1 Trigonometric Identities

1.1 Double-Angle Formulas

You can use the formula for $\cos(2x)$ with the identity $\sin^2 x + \cos^2 x = 1$ to produce other useful formulas.

$$\begin{aligned}
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) && ** \\
&= \cos^2(x) - (1 - \cos^2(x)) \\
&= 2\cos^2(x) - 1 && ** \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= (1 - \sin^2(x)) - \sin^2(x) \\
&= 1 - 2\sin^2(x) && **
\end{aligned}$$

1.2 Half-Angle Formulas

You can then use the double-angle formulas to derive the following.
These are the ones most useful for integral calculus.
Memorizing the original double-angle formulas allows one to derive these easily.

$$\begin{aligned}
\cos^2(x) &= \frac{1 + \cos(2x)}{2} \\
\sin^2(x) &= \frac{1 - \cos(2x)}{2}
\end{aligned}$$

2 Limits

2.1 e

The function e is defined as a continuous, differentiable function $f(x)$ that satisfies $f'(x) = f(x)$ for all x and $f(0) = 1$.

$$e = \lim_{n \to 0} (1 + n)^{\frac{1}{n}}$$

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

3 Integrals

3.1 Improper Integral Summary

Integral	$p \leq 1$	$p > 1$	Value
$\int_0^1 \frac{1}{x^p}$	divergent	convergent	$\frac{1}{1-p}$
$\int_1^\infty \frac{1}{x^p}$	divergent	convergent	$\frac{1}{p-1}$

3.2 Comparison Theorem

If f and g are continuous and $f(x) \geq g(x) \geq 0$ for $x \geq a$ (there is some a where f is now always larger than g) then,

If $\int_a^\infty f(x)dx$ is convergent then the “smaller” integral $\int_a^\infty g(x)dx$ must be convergent too.

If $\int_a^\infty g(x)dx$ is divergent then the “larger” integral $\int_a^\infty f(x)dx$ must be divergent too.

4 Sequences

4.1 Precise Limit Definition

Say we have an arbitrary number $\epsilon > 0$ as a “tolerance band” from the limit L. Assume the sequence converges. There will be some integer N where every $n > N$ holds $|a_n - L| < \epsilon$.

This allows subsequent terms in the sequence to oscillate around the limit L, so long as they remain in our tolerance band ϵ.

4.2 Convergence

A sequence is convergent if:

- Its limit exists.
- We can make a_n closer and closer to L by increasing n.

4.3 Limit Theorems

- If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then the sequence has the same limit. Essentially, if a function has the same value as the sequence for every integer, then its limit is the same.

- Given an arbitrary value, there will be a number N where every $a_n, n > N$ is larger than the arbitrary value, if the sequence diverges to infinity.

- $\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p$ if $p > 0$ and $a_n > 0$

- $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$. If the limit of the absolute terms of the sequence is 0, then the limit of the terms is 0.

- If the terms of a convergent sequence ($\lim a_n = L$) are applied to a continuous function, then the result is convergent too.
$$\lim_{n\to\infty} f(a_n) = f(L)$$

4.4 Squeeze Theorem

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$
then
$$\lim_{n\to\infty} b_n = L$$

4.5 r^n sequences

Sequences defined as r^n are convergent if $-1 < r \le 1$.
$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

4.6 Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

5 Series

5.1 Definition

A series is simply the sum of terms in a sequence.
An infinite series (often simply just called a "sum" or "series") is what we get when we sum an infinite number of terms in a sequence.
A partial sum is what we get when we sum a finite number of terms in a sequence, s_3 for a_n is the sum of $a_1, a_2,$ and a_3.

A series s_n forms its own sequence s_n.
As we increase n in s_n we get closer and closer to the limit—the infinite sum—of the series.

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

$$s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \ldots$$

Suppose $\sum a_n = 3$ and s_n is the nth partial sum of the series.
Since $\sum a_n$ converges on 3, $\lim_{n->\infty} a_n = 0$ must be true.
This is a fundamental theorem, if a series is convergent then the limit of its sequence is 0.
However, this does not mean that if the limit of a sequence is 0 that its series is convergent.
Since $\sum a_n$ converges on 3, $\lim_{n->\infty} s_n = 3$.

