Supplementary Material for

"Prediction of General Small Area Parameters for Unit-Level Count Data"

In this supplement, we present extended simulation output. The primary purpose of this section of the supplement is to address the secondary objectives of the simulation study. The secondary objectives are described in the first paragraph of Section 4 of the main document. We also use this supplement to present output that we did not have space to include in the main document. In Section 1, we present the root mean square error and bias of the predictors in the absolute (not relative) scale. In Section 2, we present output using the R function `glmer`. Sections 3-5 concern the secondary objectives of the simulation. In section 3, we evaluate the choice of $T$ for the Monte Carlo SIR algorithm used to compute the predictors for the GLMM model. In Section 4, we compare the computing times of the Gam-Pois and GLMM procedures. In Section 5, we conduct a simulation to evaluate the quality of the estimator of the leading term in the mean square error of the Gam-Pois predictor. In Section 6, we present results for $L = 100$.

# 1 Raw (not relative) RMSE and RB

In the main manuscript, we present the relative root mean square errors and relative biases of the predictors. In this supplement, we present the absolute versions of those quantities. Specifically, we present the Monte Carlo (MC) root mean square error and the MC bias. Let $\hat{\theta}_i^{(m)}$ and $\theta_i^{(m)}$, respectively, denote a predictor and true parameter obtained in MC simulation $m$. We define the MC RMSE for area $i$ as

$$RMSE_i = \sqrt{\frac{1}{M}\sum_{m=1}^{M}(\hat{\theta}_i^{(m)} - \theta_i^{(m)})^2}.$$

We define the MC bias for area $i$ as

$$Bias_i = \frac{1}{M}\sum_{m=1}^{M}(\hat{\theta}_i^{(m)} - \theta_i^{(m)}).$$

We report the average RMSE and average absolute bias defined, respectively, as

$$\text{RMSE} = \frac{1}{D}\sum_{i=1}^{D} RMSE_i \tag{S1}$$

and

$$\text{Bias} = \frac{1}{D} \sum_{i=1}^{D} \mid Bias_i \mid .$$ (S2)

Tables 1 and 3 report the Bias and RMSE for the simulations in which the gamma-Poisson model is the true model. Tables 2 and 4 report the Bias and RMSE for the simulations in which the GLMM is the true model. For Tables 1 and 2, $L = 100$, and for Tables 3 and 4, $L = 1000$. The MC sample size is $M = 5000$ when $L = 100$ and is $M = 500$ when $L = 1000$.

Table 1: RMSE and Bias when gamma-Poisson is true model. $L = 100$

| Predictor | $\alpha$ | Mean RMSE | Mean Bias | Med. RMSE | Med. Bias | IQR RMSE | IQR Bias |
|---|---|---|---|---|---|---|---|
| Gam-Pois | 5.000 | 1.191 | 0.013 | 0.836 | 0.009 | 1.292 | 0.015 |
| Gam-Pois-Alt | 5.000 | 1.185 | 0.013 | | | | |
| GLMM | 5.000 | 1.196 | 0.018 | 0.839 | 0.015 | 1.296 | 0.016 |
| PI | 5.000 | 1.191 | 0.032 | 0.857 | 0.135 | 1.511 | 0.682 |
| Direct | 5.000 | 4.198 | 0.052 | 3.280 | 0.627 | 5.266 | 1.128 |
| Gam-Pois | 0.500 | 0.378 | 0.004 | 0.306 | 0.003 | 0.519 | 0.006 |
| Gam-Pois-Alt | 0.500 | 0.376 | 0.004 | | | | |
| GLMM | 0.500 | 0.385 | 0.019 | 0.309 | 0.008 | 0.524 | 0.008 |
| PI | 0.500 | 0.383 | 0.008 | 0.337 | 0.125 | 0.635 | 0.213 |
| Direct | 0.500 | 0.740 | 0.008 | 0.671 | 0.105 | 1.031 | 0.143 |

