10

Marriage and Divorce

Mathematics required:
same as the previous chapter

10.1 Introduction

About two thirds of marriages in the United States end in divorce within a 40-year period, with huge social and economic consequences. I cannot think of any form of legal contract, other than the marital vow, that sane adults would willingly enter into knowing full well such a high potential failure rate. The divorce rate for second marriages is even higher, about 75%; one would have thought that the participants would have become wiser from their first experience! Previous attempts at predicting marital dissolution tended to be based on mismatches in the couple's personality or modes of communication. These have not proved to be too successful.

Professor John Gottman of the University of Washington (Figure 10.1) does ground-breaking research on marriage, divorce, and repair. Gottman's work contradicts the "men are from Mars, women are from Venus" school of relationships, which holds that a lack of understanding of gender differences in communication styles is at the root of marital problems. His prediction of which couples would divorce within a four-year period is 94% accurate. Based on decades of experience interviewing couples, Gottman (1994) identified five types of married couples: Volatiles, Validators, Avoiders, Hostiles, and Hostile-detacheds. The first three produce stable marriages, while the last two produce unstable (high-risk) marriages. A Volatile couple tends to be quite romantic and passionate, but there are also heated arguments and fights. The Validating couple is calmer and intimate; these couples appear to place a high degree of value on shared experience, not on individuality. The Avoider couples avoid the pain of confrontation and conflict. The spouses interact with each other only in the positive range of their emotions. The Validating couple is the classic example of a successful marriage and is well known, but the surprise finding is that two other types of marriages are also stable. Another one of Gottman's unexpected

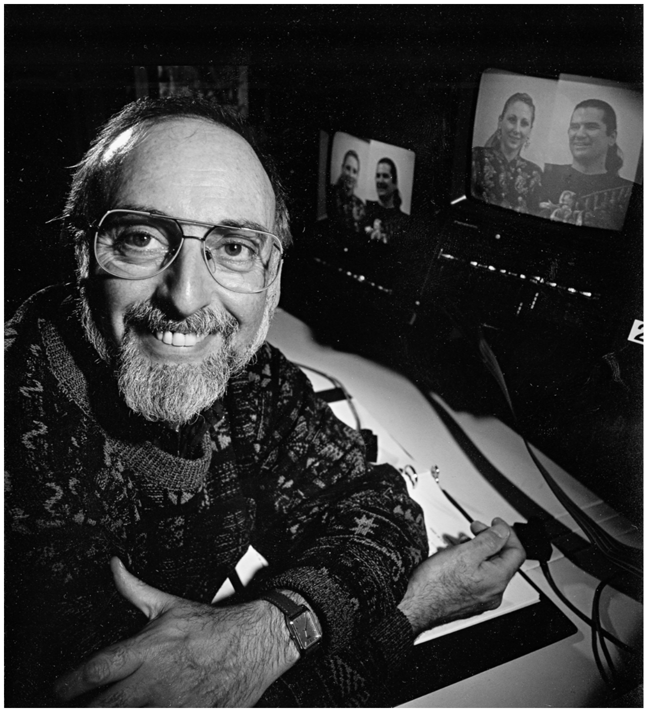

Figure 10.1. Professor John Gottman in his “Love Lab” at the University of Washington. (Courtesy of UW News and Information.)

findings is that anger is not the most destructive emotion in a marriage, since both happy and unhappy couples fight.

The types of marital interaction are deduced from interviews with the couples and subsequent observation (videotaped) and coding. The couple is asked to choose a problem area to discuss in a 15-minute session; details of the exchange are tracked by video cameras, and the emotions of both partners measured by electrodes taped to their foreheads and chests. The problem area could be the frequency of sex, money, in-laws, etc. The videotapes of the interactions as the couple works through the conflict are then coded. There are a number of different coding systems, but the objective is to measure positive versus negative responses in each spouse as he/she speaks in turn. Since all couples, even happily married ones, have some amount of negative interaction, and all couples, even unhappily married ones, have some degree of positive interaction, some averaging or smoothing of the scores is necessary. In the Rapid Couples Interaction Scoring System (RCISS) used by Gottman (1979), the total number of positive RCISS

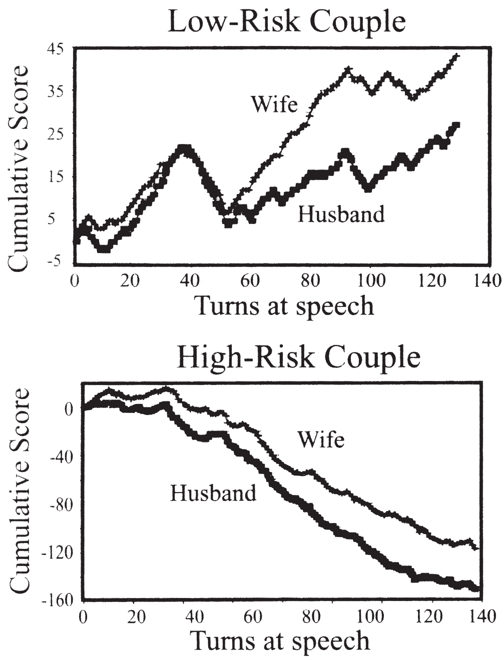

Figure 10.2. The cumulative RCISS scores of husband and wife for a typical low-risk couple (top panel) and for a high-risk couple (lower panel). (From Gottman et al. [2002].)

