11

Chaos in Deterministic Continuous Systems, Poincaré and Lorenz

Mathematics introduced:

system of three coupled ordinary differential equations; linear and nonlinear stability; periodic and aperiodic solutions; deterministic chaos; sensitivity to initial conditions

11.1 Introduction

Unlike quantum mechanics, the world of Newtonian mechanics is described by deterministic equations: given a precise initial condition of a particle's position and velocity, these equations predict the precise trajectory of that particle for all later times. This does not mean, however, that a small difference in the initial condition will not lead to wildly different later trajectories.

These "chaotic," unpredictable behaviors were accidentally discovered by Henri Poincaré and Edward Lorenz in different contexts.

11.2 Henri Poincaré

For two hundred years after Isaac Newton (1642–1727) discovered his law of gravitation and law of motion (mass times acceleration is equal to force), it was not recognized that planetary motion governed by these two deterministic laws may yield "unpredictable" trajectories. The two-body problem (e.g., a planet revolving around a sun) was solved by Newton analytically. It yields an elliptic orbit for the planet, as observed by Johannes Kepler (1571–1630). The three-body problem, involving, e.g., a sun, a planet, and a moon, is much more difficult. We now know that this problem cannot be solved exactly (in closed form).

In 1885, to celebrate the 60th birthday of Oscar II, King of Sweden and Norway, a contest was proposed by the mathematician Magnus Gösta Mittag-Liffler (1846–1927). The problems, four of them, were proposed by his teacher Karl Weierstrass at the University of Berlin. One of the problems was on celestial mechanics and the stability

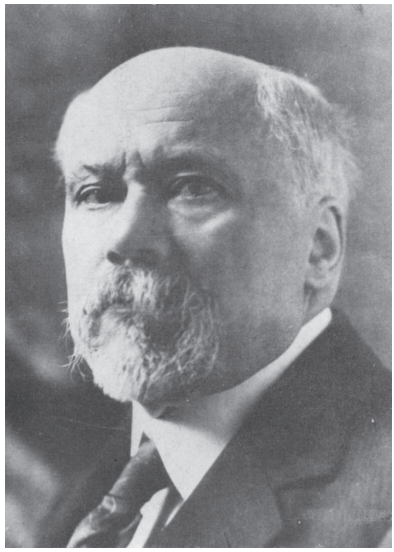

Figure 11.1. Henri Poincaré (1854–1912).

of the solar system:

> Given a system of arbitrarily many mass points that attract each other according to Newton's laws, assuming that no two points ever collide, give the coordinates of the individual points for all time as the sum of a uniformly convergent series whose terms are made up of known functions.

The contest attracted an entry from a brilliant young professor, Henri Poincaré (1854–1912) of the University of Paris (Figure 11.1).

Poincaré was probably the last universalist in mathematics, equally at home in pure and applied mathematics. His philosophy towards mathematics, however, is more akin to that of an applied mathematician, in that he believed in the important role of intuition in mathematics rather than in treating mathematics as deducible from logic. Poincaré had poineered in his doctoral thesis the geometric view for understanding differential equations qualitatively using phase spaces and trajectories, which is still being used to analyze systems of ordinary equations (called dynamical systems). He was anxious to try out his novel approach to celestial mechanics and submitted a lengthy entry on the three-body problem. Although it did not provide a complete solution to the question posed, it was deemed by Weierstrass as "nevertheless of such importance that its publication will inaugurate a new era in the history of celestial mechanics." Poincaré won the gold medal prize.

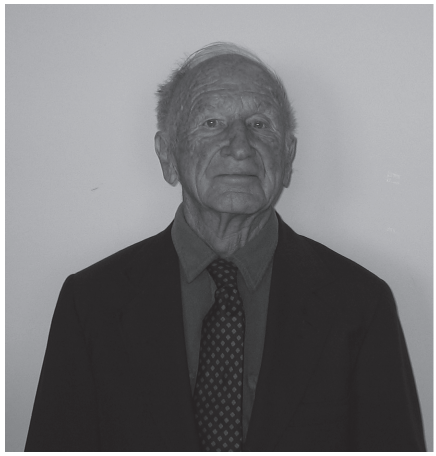

Figure 11.2. Professor Edward Lorenz (1917–2008). (Courtesy of E. N. Lorenz.)