Table 2: RMSE and Bias when GLMM is true model. $L = 100$

| Predictor | $\sigma_b^2$ | Mean RMSE | Mean Bias | Med. RMSE | Med. Bias | IQR RMSE | IQR Bias |
|---|---|---|---|---|---|---|---|
| Gam-Pois | 0.500 | 0.756 | 0.008 | 0.750 | 0.008 | 0.717 | 0.012 |
| Gam-Pois-Alt | 0.500 | 0.753 | 0.008 | | | | |
| GLMM | 0.500 | 0.747 | 0.015 | 0.743 | 0.015 | 0.712 | 0.010 |
| PI | 0.500 | 0.743 | 0.009 | 0.755 | 0.122 | 1.219 | 0.970 |
| Direct | 0.500 | 1.197 | 0.012 | 1.387 | 0.124 | 1.841 | 0.705 |
| Gam-Pois | 1.500 | 1.020 | 0.012 | 0.994 | 0.012 | 0.973 | 0.020 |
| Gam-Pois-Alt | 1.500 | 1.016 | 0.012 | | | | |
| GLMM | 1.500 | 1.008 | 0.013 | 0.985 | 0.013 | 0.965 | 0.013 |
| PI | 1.500 | 1.003 | 0.016 | 1.007 | 0.135 | 1.401 | 0.933 |
| Direct | 1.500 | 2.653 | 0.030 | 2.901 | 0.193 | 3.935 | 0.973 |

Table 3: RMSE and Bias when gamma-Poisson model is true model. $L = 1000$

| Predictor | $\sigma_b^2$ | Mean RMSE | Mean Bias | Med. RMSE | Med. Bias | IQR RMSE | IQR Bias |
|---|---|---|---|---|---|---|---|
| Gam-Pois | 5.000 | 1.174 | 0.045 | 0.822 | 0.032 | 1.278 | 0.050 |
| Gam-Pois-Alt | 5.000 | 1.174 | 0.045 | | | | |
| GLMM | 5.000 | 1.181 | 0.048 | 0.825 | 0.036 | 1.282 | 0.051 |
| PI | 5.000 | 1.180 | 0.048 | 0.846 | 0.131 | 1.504 | 0.680 |
| Direct | 5.000 | 4.164 | 0.140 | 3.225 | 0.632 | 5.141 | 1.108 |
| Gam-Pois | 0.500 | 0.372 | 0.013 | 0.302 | 0.011 | 0.514 | 0.017 |
| Gam-Pois-Alt | 0.500 | 0.372 | 0.013 | | | | |
| GLMM | 0.500 | 0.380 | 0.019 | 0.305 | 0.013 | 0.518 | 0.019 |
| PI | 0.500 | 0.379 | 0.015 | 0.334 | 0.125 | 0.633 | 0.212 |
| Direct | 0.500 | 0.724 | 0.027 | 0.663 | 0.106 | 1.023 | 0.139 |

Table 4: RMSE and Bias when GLMM is true model. $L = 1000$

| Predictor | $\sigma_b^2$ | Mean | | Med. | | IQR | |
|---|---|---|---|---|---|---|---|
| | | RMSE | Bias | RMSE | Bias | RMSE | Bias |
| Gam-Pois | 0.500 | 0.753 | 0.025 | 0.750 | 0.026 | 0.712 | 0.028 |
| Gam-Pois-Alt | 0.500 | 0.752 | 0.025 | | | | |
| GLMM | 0.500 | 0.743 | 0.026 | 0.741 | 0.027 | 0.705 | 0.028 |
| PI | 0.500 | 0.743 | 0.025 | 0.757 | 0.121 | 1.213 | 0.967 |
| Direct | 0.500 | 1.192 | 0.043 | 1.387 | 0.122 | 1.821 | 0.696 |
| Gam-Pois | 1.500 | 1.010 | 0.031 | 0.984 | 0.032 | 0.961 | 0.033 |
| Gam-Pois-Alt | 1.500 | 1.009 | 0.031 | | | | |
| GLMM | 1.500 | 0.999 | 0.032 | 0.975 | 0.032 | 0.954 | 0.033 |
| PI | 1.500 | 0.999 | 0.031 | 1.002 | 0.139 | 1.394 | 0.927 |
| Direct | 1.500 | 2.678 | 0.099 | 2.942 | 0.199 | 3.824 | 0.961 |

