14

Collapsing Bridges: Broughton and Tacoma Narrows

Mathematics introduced:

the wave equation and its solution; the concept of resonance in the context of a forced partial differential equation

14.1 Introduction

We wish to model the oscillations of suspension bridges under forcing. The forcing could come from wind, as in the case of the collapse of the Tacoma Narrows Bridge in 1940, or as a result of a column of soldiers marching in cadence over a bridge, as in the collapse of the Broughton Bridge near Manchester, England, in 1831.

These disasters have often been cited in textbooks on ordinary differential equations as examples of *resonance*, which happens when the frequency of forcing matches the natural frequency of oscillation of the bridge, with no discussion given on how the natural frequency is determined, or even where the ordinary differential equation used to model this phenomenon comes from. The modeling of bridge vibration by a partial differential equation, although still simple-minded, is a big step forward in connecting to reality.

14.2 Marching Soldiers on a Bridge: A Simple Model

In 1831, a column of soldiers marched in cadence over the Broughton Bridge near Manchester. The suspension bridge moved up and down so violently that a pin anchoring the bridge came loose and the bridge collapsed. Whether the bridge's collapse was caused by the synchronized steps of the marching or was simply a result of weight overload is not clear. This incident nevertheless led from that time on to the order to "break steps" when soldiers approach bridges. We study here the possibility that the periodic forcing may lead to resonance.

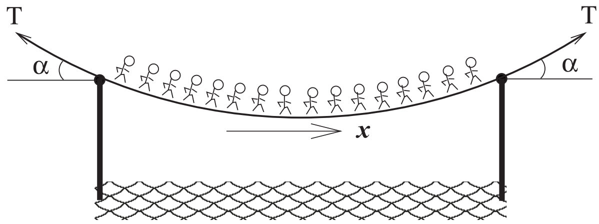

Figure 14.1. A schematic of our simple suspension bridge. (Drawing by E. Hinkle.)

When a column of soldiers marches in unison over a bridge, a vertical force

$$f(x, t)$$

is exerted on the bridge that is periodic in time t, with a period P determined by the time interval between steps (Figure 14.1). In this one-dimensional problem, with x measured along the length of the bridge, we do not distinguish left-foot steps from right-foot steps. In reality, these left–right steps create additional torsional vibrations over the width of the bridge, which, in some cases, may be more important in causing collapse. This aspect of the problem can be handled by introducing another space dimension into the model but will be ignored here.

Specifically, we will model the bridge as an elastic string of length L, suspended at only $x = 0$ and $x = L$. (We know, of course, that bridges do not behave like guitar strings. Nevertheless, this simplification allows us to skip most of the structural mechanics that one needs to know and yet still retain most of the ingredients we need to illustrate the mathematical problem of resonance.) We consider the vertical displacement $u(x, t)$ of the string (i.e., bridge) from its equilibrium position, where x is the distance from the left suspension point, and t is time. We consider a small section of the string between x and $x + \Delta x$. See Figure 14.2.

We apply Newton's law of motion,

$$ma = F$$

(mass times acceleration balancing force), to the vertical motion of this small section of the string. Its mass m is $\rho A \Delta x$, where ρ is the density of the material of the string, and A its cross-sectional area. The acceleration in the vertical direction is

