Stability of Ice-Age Ice Sheets

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Abstract. The stability of large ice sheets is investigated by using the present-day theory of the flow of ice in glaciers and ice sheets. The type of instability considered is that first mentioned by Bodvarsson. It is concluded that a small Arctic ice cap can become unstable and expand into a large ice age ice sheet as a result of moderate changes in the regime of the ice cap. A large continental ice sheet can also become unstable and shrink to nothing if the snow accumulation is reduced or the ablation rate increased. The results obtained fit well into the Ewing-Donn theory of ice ages. There is the possibility that the inherent instability of ice age ice sheets is in itself sufficient to explain both the formation and the disappearance of these ice sheets.

Introduction. In this paper we consider a problem important to any theory of the ice ages, namely, the stability of continental ice sheets. We wish to show from the mechanics of ice flow that a small ice cap situated in high latitudes on a continental land mass may be unstable in the sense that, if its width exceeds a critical size, the ice cap will grow unchecked until it reaches lower latitudes and is of continental dimensions. Further, it can be shown that once an ice cap reaches this size, another instability may set in if the rate of accumulation decreases or the rate of ablation increases. The ice sheet may then shrink to a small size or disappear.

We make the observation that an ice cap will grow when the snowfall on it increases or the melting at its edge decreases and that it will shrink when the snowfall decreases or the melting increases. This fact has been noted many times before. The new aspect introduced here is the use of the recently developed theory of the mechanics of glaciers and ice sheets to calculate the sensitivity of an ice cap to changes in rates of accumulation and ablation.

The type of instability with which we are concerned has already been pointed out by *Bodvarsson* [1955], who discovered this behavior while investigating ice sheet and glacier profiles. His work was based on boundary conditions rather different from those which would be used now.[2]

The instability discussed here is different in nature from that recently analyzed by *Nye* [1960]. Nye considered the problem of determining the manner in which a glacier or ice sheet approaches a stable steady-state profile. He showed that, in ablation regions, unstable behavior may occur before an equilibrium profile is finally reached. It is implicit in Nye's theory that there are always stable steady-state profiles which a glacier or ice sheet will approach. In other words, if the accumulation or ablation conditions change slightly, a glacier or ice sheet will assume a new steady-state shape of slightly different thickness and width. Nye was not primarily concerned with the problem that a steady-state profile might be in unstable equilibrium.

It may be useful to point out the reason why Nye's treatment leads to profiles of stable equilibrium whereas ours sometimes produces profiles that are unstable. Nye assumed that the accumulation area of a glacier or ice cap extends out to a fixed distance D from the center, as is shown in Figure 1*a*. Ablation occurs at all distances greater than D. Consider the successively larger ice caps labeled 1, 2, and 3 in Figure 1*a*.

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[2] Bodvarsson used as the sliding velocity of an ice cap or glacier over its bed a quantity which is proportional to the stress acting at the bed and inversely proportional to the thickness of ice over the bed. In other words, he assumed $U \propto dh/dx$, whereas we shall use $U \propto h^2(dh/dx)^2$.

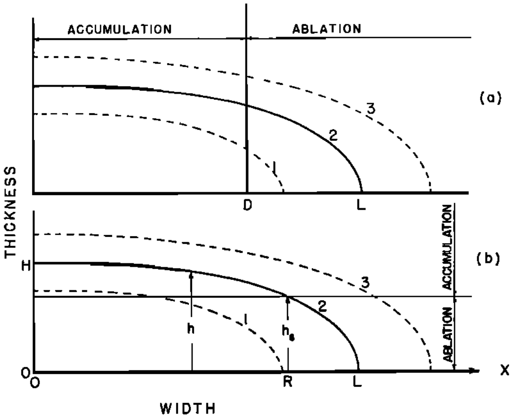

Fig. 1. Cross sections of one-half of ice sheets. (*a*) Accumulation area occurring out to a distance *D* from the center. (*b*) Accumulation area occurring above an elevation h_s.

It can be seen that regardless of the size of the ice cap the total accumulation area, and hence the total accumulation, remains constant, whereas the ablation area, and hence the total ablation, increases as the ice cap is made larger. Suppose curve 2 of Figure 1*a* represents an equilibrium ice cap, i.e., and ice cap whose total accumulation is balanced exactly by its total ablation. This ice cap obviously is in stable equilibrium. The total ablation of a larger ice cap (curve 3) would be greater than its total accumulation, a condition which would cause the ice cap to shrink. Similarly, the total ablation of a smaller ice cap (curve 1) would be less than the total accumulation, and the ice cap would grow towards the equilibrium size.