The conclusions based on the absolute bias and RMSE are the same as the conclusions based on the relative measures reported in the main document. When the gamma-Poisson model is true, the Gam-Pois-Alt predictor is most efficient for the mean, and the Gam-Pois predictor is more efficient than the GLMM predictor. The GLMM predictor is more efficient than the Gam-Pois predictor when the GLMM model is true. The PI predictor is more efficient than the GLMM predictor for the mean when $L = 100$, but the PI predictor is inefficient for the median and for the IQR. Increasing $L$ improves the efficiency of the Gam-Pois and GLMM predictors. The loss of efficiency from incorrect use of the GLMM or Gam-Pois predictor when the other model is true is negligible relative to the loss of efficiency from the use of the direct estimator. The bias is a small fraction of the RMSE, indicating that the variance is more important than the bias.

## 2   Output Using GLMER

We repeat the simulation, where we use the R function glmer to calculate the GLMM and PI predictors. The %RRMSE and %RB based on glmer are provided in Tables 5 and 6 below.

The use of `glmer` improves the GLMM and PI predictors slightly. Overall, the conclusions based on `glmer` are the same as the results obtained from the IRLS algorithm presented in the main document.

Table 5: Relative bias and relative root mean square error of the alternative predictors when the gamma-Poisson model is the true model. $L = 100$.

| | $\alpha$ | Mean | | Med | | IQR | |
|---|---|---|---|---|---|---|---|
| | | RRMSE | RB | RRMSE | RB | RRMSE | RB |
| Gam-Pois | 5.000 | 17.530 | 0.189 | 20.507 | 0.214 | 19.527 | 0.195 |
| Gam-Pois-Alt | 5.000 | 17.445 | 0.189 | | | | |
| GLMM | 5.000 | 17.616 | 0.187 | 20.580 | 0.213 | 19.590 | 0.195 |
| PI | 5.000 | 17.532 | 0.290 | 21.036 | 3.311 | 22.931 | 10.478 |
| Direct | 5.000 | 61.233 | 0.695 | 80.507 | 15.357 | 79.954 | 16.836 |
| Gam-Pois | 0.500 | 55.765 | 0.582 | 109.058 | 1.298 | 64.523 | 0.776 |
| Gam-Pois-Alt | 0.500 | 55.501 | 0.580 | | | | |
| GLMM | 0.500 | 56.890 | 0.650 | 110.213 | 3.695 | 65.272 | 3.413 |
| PI | 0.500 | 56.792 | 3.554 | 118.525 | 39.865 | 78.630 | 28.455 |
| Direct | 0.500 | 107.558 | 1.121 | 235.484 | 36.186 | 126.890 | 17.814 |

Table 6: Relative bias and relative root mean square error of the alternative predictors when the Poisson-GLMM model is the true model. $L = 100$.

| | $\alpha$ | Mean | | Med | | IQR | |
|---|---|---|---|---|---|---|---|
| | | RRMSE | RB | RRMSE | RB | RRMSE | RB |
| Gam-Pois | 0.500 | 24.552 | 0.258 | 28.902 | 0.320 | 25.219 | 0.379 |
| Gam-Pois-Alt | 0.500 | 24.442 | 0.259 | | | | |
| GLMM | 0.500 | 24.254 | 0.258 | 28.608 | 0.309 | 25.019 | 0.283 |
| PI | 0.500 | 24.146 | 0.432 | 29.033 | 4.385 | 43.018 | 34.345 |
| Direct | 0.500 | 38.861 | 0.426 | 53.270 | 4.769 | 64.620 | 24.709 |
| Gam-Pois | 1.500 | 20.107 | 0.222 | 22.858 | 0.262 | 24.267 | 0.508 |
| Gam-Pois-Alt | 1.500 | 20.024 | 0.222 | | | | |
| GLMM | 1.500 | 19.834 | 0.222 | 22.621 | 0.248 | 24.071 | 0.275 |
| PI | 1.500 | 19.741 | 0.250 | 23.101 | 3.190 | 35.105 | 23.540 |
| Direct | 1.500 | 51.370 | 0.586 | 65.581 | 4.547 | 96.024 | 24.215 |

