A Simple Gaussian Location Model

a.) $p(\theta,\omega) \propto \omega^{(d+1)/2-1} \exp\left\{-\omega \cdot \frac{k(\theta-\mu)^2}{2}\right\} \exp\left\{-\omega \cdot \frac{\eta}{2}\right\}$

$$p(\theta) \propto \int_0^\infty \omega^{(d+1)/2-1} \exp\left\{-\omega\left[\frac{k(\theta-\mu)^2+\eta}{2}\right]\right\}$$ then using gamma kernel

$$= \frac{\Gamma\left(\frac{d+1}{2}\right)}{\left[\frac{k(\theta-\mu)^2+\eta}{2}\right]^{\frac{d+1}{2}}} \propto \left[\frac{\eta}{2}+\frac{k(\theta-\mu)^2}{2}\right]^{-\frac{(d+1)}{2}} \propto \left[1+\frac{1}{\eta}\cdot\frac{(\theta-\mu)^2}{(1/k)}\right]^{-\frac{(d+1)}{2}}$$

$$= \left[1+\frac{1}{\eta d}\cdot\frac{(\theta-\mu)^2}{\left(\frac{1}{k}\right)\left(\frac{1}{d}\right)}\right]^{-\frac{(d+1)}{2}} = \left[1+\frac{1}{d}\cdot\frac{1}{\eta\left(\frac{1}{k}\right)\left(\frac{1}{d}\right)}(\theta-\mu)^2\right]^{-\frac{(d+1)}{2}}$$

$$\boxed{m=\theta,\ s^2=\frac{\eta}{kd},\ \nu=d}$$

b.) $p(\theta,\omega|y) \propto p(y|\theta,\omega)\,p(\theta,\omega)$

$$= p(y|\theta,\omega)\,p(\omega)\,p(\theta|\omega)$$

$$= \left(\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\omega^{1/2}\exp\left\{-\omega\cdot\left(\frac{S_y+n(\bar{y}-\theta)^2}{2}\right)\right\}\right)\left(\frac{\left(\frac{\eta}{2}\right)^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\omega^{\frac{d}{2}-1}\exp\left\{-\omega\cdot\frac{\eta}{2}\right\}\right)\left(\frac{1}{\sqrt{2\pi}}\sqrt{\omega k}\exp\left\{\frac{-1}{2}\omega k(\theta-\mu)^2\right\}\right)$$

$$\propto \omega^{\left(\frac{n+d+1}{2}\right)-1}\exp\left\{-\omega\cdot\frac{1}{2}\left[\underbrace{S_y+n(\theta-\bar{y})^2+\eta+k(\theta-\mu)^2}\right]\right\}$$

$$\longrightarrow S_y+n(\theta-\bar{y})^2+k(\theta-\mu)^2+\eta$$
$$= S_y+n(\theta^2-2\theta\bar{y}+\bar{y}^2)+k(\theta^2-2\theta\mu+\mu^2)+\eta$$
$$= (n+k)\theta^2-2(n\bar{y}+k\mu)\theta+n\bar{y}^2+k\mu^2+\eta+S_y$$
$$= (n+k)\left(\theta-\frac{n\bar{y}+k\mu}{n+k}\right)^2+n\bar{y}^2+k\mu^2+\eta+S_y-\frac{(n\bar{y}+k\mu)^2}{n+k}$$

$$= \omega^{\left(\frac{n+d+1}{2}\right)-1}\exp\left\{-\omega\cdot\frac{1}{2}\left[(n+k)\left(\theta-\frac{n\bar{y}+k\mu}{n+k}\right)^2+n\bar{y}^2+k\mu^2+\eta+S_y-\frac{(n\bar{y}+k\mu)^2}{n+k}\right]\right\}$$

