If $(x_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}^n$, then for every $k\in\mathbb{N}$ we have

$$x_k = \begin{pmatrix} x_{k,1} \\ \vdots \\ x_{k,n} \end{pmatrix} \in \mathbb{R}^n ,$$

which means that, for every $j\in\{1,\dots,n\}$, we can consider the real-valued sequence of $j$-th coordinates $(x_{k,j})_{k\in\mathbb{N}}$.

<u>Theorem 1.</u> Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ and $x\in\mathbb{R}^n$. Then $(x_k)_{k\in\mathbb{N}}$ converges to $x$ if and only if all its coordinate sequences $(x_{k,j})_{k\in\mathbb{N}}$ converge to $x_j$, $j\in\{1,\dots,n\}$.

<u>Proof.</u> The proof follows immediately from the inequalities

$$|x_{k,j}-x_j| \le \left(\sum_{\nu=1}^{n}(x_{k,\nu}-x_\nu)^2\right)^{\frac{1}{2}} = \|x_k - x\|$$
$$\le \sum_{\nu=1}^{n}|x_{k,\nu}-x_\nu| , \quad j\in\{1,\dots,n\}. \qquad \square$$

<u>Examples.</u> i) Let $(x_k)_{k\in\mathbb{N}}$ be a sequence

of vectors in $\mathbb{R}^3$ defined by

$$x_k = \left(\frac{1}{k}, 1 + \frac{1}{k^2}, e^{-k}\right)^T, \quad k \in \mathbb{N}.$$

Theorem 1 allows us to study the convergence of $(x_k)_{k\in\mathbb{N}}$ by studying the coordinates separately:

$$x_{k,1} = \frac{1}{k}, \quad x_{k,2} = 1 + \frac{1}{k^2}, \quad x_{k,3} = e^{-k}.$$

We observe: $\lim_{k\to\infty} x_{k,1} = 0$, $\lim_{k\to\infty} x_{k,2} = 1$

and $\lim_{k\to\infty} x_{k,3} = 0$, so we conclude

$$\lim_{k\to\infty} x_k = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

ii) Let us consider $x_k = \left((-1)^k, \frac{1}{k}\right)^T \in \mathbb{R}^2$, $k \in \mathbb{N}$.

We observe $x_{k,1} = (-1)^k$, $k \in \mathbb{N}$, and this sequence is not convergent. Hence, using Theorem 1, we conclude that $(x_k)_{k\in\mathbb{N}}$ is not convergent.

Next we turn to important "topological" properties of the Euclidean space.

<u>Definition.</u> A set $A \subset \mathbb{R}^n$ is <u>bounded</u> if there exists a positive number $r > 0$ such that $\|x\| \leq r$ for all $x \in A$.

<u>Examples.</u> i) The whole space $\mathbb{R}^n$ is not bounded because for every $r > 0$ we find a vector $x \in \mathbb{R}^n$ with $\|x\| > r$.

ii) The set $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ is not bounded.

iii) Open, half-open and closed intervals of the form $(a,b)$, $(a,b]$, $[a,b)$ and $[a,b]$ with $a, b \in \mathbb{R}$ are bounded subsets of $\mathbb{R}$.

iv) For $a \leq b$ and $c \leq d$ the set $[a,b] \times [c,d]$ is a bounded subset of $\mathbb{R}^2$.

<u>Definition.</u> Let $r > 0$ and $y \in \mathbb{R}^n$. We define the <u>open ball</u> centered at $y$ with radius $r$ by

$$U_r(y) := \{x \in \mathbb{R}^n : \|x - y\| < r\}.$$

We define the <u>closed ball</u> centered at $y$ with radius $r$ by

$$B_r(y) := \{x \in \mathbb{R}^n : \|x - y\| \leq r\}.$$

Moreover, a set $A \subset \mathbb{R}^n$ is called <u>open</u> if for every $x \in A$ there exists some $r > 0$ such that $U_r(x) \subset A$. A set $A \subset \mathbb{R}^n$ is <u>closed</u> if its complement $A^c = \mathbb{R}^n \setminus A$ is open.

<u>Examples.</u> i) Open intervals $(a,b)$ are open subsets of $\mathbb{R}$, but intervals of the form $(a,b]$ or $[a,b)$ are neither open nor closed.

ii) The empty set and $\mathbb{R}^n$ both are open and closed subsets of $\mathbb{R}^n$. Such sets are called "clopen".

iii) Open balls are open and closed balls are closed.

iv) The upper half-plane $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 > 0 \right\}$ is an open subset of $\mathbb{R}^2$.

<u>Remark.</u> It is important to observe that the intersection of finitely many open sets gives an open set, and that the union of an arbitrary number of open sets gives an open set.

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<u>Definition.</u> Let $A \subset \mathbb{R}^n$ be an arbitrary subset.

A point $x \in \mathbb{R}^n$ is a

i) <u>limit point</u> of $A$ if for every $\varepsilon > 0$

$$(A \setminus \{x\}) \cap U_\varepsilon(x) \neq \emptyset,$$

ii) <u>boundary point</u> of $A$ if for every $\varepsilon > 0$

$$A \cap U_\varepsilon(x) \neq \emptyset \text{ and } A^c \cap U_\varepsilon(x) \neq \emptyset,$$

iii) <u>interior point</u> of $A$ if there exists $\varepsilon > 0$ such that $U_\varepsilon(x) \subset A$,

iv) <u>isolated point</u> of $A$ if there exists $\varepsilon > 0$ such that $A \cap U_\varepsilon(x) = \{x\}$.

Moreover, we define

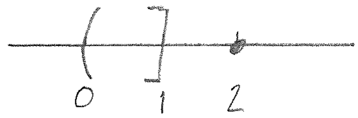
$A' := \{x \in \mathbb{R}^n : x \text{ is a limit point of } A\}$ "set of limit points of $A$",

$\mathring{A} := \{x \in \mathbb{R}^n : x \text{ is an interior point of } A\}$ "set of interior points of $A$",

$\partial A := \{x \in \mathbb{R}^n : x \text{ is a boundary point of } A\}$ "boundary of $A$",

$\overline{A} := A \cup \partial A$ "closure of $A$".

Examples. i) $n = 1$, $A := (0, 1] \cup \{2\}$

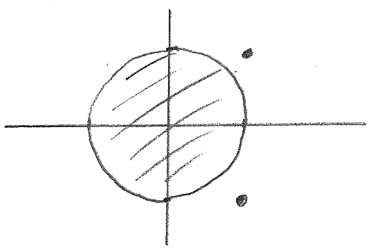


Then we have $A' = [0,1]$, $\mathring{A} = (0,1)$,

$\partial A = \{0,1,2\}$, $\bar{A} = [0,1] \cup \{2\}$,

and $x = 2$ is an isolated point.

ii) $n = 2$, $A := \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\right\} \cup \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$



Then we have $A' = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\right\}$,

$\mathring{A} = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\right\}$,

$\partial A = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\right\} \cup \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$,

$\bar{A} = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\right\} \cup \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$,

and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are isolated points.

Remarks. i) It follows from the definitions that a set $A \subset \mathbb{R}^n$ is open if and only if $A = \mathring{A}$.