# 3 Evaluation of the choice of $T$ in the SIR algorithm

We evaluate the choice of $T$ for the sampling importance resampling algorithm. We simplify the output and only present results for the GLMM model and procedure. Table 7 compares the RRMSE and RB of the alternative predictors. The RRMSE and RB are defined in (22) and (23) of the main document. Table 8 contains the corresponding absolute measures, defined as RMSE and Bias in (S1) and (S2) of Section 1.1 of this supplement. We use "GLMM, T = X" to denote the GLMM predictor with $T = X$. We consider $T = 200$ and $T = 2000$. For all simulations in this section, we set $L = 100$.

The main conclusion from Table 7 and Table 8 is that the effect of increasing $T$ is minimal. For each combination of $\sigma_b^2$ and small area parameter of interest, the RRMSE based on $T = 200$ is nearly the same as the RRMSE based on $T = 2000$. That said, it is counter-intuitive that the RRMSE for $T = 2000$ tends to exceed the RRMSE for $T = 200$. We conjecture that this counter-intuitive result is due to the bias of the IRLS estimators of the fixed parameters. To support the conjecture, we repeated the comparison of $T = 200$

to $T = 2000$ using the true values of the fixed parameters instead of the estimated values. When we use the true parameters, we find that the RRMSE for $T = 200$ tends to exceed the RRMSE for $T = 2000$, as expected. We therefore think that the bias of the estimators of the fixed parameters is the main factor that leads to the slight increase in RRMSE when $T$ increases from 200 to 2000.

In our discussion of the comparison of $T = 200$ to $T = 2000$, we have focused on the relative root mean square error. The conclusions based on the raw root mean square error are the same as the conclusions based on the relative root mean square error. The raw root mean square errors and biases for $T = 200$ are the same as those for $T = 2000$ through the third decimal place. We focus on mean square error instead of bias because the contribution from the bias to the overall mean square error of the predictor is negligible.

In summary, increasing $T$ leads to a slight increase in the RRMSE and RB of the predictor. The evaluation of the choice of $T$ supports the value of $T = 200$, used in the main manuscript.

Table 7: Relative bias and relative root mean square error of the alternative predictors when the GLMM is the true model.

|  | $\sigma_b^2$ | Mean | | Med | | IQR | |
|---|---|---|---|---|---|---|---|
|  |  | RRMSE | RB | RRMSE | RB | RRMSE | RB |
| GLMM, $T = 200$ | 1.500 | 19.946 | 0.448 | 22.763 | 0.479 | 24.578 | 0.639 |
| GLMM, $T = 2000$ | 1.500 | 19.953 | 0.442 | 22.768 | 0.487 | 24.596 | 0.645 |
| GLMM, $T = 200$ | 0.500 | 24.477 | 0.669 | 28.949 | 0.790 | 25.197 | 0.659 |
| GLMM, $T = 2000$ | 0.500 | 24.479 | 0.671 | 28.940 | 0.799 | 25.200 | 0.672 |

Table 8: Absolute bias and root mean square error of the alternative predictors when the GLMM is the true model.

| | $\sigma_b^2$ | Mean | | Med | | IQR | |
| --- | --- | --- | --- | --- | --- | --- | --- |
| | | RMSE | Bias | RMSE | Bias | RMSE | Bias |
| GLMM, $T = 200$ | 1.500 | 1.007 | 0.022 | 0.986 | 0.021 | 0.982 | 0.026 |
| GLMM, $T = 2000$ | 1.500 | 1.008 | 0.022 | 0.986 | 0.021 | 0.983 | 0.026 |
| GLMM, $T = 200$ | 0.500 | 0.746 | 0.020 | 0.743 | 0.020 | 0.708 | 0.018 |
| GLMM, $T = 2000$ | 0.500 | 0.746 | 0.020 | 0.743 | 0.020 | 0.708 | 0.019 |