$$= \omega^{\left(\frac{n+d+1}{2}\right)-1}\exp\left\{-\omega\cdot\frac{(n+k)\left(\theta-\frac{n\bar{y}+k\mu}{n+k}\right)^2}{2}\right\}\exp\left\{-\omega\cdot\left(\frac{\eta+S_y+nk(\mu-\bar{y})^2/(n+k)}{2}\right)\right\}$$

$$n\bar{y}^2+k\mu^2-\frac{(n\bar{y}+k\mu)^2}{n+k}$$
$$= \frac{(n+k)n\bar{y}^2+k\mu^2(n+k)-(n\bar{y}+k\mu)^2}{n+k}$$
$$= \cancel{n^2\bar{y}^2}+kn\bar{y}^2+kn\mu^2+\cancel{k^2\mu^2} - \left(\cancel{n^2\bar{y}^2}+\cancel{k^2\mu^2}+2n\bar{y}k\mu\right)/(n+k)$$
$$= (kn\bar{y}^2+kn\mu^2-2n\bar{y}k\mu)/(n+k)$$
$$= nk(\bar{y}^2+\mu^2-2\mu\bar{y})/(n+k)$$
$$= nk(\mu-\bar{y})^2/(n+k)$$

$$\boxed{\begin{aligned}\mu^* &= \frac{n\bar{y}+k\mu}{n+k}\\ k^* &= n+k\\ d^* &= n+d\\ \eta^* &= \eta+S_y+\frac{nk(\bar{y}-\mu)^2}{n+k}\end{aligned}}$$

c.) $p(\theta|y,\omega) \sim \boxed{\text{Normal}\left(\frac{n\bar{y}+k\mu}{n+k},\ \frac{1}{\sqrt{n+k}}\right)}$

d.) $p(\omega|y) \propto \int \underbrace{\omega^{\left(\frac{n+d+1}{2}\right)-1} \exp\left\{-\omega \cdot \frac{1}{2}\left(\eta + S_y + nk(\mu-\bar{y})^2/(n+k)\right)\right\}}_{\text{doesn't have } \theta \text{ in it}} \underbrace{\exp\left\{\frac{-\omega(n+k)}{2}\left(\theta - \frac{n\bar{y}+k\mu}{n+k}\right)^2\right\}}_{\text{looks like Normal dist w/ variance } \frac{1}{\omega(n+k)}} d\theta$

$$\propto \omega^{\left(\frac{n+d+1}{2}\right)-1} \exp\left\{-\omega \cdot \frac{1}{2}\left(\eta + S_y + nk(\mu-\bar{y})^2/(n+k)\right)\right\} \omega(n+k)$$

$$\propto \omega^{\frac{n+d+1}{2}} \exp\left\{-\omega \cdot \frac{1}{2}\left(\eta + S_y + nk(\bar{y}-\mu)^2/(n+k)\right)\right\}$$

$$\to \text{Gamma}\left(\frac{1}{2}(n+d), \frac{1}{2}\left(\eta + S_y + nk(\bar{y}-\mu)^2/(n+k)\right)\right)$$

or $\boxed{\text{Gamma}\left(\frac{d^*}{2}, \frac{\eta^*}{2}\right)}$

e.) Matching the form of part (a.), we have $p(\theta|y) \sim \boxed{t\text{-dist}\left(d^*, \mu^*, \frac{\eta^*}{k^* d^*}\right)}$

f.) $k, d, \eta \to 0$ $\boxed{\text{No!}}$

g.) $\boxed{\text{No}}$

h.) $\boxed{\text{true}}$

The Conjugate Gaussian Linear Model

$$(y|\beta,\sigma^{-2}) \sim N(X\beta, (\omega\Lambda)^{-1})$$

$$(\beta|\omega) \sim N(m, (\omega K)^{-1}) \qquad \sigma^{-2} = (\omega\Lambda)^{-1}$$