speaker codes (where the spouse says something positive) minus the total number of negative speaker codes was plotted for each spouse as a function of turns of speech (in essence, time). Two examples are given in Figure 10.2. The average slope of these cumulative scores is then used to classify the couples into high risk or low risk. Since the slope of the cumulated score is the same as the running mean of the score at each time, we shall now refer to (the slope of) the scores as measuring the (running time) average positivity or negativity of the spouse as a function of time.

The five types of couples seem to have different interaction styles. Figure 10.3 summarizes the different influence functions between husband and wife by fitting the empirical data into a two-slope functional form, as we will explain in a moment.

Validators have an influence function that tends to create a positive response in a spouse if the partner's behavior is positive and a negative response if the partner is negative. Volatiles and Conflict-avoiders have influence functions that appear to be one half of the Validators', with Volatiles having the right half of the curve close to the zero slope and the Avoiders having a near zero slope on the left half. Avoiders, by

Figure 10.3. Influence functions. The left column is the theoretical influence function that summarizes the observation. The right two columns are best fits to the data using a two-slope linear function, with the numerical value of the slopes indicated. (Based on data in Cook et al. [1995].)

definition, tend not to interact with each other when they are negative; they influence one another only with positivity. On the other hand, the Volatile spouses influence one another primarily with negativity.

Hostile couples appear to have mixed a Validator husband influence function with an Avoider wife influence function. The Hostile-detached couples mix a Validator husband influence function with a Volatile wife influence function.

With three possible interaction styles for each of the two spouses in a marriage, there could in principle be nine, instead of five, types of marriages. Three of these have *matched* interaction styles (the same style for husband and wife) and six are *mismatched*. Four of the six mismatched marriages, e.g., one that mixes a Volatile style with an Avoider style, were not found in Gottman's data. Perhaps they are just too different for the relationship to survive, even temporarily, through courtship.

Of the five types of marriages observed, the stable marriages all seem to have matched styles of interaction between husband and wife, while the unstable marriages seem to have mismatched styles. Hence it was suggested (Cook et al., 1995) that perhaps the unstable marriages are simply failures to create a stable adaptation to a marriage that is matched, either Volatile, Validating, or Avoiding. For example, a person who is more suited to a Volatile or an Avoiding marriage may have married one who wishes for a Validating marriage, resulting in an unstable marriage.

Since there are many factors that can influence marital stability, it is not clear at this point that influence-style matches or mismatches are the determining factor. Another important factor appears to be an individual's own natural disposition, which measures how one reacts in the absence of spousal interaction. For Volatile couples, both husband and wife tend to have very positive personalities, followed by Validators and Avoiders. Volatile couples have stable marriages perhaps not because their influence styles are "matched" but because they are naturally very positive individuals; their marriage survives *despite* their influence style, which shows that they affect each other negatively. The husband and wife in the unstable marriages tend to have very negative natural dispositions, and *this* may be the reason for their marital problems.

10.2 Mathematical Modeling

We would like to mathematically model the interplay between the two factors affecting marriage and its dissolution: the natural disposition of husband and wife and the style of interaction each has. The aim of the mathematical model is to use the parameters distilled from the taped interviews to predict the long-term behavior of the marriage.

Let $x(t)$ be a measure of the husband's positivity (e.g., happiness) and $y(t)$ be the corresponding measure for the wife. In terms of the RCISS-coded scores, we are referring to its running mean and not the cumulative scores.

Self-Interaction

In the absence of marital interaction, e.g., when each was single, each spouse tends to his/her own "uninfluenced steady state": x_0 and y_0. This process is modeled by

$$\frac{dx}{dt} = r_1(x_0 - x),$$

$$\frac{dy}{dt} = r_2(y_0 - y).$$

The equilibrium (steady state) solution is

$$x^* = x_0, \quad y^* = y_0,$$

and perturbations from this equilibrium return to the equilibrium at an exponential decay rate of r_1 for the husband and r_2 for the wife. This aspect of the model can be seen by solving the above uncoupled ordinary differential equations in terms of

$$u = x - x_0, \quad v = y - y_0:$$

$$\frac{du}{dt} = -r_1 u, \quad \frac{dv}{dt} = -r_2 v,$$

giving

$$u(t) = u(0)e^{-r_1 t},$$

$$v(t) = v(0)e^{-r_2 t}.$$

The parameters for the uninfluenced states, x_0, y_0, r_1, and r_2, are determined from the subset of data points of the couple when the score of one of the spouses is zero. This happens in about 15% of the data (Cook et al., 1995). Presumably we can also determine these parameters if each spouse is interviewed alone.