# 4    Comparison of Computing Times of Alternative Procedures

We compare the computing time of the GLMM procedure to the computing time of the Gam-Pois procedure. When implementing the GLMM procedure we split the calculations for the sampling importance resampling procedure onto three cores that are run in parallel. Table 9 contains the computing times for the four simulation models and the two procedures. The computing time of the Gam-Pois procedure is approximately half the computing time of the GLMM procedure. The comparison of the computing times of the two procedures supports the conclusion that a benefit of the gamma-Poisson model is computational simplicity.

Table 9: Computing times (seconds) of two procedures for four simulation models.

| Model | Gam-Pois | GLMM |
| --- | --- | --- |
| GLMM, $\sigma_b^2 = 0.5$ | 4.28 | 8.35 |
| GLMM, $\sigma_b^2 = 1.5$ | 3.98 | 8.91 |
| Gamma-Poisson, $\alpha = 5$ | 3.83 | 8.58 |
| Gamma-Poisson, $\alpha = 0.5$ | 4.08 | 9.16 |

# 5 Simulations to Evaluate the Quality of the Estimator of the Leading Term in the MSE

We conduct a simulation to evaluate the quality of the estimator of the leading term in the MSE of the Gam-Pois predictor. The estimator of the leading term is an estimator of the mean square error (MSE) of the best predictor, where the best predictor is defined as $\hat{\theta}_i^B = E[\theta_i \mid \boldsymbol{y}_{is}; \alpha, \beta, \boldsymbol{\gamma}]$. We calculate a Monte Carlo (MC) approximation for the best predictor as

$$\hat{\theta}_{i,L}^B = \hat{\theta}_{i,L}(\boldsymbol{y}_{is}, \alpha, \beta, \boldsymbol{\gamma}) = L^{-1} \sum_{\ell=1}^{L} \theta_i^{(\ell)},$$

where $\theta_i^{(\ell)} \sim f(\theta_i \mid \boldsymbol{y}_{is}, \alpha, \beta, \boldsymbol{\gamma})$. The estimator of the leading term, evaluated at the true parameters, is defined as

$$\hat{g}_{1i}(\boldsymbol{y}_{is}, \alpha, \beta, \boldsymbol{\gamma}) = L^{-1} \sum_{\ell=1}^{L} (\theta_i^{(\ell)} - \hat{\theta}_{i,L}^B)^2. \tag{S3}$$

We evaluate the estimator of the leading term for the gamma-Poisson simulation model with $\alpha = 5$. We let $\hat{\theta}_{i,L}^{B,m}$ denote the best predictor of parameter $\theta_i^{(m)}$ calculated in MC simulation $m$, where $m = 1, \ldots, M$, and $M = 500$. We let $\hat{g}_{1i}^{(m)}$ denote the estimator of the leading term (S3) obtained in MC simulation $m$. We define the % relative bias of the estimator of the leading term for area $i$ as

$$RB_i = 100 \left( \frac{M^{-1} \sum_{m=1}^{M} \hat{g}_{1i}^{(m)}}{M^{-1} \sum_{m=1}^{M} (\hat{\theta}_{i,L}^{B,m} - \theta_i^{(m)})^2} - 1 \right).$$