$$\omega \sim Ga\left(\frac{d}{2}, \frac{\eta}{2}\right)$$

a.) $p(\beta|y,\omega) \propto p(\beta|\omega)\, p(y|\beta,\omega)$

$$\propto \det(2\pi(\omega K)^{-1})^{-1/2} \exp\left\{\frac{-1}{2}(\beta-m)'(\omega K)(\beta-m)\right\}\left(\det(2\pi(\omega\Lambda)^{-1})^{-1/2} \exp\left\{\frac{-1}{2}(y-X\beta)'(\omega\Lambda)(y-X\beta)\right\}\right)$$

$$\propto \exp\left\{\frac{-1}{2}\left[(\beta-m)'\omega K(\beta-m) + (y-X\beta)'\omega\Lambda(y-X\beta)\right]\right\}$$

$$= \exp\left\{\frac{-\omega}{2}\left[y'\Lambda y - \underline{\beta'X'\Lambda y} + \underline{\beta'X'\Lambda X\beta} - \underline{y'\Lambda X\beta} + \underline{\beta'K\beta} - \underline{m'K\beta} + \underline{\beta'K\beta} - \underline{\beta'Km} + m'Km\right]\right\}$$

$$\propto \exp\left\{\frac{-1}{2}\left[\beta'(X'\omega\Lambda X + \omega K)\beta - 2\beta'(X'\omega\Lambda y + \omega Km)\right]\right\}$$ then complete the square

$\to$ Normal$(A^{-1}b, A^{-1})$ w/ $A = X'\omega\Lambda X + \omega K$, $b' = y'\omega\Lambda X + m'\omega K$

$$\boxed{\text{Normal}\left((X'\Lambda X + K)^{-1}(X'\Lambda' y + K'm), \frac{1}{\omega}(X'\Lambda X + K)^{-1}\right)}$$

b.) $p(\omega, \beta | y) \propto p(\omega) p(\beta|\omega) p(y|\omega,\beta)$

$$\propto \left(\omega^{\frac{d}{2}-1} \exp\left\{\frac{-\eta}{2}\omega\right\}\right)\left(\frac{1}{|\sqrt{2\pi(\omega K)^{-1}}|}\exp\left\{\frac{-1}{2}(\beta-m)'(\omega K)(\beta-m)\right\}\right)\left(\frac{1}{|\sqrt{2\pi(\omega\Lambda)^{-1}}|}\exp\left\{\frac{-1}{2}(y-X\beta)'(\omega\Lambda)(y-X\beta)\right\}\right)$$

$$\propto \omega^{\frac{n}{2}+\frac{p}{2}+\frac{d}{2}-1}\exp\left\{\frac{-1}{2}\left[\eta\omega+(\beta-m)'(\omega K)(\beta-m)+(y-X\beta)'\omega\Lambda(y-X\beta)\right]\right\}$$

$$\propto \omega^{\frac{n}{2}+\frac{p}{2}+\frac{d}{2}-1}\exp\left\{\frac{-\omega}{2}\left[\eta+y'\Lambda y+m'Km\right]\right\}\exp\left\{\frac{-1}{2}\left[\beta'(X'\omega\Lambda X+\omega K)\beta-2(y'\omega\Lambda X+m'\omega K)\beta\right]\right\} d\beta$$

To get the marginal, marginalize the β.

$$p(\omega|y) \propto \omega^{\frac{n}{2}+\frac{p}{2}+\frac{d}{2}-1}\exp\left\{\frac{-\omega}{2}(\eta+y'\Lambda y+m'Km)\right\}\int_\beta \exp\left\{\frac{-1}{2}\left[\beta'\underbrace{(X'\omega\Lambda X+\omega K)}_{A}\beta-2\underbrace{(y'\omega\Lambda X+m'\omega K)}_{b}\beta\right]\right\}d\beta$$

kernel of $N(A^{-1}b, A^{-1})$

$$\propto \omega^{\frac{n}{2}+\frac{p}{2}+\frac{d}{2}-1}\exp\left\{\frac{-\omega}{2}(\eta+y'\Lambda y+m'Km)\right\}\det(2\pi A^{-1})^{1/2}\exp\left\{\frac{1}{2}b'Ab\right\}$$