In contrast to Nye's assumption that the accumulation and ablation areas are determined by a horizontal distance *D*, our assumption is that the accumulation area is found above an elevation h_s (which will be considered later in the paper to be a function of the horizontal distance x) and the ablation area is found below this elevation, as is shown in Figure 1*b*. Under these conditions the accumulation area (and hence the total accumulation) is no longer constant but rather depends on the size of the ice cap. The larger the ice cap the larger the total accumulation. Suppose in Figure 1*b* that curve 2 represents an equilibrium profile. It is now open to question whether this profile is of stable equilibrium. The total accumulation of a larger ice cap, represented by curve 3 in Figure 1*b*, could be greater than its total ablation, in which case the ice cap would grow rather than shrink toward the equilibrium size. Similarly, a smaller ice cap, curve 1, could have a total ablation greater than its total accumulation, and hence it would shrink and ultimately disappear.

Preliminary considerations. There are three main variables which are involved in the determination of the size of an ice cap or ice sheet. These are the accumulation rate in the accumulation area which exists at the higher elevations of an ice cap (see Fig. 1*b*), the ablation rate in the ablation area of the lower elevations, and the elevation of the border separating the accumulation area from the ablation area. To simplify the analysis it will be assumed that the accumulation rate at every point in the accumulation area is equal to the *average* accumulation rate a and that everywhere in the ablation area the ablation rate is the *average* ablation rate $\bar{a}$. This assumption does not involve too great a loss in the accuracy of the results, since it is known that accumulation and ablation rates usually are slowly varying functions of the distance x from the center of an ice sheet. Thus the average ablation or accumulation rate is almost always of the same order of magnitude as the actual accumulation or ablation rate.

One can expect that the average accumulation and ablation rates will be a function of the size of the ice cap or ice sheet. Conversely, one can also expect that the size of an ice cap depends on the magnitude of its accumulation and ablation rates. To disentangle these two functional relationships we shall assume in this analysis that the accumulation rate and ablation rate are constant and independent of the size of the ice sheet. The effect of a change in ice sheet size on the accumulation and ablation rates can be examined later by means of the equations obtained upon the assumption of constant rates.

Once the average accumulation and ablation rates are specified, an ice sheet profile can be calculated from the principle of conservation of mass [*Nye*, 1959]. Only two-dimensional ice sheets (i.e., ice sheets which have one axis very much longer than the other and whose profiles, therefore, are essentially independent of distance along the long axis) will be considered in this paper. As Nye has shown, the profile of a circular ice sheet is essentially the same as that of a two-dimensional ice sheet, and thus the results obtained from two-dimensional ice sheets apply equally well to circular ice sheets.

An ice sheet profile is calculated [*Nye*, 1959] by equating the volume of ice passing through a cross section of the ice sheet to the total amount of ice accumulation between that cross section and the center of the ice sheet. If the average accumulation and ablation rates are fixed, and if the elevation of the border separating the accumulation and the ablation areas is likewise specified, the analysis of the following section shows that *only one possible equilibrium ice sheet profile exists.*

Snow line of fixed elevation. We define an elevation h_s (called hereinafter the elevation of the snow line) to be an elevation such that any part of an ice sheet which lies above h_s is in an accumulation zone and any part which lies below h_s is in an ablation zone. The base of the ice sheet is assumed to be flat and at sea level. In this section it is assumed that the value of h_s is constant.

Let U represent the average velocity of ice passing through a cross section situated at a distance x from the center of the ice sheet. The total volume of ice passing through this cross section per unit time (for a unit length of ice sheet) is Uh, where h is the height of the profile above the bed. Let A represent the accumulation or ablation at any point on the upper surface. For steady-state conditions one has the equation

$$Uh = \int_0^x A\, dx \tag{1}$$

Nye assumes that the average velocity U is given by the equation

$$U = B\tau^m \tag{2}$$

where B is a constant, m is a constant whose value is of the order of 2 to 2.5, and τ is the shear stress[3] acting at the bed of an ice sheet. The value of τ is given by

$$\tau = -\rho g h (dh/dx) \tag{3}$$

[3] In Nye's analysis it is assumed that the shear stress is the dominant stress and that longitudinal stresses are unimportant. It is possible [*Weertman*, 1961*a*] to calculate ice sheet profiles when longitudinal stresses are large. Since the profiles so obtained are almost the same as those found by Nye, there is no need to use this more complicated theory in the above analysis.

where ρ is the density of ice, g is the gravitational acceleration, and dh/dx is the slope of the upper ice surface. It is reasonable to use the velocity given by (2), since a velocity of this form is to be expected both from an analysis of the differential flow of ice within an ice mass [*Nye*, 1959] and from a theoretical study of the sliding of ice [*Weertman*, 1957] over a glacier or ice-cap bed. Since the value of m in (2) is approximately 2, we shall assign this value to it now in order to avoid some cumbersome expressions in later sections of this paper. When (2) and (3) are substituted into (1) and m is set equal to 2, one obtains

$$B(\rho g)^2 h^3 (dh/dx)^2 = \int_0^x A\, dx \tag{4}$$

The solution of this equation is the profile of an equilibrium ice sheet.