Figure 1 presents boxplots of $RB_i$ for $i = 1, \ldots, 100$ for the three parameters defined as the mean, the median, and the inter-quartile range. The relative biases of the estimators of the leading term are consistently between -20% and 20%.
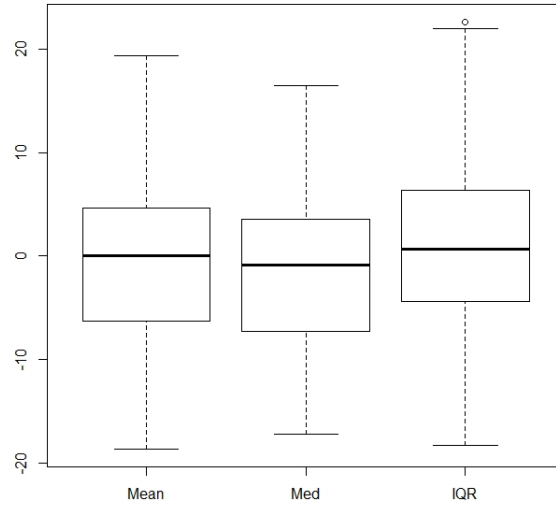
Figure 1: Boxplots of relative bias $(RB_i)$ of estimator of leading term in MSE for mean, median (Med), and inter-quartile range (IQR).

# 6 Results for $L = 100$

Table 10: %RB and %RRMSE of alternative predictors when the true model is the gamma-Poisson model and $L = 100$

|              | $\alpha$ | Mean | | Med. | | IQR | |
|--------------|-------|--------|-------|---------|--------|---------|--------|
|              |       | %RRMSE | %RB   | %RRMSE  | %RB    | %RRMSE  | %RB    |
| Gam-Pois     | 5.000 | 17.544 | 0.192 | 20.538  | 0.231  | 19.559  | 0.232  |
| Gam-Pois-Alt | 5.000 | 17.459 | 0.192 |         |        |         |        |
| GLMM         | 5.000 | 17.621 | 0.268 | 20.606  | 0.378  | 19.614  | 0.248  |
| PI           | 5.000 | 17.541 | 0.475 | 21.058  | 3.324  | 23.018  | 10.639 |
| Direct       | 5.000 | 61.162 | 0.758 | 80.339  | 15.371 | 79.867  | 16.787 |
| Gam-Pois     | 0.500 | 55.560 | 0.641 | 108.581 | 1.200  | 64.293  | 0.775  |
| Gam-Pois-Alt | 0.500 | 55.313 | 0.646 |         |        |         |        |
| GLMM         | 0.500 | 56.687 | 2.774 | 109.498 | 2.934  | 64.926  | 0.938  |
| PI           | 0.500 | 56.311 | 1.122 | 119.499 | 44.742 | 78.893  | 26.475 |
| Direct       | 0.500 | 107.935| 1.206 | 237.138 | 37.307 | 127.339 | 17.519 |

Table 11: %RB and %RRMSE of alternative predictors when the true model is the Poisson-GLMM model and $L = 100$

|  | $\sigma_b^2$ | Mean | | Med. | | IQR | |
|---|---|---|---|---|---|---|---|
|  | | %RRMSE | %RB | %RRMSE | %RB | %RRMSE | %RB |
| Gam-Pois | 0.500 | 24.592 | 0.278 | 28.913 | 0.315 | 25.248 | 0.437 |
| Gam-Pois-Alt | 0.500 | 24.482 | 0.276 | | | | |
| GLMM | 0.500 | 24.307 | 0.491 | 28.633 | 0.567 | 25.060 | 0.362 |
| PI | 0.500 | 24.185 | 0.311 | 29.123 | 4.702 | 43.103 | 34.373 |
| Direct | 0.500 | 38.882 | 0.402 | 53.443 | 4.773 | 64.769 | 24.815 |
| Gam-Pois | 1.500 | 20.129 | 0.243 | 22.884 | 0.289 | 24.332 | 0.495 |
| Gam-Pois-Alt | 1.500 | 20.032 | 0.245 | | | | |
| GLMM | 1.500 | 19.880 | 0.263 | 22.672 | 0.294 | 24.130 | 0.317 |
| PI | 1.500 | 19.779 | 0.309 | 23.167 | 3.107 | 35.245 | 23.626 |
| Direct | 1.500 | 52.235 | 0.598 | 66.716 | 4.448 | 98.095 | 24.274 |