$\to \propto \det(A)^{-1/2}$

$\propto \omega^{-p/2}\det(\dots)$

$$\propto \omega^{\frac{n}{2}+\frac{d}{2}-1}\exp\left\{\frac{-\omega}{2}(\eta+y'\Lambda y+m'Km)\right\}\exp\left\{\frac{1}{2}(y'\omega\Lambda X+m'\omega K)'(X'\omega\Lambda X+\omega K)(y'\omega\Lambda X+m'\omega K)\right\}$$

$$\propto \omega^{\frac{n}{2}+\frac{d}{2}-1}\exp\left\{\frac{-\omega}{2}\left(\eta+y'\Lambda y+m'Km-(y'\Lambda X+m'K)'(X'\Lambda X+K)(y'\Lambda X+m'K)\right)\right\}$$

$$\to \boxed{\text{Gamma}\left(\frac{n}{2}+\frac{d}{2}, \frac{1}{2}\left(\eta+y'\Lambda y+m'Km-(y'\Lambda X+m'K)'(X'\Lambda X+K)(y'\Lambda X+m'K)\right)\right)}$$ call 2nd param η^*.

c.) $p(\beta|y) \propto \int p(\beta,\omega|y)\,d\omega$

$$\propto \int \underbrace{p(\beta|\omega,y)}_{N(A^{-1}b,A^{-1})}\underbrace{p(\omega|y)}_{Ga(\dots)}\,d\omega$$

$$\propto \int \omega^{\frac{n}{2}+\frac{p}{2}+\frac{d}{2}-1}\exp\left\{\frac{-1}{2}(y-X\beta)'\omega\Lambda(y-X\beta)\right\}\exp\left\{\frac{-1}{2}(\beta-m)'\omega K(\beta-m)\right\}\exp\left\{\frac{-\omega\eta}{2}\right\}d\omega$$

Kernel of a gamma in ω.

$$\propto \frac{\Gamma\left(\frac{n}{2}+\frac{p}{2}+\frac{d}{2}\right)}{\left(\frac{1}{2}\left[(y-X\beta)'\Lambda(y-X\beta)+(\beta-m)'K(\beta-m)+\eta\right]\right)^{\frac{n}{2}+\frac{p}{2}+\frac{d}{2}}}$$

$$\propto \left(\frac{1}{2}\left[(y-X\beta)'\Lambda(y-X\beta)+(\beta-m)'K(\beta-m)+\eta\right]\right)^{-\left(\frac{n+p+d}{2}\right)}$$

$$\propto \left(\frac{1}{2}\left[\beta'(X'\Lambda X+K)\beta-2(y'\Lambda X+m'K)\beta\right]\right)^{(\dots)}$$

Now let $\Lambda^* = X'\Lambda X + K$ and $\mu^* = (\Lambda^*)^{-1}(X'\Lambda y + K'm)$

$$\propto \left((\beta-\mu^*)'\Lambda^*(\beta-\mu^*)+\eta^*\right)^{(\dots)}$$ from completing the square

$$\propto \left((\beta-\mu^*)'\frac{\Lambda^*}{\eta^*}(\beta-\mu^*)+1\right)^{-\left(\frac{n+p+d}{2}\right)}$$ divide through by η^*

$$\propto \left(\frac{1}{n+d} (\beta - \mu^*)' \frac{(n+d)\Lambda^*}{\eta^*} (\beta - \mu^*) + 1 \right)^{-\frac{(n+p+d)}{2}}$$ need to get degrees of freedom

let $v^* = n+d$

$$\propto \left(\frac{1}{v^*} (\beta - \mu^*)' \left(\frac{v^* \Lambda^*}{\eta^*} \right) (\beta - \mu^*) + 1 \right)^{-\frac{(v^*+p)}{2}}$$