While his paper was being readied for publication in *Acta Mathematica*, Poincaré discovered that the result on the stability of the three-body problem was in error. The press was stopped and copies already distributed were retrieved and destroyed, while Poincaré frantically worked on a massive revision of his submitted work. He eventually came to accept the possibility that the deterministic world of Newtonian mechanics admits the possibility of unpredictable solutions even for a system as simple as three gravitating bodies. In one geometric case he had not considered previously, Poincaré now found orbits "so tangled that I cannot even begin to draw them." Moreover, he discovered that "It may happen that small differences in the initial positions may lead to enormous differences in the final phenomena. Prediction becomes impossible." We now recognize this as the hallmark of a chaotic dynamical system, although the quoted sentence by Poincaré was a philosophical remark. Poincaré's mathematical discovery was largely neglected outside the celestial mechanics area until Edward Lorenz demonstrated the same ultrasensitivity to initial conditions in a simpler set of three ordinary differential equations.

11.3 Edward Lorenz

Edward Lorenz (Figure 11.2), a professor of meteorology at MIT, published his seminal paper on chaos in 1963 in the *Journal of Atmospheric Sciences*. That paper's impact on mathematics extends well beyond its original focus on weather prediction.

Lorenz discovered this "chaotic" behavior in 1959 (before the mathematical use of the word chaos was coined) when he was numerically integrating a truncated model of weather. Lorenz has the habit of doing all the programming and analyzing rows of numerical output personally, unlike most other professors then and now, who delegate such "routine" chores to postdocs and graduate students. He actually had one desk-size "personal" computer, a Royal McBee, installed in this office for his own use. On this one occasion he wanted to examine more closely a particular segment of the printed output of a previous run. He thought he could restart the run in the middle by typing into the computer the model output as the initial condition. To his surprise he found that the restarted run diverged from the original run. He traced this discrepancy to the fact that while the computer had six digits of accuracy, the printed output had only three. That discrepancy in the last few digits of the solution led to major differences in the future behavior of the solution. Later Lorenz used this sensitivity to initial data as the basis for his argument that weather cannot be predicted in detail for more than a couple of weeks in advance. This is because, among other inaccuracies in our knowledge, we cannot measure wind and temperature initial conditions to very high degrees of accuracy. Lorenz found the same behavior later in a simplified three-component system.

Chaotic solutions are actually quite rare in differential equations. Their existence requires at least three degrees of freedom (three coupled first-order ordinary differential equations or a single second-order ordinary differential equation with an imposed forcing frequency) and nonlinearity. The simplest such model is the three-component truncated model of the Rayleigh–Bénard convection of Lorenz (1963). Even in such models, chaotic solutions exist under a very restricted set of model parameters. Such models are not solvable analytically. This may account for why chaotic solutions remained hidden for so long in the deterministic solutions to Newton's equation.

In 1990, Lorenz was invited to give a series of three Danz lectures at the University of Washington. (I had the distinct honor of introducing him to an audience of 500 in Kane Hall.) The lecture notes were later, in 1993, published in book form under the title *The Essence of Chaos*. In it Lorenz described how he arrived at the set of parameter values that give rise to chaotic solutions. Unlike Poincaré, who was not looking for chaos in realistic three-body systems such as the sun, earth, and moon and who had to reluctantly admit that a chaotic solution is possible under some unusual arrangement of parameters, Lorenz was at the time actively looking for model parameters that could produce fluctuations like those of temperature or pressure in real weather, which are neither steady nor periodic. He was using his model equations not to reproduce real weather but to produce a time series (a sequence of numbers) to test some statistical schemes of weather prediction. If his time series converged to a fixed point (an equilibrium) it would not be of any use to him because predicting the subsequent evolution of his "weather" would be unrealistically easy. Lorenz would change the constants in his equations to get more interesting "data." A periodic oscillation also was not satisfactory. It took many adjustments to the parameters for the time series to look interesting to Lorenz, i.e., aperiodic. Whether the parameters were realistic was not a concern to Lorenz.

11.4 The Lorenz Equations

As mentioned above, for (continuous) first-order ordinary differential equations, chaos normally does not exist in the solution unless there are at least three such equations nonlinearly coupled together. In the late 1950s, Professor Edward Lorenz was looking for a simple system of equations whose solution may possess aperiodic behavior. He was interested in having such a simple set to generate artificial data to test some statistical methods used in numerical weather prediction. A colleague showed Lorenz a seventh-order system (a system involving seven first-order ordinary differential equations) that was a spectrally truncated model for thermal convection in a fluid confined between two flat plates, the lower plate being maintained at a higher temperature than the upper one. The numerical solution of that seven-component system appeared to possess aperiodic behavior, which Lorenz called "deterministic nonperiodic flow." Since four of the variables eventually approach zero, Lorenz set these to zero and arrived at the following simpler three-component system, the now famous *Lorenz equations*:

