1 Definitions

1.1 Row Echelon Form

Row Echelon Form:

- A nonzero row must have a leftmost "leading" 1. $\begin{bmatrix} 0100 \end{bmatrix}$
- A nonzero row below another nonzero row must have its leading 1 farther to the right.
- Any zero rows must be grouped together at the bottom of the matrix.
- This form is not unique.

$$\begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form:

- Any column with a leading 1 must be zero elsewhere.
- This form is unique for any system.

$$\begin{bmatrix} 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.2 Pivot Positions and Columns

The position of a leading 1 is a pivot position of its matrix. Columns with a pivot position are pivot columns.

1.3 Leading and Free Variables

Leading: Corresponding to a leading 1 in an augmented matrix.
Free: The remaining variables. Can be assigned a parameter.

1.4 Trivial Solution

A zero vector. If $Ax = 0$ has only the trivial solution then x must be something like,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

1.5 Homogeneous System

A matrix equation equal to a zero vector. All constant terms are 0. $Ax = 0$. A homogeneous system must be consistent. It will have either only the trivial solution or will have infinitely many solutions.

1.6 Consistent

A system is consistent if it has one or infinitely many solutions. There is no other option for a consistent system. An inconsistent system has no solutions.
A single linear equation with two or more unknowns must have infinitely many solutions.

1.7 Symmetric Matrix

A square matrix A where $A = A^T$. Thus, $(A)_{ij} = (A)_{ji}$.

$$\begin{bmatrix} 1 & 9 \\ 9 & 2 \end{bmatrix}$$

1.8 Skew-symmetric Matrix

A square matrix A where $A^T = -A$.
All the main diagonal entries must be 0.

$$\begin{aligned} -(A_{ij}) &= (A^T)_{ij} \\ -(A_{ij}) &= A_{ji} \\ -(A_{ii}) &= A_{ii} && \text{On the diagonal, } i = j \\ A_{ii} &= 0 && \text{0 is the only value that will hold} \end{aligned}$$

1.9 Linear Combination

The sum of multiple, equally sized matrices with multiple scalar coefficients can be expressed as $c_1A_1 + c_2A_2 + \cdots + c_rA_r$.
This can be used to express matrix products. A is an $m \times n$ matrix and x is an $n \times 1$ column vector.

$$Ax = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{mn} \end{bmatrix}$$

1.10 Column-Row Expansion

$$\begin{aligned}
AB &= \left[\begin{array}{c|c} 2 & 3 \\ 1 & 4 \end{array}\right] \left[\begin{array}{ccc} 1 & 4 & 6 \\ \hline 6 & 3 & 5 \end{array}\right] \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 6 & 3 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 8 & 12 \\ 1 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 18 & 9 & 15 \\ 24 & 12 & 20 \end{bmatrix} \\
&= \begin{bmatrix} 20 & 17 & 27 \\ 25 & 16 & 26 \end{bmatrix}
\end{aligned}$$

1.11 Trace

$\operatorname{tr}(A)$ of a square matrix A is defined by the sum of the entries on the main diagonal of A.

$$\begin{aligned}
\operatorname{tr}(AB) &\neq \operatorname{tr}(A)\operatorname{tr}(B) \\
\operatorname{tr}(A^T) &= \operatorname{tr}(A) \\
\operatorname{tr}(cA) &= c\operatorname{tr}(A)
\end{aligned}$$

1.12 Nonsingular Matrix

A matrix that is invertible.

1.13 Determinant and Cofactor Expansion

The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

The determinant of a square matrix can be determined by choosing a row or column and multiplying each entry by the corresponding cofactor. For instance, the determinant of the matrix A can be found by choosing the first row, $\det(A) = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$.

1.14 2×2 Inverse

The inverse of a 2×2 matrix is $\dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

1.15 Minor

The minor of entry A_{ij} is the determinant of the submatrix A with row i and column j deleted.

1.16 Cofactor

The cofactor of entry A_{ij} is the minor of entry $A_{ij} * (-1)^{i+j}$. If $i + j$ is even, the minor retains its sign.

1.17 Row Equivalency

Two matrices are row equivalent if either can be obtained from the other by a sequence of elementary row operations.
If A and B are row equivalent, and if B and C are row equivalent, then A and C are row equivalent.

1.18 Elementary Matrix

A matrix that can be obtained from an identity matrix by a single elementary row operation.

2 Equivalence Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent. That is, if one is true, the rest is true, as they are logically equivalent.

- A is invertible.
- $Ax = 0$ has only the trivial solution.
- The reduced row echelon form of A is I_n.
- A is expressible as a product of elementary matrices. $A = E_n E_{n-1} \dots E_1 I_n$.
- $Ax = b$ has exactly one solution for every $n \times 1$ matrix b. $x = A^{-1}b$.
- $\det(A) \neq 0$.
- $\lambda = 0$ is not an eigenvalue of A.