The average accumulation rate a and the average ablation rate $\bar{a}$ are defined by

$$\begin{aligned} a &= R^{-1} \int_0^R A\, dx \\ \bar{a} &= -(L - R)^{-1} \int_R^L A\, dx \end{aligned} \tag{5}$$

where R is the distance from the ice sheet center to the border between the accumulation and ablation areas and L is the width of the ice sheet (Fig. 1). The term $\bar{a}$ is defined so that it is a positive quantity. If equations 5 are substituted into (4) and if the requirement for an ice sheet to be in equilibrium,

$$aR = \bar{a}(L - R) \tag{6}$$

is satisfied, the following equations are obtained for the equilibrium profile:

$$h^{5/2} = H^{5/2} - (c/a)(ax)^{3/2}$$
$$(\text{when } h > h_s \quad \text{and} \quad 0 \le x \le R) \tag{7}$$

$$h^{5/2} = h_s^{5/2} + (c/\bar{a})$$
$$\cdot \{[aR - \bar{a}(x - R)]^{3/2} - [aR]^{3/2}\}$$
$$(\text{when } h < h_s \quad \text{and} \quad R \le x \le L) \tag{8}$$

where H is the thickness of the ice cap or ice sheet at its center and c is equal to $(5/3)/B^{1/2}\rho g$. The distance R is given by

$$R = a^{-1}(\bar{a}/c)^{2/3} h_s^{5/3} \tag{9}$$

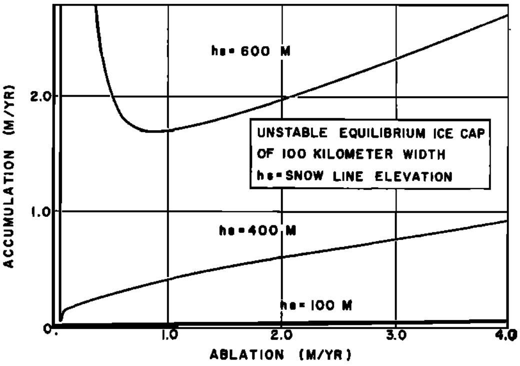

Fig. 2. Curves of accumulation versus ablation for an unstable equilibrium ice cap of 100-km. width for three different snow line elevations.

the thickness H by

$$H = (1 + \bar{a}/a)^{2/5} h_s \qquad (10)$$

and the distance L from center to edge by

$$\begin{aligned} L &= (1 + \bar{a}/a)^{1/3} (\bar{a} c^2)^{-1/3} H^{5/3} \\ &= \bar{a}^{-1} (1 + \bar{a}/a)(\bar{a}/c)^{2/3} h_s^{5/3} \\ &= (1 + a/\bar{a}) R \qquad (11) \end{aligned}$$

The profile determined by (7) through (11) is one of unstable equilibrium. For example, both the width L and the thickness H decrease with increasing accumulation rate or lowering snow line. Now an increase in the accumulation rate or a lowering of the snow line obviously increases the volume of an ice sheet. However, since the new equilibrium profile, determined by (7) through (11), resulting from this change has a smaller total volume, a new equilibrium profile will not be approached if the accumulation rate is increased or the snow line lowered. The ice sheet will grow indefinitely. Similarly, if the accumulation rate is decreased on an ice sheet in equilibrium or the snow line is raised, an ice sheet or ice cap will shrink until it disappears, since the new equilibrium profile contains a greater volume of ice than the original equilibrium profile.

In Figures 2 and 3 are shown graphs of the values of accumulation versus ablation rates which satisfy (7) through (11) for various values of the elevation of the snow line and when the width of the ice cap is either 100 or 3000 km (these widths are the total width = $2L$). A value of $c = 2$ (meter-years)$^{1/2}$ was used in making the calculations. This value of c is obtained from a comparison of theoretical ice sheet profiles with measured profiles [*Weertman*, 1961*a*]. Values of accumulation and ablation rates which lie above a given curve in Figures 2 and 3 would lead to an ice sheet, which is originally 100 or 3000 km wide, growing indefinitely in size. If the accumulation or ablation rates are below the curve, the ice cap or ice sheet will shrink away to nothing. From Figure 2 it can be seen that rather small rates of accumulation will lead to the unstable growth of an ice cap 100 km wide, if the snow line is as low as 100 m. On the other hand, if the snow line is at 600 m, a very high rate of accumulation ($\sim$2 m/yr) will be required to maintain the ice cap or cause it to grow.