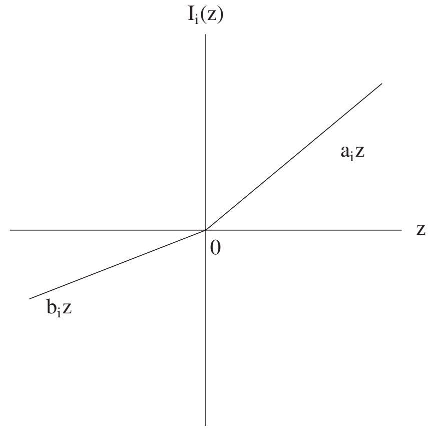

Figure 10.4. The theoretical influence function.

Marital Interactions

Cook et al. (1995) identified three major styles of marital interactions for husband, x, and for wife, y. We attribute the type of style of the influence function to the influencer. Here $I_2(x)$ is the influence exerted on the wife by the husband's emotions, while $I_1(y)$ is the influence of the wife on the husband (see Figure 10.4).

For all possible styles, the governing mathematical model is

$$\frac{dx}{dt} = r_1(x_0 - x) + I_1(y), \tag{10.1}$$

$$\frac{dy}{dt} = r_2(y_0 - y) + I_2(x), \tag{10.2}$$

where

$$I_i(z) = \begin{cases} a_i z & \text{if } z > 0, \\ b_i z & \text{if } z < 0. \end{cases}$$

Here i can be either 1 or 2, and z stands for either x or y.

In a *validating* style of interaction, one spouse influences the other across both the negative and positive ranges of emotions, with $a \cong b$; i.e., the slopes in the interaction function are approximately the same (see Figure 10.3). In a *conflict avoiding* style of interaction, the spouse who adopts this style avoids interacting with the other spouse through the negative range of his/her emotions, with $b/a \ll 1$. The spouse who

TABLE 10.1
Uninfluenced steady states and rates of relaxation for each spouse in five types of marriages, based on data from Cook et al. (1995)

	Husband's steady state			*Wife's steady state*		
Group	r_1	x_0	x^*	r_2	y_0	y^*
Low-risk couples						
Volatile	0.67	.68	.75	0.80	.68	.61
Validating	0.63	.38	.56	0.86	.52	.59
Avoiding	0.82	.26	.53	0.75	.46	.60
Average	0.71	.44	.61	0.80	.55	.60
High-risk couples						
Hostile	0.68	.10	.03	.51	−.64	−.45
Hostile-detached	0.60	−.42	−.50	.54	−.24	−.62
Average	0.64	−.16	−.24	.53	−.44	−.54

has a *volatile* style interacts with the other mainly in the negative range of his/her emotion, with $a/b \ll 1$.

10.3 Data

The parameters that enter into our model were determined empirically using codings of the interviews. They are converted from the discrete form used by Cook et al. (1995) into the form useful for our continuous model and listed in Table 10.1. x_0 and x^*, y_0 and y^* are in units of the mean RCISS scores, while r_1 and r_2 are in units of the RCISS scores per unit time (turns of speech).

It is seen that the uninfluenced steady states (i.e., x_0, y_0) appear to separate the high-risk couples from the low-risk couples. At least one of the spouses is highly negative in the high-risk couples. The wife is highly negative in the Hostile couple, while the husband is highly negative in the case of the Hostile-detached couple. On the other hand, the low-risk couples all have positive dispositions to various degrees. Both husband and wife in a Volatile couple have very positive natural dispositions, followed by the Validator couple, which is followed by the Avoider couple. It appears that the Volatile marriage is highly "regulated" by the couple's uninfluenced state, and the marriage is successful *despite* their negative marital interaction style. The spouses of a Validator couple are somewhat less positive than the Volatiles and they make up for it by interacting with each other positively as well as negatively. The Avoiders are less secure because they are even less positive than the Validators. They keep their marriages intact by avoiding conflict.

10.4 An Example of a Validating Couple

Before we proceed to solve the general case, let us consider the classic example of a *Validating* couple. It is known that their marriage is "regulated" (low risk). We want to know why, mathematically.

The couple has a *matched* style of interaction, with $a_i \sim b_i$. We shall in fact take $a_i = b_i$ for simplicity in this example. Thus

$$\frac{dx}{dt} = r_1(x_0 - x) + a_1 y, \tag{10.3}$$

$$\frac{dy}{dt} = r_2(y_0 - y) + a_2 x. \tag{10.4}$$

The husband is influenced by the wife's positivity (or negativity) with the rate of change due to this influence being a_1. Similarly, the wife is influenced by the husband's emotions, with a_2 being the constant of proportionality. Curiously, the set (10.3) and (10.4), governing the marital interactions, is of exactly the same form as the set of equations governing the *arms race* between two warring states (see exercise 1 of chapter 9)!