5.2 Example

Take the sequence $a_n = \frac{1}{2^n}$.
This gives us $a_1 = \frac{1}{2}, a_2 = \frac{1}{4},$ etc $\ldots$.
s_2 would then be $a_1 + a_2 = \frac{3}{4}$.
This forms a sequence from the series, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$.
This sequence converges on 1 the more terms we add.
This is the infinite sum.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{3}{4} + \cdots + \frac{1}{2^n} + \cdots = 1$$

The sum of a series is $s = \lim_{n\to\infty} s_n$.
The series will be divergent—and not have a sum—if the sequence s_n diverges.

5.3 Geometric Series

A geometric series occurs when each term of the sequence is multiplied by the preceding one by a common ratio.

$$a \neq 0 \quad \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \ldots$$

This series is convergent if $|r| < 1$.
The partial sum is defined by the following.

$$s_n = \frac{a(1-r^n)}{1-r}$$

The sum of a convergent geometric series is defined by the following.

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

5.4 Sum and Difference Theorem

The following are all convergent series if $\sum a_n$ and $\sum b_n$ are both convergent.

- $\sum ca_n = c \sum a_n$
- $\sum (a_n + b_n) = \sum a_n + \sum b_n$
- $\sum (a_n - b_n) = \sum a_n - \sum b_n$

If $\sum a_n$ was convergent and $\sum b_n$ was divergent, you could use this theorem to show the divergence of $\sum (a_n + b_n)$. By the theorem, $\sum (a_n + b_n) - a_n$ would be convergent if both $\sum a_n$ and $\sum b_n$ were. This is equal to just $\sum b_n$. However, since we know $\sum b_n$ is divergent, $\sum (a_n + b_n)$ must also be divergent.

If we were given a power series $\sum (c_n + d_n)x^n$ and told that told that $\sum c_n x^n$ has a radius of convergence 2 and that $\sum d_n x^n$ has a radius of covergent 3, we can determine the radius of convergence for the whole series.
$\sum c_n x^n$ only converges if $|x| < 2$ and $\sum d_n x^n$ only converges if $|x| < 3$. We know from this theorem that both series must converge for the sum to be convergent. Thus $\sum (c_n + d_n)x^n$ is convergent for $|x| < 2$ and has a radius of convergence 2.

5.5 Test for Divergence

If the series is convergent then the limit of a_n will be 0.
However, we cannot conclude that a series if convergent just because a_n has a limit of 0.

$$\text{If } \sum a_n \text{ converges, then } \lim_{n \to \infty} a_n = 0$$

Even though we cannot make conclusions about the series being convergent, we can check if the series is divergent.

$$\sum a_n \text{ divergent if } \lim_{n \to \infty} a_n \neq 0 \text{ or the limit does not exist.}$$

5.6 Integral Test

If f is continuous, positive, and decreasing on $[1, \infty)$ then let $a_n = f(n)$.

$$\text{If } \int_1^{\infty} f(x)dx \text{ is convergent, then } \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

$$\text{If } \int_1^\infty f(x)dx \text{ is divergent, then } \sum_{n=1}^\infty a_n \text{ is divergent.}$$

5.7 p-series Test

$$\sum_{n=1}^\infty \frac{1}{n^p} \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1.$$

5.8 Error with the Integral Test

Estimating an infinite sum with a finite number of terms yields a remainder $R_n = s - s_n$. This remainder is our error.

$$\int_{n+1}^\infty f(x)dx \leq R_n \leq \int_n^\infty f(x)dx$$

$$s_n + \int_{n+1}^\infty f(x)dx \leq s \leq s_n + \int_n^\infty f(x)dx$$

5.9 Comparison Tests

If $a_n \leq b_n$ for all n, then $\sum a_n$ will be convergent if $\sum b_n$ is convergent.
If $a_n \geq b_n$ for all n, then $\sum a_n$ will be divergent if $\sum b_n$ is divergent.
Essentially, if a bigger series is convergent, then smaller series must be as well.
If a smaller series is divergent, then larger series must be as well.

