1 Limits

1.1 e

The function e is defined as a continuous, differentiable function $f(x)$ that satisfies $f'(x) = f(x)$ for all x and $f(0) = 1$.

$$e = \lim_{n \to 0} (1+n)^{\frac{1}{n}}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

2 Integrals

2.1 Improper Integral Summary

Integral	$p \leq 1$	$p > 1$	Value
$\int_0^1 \frac{1}{x^p}$	divergent	convergent	$\frac{1}{1-p}$
$\int_1^\infty \frac{1}{x^p}$	divergent	convergent	$\frac{1}{p-1}$

2.2 Comparison Theorem

If f and g are continuous and $f(x) \geq g(x) \geq 0$ for $x \geq a$ (there is some a where f is now always larger than g) then,

If $\int_a^\infty f(x)dx$ is convergent then the "smaller" integral $\int_a^\infty g(x)dx$ must be convergent too.

If $\int_a^\infty g(x)dx$ is divergent then the "larger" integral $\int_a^\infty f(x)dx$ must be divergent too.

3 Sequences

3.1 Precise Limit Definition

Say we have an arbitrary number $\epsilon > 0$ as a "tolerance band" from the limit L. Assume the sequence converges. There will be some integer N where every $n > N$ holds $|a_n - L| < \epsilon$.

This allows subsequent terms in the sequence to oscillate around the limit L, so long as they remain in our tolerance band ϵ.

3.2 Convergence

A sequence is convergent if:

- Its limit exists.
- We can make a_n closer and closer to L by increasing n.

3.3 Limit Theorems

- If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then the sequence has the same limit. Essentially, if a function has the same value as the sequence for every integer, then its limit is the same.
- Given an arbitrary value, there will be a number N where every $a_n, n > N$ is larger than the arbitrary value, if the sequence diverges to infinity.
- $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$
- $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$. If the limit of the absolute terms of the sequence is 0, then the limit of the terms is 0.
- If the terms of a convergent sequence ($\lim a_n = L$) are applied to a continuous function, then the result is convergent too.

$$\lim_{n\to\infty} f(a_n) = f(L)$$

3.4 Squeeze Theorem

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = L$$

3.5 r^n sequences

Sequences defined as r^n are convergent if $-1 < r \le 1$.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

3.6 Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

4 Series

4.1 Definition

A series is simply the sum of terms in a sequence.
An infinite series (often simply just called a "sum" or "series") is what we get when we sum an infinite number of terms in a sequence.
A partial sum is what we get when we sum a finite number of terms in a sequence, s_3 for a_n is the sum of $a_1, a_2,$ and a_3.
A series s_n forms its own sequence s_n.
As we increase n in s_n we get closer and closer to the limit—the infinite sum—of the series.

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

$$s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \dots$$

4.2 Example

Take the sequence $a_n = \frac{1}{2^n}$.
This gives us $a_1 = \frac{1}{2}, a_2 = \frac{1}{4},$ etc....
s_2 would then be $a_1 + a_2 = \frac{3}{4}$.
This forms a sequence from the series, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots$.
This sequence converges on 1 the more terms we add.
This is the infinite sum.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{3}{4} + \cdots + \frac{1}{2^n} + \cdots = 1$$

The sum of a series is $s = \lim_{n\to\infty} s_n$.
The series will be divergent—and not have a sum—if the sequence s_n diverges.

4.3 Geometric Series

A geometric series occurs when each term of the sequence is multiplied by the preceding one by a common ratio.

$$a \neq 0 \quad \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \dots$$

This series is convergent if $|r| < 1$.
The partial sum is defined by the following.

$$s_n = \frac{a(1-r^n)}{1-r}$$

The sum of a convergent geometric series is defined by the following.

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

4.4 Test for Divergence

If the series is convergent then the limit of a_n will be 0.
However, we cannot conclude that a series if convergent just because a_n has a limit of 0.

$$\text{If } \sum a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0$$

Even though we cannot make conclusions about the series being convergent, we can check if the series is divergent.

$$\sum a_n \text{ divergent if } \lim_{n\to\infty} a_n \neq 0 \text{ or the limit does not exist.}$$

4.5 Integral Test

If f is continuous, positive, and decreasing on $[1,\infty)$ then let $a_n = f(n)$.

$$\text{If } \int_1^{\infty} f(x)dx \text{ is convergent, then } \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

$$\text{If } \int_1^{\infty} f(x)dx \text{ is divergent, then } \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

4.6 p-series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

4.7 Error with the Integral Test

Estimating an infinite sum with a finite number of terms yields a remainder $R_n = s - s_n$. This remainder is our error.

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_{n}^{\infty} f(x)dx$$

4.8 Comparison Tests

If $a_n \leq b_n$ for all n, then $\sum a_n$ will be convergent if $\sum b_n$ is convergent.
If $a_n \geq b_n$ for all n, then $\sum a_n$ will be divergent if $\sum b_n$ is divergent.
Essentially, if a bigger series is convergent, then smaller series must be as well.
If a smaller series is divergent, then larger series must be as well.

4.9 Limit Comparison Test ($c > 0$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

If this holds ($c > 0$) then either both $\sum a_n$ and $\sum b_n$ converge, or they both diverge.

4.10 Alternating Series Test

An alternating series is one defined by the following.

$$\sum (-1)^{n-1} b_n \quad b_n > 0$$

If the following are satisfied then the series will be convergent.

- $b_{n+1} \leq b_n$ for all n
- $\lim_{n\to\infty} b_n = 0$

4.11 Alternating Series Estimation Theorem

If the Alternating Series Test is satisfied then the following holds.

$$|R_n| = |s - s_n| \leq b_{n+1}$$

This lets us find a desired error by simply computing a value in the sequence.

4.12 Absolute and Conditional Convergence

A series is absolutely convergent if $\sum |a_n|$ is convergent.
A series is convergent if it is absolutely convergent.
A series is conditionally convergent if $\sum a_n$ is convergent but not $\sum |a_n|$.

4.13 Ratio and Ratio Test

Take a series $\sum a_n$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

or

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

If $L < 1$ then the series is absolutely convergent.
If $L > 1$ or $L = \infty$ then the series is divergent.
If $L = 1$ then the nothing about the series can be concluded.