The long-term equilibrium state for a couple is found by setting the right-hand sides of (10.3) and (10.4) to zero. The equilibrium solution is denoted by (x^*, y^*):

$$r_1(x_0 - x^*) + a_1 y^* = 0, \quad r_2(y_0 - y^*) + a_2 x^* = 0.$$

The solution is easily found to be

$$\begin{aligned} x^* &= \left[x_0 + \frac{a_1}{r_1} y_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right], \\ y^* &= \left[y_0 + \frac{a_2}{r_2} x_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right]. \end{aligned} \tag{10.5}$$

This single equilibrium turns out to be a *stable node* if

$$0 < (a_1 a_2)/(r_1 r_2) < 1.$$

Otherwise it is an *unstable saddle* (see chapter 9).

Generally, $r_1/a_1 > 1$, and $r_2/a_2 > 1$, since one responds to one's own feelings more quickly than to the spouse's feelings. Consequently, we are generally dealing with the stable case,

$$a_i/r_i < 1, \quad b_i/r_i < 1, \tag{10.6}$$

of a single equilibrium. (Incidentally, it is easy to understand why the case with $r_1/a_1 < 1$ and $r_2/a_2 < 1$ is unstable. The two spouses feed on each other's emotions without much of a moderating mechanism: the wife being sad makes the husband sadder, which in turn makes the wife even sadder, and so on. A similar chain of events happens when a spouse is happy. We will not consider this case in what follows. This unstable case, however, is applicable to *arms race* modeling.)

The long-term effects of marital interaction on the positivity (i.e., happiness) of each spouse in a Validating marriage are twofold: additive and magnifying. If both husband and wife are naturally positive people, i.e., $x_0 > 0, y_0 > 0$, their happiness is enhanced by the positivity of their spouse. This influence is additive, i.e.,

$$\text{Husband: Single:} \quad x_0 \to \text{ married: } \quad x_0 + \frac{a_1}{r_1} y_0,$$

$$\text{Wife: Single:} \quad y_0 \to \text{ married: } \quad y_0 + \frac{a_2}{r_2} x_0.$$

The sum, $x_0 + \frac{a_1}{r_1} y_0$, is also magnified by the factor $1/[1 - \frac{a_1 a_2}{r_1 r_2}] > 1$ in marriage. The result is that in marriage:

$$x^* = \left[x_0 + \frac{a_1}{r_1} y_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right] \gg x_0,$$

$$y^* = \left[y_0 + \frac{a_2}{r_2} x_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right] \gg y_0.$$

That is, the spouses are both much happier in marriage than if they were single. The Validating couple is said to have a *regulated* (low-risk) marriage. On the other hand, if the husband and wife both have negative "uninfluenced" steady state, i.e., $x_0 < 0, y_0 < 0$, the Validating marriage would make them more unhappy, i.e., $x^* < x_0$, $y^* < y_0$. This marriage is probably *unregulated* (high risk).

If x_0 and y_0 are of opposite sign, the marriage can still be successful under certain conditions. For example, if the husband is negative, i.e., $x_0 < 0$, but the wife is very positive, through marital interactions the husband's steady state, x^*, can be made positive if $x_0 + \frac{a_1}{r_1} y_0 > 0$. The wife's steady state, y^*, can still be positive if $y_0 + \frac{a_2}{r_2} x_0 > 0$. The marriage is probably successful.

Exact Solution

Since (10.3) and (10.4) are linear, they can in fact be solved exactly. The exact solution, with c_1 and c_2 being arbitrary constants determinable from initial conditions, is

$$x(t) = x^* + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

$$y(t) = y^* + c_1 \left(\frac{r_1}{a_1} + \frac{\lambda_1}{a_1}\right) e^{\lambda_1 t} + c_2 \left(\frac{r_1}{a_1} + \frac{\lambda_2}{a_1}\right) e^{\lambda_2 t},$$

where

$$\lambda_{1,2} = \frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}, \quad p \equiv -(r_1 + r_2), \;\; q \equiv r_1 r_2 - a_1 a_2.$$

The solution is stable if $\lambda_1 < 0$, $\lambda_2 < 0$, and unstable if at least one of them is positive. Since p is negative, $\lambda_{1,2}$ are both negative if $q > 0$. That is the same as the stability criterion (10.6) mentioned previously. For the case of $q > 0$, the solution will eventually tend to x^*, y^*, the equilibrium solution given by (10.5).

10.5 Why Avoiding Conflicts Is an Effective Strategy in Marriage

When the spouses in a marriage do not have as positive a natural disposition as the Validating couple, adopting a conflict-avoiding style of interaction may be effective in maintaining a successful marriage. This is a surprising finding since traditionally psychologists have thought that sweeping marital problems under the rug, where they fester, may be detrimental to the health of a marriage. We model an Avoider couple by the following model:

$$\frac{dx}{dt} = r_1(x_0 - x) + a_1 I(y),$$

$$\frac{dy}{dt} = r_2(y_0 - y) + a_2 I(x),$$

where $I(y)$ is defined as $I(y) = y$ if $y > 0$, $I(y) = 0$ if $y < 0$, and similarly for $I(x)$.