$$a = \frac{\partial^2}{\partial t^2} u.$$

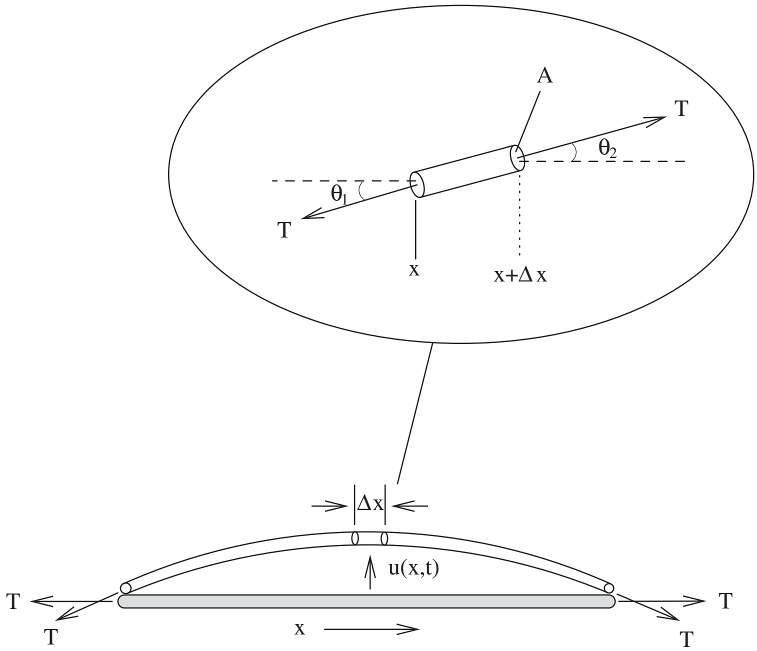

Figure 14.2. A stretched elastic string. (Drawing by E. Hinkle.)

The force should be the vertical component of the tension, plus other forces such as gravity and air friction.

The net vertical component of tension is

$$\begin{aligned} &T\sin\theta_2 - T\sin\theta_1 \\ &\quad \cong T[\theta_2 - \theta_1] \\ &\quad \cong T[u_x(x+\Delta x, t) - u_x(x, t)], \end{aligned}$$

assuming that the angles θ_1 and θ_2 are small. The subscript x denotes partial derivative with respect to x. Putting these all together, we have

$$\rho A \Delta x \frac{\partial^2}{\partial t^2} u = TA[u_x(x+\Delta x, t) - u_x(x, t)] + \rho A \Delta x \cdot f, \tag{14.1}$$

where f represents all additional force per unit mass. Equation (14.1) is

$$\frac{\partial^2}{\partial t^2} u = \frac{T}{\rho}\frac{1}{\Delta x}[u_x(x+\Delta x, t) - u_x(x, t)] + f,$$

which becomes, as $\Delta x \to 0$:

$$\frac{\partial^2}{\partial t^2}u = c^2 \frac{\partial^2}{\partial x^2}u + f, \tag{14.2}$$

where $c^2 \equiv T/\rho$.

The tension along the bridge, T, is assumed to be uniform and is therefore equal to the force per unit area exerted on the suspension point $x = 0$ or $x = L$. Since the weight of the bridge is borne by these two suspension points, the vertical force exerted on each is half the weight of the bridge, and this should be equal to the projection of T in the vertical direction (see Figure 14.1):

$$T \sin \alpha = \frac{1}{2}(\rho L A)g/A = \frac{1}{2}\rho L g,$$

where α is the angle from the horizontal to the tangent of the bridge at the suspension point, $g = 980\,\text{cm/s}^2$, ρ is the density of the bridge material, and A is the cross section of the bridge. Thus

$$\boxed{c^2 \equiv T/\rho = \frac{1}{2}Lg/\sin\alpha.} \tag{14.3}$$

Since the static weight of the bridge is balanced by the tension, the forcing f in (14.2) represents unbalanced vertical acceleration due to the marching soldiers. The system we need to solve is, with $u(x, t)$ being the vertical displacement of the bridge with respect to its equilibrium position:

$$\boxed{\begin{aligned} &\text{PDE:} \quad u_{tt} = c^2 u_{xx} + f(x,t), \quad 0 < x < L,\ t > 0, \\ &\text{BCs:} \quad u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \\ &\text{ICs:} \quad u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < L. \end{aligned}} \tag{14.4}$$

The simplest expression for the periodic force exerted by a column of marching soldiers is probably

$$\begin{aligned} &f(x,t) = a\sin(\omega_D t)\sin(\pi x/L), \quad 0 < x < L, \\ &\omega_D = 2\pi/P. \end{aligned} \tag{14.5}$$

(Actually, this is meant to be the force *anomaly*, i.e., the difference between the force exerted by the marching soldiers and their static weight. This is why (14.5) can take on positive and negative values. The force due to the static weight of the soldiers, if it is a significant fraction of the weight of the bridge, can be incorporated into the weight of the bridge in our earlier calculation of the tension T. Nevertheless, the parameter c^2 in (14.3) should not be affected, amazingly!) Note that the assumed forcing implies that the soldiers march in synch. That is, the anomalous forcing is of the same sign across the span of the bridge.

Solution

There is a general method for solving boundary value problems called the eigenfunction expansion method. However, since knowledge of partial differential equations is not a prerequisite for this course, we shall proceed more intuitively.