Stability and instability considerations. In the previous section it was shown that, if accumulation and ablation rates are fixed, and if the snow line elevation is fixed, only one possible equilibrium ice sheet profile exists and this profile is of unstable equilibrium. Stability can be introduced only if a, $\bar{a}$, or h_s is some appropriate function of ice sheet size. For example, suppose the snow line elevation is given by

$$h_f = x^{3/5} c^{2/5} / \bar{a}^{1/5} = R^{3/5} c^{2/5} / \bar{a}^{1/5} \qquad (12)$$

This equation is simply (9) rearranged. We have labeled the snow line elevation of this equation h_f in order to emphasize the fact that it is a snow line elevation which obeys a very particular equation. We shall call this particular snow line elevation the elevation of the equilibrium firn line. When the snow line elevation is

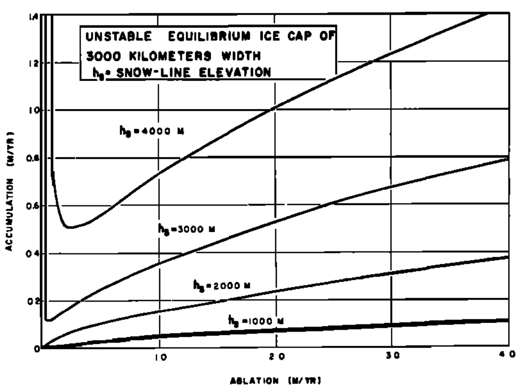

Fig. 3. Curves of accumulation versus ablation for an unstable ice sheet of 3000-km width for four different snow line elevations.

given by (12), an infinity of equilibrium profiles exist. An ice sheet will always be able to approach an equilibrium profile if, initially, it is not in an equilibrium condition.

In general, the snow line will not follow an equation such as (12). The actual snow line is known to rise approximately linearly as a function of decreasing latitude [*Charlesworth*, 1957]. Such a linear rise for h_s is shown in Figure 4. Also shown schematically in this figure is h_f given by (12), with $\bar{a}$ and a held constant. An equilibrium profile can occur only when these two curves touch each other, i.e., when $h_s = h_f$. No equilibrium profile is possible if the two curves do not meet; if they are tangent to each other, one profile is possible; if they cross each other, as is shown in the figure, two profiles are possible.

The question of the stability of an equilibrium profile which occurs at the crossing of the two curves of Figure 4 can be resolved by a consideration of ice sheet profiles which are slightly larger or slightly smaller than the equilibrium profile. For example, in the case of a somewhat larger ice sheet the distance R out to the border between the accumulation and ablation areas will be larger than the corresponding quantity for the equilibrium ice sheet. For this larger value of R the elevations h_s and h_f in Figure 4 will not be equal. If h_s is less than h_f the larger ice cap or ice sheet will grow and if h_s is greater than h_f the ice sheet will shrink. In order for an ice sheet or ice cap to be stable it must tend to shrink if it becomes larger than the equilibrium size and tend to grow if it becomes smaller than this size. If it is unstable the opposite will occur. Thus the left-hand intersection of Figure 4 represents an ice cap in unstable equilibrium and the right-hand intersection represents a stable ice cap. An ice sheet with a value of R smaller than that of the left-hand intersection will shrink until it disappears. If its value of R is larger than that of the left-hand intersection, the sheet will grow until it reaches the equilibrium profile represented by the right-hand intersection.

Plots similar to that of Figure 4 can be obtained by considering h_s to be fixed and assuming that a or $\bar{a}$ is a function of ice sheet size. In order for stability to exist the accumulation rate of an ice sheet would have to decrease or the ablation rate increase with increasing size of the ice sheet. Since it is virtually impossible to discover the functional relationship between accumulation or ablation rates and the size of an ice-age ice sheet, we shall consider the influence of accumulation and ablation only through the equations obtained when h_s is allowed to vary.

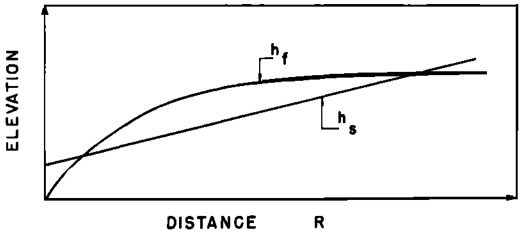

Fig. 4. Schematic plot of the elevation of the snow line h_s and the elevation of the equilibrium firn line h_f of an equilibrium ice cap versus distance R from the center of an ice cap to the boundary between the accumulation and ablation areas.

Stability of Antarctic and Greenland ice sheets. Before proceeding any further we should like to make a digression in order to point out the fact that the ice sheet of Antarctica (and, to a lesser extent, Greenland) is a rather exceptional case in the analysis. We are assuming that we are dealing with an ice sheet which lies on a land base extending to infinity. In Antarctica, and, to a lesser extent, in Greenland, the ice sheet terminates at the water's edge where the sea is an agent providing for an infinite amount of ablation. Thus even if h_s is at sea level and there is no ablation area on the ice sheet itself, a stable ice sheet can exist because there is no more land on which an inherently unstable ice sheet can spread. The profile of such an ice sheet can be obtained from (7) through (11) by allowing the ablation to be very large at the edge of the ice sheet and, from (11), by setting $\bar{a}^{2/3}h_s^{5/3} = Lc^{2/3}$, where $2L$ is the width of the land mass. In this situation $R \cong L$, and the ablation starts virtually at the edge of the ice sheet.