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \sigma(-x+y), \\ \frac{dy}{dt} &= rx - y - xz, \\ \frac{dz}{dt} &= -bz + xy. \end{aligned}} \tag{11.1}$$

The variable $x(t)$ measures the rotational speed of the convection cell, with its sign indicating clockwise or anticlockwise rotation. The horizontal temperature variation is projected onto a single sinusoidal spatial

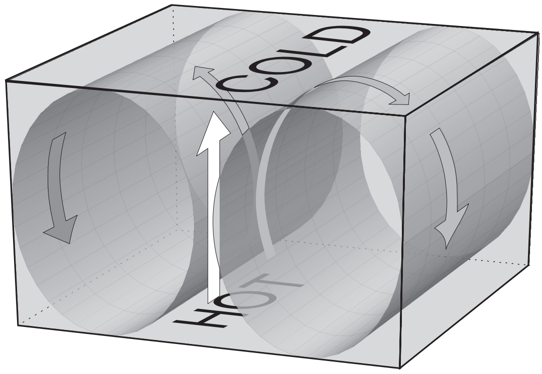

Figure 11.3. Two-dimensional convection cell. (Drawing by Wm. Dickerson.)

pattern, with $y(t)$ measuring the difference in the temperature between the rising branch and the sinking branch of the convection cell. See Figure 11.3. In the absence of convection, the static heat conduction state has $x(t) \equiv 0$ and $y(t) \equiv 0$, with the vertical temperature being a linear function joining the temperature on the bottom plate to that on the upper plate. The variable $z(t)$ measures the deviation of the temperature from the linear conduction vertical profile, and $z(t) \equiv 0$ when there is no convection. The parameters in the equations are all positive and real. The Prandtl number σ describes the property of the fluid and the parameter b describes the geometry. The Rayleigh number r measures the specified temperature difference between the lower and upper plates. It is normalized so that $r = 1$ corresponds to the critical value for convection to first occur.

By heating the lower plate, r increases, and after exceeding a certain critical value ($r = 1$), convection of the fluid is needed to transport the heat from the lower plate to the upper plate ($x \not\equiv 0$). Before that, heat is simply conducted by a static fluid ($x \equiv 0$).

These results can be inferred from an examination of the equilibrium points of the Lorenz equations. Setting the right-hand side of (11.1) to zero, we have

$$\sigma(x^* - y^*) = 0,$$

$$rx^* - y^* - x^*z^* = 0,$$

$$-bz^* + x^*y^* = 0.$$

From the first equation we have $y^* = x^*$. Substituting it into the second and third equations, we get

$$x^*(r - 1 - z^*) = 0,$$
$$-bz^* + x^{*2} = 0.$$

Thus either $x^* = 0$ or $z^* = r - 1$. If the former, then $y^* = x^* = 0$ and $bz^* = x^{*2} = 0$, resulting in one equilibrium:

$$P_1 = (0, 0, 0).$$

If the latter, then $x^* = \pm\sqrt{bz^*} = \pm\sqrt{b(r-1)} = y^*$, giving two more equilibria:

$$P_2 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r - 1) \text{ and}$$
$$P_3 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r - 1).$$

Since we are only interested in real equilibrium points, there is only one such equilibrium when $r < 1$:

$$P_1 = (0, 0, 0),$$

and this equilibrium is stable (see later). This is the static conduction state. When $r > 1$, P_1 becomes unstable and convection begins. It evolves into one or the other stable equilibrium pattern, P_2 or P_3. These eventually also become unstable for $r > r_c$. This is discussed further below.

To find the stability of an equilibrium (x^*, y^*, z^*), we perturb around it by assuming

$$(x(t), y(t), z(t)) = (x^* + u(t), y^* + v(t), z^* + w(t)),$$

with

$$\mathbf{u}(t) \equiv (u(t), v(t), w(t))$$

being the small perturbation.