The solution $u(x, t)$ is a function of both space and time. Given that the equation (14.4) is linear and contains a forcing term with a known x-structure:

$$X_1(x) \equiv \sin(\pi x/L), \quad 0 < x < L,$$

we shall try a solution of the form

$$u(x, t) = T_1(t)X_1(x). \tag{14.6}$$

Furthermore, (14.6) satisfies the boundary condition of u vanishing at $x = 0$ and $x = L$. We therefore do not need to worry about that boundary condition anymore. Substituting (14.6) into the partial differential equation in (14.4), we obtain

$$T_1''(t)X_1(x) = c^2 T_1(t)X_1''(x) + a\sin(\omega_D t)X_1(x). \tag{14.7}$$

The prime denotes differentiation with respect to the arguments. Thus

$$X_1''(x) = \frac{d^2}{dx^2}X_1(x) = -(\pi/L)^2 X_1(x).$$

Canceling out $X_1(x)$ in (14.7), we are left with

$$\boxed{\frac{d^2}{dt^2}T_1(t) + \omega_1^2 T_1(t) = a\sin(\omega_D t).} \tag{14.8}$$

The "natural frequency" ω_1 of the bridge is given by

$$\omega_1 = c\pi/L. \tag{14.9}$$

We see that the natural frequency would have been different if the forcing structure $X_1(x)$ had a shorter wavelength. Equation (14.8) is the ordinary differential equation for the forced oscillator described in some physics textbooks. Here we have given a physical derivation of how the natural frequency of the oscillator is determined; it is related to the spatial structure of the oscillation (14.6). This piece of information is not available if the bridge is modeled by an ordinary differential equation.

The solution to Eq. (14.8) consists of particular plus homogeneous solutions. The homogeneous solution is

$$A_1 \sin \omega_1 t + B_1 \cos \omega_1 t,$$

while the particular solution can be obtained by trying

$$D \sin(\omega_D t)$$

and finding $D = a/(-\omega_D^2 + \omega_1^2)$ upon substituting into (14.8):

$$\begin{aligned} T_1(t) &= A_1 \sin \omega_1 t + B_1 \cos \omega_1 t + \frac{a \sin(\omega_D t)}{\omega_1^2 - \omega_D^2} \\ &= \frac{T_1'(0)}{\omega_1} \sin \omega_1 t + T_1(0) \cos \omega_1 t + \frac{a}{\omega_1^2 - (\omega_D)^2} \left[\sin \omega_D t - \frac{\omega_D}{\omega_1} \sin \omega_1 t \right]. \end{aligned}$$

Applying the initial conditions from Eq. (14.4), we have $T_1(0) = 0$, $T_1'(0) = 0$. Finally, the solution is

$$u(x, t) = \frac{a}{\omega_1^2 - \omega_D^2} \left[\sin(\omega_D t) - \frac{\omega_D}{\omega_1} \sin \omega_1 t \right] \sin \frac{\pi x}{L}. \tag{14.10}$$

Resonance

The solution (14.10) involves the interference of a forced frequency ω_D with a fundamental frequency ω_1. When the two frequencies get close to each other, the numerator and the denominator of (14.10) both approach zero. Their ratio as $\omega_D \to \omega_1$ is obtained by l'Hôpital's rule to be (see Appendix A.2)

$$u(x,t) = a\left[\frac{-t\cos\omega_1 t}{2\omega_1} + \frac{\sin\omega_1 t}{2\omega_1^2}\right]\sin\left(\frac{\pi x}{L}\right). \tag{14.11}$$

The oscillation grows in amplitude linearly in time, leading, presumably, to the collapse of the bridge.

The fundamental frequency ω_1 of the bridge is given by

$$\omega_1 = c\pi/L = \pi\sqrt{\frac{1}{2}g/(L\sin\alpha)}.$$

Thus the natural period P_1 is given by

$$P_1 \equiv 2\pi/\omega_1 = \sqrt{8L\sin\alpha/g},$$

which is about 1 second for a bridge 10 meters long, if the bridge deck is nearly horizontal, say $\alpha \sim 10^\circ$:

$$P_1 = \sqrt{8L\sin\alpha/g} \sim 1.1 \text{ second}.$$

This is close to the probable forcing period P, and resonance is likely. Note that there is no need for an exact match of the two frequencies to get an enhanced response. Try to convince yourself that the oscillation is magnified when ω_1 is close to ω_D in the solution of Eq. (14.10).

