ii) A point $x \in \mathbb{R}^n$ is a limit point of the set $A \subset \mathbb{R}^n$ if and only if there is a sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in A \setminus \{x\}$ such that $x_k \to x$.

iii) For a set $A \subset \mathbb{R}^n$ the following are equivalent:

a) A is closed b) $A' \subset A$ c) $A = \overline{A}$

d) for every sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \to x$, $k \to \infty$, and $x_k \in A$ for all $k \in \mathbb{N}$ we have $x \in A$.

Definition. A subset of $\mathbb{R}^n$ is called compact if it is closed and bounded.

Remark. It is not difficult to show that a set $A \subset \mathbb{R}^n$ is compact if and only if every sequence $(x_k)_{k\in\mathbb{N}}$ in A has a convergent subsequence with limit in A.

Examples. i) Closed balls $B_r(x) \subset \mathbb{R}^n$ are compact.

ii) One can show that $K_1 \times K_2 \times \cdots \times K_n \subset \mathbb{R}^n$ is compact if $K_1, \dots, K_n$ are compact subsets of $\mathbb{R}$.

The central objects in multivariate calculus are functions $f: D \to \mathbb{R}^m$ with domain $D \subset \mathbb{R}^n$ for some integers $n, m \in \mathbb{N}$.
Such a function is real-valued if $m = 1$, and vector-valued if $m \geq 2$.

Moreover, for $x \in D$ we write

$$f(x) = (f_1(x), \dots, f_m(x))^T,$$

where $f_1, \dots, f_m : D \to \mathbb{R}$ are the components of f. We recall that the range of a function is given by

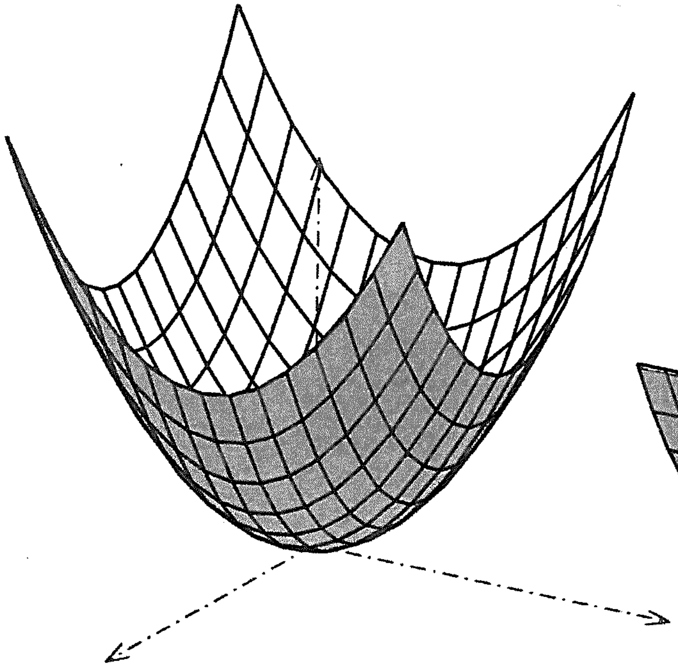
$$f(D) = \{f(x) \in \mathbb{R}^m : x \in D\},$$

and its graph is defined by

$$\{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m : x \in D\}.$$

<u>Examples.</u> In the case $n = 2$ and $m = 1$ graphs can be visualised as surfaces in a three-dimensional space. For instance:

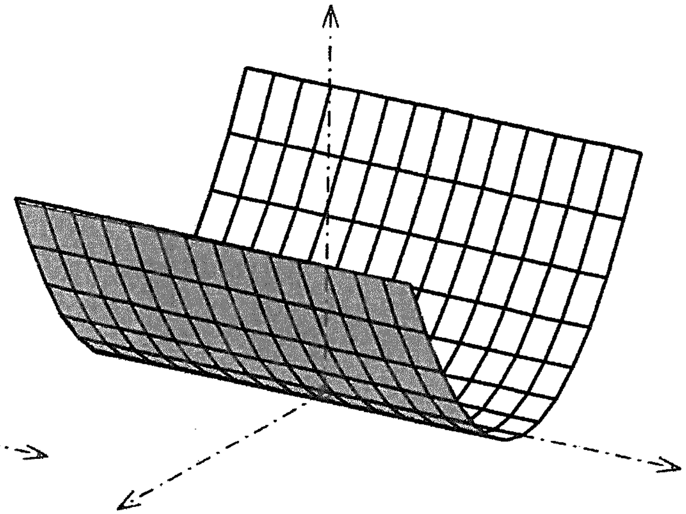
$$f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) := x^2 + y^2,$$
$$g: \mathbb{R}^2 \to \mathbb{R}, \quad g(x,y) := x^2,$$
$$h: \mathbb{R}^2 \to \mathbb{R}, \quad h(x,y) := x^2 - y^2.$$