The Lorenz system (11.1) becomes, upon dropping the products of u, v, w, which are small,

$$\frac{d}{dt}\mathbf{u}(t) = A\mathbf{u}(t), \tag{11.2}$$

where the constant matrix A is given by

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z^* & -1 & -x^* \\ y^* & x^* & -b \end{bmatrix}.$$

This system of linear ordinary differential equations with constant coefficients possesses solutions of the form

$$\mathbf{u}(t) = \mathbf{u}(0)e^{\lambda t}.$$

Substituting into Eq. (11.2) yields

$$\lambda\mathbf{u}(0) = A\mathbf{u}(0),$$

which is

$$(A - \lambda I)\mathbf{u}(0) = 0,$$

where I is a 3×3 unit matrix. The above algebraic system has the trivial solution

$$\mathbf{u}(t) = \mathbf{u}(0) \equiv 0.$$

Nontrivial solutions are possible only if

$$\det(\mathbf{A} - \lambda I) = 0.$$

That is,

$$\det \begin{bmatrix} -\sigma - \lambda & \sigma & 0 \\ r - z^* & -1 - \lambda & -x^* \\ y^* & x^* & -b - \lambda \end{bmatrix} = 0. \tag{11.3}$$

$(\mathbf{x^*}, \mathbf{y^*}, \mathbf{z^*}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$
Equation (11.3) can be written as

$$(\lambda + b)(\lambda + \sigma)(\lambda + 1) - (\lambda + b)\sigma r = 0.$$

Factoring out $(\lambda + b)$:

$$(\lambda + b)[(\lambda + \sigma)(\lambda + 1) - \sigma r] = 0.$$

The three *eigenvalues* are

$$\lambda = \lambda_1 = -b,$$

$$\lambda = \lambda_2 = \frac{1}{2}[-(\sigma + 1) + \sqrt{(\sigma + 1)^2 - 4\sigma(1 - r)}],$$

$$\lambda = \lambda_3 = \frac{1}{2}[-(\sigma + 1) - \sqrt{(\sigma + 1)^2 - 4\sigma(1 - r)}].$$

The last two are the roots of the quadratic equation inside the square brackets. For $r < 1$, both λ_2 and λ_3 are real and negative. Since λ_1 is real and negative, the equilibrium point $P_1 = (0, 0, 0)$ is stable. For $r > 1$, λ_2 becomes positive, while λ_1 and λ_3 remain negative. *Therefore P_1 becomes unstable for $r > 1$.*

$(x^*, y^*, z^*) = P_2$ or P_3

Equation (11.3) becomes the cubic equation, with no obvious factors:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{11.4}$$

where

$$a_2 \equiv \sigma + b + 1,$$

$$a_1 \equiv b + \sigma(b + 1 - r + z^*) + x^{*2},$$

$$a_0 \equiv \sigma[b + x^{*2} + x^* y^* - b(r - z^*)].$$

They are, for either P_2 or P_3:

$$a_2 = \sigma + b + 1,$$

$$a_1 = b(r + \sigma),$$

$$a_0 = 2\sigma b(r - 1).$$

Presumably there exists a critical value of r, which we call r_c, below which the equilibrium (P_2 or P_3) is stable but above which it is unstable.

That is,

$$\operatorname{Re}\lambda < 0 \quad \text{for } 1 < r < r_c,$$

$$\operatorname{Re}\lambda > 0 \quad \text{for } r > r_c.$$

The boundary between stability and instability is given by

$$\operatorname{Re}\lambda = 0 \quad \text{for } r = r_c.$$

To find r_c, we write

$$\lambda = i\omega$$

and assume ω to be real. Equation (11.4) becomes

$$-i\omega^3 - a_2\omega^2 + ia_1\omega + a_0 = 0.$$

Separating out the real and imaginary parts of the above equation, we get two equations:

$$-a_2\omega^2 + a_0 = 0,$$

$$-\omega^3 + a_1\omega = 0.$$

The first yields the condition

$$\omega^2 = a_0/a_2.$$

When substituted into the second, the latter becomes

$$\omega(a_0/a_2 - a_1) = 0.$$

Since ω cannot be zero (because it won't satisfy the first equation), we must have

$$a_1a_2 = a_0 \quad \text{when } r = r_c.$$

This is a surprisingly simple result for a cubic equation:

> The product of the coefficients of the λ^2 and λ terms is equal to the constant term at the boundary where $\operatorname{Re}\lambda$ is about to become positive.