Profile of an ice sheet with constant accumulation and ablation rates and a variable snow line. In this section we calculate the profile of an ice sheet which has a fixed rate of accumulation and ablation, but whose snow line is a function of latitude. We know, of course, that the snow line in high latitudes lies at low elevations and that it rises to elevations of the order of 4 km in temperate latitudes [*Matthes*, 1942; *Charlesworth*, 1957]. The rise appears to be linear [Fig. 6 on p. 9 of *Charlesworth*, 1957]. Thus we can write for h_s

$$h_s = \bar{h} + sx \tag{13}$$

where x is measured toward the equator, s is a constant of the order of 10^{-3}, and $\bar{h}$ is the value of h_s in polar regions (at $x = 0$). Its value [*Matthes*, 1942; *Charlesworth*, 1957] seems to lie in the range from 100 to 1000 meters.

Suppose we consider an ice sheet whose center lies at $x = 0$, and the snow line rises in either direction from the center of the sheet by an equation of the form of (13). The analysis of the section of a snow line of fixed elevation is still valid, with the exception of (9), which, in order to take (13) into account, should be replaced by

$$h_s = \bar{h} + sR = a^{3/5}(c/\bar{a})^{2/5}R^{3/5} = h_f \tag{14}$$

There may be two values, one value, or no value of R which satisfies this equation. These values of R represent the points of intersection of the curves h_f and h_s shown in Figure 4. If the two curves of that figure never meet (this occurs when a is small, or $\bar{a}$ or s is very large) there is no possible equilibrium ice sheet, and any ice sheet already in existence will shrink and disappear. If s is equal to zero, only one intersection point exists. This is the case considered in a previous section, and there we found that the ice sheet is not stable. If s is increased from zero to some finite value, two intersections will occur. The intersection with the smaller value of R gives a profile that approximates the case in which s is equal to zero and clearly corresponds to an unstable ice cap. Consider the intersection at the larger value of R. Let us take the case in which R is so large that $sR > \bar{h}$. In this situation, R is approximately equal to

$$R = (1/s)^{5/2}(c/\bar{a})a^{3/2} \tag{15}$$

and h_s, H, and L become

$$h_s = (1/s)^{3/2}(c/\bar{a})a^{3/2} \tag{16}$$

$$H = (1/s)^{3/2}(1 + a/\bar{a})^{2/5}(a/\bar{a})^{3/5}a^{1/2}c \tag{17}$$

$$L = (1/s)^{5/2}(1 + a/\bar{a})(c/\bar{a})a^{3/2} \tag{18}$$

The magnitudes of H and L in these equations increase with increasing accumulation rate and decreasing ablation rate and decrease with decreasing a and increasing $\bar{a}$. Hence, the profile can be a stable one. If the rates of accumulation and ablation are changed, the ice sheet will be able to approach a new equilibrium profile.

Effect of change in position of center of ice sheet. The center of an ice sheet located on a continent in the northern hemisphere will shift southward as the ice sheet grows. Because of this migration of the center, the analysis of the previous section has to be modified slightly. The height of the equilibrium firn line still is

$$h_f = a^{3/5}(c/\bar{a})^{2/5}R^{3/5}$$

where R is measured from the center, but the snow line elevation at R is now

$$h_s = \bar{h} + s(L' + R) \tag{19}$$

where L' is the width of the northern half of the ice sheet. If the rate of accumulation is about the same in the northern and southern halves of the ice sheet then according to (11) the width of the two halves will be approximately the same.[4]

If h_s is set equal to h_f and $L = L' \cong R$ we obtain

$$\bar{h} + 2sR = a^{3/5}(c/\bar{a})^{2/5}R^{3/5} \tag{20}$$

which replaces (14) of the previous section. Using this equation we find that

$$R = (1/2s)^{5/2}(c/\bar{a})a^{3/2} \tag{21}$$

$$h_s = (1/2s)^{3/2}(c/\bar{a})a^{3/2} \tag{22}$$

$$H = (1/2s)^{3/2}(1 + a/\bar{a})^{2/5}(a/\bar{a})^{3/5}a^{1/2}c \tag{23}$$

$$L = (1/2s)^{5/2}(1 + a/\bar{a})(c/\bar{a})a^{3/2} \tag{24}$$

[4] By center of an ice sheet we mean the position of greatest elevation where $h = H$. According to (11) whenever $\bar{a}$ is larger than a (the more commonly occurring relationship) then $L \cong (ac^2)^{-1/3}H^{5/3}$ and $L' \cong (a^*c^2)^{-1/3}H^{5/3}$, where a^* is the accumulation rate in the northern half. Since H must be the same for the northern and southern halves we see that $L/L' \cong (a^*/a)^{1/3}$. Since the ratio L/L' depends only on a one-third power of a^*/a the widths of the southern and northern halves will almost equal each other if a^* is of the same order of magnitude as a. The analysis is easily modified to account for the case in which L differs greatly from L'. When L approximates L' it is also almost equal to R.

The northern half of the ice sheet is stable, of course, in the sense that the Antarctic ice sheet is stable, since it cannot spread out indefinitely in a northerly direction.