$\rightarrow$ $\boxed{\text{t-distribution}}$ w/ df $= n+d$, mean $(\Lambda^*)^{-1}(X'\Lambda y + K'm)$, and variance $\left(\frac{v^*\Lambda^*}{\eta^*}\right)^{-1}$ where $\Lambda^* = X'\Lambda X + K$

A heavy-tailed error model

a.) $p(y_i \mid \beta, \omega) = \int p(y_i, \lambda_i \mid \beta, \omega)\, d\lambda_i$

$= \int p(y_i \mid \lambda_i, \beta, \omega)\, p(\lambda_i)\, d\lambda_i \qquad y_i \sim N(x_i'\beta, (\omega\Lambda)^{-1}_{ii}) \rightarrow \frac{1}{\omega \lambda_i}$

$\propto \int \sqrt{\frac{\omega\lambda_i}{2\pi}} \exp\left\{\frac{-\omega\lambda_i}{2}(y_i - x_i'\beta)^2\right\} \lambda_i^{\frac{h}{2}-1} \exp\left\{-\lambda_i\left(\frac{h}{2}\right)\right\} d\lambda_i$

$\propto \int \lambda_i^{\frac{1}{2}+\frac{h}{2}-1} \exp\left\{-\lambda_i\left(\frac{\omega}{2}(y_i - x_i'\beta)^2 + \frac{h}{2}\right)\right\} d\lambda_i$

$\text{Gamma}\left(\frac{1}{2}+\frac{h}{2}, \frac{\omega}{2}(y_i - x_i'\beta)^2 + \frac{h}{2}\right)$

$\propto \left[\frac{\omega}{2}(y_i - x_i'\beta)^2 + \frac{h}{2}\right]^{-\frac{(h+1)}{2}} \Gamma\left(\frac{h+1}{2}\right)$ divide thru by $\frac{2}{h}$

$\propto \left[\frac{\omega}{h}(y_i - x_i'\beta)^2 + 1\right]^{-(h+1)/2}$

$\rightarrow \boxed{t\text{-dist}\left(\nu = h, \mu = x_i'\beta, s^2 = \frac{1}{\omega}\right)}$

b.) $p(\lambda_i \mid \underline{y}, \beta, \omega) \propto p(y_i \mid \lambda_i, y, \beta)\, p(\lambda_i \mid y, \beta)$

$= p(y_i \mid \lambda_i, y, \beta)\, p(\lambda_i)$

$\propto (\omega\lambda_i)^{1/2} \exp\left\{\frac{-1}{2}\omega\lambda_i(y_i - x_i'\beta)^2\right\} \lambda_i^{\frac{h}{2}-1} \exp\left\{-\lambda_i\left(\frac{h}{2}\right)\right\}$

$\propto \lambda_i^{\frac{h+1}{2}} \exp\left\{-\frac{1}{2}\lambda_i\left(\omega(y_i - x_i'\beta)^2 + \frac{h}{2}\right)\right\}$

$\rightarrow \boxed{\text{Gamma}\left(\frac{h+1}{2}, \frac{1}{2}\omega(y_i - x_i'\beta)^2 + \frac{h}{2}\right)}$

c.) Gibbs sampler. $\Lambda^* = (X'\Lambda X + K) \qquad a^* = X'\Lambda y + K'm$

$p(\beta \mid y, \omega, \Lambda) \sim N(\mu^*, V^*)$ w/ $V^* = \frac{1}{\omega}(\Lambda^*)^{-1}$; $\mu^* = (\Lambda^*)^{-1}(X'\Lambda y + K'm)$

$p(\omega \mid y, \Lambda) \sim Ga\left(\frac{n+d}{2}, \eta^*\right)$ where $\eta^* = \frac{1}{2}(\eta + y'\Lambda y + m'Km - (a^*)(\Lambda^*)(a^*)')$

$p(\lambda_i \mid y, \beta, \omega) \sim Ga\left(\frac{h+1}{2}, \frac{1}{2}\omega(y_i - x_i'\beta)^2 + \frac{h}{2}\right)$