To find the equilibrium solution, we set the right-hand sides of the above two equations to zero, i.e.,

$$r_1(x_0 - x^*) + a_1 I(y^*) = 0,$$

$$r_2(y_0 - y^*) + a_2 I(x^*) = 0.$$

If $x^* > 0$, $y^* > 0$, then $I(x^*) = x^*$, $I(y^*) = y^*$, and so the equilibrium solution is the same as that for the Validating couple:

$$x^* = \left[x_0 + \frac{a_1}{r_1}y_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right] > 0,$$

$$y^* = \left[y_0 + \frac{a_2}{r_2}x_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right] > 0.$$

On the other hand, if $x^* < 0$, $y^* < 0$, then $I(x^*) = 0$, $I(y^*) = 0$, and we have

$$x^* = x_0 < 0, \quad y^* = y_0 < 0.$$

It can be seen that if the couple are individually negative (i.e., $x_0 < 0$, $y_0 < 0$), they are no less unhappy in marriage ($x^* = x_0$, $y^* = y_0$). However, if they are naturally positive (i.e., $x_0 > 0$, $y_0 > 0$), they are much happier in marriage ($x^* > x_0 > 0$, $y^* > y_0 > 0$). Thus a conflict-avoiding style is a "safer" style of marital interaction for the couples whose natural dispositions are not as high as those of Validating couples.

The conflict-avoiding marriage is also a successful one even if one spouse is negative, provided that the other spouse is more positive than the negative spouse is negative. By avoiding interaction when one is negative, the couple feeds on the positive spouse's positivity and is not affected by the negative spouse's negativity. The steady state is the same as that for the Validating couple if at least one spouse is positive (if $x_0 + \frac{a_1}{r_1}y_0 > 0$, $y_0 + \frac{a_2}{r_2}x_0 > 0$).

10.6 Terminology

This is probably a good time to define what we mean by a "successful" or "failed" marriage, and by a "stable" or "unstable" marriage. The words "stable" and "unstable" have different meanings to a mathematician and to a behavioral psychologist. We say a marriage is successful ("regulated") if the couple is happier in marriage than if single. We may want to relax this condition a little, to say that each spouse is at least as happy in marriage as they would be if single. Thus, the condition for a successful marriage is:

$$x^* \gtrsim x_0, \quad y^* \gtrsim y_0.$$

As we have found, the equilibrium (x^*, y^*) is stable. The solution will inevitably tend to this equilibrium, unless actions not described by our model intervene. If the marriage is tending to a steady state with $x^* < 0$, $y^* < 0$ (both husband and wife unhappy), one or both spouses may wake up one day and make a decision that he/she does not want this marriage to continue this way any longer. This action of dissolving the marriage is not contained in our model. We have no reason to expect that this action will manifest itself as an "instability" of our solution. Instead, we simply predict, based on the fact that the long-term state to which the marriage is tending is so negative, that the probability is high that one or both of the spouses will take the action to dissolve the marriage.

10.7 General Equilibrium Solutions

We now solve (10.1) and (10.2) for the equilibrium solution obtained by setting ($\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$). This equilibrium is given by

$$\boxed{\begin{aligned} r_1(x_0 - x^*) + I_1(y^*) = 0, \\ r_2(y_0 - y^*) + I_2(x^*) = 0. \end{aligned}} \tag{10.7}$$

There is only a single equilibrium solution (x^*, y^*) of (10.7). When (10.6) is satisfied, this equilibrium is stable. This stable equilibrium can be found in any one of the four quadrants depending on the parameters, x_0, y_0, r_1, r_2, and a_i and b_i. The successful marriages are generally found in the first quadrant, where $x^* > 0$, $y^* > 0$; failed marriages are generally located in the third quadrant, where $x^* < 0$, $y^* < 0$.

1. *First quadrant, successful marriages*
   For $x^* > 0$, $y^* > 0$, (10.7) becomes

$$x^* = x_0 + \frac{1}{r_1} I_1(y^*) = x_0 + \frac{1}{r_1} a_1 y^*,$$

$$y^* = y_0 + \frac{1}{r_2} I_2(x^*) = y_0 + \frac{1}{r_2} a_2 x^*.$$

   So, solving them simultaneously yields

$$\boxed{\begin{aligned} x^* &= \left[x_0 + \frac{a_1}{r_1} y_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right] > 0, \\ y^* &= \left[y_0 + \frac{a_2}{r_2} x_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right] > 0, \end{aligned}} \tag{10.8}$$

In order for x^*, y^* to both be positive, we need

$$x_0 + \frac{a_1}{r_1} y_0 > 0, \quad y_0 + \frac{a_2}{r_2} x_0 > 0 \tag{10.9}$$

(assuming that (10.6) is satisfied).

For the case where (10.9) is true, we have

$$x^* \gg x_0 > 0, \quad y^* \gg y_0 > 0.$$

Both the additive and magnifying effects of marital interaction are present. The marriage is a successful one.

One surprising conclusion from the above result is that the marriage is successful *no matter what interaction style each spouse adopts*, as long as each's natural disposition is positive ($x_0 > 0$, $y_0 > 0$). This is consistent with the data in Table 10.1.

