1 Definitions

1.1 Row Echelon Form

Row Echelon Form:

- A nonzero row must have a leftmost "leading" 1. $\begin{bmatrix} 0100 \end{bmatrix}$
- A nonzero row below another nonzero row must have its leading 1 farther to the right.
- Any zero rows must be grouped together at the bottom of the matrix.
- This form is not unique.

$$\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form:

- Any column with a leading 1 must be zero elsewhere.
- This form is unique for any system.

$$\begin{bmatrix} 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.2 Pivot Positions and Columns

The position of a leading 1 is a pivot position of its matrix. Columns with a pivot position are pivot columns.

1.3 Leading and Free Variables

Leading: Corresponding to a leading 1 in an augmented matrix.
Free: The remaining variables. Can be assigned a parameter.

1.4 Trivial Solution

A zero vector. If $Ax = 0$ has only the trivial solution then x must be something like,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

1.5 Homogeneous System

A matrix equation equal to a zero vector. All constant terms are 0. $Ax = 0$. A homogeneous system must be consistent. It will have either only the trivial solution or will have infinitely many solutions.

1.6 Consistent

A system is consistent if it has one or infinitely many solutions. There is no other option for a consistent system. An inconsistent system has no solutions.
A single linear equation with two or more unknowns must have infinitely many solutions.

1.7 Symmetric Matrix

A square matrix A where $A = A^T$. Thus, $(A)_{ij} = (A)_{ji}$.

$$\begin{bmatrix} 1 & 9 \\ 9 & 2 \end{bmatrix}$$

1.8 Skew-symmetric Matrix

A square matrix A where $A^T = -A$.
All the main diagonal entries must be 0.

$$\begin{aligned} -(A_{ij}) &= (A^T)_{ij} \\ -(A_{ij}) &= A_{ji} \\ -(A_{ii}) &= A_{ii} && \text{On the diagonal, } i = j \\ A_{ii} &= 0 && \text{0 is the only value that will hold} \end{aligned}$$

1.9 Linear Combination

The sum of multiple, equally sized matrices with multiple scalar coefficients can be expressed as $c_1A_1 + c_2A_2 + \cdots + c_rA_r$.
This can be used to express matrix products. A is an $m \times n$ matrix and x is an $n \times 1$ column vector.

$$Ax = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{mn} \end{bmatrix}$$

1.10 Column-Row Expansion

$$
\begin{aligned}
AB &= \left[\begin{array}{c|c} 2 & 3 \\ 1 & 4 \end{array}\right] \left[\begin{array}{ccc} 1 & 4 & 6 \\ \hline 6 & 3 & 5 \end{array}\right] \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 6 & 3 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 8 & 12 \\ 1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 18 & 9 & 15 \\ 24 & 12 & 20 \end{bmatrix} \\
&= \begin{bmatrix} 20 & 17 & 27 \\ 25 & 16 & 26 \end{bmatrix}
\end{aligned}
$$

1.11 Trace

$\mathrm{tr}(A)$ of a square matrix A is defined by the sum of the entries on the main diagonal of A.

$$
\begin{aligned}
\mathrm{tr}(AB) &\neq \mathrm{tr}(A)\,\mathrm{tr}(B) \\
\mathrm{tr}(A^T) &= \mathrm{tr}(A) \\
\mathrm{tr}(cA) &= c\,\mathrm{tr}(A)
\end{aligned}
$$

2 Equivalence Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent. That is, if one is true, the rest is true, as they are logically equivalent.

- A is invertible.
- $Ax = 0$ has only the trivial solution.
- The reduced row echelon form of A is I_n.
- A is expressible as a product of elementary matrices. $A = E_n E_{n-1} \ldots E_1 I_n$.
- $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.
- $\det(A) \neq 0$.
- $\lambda = 0$ is not an eigenvalue of A.

3 Determinant Properties

3.1 Adjoint Matrices

We know the following:

$$
\begin{aligned}
A\,\mathrm{adj}(A) &= \det(A)I \\
\mathrm{adj}(A) &= A^{-1}\det(A)I
\end{aligned}
$$

We can then find the determinant of the adjoint of a matrix in terms of the determinant of the original matrix.

$$
\begin{aligned}
A &= \mathrm{adj}(B) \\
A &= B^{-1}\det(B)I \\
\det(A) &= \det(B^{-1})\det(\det(B))\det(I) \\
\det(A) &= \det(B)^{-1}\det(B)^{n}1 \\
\det(A) &= \det(B)^{n-1}
\end{aligned}
$$

4 Theorems

4.1 Free Variable Theorem and Homogeneous Linear System Theorems

These theorems apply only to homogeneous linear systems (HLS).

- An HLS in reduced row echelon form with n unknowns and r nonzero rows (thus, r leading 1s) has $n - r$ free variables.
- An HLS with more unknowns than equations has infinitely many solutions.
- An HLS of n equations with n leading 1s in reduced row echelon form has only the trivial solution (see equivalence theorem).