MTH1035 Calculus of Several Variables 25/26

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Course structure:

- Three hours of lectures each week
- Two hours of tutorials each week starting in week 2

Assessment:

- Written exam at the end: 80%
- Continuous assessment: 20%

  Approx. four formative assessment tests throughout the semester.
- Category 3 "best marks" module:
  no resit for the CA

final grade = $\max\{0.8\,\text{Exam} + 0.2\,\text{CA},\ \text{Exam}\}$.

Chapter 1. The Euclidean space $\mathbb{R}^n$, convergence and continuity

Let $n \in \mathbb{N}$, we define the Euclidean space $\mathbb{R}^n$ as

$$\mathbb{R}^n := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, x_2, \dots, x_n \in \mathbb{R} \right\}.$$

For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ we also write $x = (x_1, \dots, x_n)^T$, and we call $x_1, \dots, x_n$ its coordinates.

The set $\mathbb{R}^n$, together with the addition

$$x + y := (x_1 + y_1, \dots, x_n + y_n)^T, \text{ where } x = (x_1, \dots, x_n)^T$$

and $y = (y_1, \dots, y_n)^T$, and the scalar multiplication

$$\lambda x := (\lambda x_1, \dots, \lambda x_n)^T, \text{ where } \lambda \in \mathbb{R} \text{ and}$$

$x = (x_1, \dots, x_n)^T$, forms a vector space over $\mathbb{R}$.

We also recall that a family of vectors $(v_1, \dots, v_n)$ with $v_i \in \mathbb{R}^n$ for every $i = 1, \dots, n$ is a basis for the space $\mathbb{R}^n$ if these vectors are linearly independent (n linearly independent vectors in $\mathbb{R}^n$ span the entire space).

The standard basis for $\mathbb{R}^n$ is given by

$(e_1, \dots, e_n)$ with $e_j := (0, \dots, 1, \dots, 0)^T$.
j-th position

We want to measure lengths of vectors and distances in $\mathbb{R}^n$.

<u>Definition.</u> A <u>norm</u> on $\mathbb{R}^n$ is a mapping $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ with the following properties:

a) If $\|x\| = 0$, then $x = 0$,

b) $\|\lambda x\| = |\lambda| \|x\|$ for $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$,

c) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in \mathbb{R}^n$.

The norm $\|x\|$ can be interpreted as the length of the vector x, and the distance between two points $x, y \in \mathbb{R}^n$ is given by $\|x - y\|$, where

$$x - y := x + (-1) y.$$

One can show that all norms on $\mathbb{R}^n$ are equivalent in the following sense: If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on $\mathbb{R}^n$, then we can find constants $c, C > 0$ such that for all $x \in \mathbb{R}^n$

$$c \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1.$$

Hence, in the following we usually only consider the <u>Euclidean norm</u> given by

$$\|x\| := \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \quad \text{for } x = (x_1, \dots, x_n)^T \in \mathbb{R}^n.$$

Moreover, a vector $x \in \mathbb{R}^n$ is called a <u>unit vector</u> if $\|x\| = 1$.

If $x \neq 0$, then $\frac{x}{\|x\|} := \frac{1}{\|x\|} \cdot x$ is a unit vector.

We also want to measure angles between vectors.

<u>Definition.</u> An <u>inner product</u> on $\mathbb{R}^n$ is a mapping $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with the following properties:

For $x, y, z \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$

a) $\langle x, y \rangle = \langle y, x \rangle$

b) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

c) $\langle x, x \rangle \geq 0$ and if $\langle x, x \rangle = 0$, then $x = 0$.