$$z = x^2 + y^2$$

$= f(x,y)$

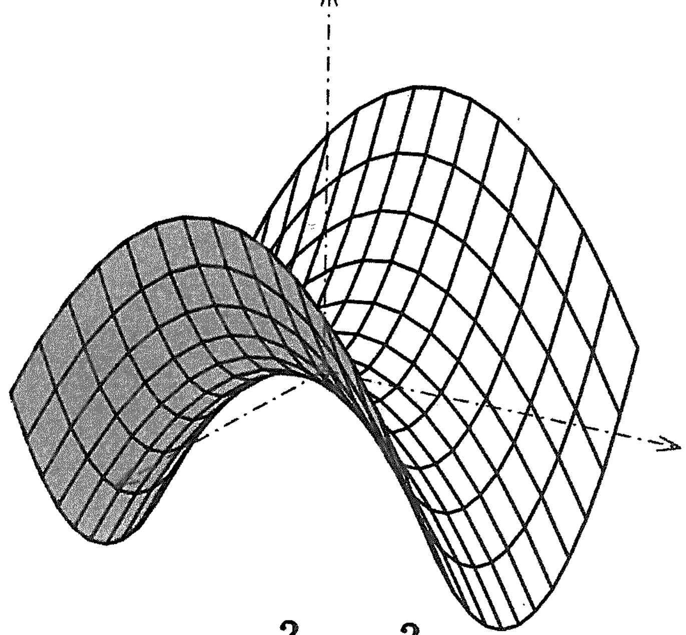
"elliptic paraboloid"



$$z = x^2$$

$= g(x,y)$

"parabolic cylinder"



$$z = x^2 - y^2$$

$= h(x,y)$.

"hyperbolic paraboloid"

<u>Definition.</u> Let $f: D \to \mathbb{R}^m$ $(D \subset \mathbb{R}^n)$ be a function, $\xi \in \mathbb{R}^n$ a limit point of D and $b \in \mathbb{R}^m$.
We say that f <u>tends to the limit b as x tends to ξ</u> if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|f(x) - b\| < \varepsilon \quad \text{for all } x \in D \cap \left(U_\delta(\xi) \setminus \{\xi\}\right).$$

In this case we write $\lim\limits_{x \to \xi} f(x) = b$ or $f(x) \to b$, $x \to \xi$.

<u>Remarks.</u> i) It is not difficult to show that, if it exists, the limit is unique.

ii) It is crucial in the definition to look at the "deleted" ball $U_\delta(\xi) \setminus \{\xi\}$ instead of $U_\delta(\xi)$. For instance, let us consider

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) := \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

then $\lim\limits_{x \to 0} f(x) = 1$, $f(0) \neq 1$.

If we used $U_\delta(\xi)$ in the definition, $\lim\limits_{x \to 0} f(x)$ would not exist.

Theorem 2. Let $f: D \to \mathbb{R}^m$ ($D \subset \mathbb{R}^n$) be a function, ξ a limit point of D and $b \in \mathbb{R}^m$.

i) If we write $f(x) = (f_1(x), \dots, f_m(x))^T$ and $b = (b_1, \dots, b_m)^T$, then we have

$$\lim_{x \to \xi} f(x) = b \iff \lim_{x \to \xi} f_i(x) = b_i \text{ for all } i = 1, \dots, m.$$

ii) We have $\lim_{x \to \xi} f(x) = b$ if and only if for every sequence $(x_k)_{k \in \mathbb{N}}$ with $x_k \in D \setminus \{\xi\}$, $k \in \mathbb{N}$, such that $x_k \to \xi$, $k \to \infty$, we have

$$\lim_{k \to \infty} f(x_k) = b.$$

iii) Let $g: D \to \mathbb{R}^m$ be another function and assume that f and g both have a limit at ξ. Then we have for all $\lambda, \mu \in \mathbb{R}$

$$\lim_{x \to \xi} (\lambda f(x) + \mu g(x)) = \lambda \lim_{x \to \xi} f(x) + \mu \lim_{x \to \xi} g(x),$$

$$\lim_{x \to \xi} f(x)^T g(x) = \lim_{x \to \xi} \langle f(x), g(x) \rangle$$

$$= \left(\lim_{x \to \xi} f(x)\right)^T \left(\lim_{x \to \xi} g(x)\right), \quad \text{and}$$

$$\lim_{x \to \xi} \frac{f(x)}{g(x)} = \frac{\lim_{x \to \xi} f(x)}{\lim_{x \to \xi} g(x)}$$

if $m=1$ and $\lim_{x\to\xi} g(x) \neq 0$.