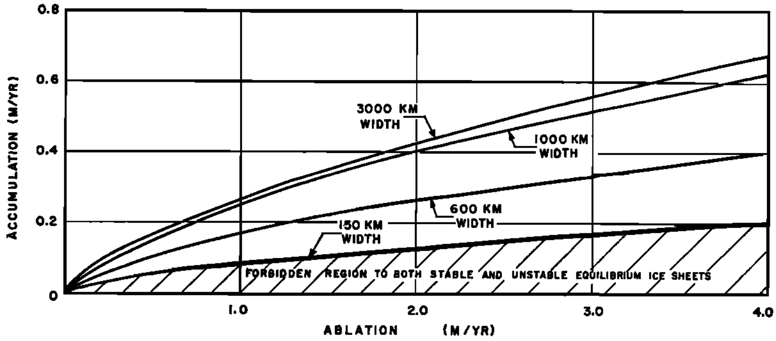

Fig. 5. Curves of accumulation versus ablation for stable equilibrium ice sheets of four different widths when the snow line elevation is a linear function of distance ($\bar{h}$ = 100 meters and $s = 10^{-3}$).

To show how sensitive the width of an ice sheet in stable equilibrium is to accumulation and ablation conditions, we have plotted curves of accumulation versus ablation for ice sheets of specific widths (Figs. 5 and 6). We have set $s = 10^{-3}$ and chosen values for $\bar{h}$ of 100 meters and 400 meters. The width in these figures is defined to be twice R, and R is determined from (21). For a fixed value of $\bar{h}$, there is a minimum-width curve below which it is impossible for an ice sheet or ice cap to exist in either stable or unstable equilibrium. (This minimum width occurs when the two curves of Figure 4 are exactly tangent to one another.) The region shown in the diagram of accumulation versus ablation where equilibrium ice sheets or ice caps cannot exist is labeled the forbidden region.

It should be noted in these figures that small changes in the rate of accumulation can change the width of an ice sheet in stable equilibrium by relatively large amounts and can even make it impossible for such an ice sheet to exist. The width of a stable ice sheet is particularly sensitive to s, the rate of change of the elevation of the snow line. To illustrate this sensitivity, we have plotted in Figure 7 curves of accumulation versus width of ice sheets in equilibrium (both stable and unstable), for a fixed value of ablation. Two different values of s were used when calculating these curves. On each curve, points to the right of the minimum represent stable equilibrium widths and points to the left unstable widths. The curve corresponding to the smaller value of s has a smaller slope in the stable region. Hence, changing the rate of accumulation results in a larger effect on the width of the ice sheet in equilibrium. The smaller s is, the more sensitive the width of a stable ice sheet is to changes in accumulation (and also in ablation).

It is known [*Charlesworth*, 1957] that during the ice age the snow line was lowered approxi-

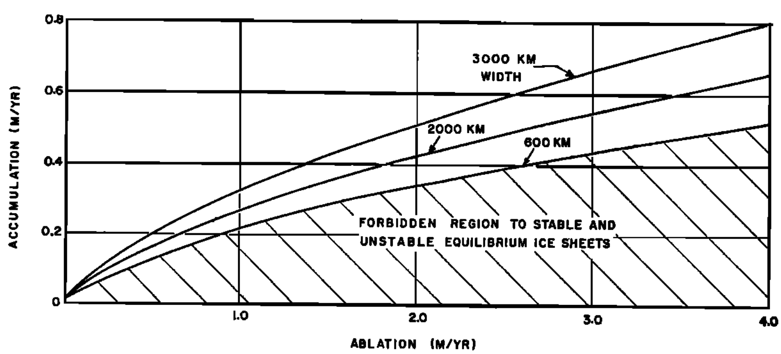

Fig. 6. Curves of accumulation versus ablation for stable equilibrium ice sheets of three different widths when the snow line elevation is a linear function of distance ($\bar{h}$ = 400 meters and $s = 10^{-3}$).

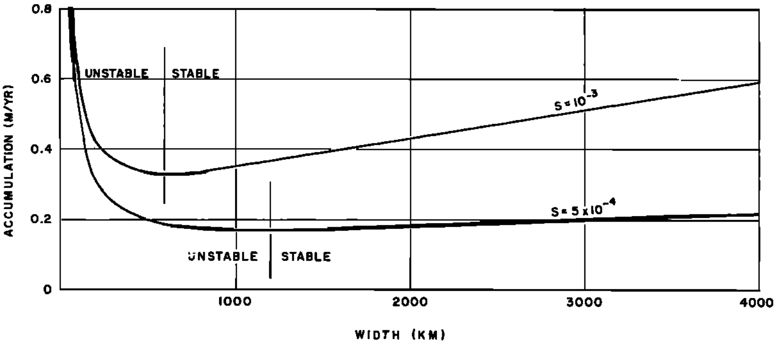

Fig. 7. Curves of accumulation versus equilibrium width of ice sheets when the snow line is a linear function of distance, the ablation rate is 2 meters/year, and $\bar{h}$ is 400 meters.

mately 1000 meters in temperate latitudes. Thus when an ice-age ice sheet reaches a large size the equation we have used for h_s can probably still be employed but with a value of s which is 30 to 50 per cent smaller than its present-day value.