5.10 Limit Comparison Test ($c > 0$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

If this holds ($c > 0$) then either both $\sum a_n$ and $\sum b_n$ converge, or they both diverge.

5.11 Alternating Series Test

An alternating series is one defined by the following.

$$\sum (-1)^{n-1} b_n \quad b_n > 0$$

If the following are satisfied then the series will be convergent.

- $b_{n+1} \leq b_n$ for all n
- $\lim_{n\to\infty} b_n = 0$

5.12 Alternating Series Estimation Theorem

If the Alternating Series Test is satisfied then the following holds.

$$|R_n| = |s - s_n| \leq b_{n+1}$$

This lets us find a desired error by simply computing a value in the sequence.
How many terms are needed to estimate the sum $\sum \frac{(-1)^n}{n^6}$ such that $|\text{error}| < 0.00005$?

$$\begin{aligned} |\text{error}| \leq b_{n+1} &< \frac{1}{20000} \\ (n+1)^6 &> 20000 \\ (5+1)^6 &> 20000 \\ 6^6 &> 20000 \\ b_6 &> 20000 \end{aligned}$$

The sixth term is less than the desired error.
Adding the sixth term will not yield more than our desired accuracy.
This means we need five terms to yield a sum with the desired accuracy.

5.13 Error

$$\begin{aligned} 0.00005 &= \frac{1}{20,000} \\ 0.00001 &= \frac{1}{100,000} \\ 0.01 &= \frac{1}{100} \\ 0.02 &= \frac{1}{50} \\ 0.05 &= \frac{1}{20} \end{aligned}$$

5.14 Absolute and Conditional Convergence

A series is absolutely convergent if $\sum |a_n|$ is convergent.
A series is convergent if it is absolutely convergent.
A series is conditionally convergent if $\sum a_n$ is convergent but not $\sum |a_n|$.

5.15 Ratio and Ratio Test

Take a series $\sum a_n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = L$$

If $L < 1$ then the series is absolutely convergent.
If $L > 1$ or $L = \infty$ then the series is divergent.
If $L = 1$ then the nothing about the series can be concluded.

5.16 Product of Convergent Series

If two series are divergent, their product is not necessarily divergent.

$$\begin{aligned} a_n = b_n &= \frac{(-1)^n}{\sqrt{n}} \\ a_n b_n &= \frac{1}{n} \end{aligned}$$

$\sum a_n$ and $\sum b_n$ are convergent by the alternating series test. $\sum a_n b_n$ is divergent.

5.17 Helpful Limit

The following applies for all real x.

$$\lim_{n\to\infty}\frac{x^n}{n!} = 0$$

5.18 Integrating and Differentiating Power Series

We know the sum of a simple geometric series with $a = 1$ and $x = r$ is the following. Paying attention to the lower bound of summation is extremely important when working with power series.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Differentiating a sum requires us to differentiate each term.

$$\begin{aligned} \sum_{n=0}^{3} x^n &= 1 + x + x^2 + x^3 \\ \frac{d}{dx}\left(\sum_{n=0}^{3}\right) x^n &= \sum_{n=0}^{3}\left(\frac{d}{dx}x^n\right) = \frac{d}{dx}1 + \frac{d}{dx}x + \frac{d}{dx}x^2 + \frac{d}{dx}x^3 \end{aligned}$$

If we differentiate the original power series and its sum we get the following.

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$
$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

This yields a general rule for differentiating power series. A similar rule works for integrating.

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$
$$\int\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

With both differentiating and integration, the **radii of convergence remains the same**. Differentiating and integrating power series does not affect their radii of convergence.

$$\int \frac{1}{1-t} = -\ln(1-t)$$

5.19 Important Maclaurin Series

These should be memorized.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad R = \infty$$
$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad R = \infty$$
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad R = \infty$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad R = 1$$
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad R = 1$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots \qquad R = 1$$