Proof. The proof of part i) follows the same idea as the proof of Theorem 1. Parts ii) and iii) are analogous to the case of real-valued functions of one variable. $\square$

Examples. i) Let $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x_1,x_2) = x_1 + x_2$, and $\xi = (a,b)^T \in \mathbb{R}^2$ with fixed $a,b \in \mathbb{R}$. We prove that f has a limit at the point ξ which is given by $a+b$. To this end, let $\varepsilon > 0$, then we have

$$\| f(x_1,x_2) - (a+b) \| = | x_1 + x_2 - (a+b) |$$

$$= | x_1 - a + x_2 - b | \leq | x_1 - a | + | x_2 - b |$$

$$\leq 2 \left((x_1-a)^2 + (x_2-b)^2 \right)^{\frac{1}{2}} = 2 \| (x_1 - a, x_2 - b)^T \|$$

$$= 2 \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \right\| < \varepsilon,$$ where the last inequality is true as long as we choose the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ close enough to $\begin{pmatrix} a \\ b \end{pmatrix}$, i.e., if $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{U}_{\frac{\varepsilon}{2}} \begin{pmatrix} a \\ b \end{pmatrix}$. Hence, if we choose $\delta := \frac{\varepsilon}{2}$, then for all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{U}_\delta \begin{pmatrix} a \\ b \end{pmatrix} \setminus \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \right\}$

we have $\| f(x_1, x_2) - (a+b) \| < \varepsilon$, in other words, $\lim_{x \to \xi} f(x) = a+b$.

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ii) Let us consider $f: \mathbb{R}^2 \setminus \{\binom{0}{0}\} \to \mathbb{R}$,

$$f(x_1, x_2) := \frac{x_1^2}{\sqrt{x_1^2 + x_2^2}} .$$

We prove $\lim_{x \to 0} f(x) = 0$, where we note that $\lim_{x \to 0} f(x)$ is shorthand for $\lim_{(x_1, x_2) \to (0,0)} f(x_1, x_2)$.

Let $\varepsilon > 0$, then we have

$$\| f(x_1, x_2) - 0 \| = \frac{x_1^2}{\sqrt{x_1^2 + x_2^2}} \leq \frac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2}}$$

$$= \sqrt{x_1^2 + x_2^2} = \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| < \varepsilon ,$$

where the last inequality is true if $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in U_\varepsilon\left(\binom{0}{0}\right)$. So we choose $\delta := \varepsilon$, and we obtain $\| f(x_1, x_2) - 0 \| < \varepsilon$ for all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in U_\delta\left(\binom{0}{0}\right) \setminus \{\binom{0}{0}\}$. As this works for every possible $\varepsilon > 0$, we conclude that

$$\lim_{x \to 0} f(x) = 0 .$$

iii) Let $f : \mathbb{R}^2 \setminus \{\binom{0}{0}\} \to \mathbb{R}$ be defined by

$f(x_1, x_2) := \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$. We prove that f does not have a limit at $\xi = \binom{0}{0}$.

According to Theorem 2 part ii), it is sufficient to find two sequences $\binom{x_{k,1}}{x_{k,2}}_{k \in \mathbb{N}}$, $\binom{y_{k,1}}{y_{k,2}}_{k \in \mathbb{N}}$ in $\mathbb{R}^2 \setminus \{\binom{0}{0}\}$ such that both sequences converge to $\binom{0}{0}$ but

$$\lim_{k \to \infty} f(x_{k,1}, x_{k,2}) \neq \lim_{k \to \infty} f(y_{k,1}, y_{k,2}).$$

We define $(x_{k,1}, x_{k,2})^T := (0, \frac{1}{k})^T$
and $(y_{k,1}, y_{k,2})^T := (\frac{1}{k}, 0)^T$, $k \in \mathbb{N}$.

Then we have $\lim_{k \to \infty} f(x_{k,1}, x_{k,2}) = 0$

but $\lim_{k \to \infty} f(y_{k,1}, y_{k,2}) = 1$.

iv) Let $f : \mathbb{R}^3 \setminus \{\left(\begin{smallmatrix} 0 \\ 0 \\ x_3 \end{smallmatrix}\right) : x_3 \in \mathbb{R}\} \to \mathbb{R}^2$ be defined by

$$f(x) = f(x_1, x_2, x_3) := \left(1 + x_2 + x_3 + \frac{x_1^2}{\sqrt{x_1^2 + x_2^2}},\ x_1 + x_3\right)^T.$$

To determine whether f has a limit at $\xi = (0,0,0)^T$, and, if so, to compute it, it is sufficient by Theorem 2 part i) to examine the individual components given by

$$f_1(x_1, x_2, x_3) = 1 + x_2 + x_3 + \frac{x_1^2}{\sqrt{x_1^2 + x_2^2}},$$

$$f_2(x_1, x_2, x_3) = x_1 + x_3 .$$

To this end, we can use Theorem 2, part iii) and Examples i) and ii) above:

$$\lim_{x \to (0,0,0)} f_1(x_1, x_2, x_3) = 1 + \lim_{x \to (0,0,0)} (x_2 + x_3) + \lim_{x \to (0,0,0)} \frac{x_1^2}{\sqrt{x_1^2 + x_2^2}}$$

$$= 1, \quad \text{and}$$

$$\lim_{x \to (0,0,0)} f_2(x_1, x_1, x_3) = 0 .$$

Hence, we conclude $\lim_{x \to (0,0,0)} f(x) = (1, 0)^T$. 25/09/25

Next we introduce the concept of continuity.

<u>Definition</u>. Let $f : D \to \mathbb{R}^m$ ($D \subset \mathbb{R}^n$) be a function. We say that f is <u>continuous</u> at the point $\xi \in D$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\| f(x) - f(\xi) \| < \varepsilon \quad \text{for all } x \in U_\delta(\xi) \cap D.$$