Application to glaciers. The analysis we have made of the stability of an ice sheet resting on a flat base can also be applied to a glacier descending a mountain or to a small ice cap which has formed in an elevated region and whose edge must descend into a lower elevation as it expands.

A glacier normally rests on a sloping bed. If the slope of the bed is β_2 the equation which describes the profile of the glacier is

$$B(\rho g)^2 h^3(\beta + dh/dx)^2 = \int_0^x A\,dx \qquad (25)$$

In this equation x is measured parallel to the bed. A profile similar to that of an ice sheet will be obtained from this equation.

Since the coordinate system for the glacier is tilted at an angle β compared with a system whose horizontal distance is parallel to the earth's surface, the snow line elevation will be given by

$$h_s = \bar{h} - \beta x \qquad (26)$$

where s has been set equal to $-\beta$ since β is less than zero. We have assumed in (26) that the elevation of the snow line in the mountains remains at a constant elevation above sea level. The value of $\bar{h}$ depends on the location of the origin of x. Typical values of β range around 10^{-2} radians, an order of magnitude larger than what we have used for s. From (18), which should be qualitatively correct when applied to glaciers, it can be seen that the length of a stable equilibrium profile depends inversely on s to a 5/2 power, if $sR > \bar{h}$. Stable equilibrium profiles for glaciers should occur at lengths which are of the order of 100 to 1000 times smaller than those of continental ice sheets. This situation results from the fact that $-\beta$ is at least 10 times larger than the values of s appropriate to large ice sheets. Since continental ice sheets had dimensions of the order of 2000 km, stable equilibrium glaciers could be expected to, and do, exist for lengths of the order of 1 to 10 km. The fact that the average ablation rate increases the further a glacier descends down a mountain also stabilizes a glacier. However, this increase is not essential to insure stabilization.

Discussion. The analysis presented in this paper suggests that a small ice cap can become unstable and grow to a large size if it exceeds a critical width. The critical nucleation size can be of the order of the dimensions of existing ice caps in the Arctic ($\sim$ 30 km) when the snow line elevation is of the order of 100 to 400 meters and the product (ablation rate)$^{2/3}$/(accumulation rate) is less than, or equal to, 1 to 10 (m/yr)$^{-1/3}$ (for example, an ablation rate of 1 m/yr and an accumulation rate of 1 m/yr). If an ice cap is less than this critical size, it should disappear completely. The fact that existing ice caps persist can be explained by the reason mentioned in the previous section or by local weather peculiarities.

In theories of the ice ages[5] it is usually assumed that an ice age starts as a result of in-

[5] Reviews of the more important work in this field are given by *Flint* [1947] and *Charlesworth* [1957].

creased accumulation or decreased ablation rates, just those conditions which would be required to make a small ice cap 'go critical.' *Ewing and Donn* [1956, 1958], for example, proposed that an ice age begins when the Arctic Ocean becomes ice free, a situation which leads to increased snowfall on the lands surrounding this body of water. It may not be necessary, however, to invoke some special event to make a small ice cap go critical. Normal weather fluctuations may produce a century of greater than normal accumulation and less than normal ablation. These conditions then may induce a small ice cap to start growing to a large size.

If a small ice cap did grow into a large ice-age ice sheet, is it possible for the sheet ever to shrink again without a significant change in the world's weather conditions (other than that change produced by the ice sheet itself)? The accumulation on the ice sheet itself would be expected to decrease as it became bigger, both because the accumulation area would be at a high elevation and because the cooling of the earth by the presence of the sheet might lead to reduced precipitation. The Antarctic ice sheet has low rates of accumulation ($\sim$ 10 to 20 cm of ice/year), as does the smaller Greenland ice sheet ($\sim$ 10 to 40 cm of ice/year), and a large ice-age ice sheet might also be expected to have such low values. An inspection of Figures 2 to 7 will show that when accumulation rates fall to these values, an ice cap may easily become unstable and shrink to nothing. If low accumulation rates once set in, and if they persist, there will be no serious problem in explaining the shrinkage of ice-age ice sheets. However, one might expect that as a large ice cap began shrinking, the accumulation rate would increase again (and the ablation rate decrease as the edge shifts poleward). Ewing and Donn circumvent this difficulty by their proposal that the Arctic Ocean, which was ice free at the beginning of the ice age, freezes over again and the ice cover persists until after the complete disappearance of the continental ice sheet. According to their theory, the snow precipitation rates are controlled by the Arctic Ocean, and low rates occur so long as the Arctic Ocean is covered with ice. If their theory is correct, there is no difficulty in understanding why an ice-age ice sheet suddenly becomes unstable and shrinks to nothing. As the ice cap grows and the Arctic Ocean freezes over, the accumulation rate decreases until the instability discussed in this paper overtakes the ice sheet, which then shrinks until it disappears.[6]

