IG\left(\frac{n+1}{2}, \frac{1}{2}\left(1 + \sum_i (\beta_i - m)'(\beta_i - m)\right)\right)$$

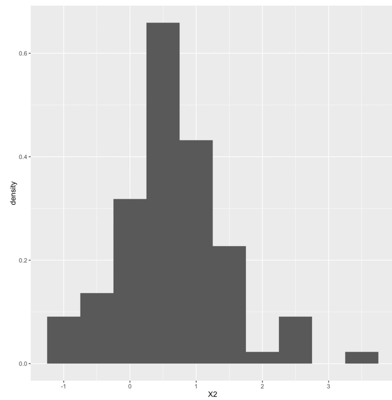
Now, code a Gibbs sampler for it.

Posterior mean estimates for:

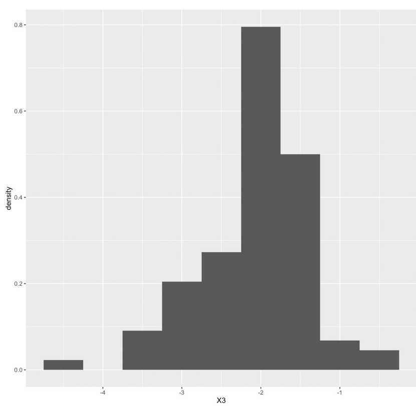
$\alpha_{0,i}$



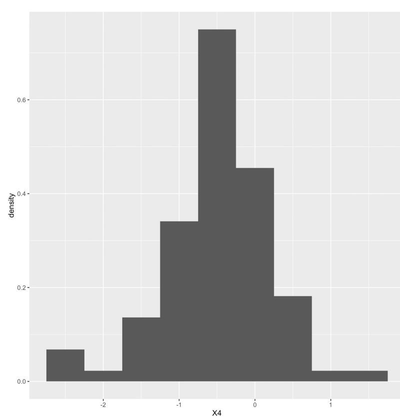
$\alpha_{1,i}$



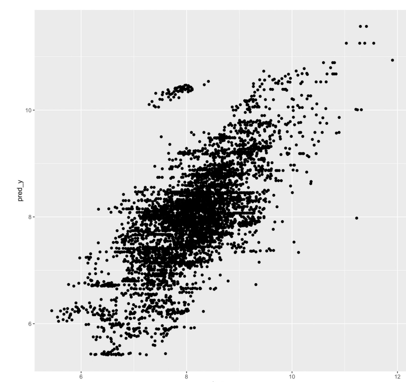
β_0



β_1



true vs. predicted



Polls

Model using

$$P(Y_{ij}=1) = \Phi(x_{ij}'\beta_i)$$

$$\beta_i \sim N(\mu_i, \Sigma_i)$$

$$\mu_i \sim N(m, s^2 I_p) \text{ here take } m=0,\ s^2 = 10^6$$

$$\Sigma_i \sim \text{Inv-Wishart}(p-1, I_p)$$

Besag (1974) lattice systems

This yields the following posteriors:

$$(\beta_i \mid \dots) \sim N(\theta, V) \quad V = (\Sigma_i^{-1} + X_i'X_i)^{-1},\ m = V^{-1}(\Sigma_i^{-1}\mu_i + X_i'Z_i)$$

$$(Z_{ij} \mid \dots) \sim \begin{cases} N(x_{ij}'\beta_i, 1)\big|_{[0,\infty)} & \text{for } Y_{ij}=1 \\ N(x_{ij}'\beta_i, 1)\big|_{(-\infty,0]} & \text{for } Y_{ij}=0 \end{cases}$$

$$(\mu_i \mid \dots) \sim N(\theta, V) \text{ w/ } V = (\Sigma_i^{-1} + s^2 I)^{-1},\ \theta = \beta_i'\Sigma_i^{-1}$$

$$(\Sigma_i \mid \dots) \sim \text{Inv-Wi}\left(n+p-1,\ I_p + \sum_{i=1}^{n}(\beta_i-\mu_i)(\beta_i-\mu_i)^T\right)$$