Extra:

- If A is a singular (non-invertible) square matrix then the linear system $Ax = 0$ has infinitely many solutions.

3 Determinant Properties

3.1 Adjoint Matrices

We know the following:

$$A\,\mathrm{adj}(A) = \det(A)I$$
$$\mathrm{adj}(A) = A^{-1}\det(A)I$$

We can then find the determinant of the adjoint of a matrix in terms of the determinant of the original matrix.

$$\begin{aligned}
A &= \operatorname{adj}(B) \\
A &= B^{-1}\det(B)I \\
\det(A) &= \det(B^{-1})\det(\det(B))\det(I) \\
\det(A) &= \det(B)^{-1}\det(B)^{n}1 \\
\det(A) &= \det(B)^{n-1}
\end{aligned}$$

4 Theorems

4.1 Free Variable Theorem and Homogeneous Linear System Theorems

These theorems apply only to homogeneous linear systems (HLS).

- An HLS in reduced row echelon form with n unknowns and r nonzero rows (thus, r leading 1s) has $n-r$ free variables.
- An HLS with more unknowns than equations has infinitely many solutions.
- An HLS of n equations with n leading 1s in reduced row echelon form has only the trivial solution (see equivalence theorem).

4.2 Inverse Theorems

- If a square matrix has a zero row or column, it is singular (not invertible). The definition of an invertible matrix is $AA^{-1} = I$. If a zero row exists, the product could not reduce down to the identity matrix.
- If B and C are both inverses of A then $B = C$. An invertible matrix has only one inverse.
- $(AB)^{-1} = B^{-1}A^{-1}$. Note the reverse order. This means that to cancel out the matrix A in $ABC = I$ we would need to multiply A^{-1} on the LHS of each side. $A^{-1}ABC = A^{-1}I \Rightarrow BC = A^{-1}I$.
- If a product of matrices is non-invertible (singular) then at least one of the factors must also be singular.
- $(A^{-1})^{-1} = A$
- $(A^n)^{-1} = A^{-n} = (A^{-1})^n$
- $(kA)^{-1} = k^{-1}A^{-1}$ with k being a scalar, thus $k^{-1} = \frac{1}{k}$.
- If AB is invertible then A and B must also both be invertible.
- If A and/or B are not invertible, then neither is AB.

4.3 Transpose Theorems

- $(A^T)^T = A$
- $(A \pm B)^T = A^T \pm B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$ (note the reverse order)
- If A is invertible then A^T is also invertible. $(A^T)^{-1} = (A^{-1})^T$.

4.4 Elementary Theorems

- Every elementary matrix is invertible.
- The inverse of any elementary matrix is also an elementary matrix.
- The product of matrix A and an elementary matrix is the matrix that results when the corresponding row operation is performed on A.
- The inverse of a matrix can be obtained by performing the same row operations that reduce the matrix to the identity to an identity matrix. If $E_1 E_0 A = I$ then $A^{-1} = E_1 E_0 I$.

4.5 Diagonal Matrix Theorems

- Invertible if all of its diagonal entries are nonzero.
- Powers of diagonal matrices (including their inverse) are easy to compute, $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$.
- If D is a diagonal matrix then the product AD can be found by multiplying the entry of each column in A by the corresponding diagonal entry (entries in first column of A multiplied by D_{11}. The product of DA can be found by multiplying the entry of each row in A by the corresponding diagonal entry (entries in first row of A multiplied by D_{11}.

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} \\ d_2 a_{21} & d_2 a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} \\ d_1 a_{21} & d_2 a_{22} \end{bmatrix}$$

4.6 Triangular Matrix Theorems

- The transpose of a triangular matrix is the opposite kind of triangular matrix (upper to lower and vise-versa).
- The product of same-side triangular matrices is the same side (product of upper triangular matrices is upper triangular).

- A triangular matrix is invertible if its diagonal entries are all nonzero.
- The inverse of a triangular matrix is the same side.
- The determinant of a triangular matrix is the product of the entries on the main diagonal, $\det(A) = a_{11}a_{22}\ldots a_{nn}$.

4.7 Symmetric Theorems

If A and B are symmetric with the same size and k is a scalar constant then the following theorems hold.

- $A \pm B$ is symmetric. This holds for skew-symmetric matrices.
- kA is symmetric. This holds for skew-symmetric matrices.
- AB is symmetric if A and B commute. This does not hold for skew-symmetric matrices.
- If A is invertible then A^{-1} is symmetric. This holds for skew-symmetric matrices.
- If A is invertible then AA^T and A^TA are also invertible.
- Every square matrix can be expressed as the sum of a symmetric and skew-symmetric matrix.