Our results answer one major criticism [*Livingstone*, 1959] of the Ewing-Donn theory. This criticism is that the theory contained no mathematically demonstrated instability which could lead to ice ages. It partially answers another serious point raised against their theory [*Livingstone*, 1959], namely, that the existence of unglaciated regions around the Arctic Ocean, such as Peary Land, argues against the theory. This latter criticism assumes tacitly that an ice age starts in the Ewing-Donn theory with *such a very heavy* rate of snowfall on the lands around the Arctic Ocean that no land can remain ice free. Now if only a modest increase in the rate of snowfall (say by a factor of 2 or 3) is required to make a small ice cap go critical, there is no reason to expect that all the lands around the Arctic Ocean will become ice covered even though an ice age can occur. Thus the existence of unglaciated regions in the Arctic is not a fatal objection to the Ewing-Donn theory.[7]

There is another way in which an instability in an ice-age ice sheet might be brought about. A small ice cap starting to grow in the far north is likely to be frozen to its bed. If such is the case, the ice cap will not be able to slide over its bed, and the effective value of B in (2) will be reduced and the value of c increased. As the ice sheet grows, it may still remain frozen to its bed until it reaches a large size. Small thickness and large accumulation rates favor a cold ice cap being frozen to its base [*Robin*, 1955; *Weertman* 1961*b*], and large thickness and small accumulation rates increase the probability that the bottom will be at the melting point. Now suppose that when the ice sheet reaches a large size, the temperature at bottom rises until the ice there is at the melting point, and thus the ice can slide over its bed. The value of c will be de-

[6] The ice-age ice sheets did not retreat in a continuous manner but rather the recession was interrupted time and again by readvances of the ice edge. This fluctuation in the dimensions of ice-age ice sheets very probably can be explained through Nye's theory [Nye, 1960] of the inherent instability of the ablation region of glaciers and ice sheets.

[7] *Ewing and Donn* [1959] have given other replies to these two criticisms.

creased. From studies made on temperate glaciers of the fraction of motion which is due to sliding and the fraction which is due to differential motion within the glaciers, one can estimate that c will be decreased by a factor of about $\frac{1}{2}$. An inspection of the equations developed in this paper will show that a decrease in c of $\frac{1}{2}$ is essentially equivalent to an increase in the ablation rate of a factor of 2. Such an increase in the ablation rate might lead to the instability of an ice-age ice sheet and its subsequent shrinkage. The decreased value of c would persist as the ice sheet decreased in size for the following reason. The bottom of a cold ice cap can be at the melting point because of the geothermal heat flowing up through the earth. In addition, if an ice cap is sliding on its bottom, the heat of sliding helps to keep the bottom at the melting point. The heat of sliding is of the same order of magnitude as the geothermal heat [*Weertman*, 1957, 1961*b*]. Because of this extra heat from the sliding process itself, once an ice cap originally frozen to its bed starts to slide it could persist in sliding even though brought back to the conditions where formerly it had been frozen.[8]

Another factor which may be responsible for initiating instability in a continental ice sheet is the isostatic sinking of the ice sheet bed as the ice sheet becomes large. The ice sheet profile can be found quite easily when this sinking occurs [*Weertman*, 1961*a*]. In such a situation the equation determining the profile is

$$B(\rho g)^2\gamma^3h^3(dh/dx)^2 = \int_0^x A\,dx \qquad (27)$$

where h is now the present height of the upper ice sheet above the original position of the bed before sinking occurred and $\gamma = (1 - \rho/\rho_r)^{-1}$ where ρ_r is the average density of rock below the ice sheet ($\rho_r \cong 3\rho$). Equation 27 can be considered to be the same as (4) but with a value of B which is $\gamma^3 \cong 3.4$ times larger. As before, this apparent increase in B can be considered to be equivalent to an increase in the ablation rate by a factor of about 1.8. Such an increase could possibly set off an instability in an ice sheet. If this mechanism is to be effective, however, the time for isostatic sinking to occur must be longer than the time required to build up a large ice sheet.

[8] *Robin* [1955] used the arguments given in this paragraph to explain catastrophic advances of glaciers.

Summary. By utilizing the present-day theory of the flow of ice in glaciers and ice sheets, we have shown that a small Arctic ice cap can expand into an ice-age sheet without large changes in the rate of snow accumulation or ice ablation, or in the elevation of the snow line. Once such an ice cap has expanded, it can become unstable again and shrink to nothing if the accumulation rate is reduced or the ablation rate increased and if the reduction or increase persists as the ice sheet shrinks. These results fit well with the recent Ewing-Donn theory of ice ages. There is a possibility that the inherent instability of ice-age ice sheets is in itself sufficient cause to account for their expansion and subsequent shrinkage.

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