2. *Third quadrant, unsuccessful marriages*
   The equilibrium solution is

$$\boxed{\begin{aligned} x^* &= \left[x_0 + \frac{b_1}{r_1} y_0\right] \Big/ \left[1 - \frac{b_1 b_2}{r_1 r_2}\right] < 0, \\ y^* &= \left[y_0 + \frac{b_2}{r_2} x_0\right] \Big/ \left[1 - \frac{b_1 b_2}{r_1 r_2}\right] < 0, \end{aligned}} \tag{10.10}$$

provided that

$$x_0 + \frac{b_1}{r_1} y_0 < 0, \quad y_0 + \frac{b_2}{r_2} x_0 < 0. \tag{10.11}$$

One conclusion is that if $x_0 < 0$, $y_0 < 0$, the marriage is doomed to failure no matter what interaction style each spouse adopts. Any marital interaction will lead to a more negative x^*, y^*. The Hostile marriage is characterized by the wife's excessive natural negativity ($y_0 < 0$). She tries to avoid conflict by interacting only in the positive range of her emotions, but she does not have very much of that. The marriage is doomed to failure regardless of the husband's interaction style. In the Hostile-detached couples, the marriage fails because the husband is so naturally negative ($x_0 < 0$). The matter is made worse by the failure of both spouses to avoid conflict.

3. *Second quadrant*

$$x^* < 0, \quad y^* > 0.$$

The equilibrium solution is

$$\boxed{\begin{aligned} x^* &= \left[x_0 + \frac{a_1}{r_1}y_0\right] \Big/ \left[1 - \frac{a_1 b_2}{r_1 r_2}\right] < 0, \\ y^* &= \left[y_0 + \frac{b_2}{r_2}x_0\right] \Big/ \left[1 - \frac{a_1 b_2}{r_1 r_2}\right] > 0, \end{aligned}} \tag{10.12}$$

provided that

$$x_0 + \frac{a_1}{r_1}y_0 < 0, \quad y_0 + \frac{b_2}{r_2}x_0 > 0.$$

Even though the husband's steady state is negative, we would still classify the marriage successful if

$$x^* > x_0, \quad y^* \cong y_0 > 0.$$

For a naturally negative husband ($x_0 \leq 0$) and a positive wife ($y_0 > 0$) we always have $x^* > x_0$ regardless of the style of interaction the wife adopts. However, in order that the wife not be too influenced by the husband's negativity, he should adopt a conflict-avoider style of interaction ($b_2 \cong 0$).

If, on the other hand, the wife is naturally negative ($y_0 < 0$) and the husband positive ($x_0 > 0$), the marriage can still be successful if a_1/r_1 is small but b_2/r_2 large. In that case, we would have

$$x^* \cong x_0 > 0, \quad y^* > y_0.$$

4. *Fourth quadrant*

$$x^* > 0, \quad y^* < 0.$$

This case is similar to case (3), except with the roles of the husband and wife reversed. That is,

$$x^* = \left[x_0 + \frac{b_1}{r_1}y_0\right] \Big/ \left[1 - \frac{b_1 a_2}{r_1 r_2}\right] > 0,$$

$$y^* = \left[y_0 + \frac{a_2}{r_2}x_0\right] \Big/ \left[1 - \frac{b_1 a_2}{r_1 r_2}\right] < 0,$$

provided that $x_0 + \frac{b_1}{r_1}y_0 > 0$ and $y_0 + \frac{a_2}{r_2}x_0 < 0$.

10.8 Conclusion

From taped interviews with couples, we attempt to deduce the parameters that we need for our mathematical model, such as the style of interaction of each spouse (a_i, b_i), their uninfluenced steady state (x_0, y_0), and each's inertia to change $(1 - r_i)$. Since it is not feasible to observe the couple for long periods of time, we rely on the mathematical model to predict where the couple will tend if the marriage is allowed to evolve with these parameters. If the long-term solution of the model is such that the husband and wife are tending to the negative quadrant $(x^* < 0, y^* < 0)$, we then predict that the current marriage is heading for dissolution if the parameters are not changed. The mathematical stability of the equilibrium is not predictive of marital stability since all equilibria in this model are stable.

The future direction of research focuses on marriage repair and, related to it, an understanding of why a couple develops a particular interaction style in marriage. These issues may need to be addressed using the mathematics of *game theory*.

10.9 Assignment

To predict the long-term outcome of a particular marriage, we need to first observe the couple and determine empirically the parameters that go into our mathematical model. For this assignment, we have five couples whose parameters have been determined. They happen to be the same as those listed in Table 10.1 and Figure 10.3 (which were actually for the mean of five types of couples).

Let $x(t)$ be the husband's RCISS score and $y(t)$ be the wife's RCISS score. RCISS scores are a measure of a spouse's positivity in marriage. The equations governing $x(t)$ and $y(t)$ are

$$\frac{dx}{dt} = r_1(x_0 - x) + I_1(y),$$

$$\frac{dy}{dt} = r_2(y_0 - y) + I_2(x),$$

where

$$I_1(z) = \begin{cases} a_i z & \text{if } z > 0, \\ b_i z & \text{if } z < 0. \end{cases}$$

The slopes a_i and b_i for various couples are given in Figure 10.3. The parameters r_1, r_2, x_0, and y_0 are given in Table 10.1.

