1 Limits

1.1 e

The function e is defined as a continuous, differentiable function $f(x)$ that satisfies $f'(x) = f(x)$ for all x and $f(0) = 1$.

$$e = \lim_{n \to 0} (1 + n)^{\frac{1}{n}}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

2 Integrals

2.1 Improper Integral Summary

Integral	$p \leq 1$	$p > 1$	Value
$\int_0^1 \frac{1}{x^p}$	divergent	convergent	$\frac{1}{1-p}$
$\int_1^\infty \frac{1}{x^p}$	divergent	convergent	$\frac{1}{p-1}$

2.2 Comparison Theorem

If f and g are continuous and $f(x) \geq g(x) \geq 0$ for $x \geq a$ (there is some a where f is now always larger than g) then,

If $\int_a^\infty f(x)dx$ is convergent then the "smaller" integral $\int_a^\infty g(x)dx$ must be convergent too.

If $\int_a^\infty g(x)dx$ is divergent then the "larger" integral $\int_a^\infty f(x)dx$ must be divergent too.

3 Sequences

3.1 Precise Limit Definition

Say we have an arbitrary number $\epsilon > 0$ as a "tolerance band" from the limit L. Assume the sequence converges. There will be some integer N where every $n > N$ holds $|a_n - L| < \epsilon$.

This allows subsequent terms in the sequence to oscillate around the limit L, so long as they remain in our tolerance band ϵ.

3.2 Convergence

A sequence is convergent if:

- Its limit exists.
- We can make a_n closer and closer to L by increasing n.

3.3 Limit Theorems

- If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then the sequence has the same limit. Essentially, if a function has the same value as the sequence for every integer, then its limit is the same.
- Given an arbitrary value, there will be a number N where every $a_n, n > N$ is larger than the arbitrary value, if the sequence diverges to infinity.
- $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$
- $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$. If the limit of the absolute terms of the sequence is 0, then the limit of the terms is 0.
- If the terms of a convergent sequence ($\lim a_n = L$) are applied to a continuous function, then the result is convergent too.
$$\lim_{n\to\infty} f(a_n) = f(L)$$

3.4 Squeeze Theorem

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = L$$

3.5 $r^n sequences$

Sequences defined as r^n are convergent if $-1 < r \leq 1$.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

3.6 Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

4 Series