PS02 - Bayesian Statistics

Due date: **01/30/2025**

Note: All problems not here should be done when doing PS01

Sources: Gelman et al. (2021); Ross (2007); DeGroot and Shervish (2012)

Question 01

Posterior inference: suppose you have a Beta$(4, 4)$ prior distribution on the probability θ that a coin will yield a 'head' when spun in a specified manner. The coin is independently spun ten times, and 'heads' appear fewer than 3 times. You are not told how many heads were seen, only that the number is less than 3. Calculate the exact posterior density (up to a proportionality constant) for θ and sketch it using R.

Question 02

Suppose that X_1 and X_2 are independent random variables, and X_i has the exponential distribution with parameter β_i $(i = 1, 2)$. Show that for each constant $k > 0$:

$$P(X_1 > kX_2) = \frac{\beta_2}{\beta_2 + k\beta_1}$$

How much is $P(X_1 \leq kX_2)$?

Question 03

Suppose that $X_1, \cdots, X_n$ form a random sample from an exponential distribution for which the value of the parameter β is unknown ($\beta > 0$). Find the Maximum Likelihood Estimator of β.

Question 04

Let $X_1, \cdots, X_n$ a random sample of size n from the distribution specified in each question below. Show that the statistic T specified is a sufficient statistic for the parameter.

A

Bernoulli distribution with parameter p, which is unknown ($0 < p < 1$) and $T = \sum_{i=1}^{n} X_i$.

B

Geometric distribution with parameter p unknown ($0 < p < 1$) and $T = \sum_{i=1}^{n} X_i$.

C

Negative binomial distribution with parameters r and p, where r is known and p is unknown ($0 < p < 1$); $T = \sum_{i=1}^{n} X_i$.

D

Normal distribution for which the mean μ is known but the variance $\sigma^2 > 0$ is unknown; $T = \sum_{i=1}^{n} (X_i - \mu)^2$.

E

The gamma distribution with parameters α and β, where the value of β is known and the value of α is unknown ($\alpha > 0$); $T = \prod_{i=1}^{n} X_i$.

F

The beta distribution with parameters α and β, where the value of β is known and the value of α is unknown ($\alpha > 0$); $T = \prod_{i=1}^{n} X_i$.