Determine the long-term values of $x(t)$ and $y(t)$ (i.e., x^* and y^*) for each of the five types of couples: Volatile, Validating, Avoider, Hostile, and Hostile-detached. (First solve x^*, y^* in algebraic form; then put in the numbers to get numerical values for each case. Do not use the values of x^*, y^* in Table 10.1.)

Solution

Find the equilibrium x^*, y^* from

$$0 = r_1(x_0 - x^*) + I_1(y^*),$$
$$0 = r_2(y_0 - y^*) + I_2(x^*).$$

From the "general equilibrium solution" section of this chapter, we have the following.

I. If $x_0 + \frac{a_1}{r_1}y_0 > 0$, $y_0 + \frac{a_2}{r_2}x_0 > 0$, then

$$x^* = \left[x_0 + \frac{a_1}{r_1}y_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right],$$

$$y^* = \left[y_0 + \frac{a_2}{r_2}x_0\right] \Big/ \left[1 - \frac{a_1 a_2}{r_1 r_2}\right].$$

II. If $x_0 + \frac{a_1}{r_1}y_0 < 0$, $y_0 + \frac{b_2}{r_2}x_0 > 0$, then

$$x^* = \left[x_0 + \frac{a_1}{r_1}y_0\right] \Big/ \left[1 - \frac{a_1 b_2}{r_1 r_2}\right],$$

$$y^* = \left[y_0 + \frac{b_2}{r_2}x_0\right] \Big/ \left[1 - \frac{a_1 b_2}{r_1 r_2}\right].$$

III. If $x_0 + \frac{b_1}{r_1}y_0 < 0$, $y_0 + \frac{b_2}{r_2}x_0 < 0$, then

$$x^* = \left[x_0 + \frac{b_1}{r_1}y_0\right] \Big/ \left[1 - \frac{b_1 b_2}{r_1 r_2}\right],$$

$$y^* = \left[y_0 + \frac{b_2}{r_2}x_0\right] \Big/ \left[1 - \frac{b_1 b_2}{r_1 r_2}\right].$$

IV. If $x_0 + \frac{b_1}{r_1} y_0 > 0$, $y_0 + \frac{a_2}{r_2} x_0 < 0$, then

$$x^* = \left[x_0 + \frac{b_1}{r_1} y_0\right] \Big/ \left[1 - \frac{b_1 a_2}{r_1 r_2}\right],$$

$$y^* = \left[y_0 + \frac{a_2}{r_2} x_0\right] \Big/ \left[1 - \frac{b_1 a_2}{r_1 r_2}\right].$$

All equilibria are stable.

From the data given, we have the following information about the various types of couples.

Volatile couple:

$$x_0 = 0.68, \quad y_0 = 0.68,$$
$$r_1 = 0.67, \quad r_2 = 0.80,$$
$$a_1 = -0.02, \quad a_2 = 0.10,$$

$$x_0 + \frac{a_1}{r_1} y_0 = 0.68 + \frac{(-0.02)}{0.67} \cdot 0.68 = 0.66 > 0,$$
$$y_0 + \frac{a_2}{r_2} x_0 = 0.68 + \frac{0.10}{0.80} \cdot 0.68 = 0.77 > 0.$$

This belongs to case (I) and so

$$x^* = 0.68/1.00 = 0.68,$$
$$y^* = 0.77/1.00 = 0.77.$$

This is a successful marriage.

Validating couple:

$$x_0 = 0.38, \quad y_0 = 0.52,$$
$$r_1 = 0.63, \quad r_2 = 0.86,$$
$$a_1 = 0.15, \quad a_2 = 0.21,$$

$$x_0 + \frac{a_1}{r_1} x_0 = 0.38 + \frac{0.15}{0.63} \cdot 0.52 = 0.50 > 0,$$
$$y_0 + \frac{a_2}{r_2} x_0 = 0.52 + \frac{0.21}{0.86} \cdot 0.38 = 0.61 > 0.$$

This belongs to case (I). $[1 - \frac{a_1 a_2}{r_1 r_2}] = 0.94$.

$$x^* = 0.50/0.94 = 0.53, \quad y^* = 0.61/0.94 = 0.65.$$

This is a successful marriage.