A Different Forcing Function

Unlike ordinary differential equation models of resonance, which assume some *given* natural frequency of the system, the partial differential equation model discussed above determines the resonant frequency by the physical parameters of the bridge (via T/ρ) and by the x-shape of the forcing function $f(x,t)$. In the previous model, it was assumed that

$$f(x,t) = a\sin(\omega_D t)\sin(\pi x/L), \quad 0 < x < L.$$

So the forcing function has the shape as the first fundamental harmonic of the homogeneous system. Consequently, resonance occurs when the forcing frequency ω_D equals the frequency ω_1 of this fundamental mode. If we had instead used

$$f(x,t) = a\sin(\omega_D t)\sin(2\pi x/L), \quad 0 < x < L,$$

for our forcing function, resonance would have occurred when the forcing frequency ω_D equalled the frequency ω_2 of the second fundamental mode. (This is because we would have assumed

$$u(x, t) = T_2(t)X_2(t), \qquad X_2(x) \equiv \sin(2\pi x/L),$$

instead of (14.6) to match the x-structure of the forcing function. As a consequence, the natural frequency in (14.9) would have to be replaced by $\omega_2 = 2c\pi/L$.)

14.3 Tacoma Narrows Bridge

Even though numerous physics and mathematics textbooks attribute the 1940 collapse of the Tacoma Narrows Bridge to "a resonance between the natural frequency of oscillation of the bridge and the frequency of wind-generated vortices that pushed and pulled alternately on the bridge structure" (Halliday and Resnick, 1988), that bridge probably did *not* collapse for this reason (see Billah and Scanlan, 1991). As observed by Professor Burt Farquharson of the University of Washington, the wind speed at the time was 42 mph, giving a frequency of forcing by the vortex shedding mechanism of about 1 Hz. Professor Farquharson also observed that the frequency of the oscillation of the bridge just prior to its destruction was about 0.2 Hz. There was a mismatch of the two frequencies, and consequently this simple resonance mechanism probably was not the cause of the bridge's collapse. The bridge collapsed due to a torsional (twisting) vibration, as can be seen in old films and in Figure 14.3.

During its brief lifetime late in 1940, the bridge, under low-speed winds of 3–4 mph, did experience vertical modes of vibration that can probably be modeled by a model similar to the one presented here. However, the bridge endured this excited vibration *safely*. In fact, the bridge's nickname, "Galloping Gertie," was gained from such vertical motions under low wind, and this phenomenon occurred repeatedly since its opening day. Motorists crossing the bridge sometimes experienced a roller-coaster-like sensation as they watched cars ahead disappear from sight, then reappear. Tourists came from afar to experience it without worrying about their safety. Although the bridge had often "galloped," it had never twisted until November 7, 1940. At higher winds of 25–35 mph, there would be no oscillation of the bridge span. On November 7, 1940, under still heavier winds of 35–40 mph, the motion of the bridge turned into torsional oscillations. One sidewalk was raised 28 ft. above the other sidewalk. This lasted for about half an hour before the center span collapsed.

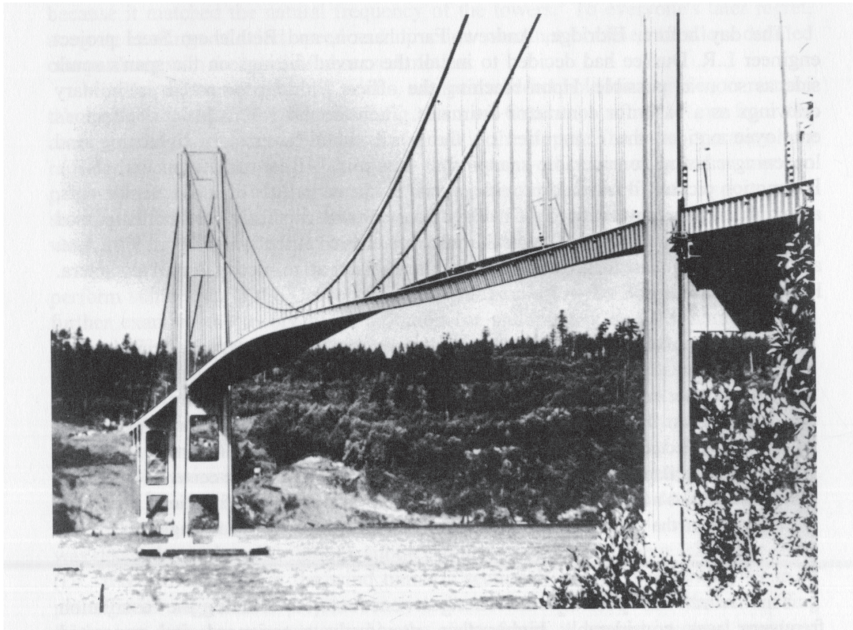

Figure 14.3. Twisting of Tacoma Narrows Bridge just prior to failure.