<u>Remark.</u> The symmetry a) and the linearity in the first argument b) together imply

linearity also in the second argument:

$$\langle x, \lambda y + \mu z\rangle \underset{a)}{=} \langle \lambda y + \mu z, x\rangle \underset{b)}{=} \lambda \langle y, x\rangle + \mu \langle z, x\rangle$$

$$\underset{a)}{=} \lambda \langle x, y\rangle + \mu \langle x, z\rangle .$$

The most important inner product on $\mathbb{R}^n$ is given by the so-called <u>standard</u> or <u>Euclidean</u> inner product

$$\langle x, y\rangle_2 := \sum_{i=1}^{n} x_i y_i = x^T y \, , \quad \text{where}$$

$x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_n)^T$.

It can be shown that any other inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ can be expressed in terms of $\langle \cdot, \cdot \rangle_2$:

$\langle x, y\rangle = \langle x, Ay\rangle_2$, where A is an $n \times n$ matrix with real entries which is symmetric ($A = A^T$) and positive definite.

In the following we usually only consider the standard inner product and simply write $\langle x, y\rangle$ instead of $\langle x, y\rangle_2$.

08/09/25

A connection between the Euclidean product and the Euclidean norm is given by

$$\sqrt{\langle x, x \rangle} = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = \|x\| , \quad x \in \mathbb{R}^n,$$

and by the <u>Cauchy-Schwarz inequality</u>

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| , \quad x, y \in \mathbb{R}^n.$$

By this inequality we have for $x, y \in \mathbb{R}^n$

$$-\|x\| \|y\| \leq \langle x, y \rangle \leq \|x\| \|y\| .$$

Hence, if $x, y \neq 0$, we have

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \|y\|} \leq 1 .$$

Since $\cos : [0, \pi] \to [-1, 1]$ is bijective, for each $x, y \in \mathbb{R}^n \setminus \{0\}$, we can find a unique number $\theta \in [0, \pi]$ such that

$$\cos \theta = \frac{\langle x, y \rangle}{\|x\| \|y\|} .$$

<u>Definition.</u> For vectors $x, y \in \mathbb{R}^n \setminus \{0\}$ we define their <u>angle</u> as the unique number $\theta \in [0, \pi]$ such that $$\cos \theta = \frac{\langle x, y \rangle}{\|x\| \|y\|} .$$

Moreover, we call x and y <u>orthogonal</u> if $\theta = \frac{\pi}{2}$, i.e., $\langle x, y \rangle = 0$.

<u>Example.</u> Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$, then $\langle e_i, e_j \rangle = \sum_{k=1}^{n} (e_i)_k (e_j)_k$

$$= \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

Hence, these vectors are pairwise orthogonal and all of them have length 1.

<u>Definition.</u> A basis $(v_1, \dots, v_n)$ of $\mathbb{R}^n$ is called <u>orthonormal</u>, if its vectors are pairwise orthogonal and each v_i has unit norm, i.e., $\|v_i\| = 1$, $i = 1, \dots, n$.

Next we introduce the notion of convergence for sequences of vectors in $\mathbb{R}^n$. The case $n=1$ corresponds to sequences of real numbers, for which we recall:

A sequence of real numbers $(a_k)_{k\in\mathbb{N}}$ is convergent with limit $a\in\mathbb{R}$ if for every $\varepsilon>0$ there exists a $k_0\in\mathbb{N}$ such that

$$|a_k-a|<\varepsilon \quad \text{for all } k\geq k_0.$$

11/09/25

In higher dimensions we use the Euclidean norm $\|\cdot\|$ instead of the modulus $|\cdot|$.

<u>Definition</u> Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of vectors $x_k\in\mathbb{R}^n$ and $x\in\mathbb{R}^n$. The sequence $(x_k)_{k\in\mathbb{N}}$ is <u>convergent</u> <u>with limit</u> x if for every $\varepsilon>0$ there exists a $k_0\in\mathbb{N}$ such that

$$\|x_k-x\|<\varepsilon \quad \text{for all } k\geq k_0.$$

In this case we write $\lim\limits_{k\to\infty} x_k = x$, or $x_k\to x$, $k\to\infty$.

Instead of working with this definition it sometimes is more convenient to work with the real-valued coordinates.