Avoiding couple:

$$x_0 = 0.26, \quad y_0 = 0.46,$$
$$r_1 = 0.82, \quad r_2 = 0.75,$$
$$a_1 = 0.15, \quad a_2 = 0.30,$$

$$x_0 + \frac{a_1}{r_1} y_0 = 0.26 + \frac{0.15}{0.81} \cdot 0.46 = 0.34 > 0,$$
$$y_0 + \frac{a_2}{r_2} x_0 = 0.46 + \frac{0.30}{0.75} \cdot 0.26 = 0.56 > 0.$$

This belongs to case (I). $[1 - \frac{a_1 a_2}{r_1 r_2}] = [1 - \frac{0.15 \times 0.30}{0.82 \times 0.75}] = 0.93$.

$$x^* = 0.34/0.93 = 0.37, \quad y^* = 0.56/0.93 = 0.60.$$

Hostile couple:

$$x_0 = 0.10, \quad y_0 = -0.64,$$
$$r_1 = 0.68, \quad r_2 = 0.51,$$
$$a_1 = 0.15, \quad a_2 = 0.15,$$
$$b_1 = -0.01, \quad b_2 = 0.14,$$

$$x_0 + \frac{b_1}{r_1} y_0 = 0.10 + \frac{(-0.01)}{0.68}(-0.64) = 0.11 > 0,$$
$$y_0 + \frac{a_2}{r_2} x_0 = -0.64 + \frac{0.15}{0.51} \cdot 0.10 = -0.61 < 0.$$

This belongs to case (IV).

$$x^* = \left[x_0 + \frac{b_1}{r_1} y_0\right] \Big/ \left[1 - \frac{b_1 a_2}{r_1 r_2}\right] = 0.11/1.14 = 0.1,$$

$$y^* = \left[y_0 + \frac{a_2}{r_2} x_0\right] \Big/ \left[1 - \frac{b_1 a_2}{r_1 r_2}\right] = -0.61/1.14 = -0.54.$$

This is a high-risk couple, but they are no more unhappy in marriage than when they are each alone.

Hostile-detached:

$$x_0 = -0.42, \quad y_0 = -0.24,$$
$$r_1 = 0.60, \quad r_2 = 0.54,$$
$$b_1 = 0.25, \quad b_2 = 0.17,$$

$$x_0 + \frac{b_1}{r_1} y_0 = -0.42 + \frac{0.25}{0.60} \cdot (-0.24) = -0.52 < 0,$$
$$y_0 + \frac{b_2}{r_2} x_0 = -0.24 + \frac{0.17}{0.54} \cdot (-0.42) = -0.37 < 0.$$

This belongs to case (III).

$$x^* = \left[x_0 + \frac{b_1}{r_1} y_0\right] \Big/ \left[1 - \frac{b_1 b_2}{r_1 r_2}\right]$$
$$= -0.52/0.87 = -0.60,$$
$$y^* = \left[y_0 + \frac{b_2}{r_2} x_0\right] \Big/ \left[1 - \frac{b_1 b_2}{r_1 r_2}\right]$$
$$= -0.37/0.87 = -0.43.$$

This is a high-risk couple; both husband and wife are very unhappy in marriage, more unhappy than when they are alone.

10.10 Exercises

1. Let $x(t)$ and $y(t)$ be measures of happiness for the husband and wife, respectively. Negative values indicate unhappiness. Let x_0 and y_0 be the "natural disposition" of the husband and wife, respectively. This is how happy they would be if they were single. During marriage, the couple develops a style of interaction that is called "validating." A model of their marriage dynamics is:

$$\frac{dx}{dt} = r_1(x_0 - x) + a_1 y,$$

$$\frac{dy}{dt} = r_2(y_0 - y) + a_2 x,$$

where a_1 measures how easily the husband is influenced by the wife's emotions, and a_2 is the corresponding quantity for the wife:

$$0 < a_1/r_1 < 1, \quad 0 < a_2/r_2 < 1.$$

a. Find out where this marriage is heading.

b. A marriage is termed "regulated" and is low risk if the long-term happiness of each spouse is enhanced by the marital interaction. Otherwise it is called "unregulated" (high risk). Give reasons for your answers to the following questions:
   i. Is the marriage "regulated" if each of the spouses is naturally happy ($x_0 > 0, y_0 > 0$)?
   ii. What if $x_0 < 0, y_0 < 0$?

2. The equations governing the marital interaction for a Validating couple, Eqs. (10.3) and (10.4), are exactly the same as those governing Richardson's model for the arms race between two warring states (exercise 1 of chapter 9). Discuss why in one case the married couple lives happily ever after and in the other case the arms race escalates into war. Use reasonable parameter values in your discussion.

3. A Hostile-detached couple has a husband who is naturally very negative ($x_0 \ll 0$) and a wife who is also negative ($y_0 < 0$). The husband interacts with a validating style while the wife interacts with a volatile style:

$$\frac{dx}{dt} = r_1(x_0 - x) + a_1 J(y),$$

$$\frac{dy}{dt} = r_2(y_0 - y) + a_2 x,$$

where $x(t)$ denotes the husband's happiness and $y(t)$ the wife's happiness. $J(y) = 0$ if $y > 0$ but $J(y) = y$ if $y < 0$. $0 < a_1/r_1 < 1, 0 < a_2/r_2 < 1$.

a. Show that if $x_0 < 0$, $y_0 < 0$, the couple is even more unhappy in marriage (at equilibrium).

b. Show that their style of interaction can lead to a successful marriage if they are naturally positive (i.e., $x_0 > 0$, $y_0 > 0$). Compare your solution for this case to that of a Validating couple worked out in class. If the solutions are the same, explain why they don't differ. If the solutions differ, also explain why.