The above discussion points to the fact that although simple linear theories of forced resonance can perhaps explain the initial excitation of certain modes of oscillation, they cannot always be counted on to explain the final collapse of bridges, which is a very nonlinear phenomenon.

Assignment

Read McKenna (1999), which describes a nonlinear model of torsional oscillations.

14.4 Exercises

1. Consider the problem of a column of soldiers marching across a suspension bridge of length L. The marching is slightly out of step, so the force exerted by the soldiers in the front of the column is opposite that in the rear. A simple model of the forcing term on the bridge is

$$f(x,t) = a\sin(2\pi t/P)\sin(2\pi x/L), \quad 0 < x < L.$$

Solve:

$$\begin{aligned} &\text{PDE:} \quad u_{tt} = c^2 u_{xx} + f(x,t), \quad 0 < x < L, \quad t > 0, \\ &\text{BCs:} \quad u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \\ &\text{ICs:} \quad u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < L. \end{aligned}$$

Discuss the criteria for resonance and sketch the shape of the mode excited.

2. The discussion in the text points to the importance of modeling the forcing function realistically. A better model for $f(x,t)$ than (14.5) is probably

$$f(x,t) = a \sin(2\pi t/P), \quad 0 < x < L,$$

which assumes that the force exerted by the soldiers marching in unison is independent of where they are on the bridge. This seemingly simpler forcing function actually has a richer eigenfunction expansion. Let

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{L}, \quad 0 < x < L,$$

where (no need for you to show this)

$$f_n(t) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \dfrac{4a}{n\pi} \sin(2\pi t/P) & \text{if } n \text{ is odd.} \end{cases}$$

a. Verify that the solution to Eq. (14.4) now becomes

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L},$$

where

$$T_n(t) = 0$$

if n is even, and

$$T_n(t) = \frac{4a/\pi}{\omega_n^2 - \omega_D^2}\left[\sin(\omega_D t) - \frac{\omega_D}{\omega_n}\sin\omega_n t\right]/n$$

if n is odd.

b. There are now chances for resonance whenever

$$P = 2\pi/\omega_n \text{ for some } n.$$

However, because the amplitude of $T_n(t)$ decreases with n, probably only the first two modes will have any real impact. To resonate the first harmonic mode, what must the forcing period P be?

c. The next nonzero fundamental mode is the third one. To resonate with this mode, what must the forcing P be?

d. A column of soldiers running in unison with a third of a second between steps may be able to induce an oscillation in the third mode. Can this mode possibly be near resonance? What would be the x shape of the resulting oscillation of the bridge? How does it compare with the x-shape of the forcing?

3. Let $N(x, t)$ be the mass of red-tide algae per unit mass of sea water. In the absence of transport by wind waves, the algae grow according to

$$\frac{\partial}{\partial t}N = rN,$$

where r is the biological production rate. In the presence of diffusive transport, which we are now considering, the growth is different at different locations. It is now governed by the following system:

$$\text{PDE:} \quad \frac{\partial}{\partial t}N = rN + D\frac{\partial^2}{\partial x^2}N, \quad 0 < x < L,$$

$$\text{BC:} \quad N(0, t) = 0, \quad N(L, t) = 0.$$

The quantity D is the coefficient of diffusion by the random wind waves. The boundary condition is meant to simulate the fact that favorable conditions for algae growth exist only in the strip $0 < x < L$. The algae will be killed quickly if they are transported beyond this strip of ocean.

a. Substitute $N(x,t) = X(x)T(t)$ into the partial differential equation and the boundary condition to find $X(x)$ and $T(t)$. Note that there are n such pairs. The general solution is a superposition of all $X_n(x)T_n(t)$.

b. Show that the algae will become extinct if $L < \pi\sqrt{D/r}$ and that there is an explosive outbreak of red tide otherwise.