Thus

$$b(r_c + \sigma)(\sigma + b + 1) = 2\sigma b(r_c - 1),$$

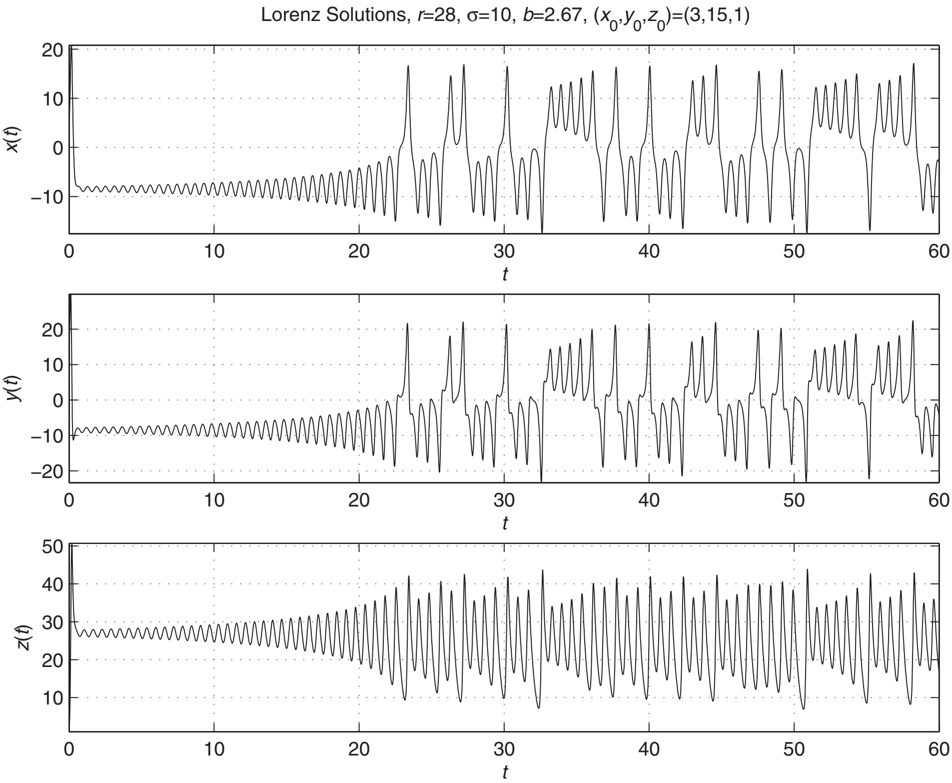

Figure 11.4. MATLAB solution of $x(t)$, $y(t)$, and $z(t)$ as a function of t for parameter values indicated.

which yields

$$\boxed{r_c = \sigma \left(\frac{\sigma + b + 3}{\sigma - b - 1} \right).} \tag{11.5}$$

One set of parameter values used by Lorenz was $\sigma = 10$ and $b = 8/3$. Then $r_c = 470/19 \cong 24.737$.

The Lorenz system is linearly unstable, for $r > r_c$, for small perturbations around each of its three equilibria. A nonlinear analysis of the Lorenz system is much more difficult. Lorenz showed numerically that for large r, the solution becomes chaotic. The value of r he used was $r = 28$. The numerical code (in MATLAB) is provided in Appendix B. Figure 11.4 shows each of $x(t)$, $y(t)$, and $z(t)$ as a function of t for these parameter values. The solution is chaotic.

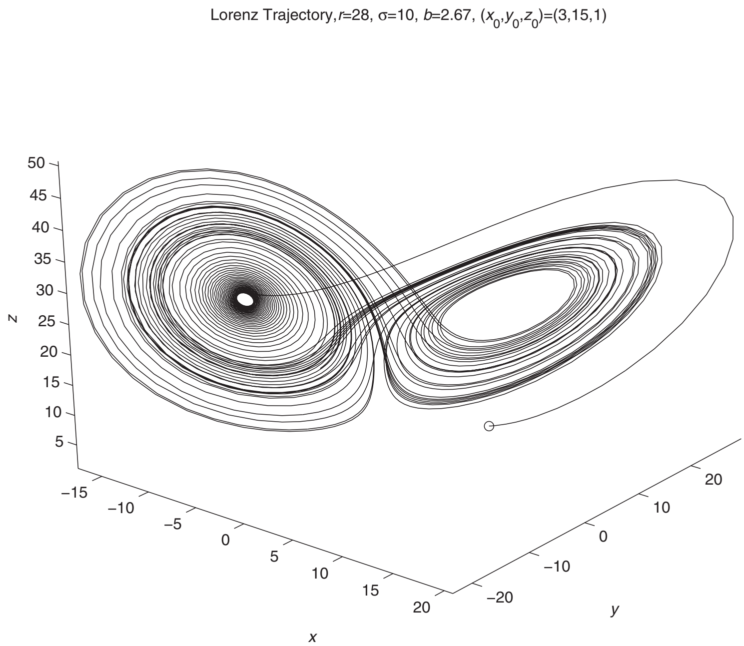

Figure 11.5. A three-dimensional plot of the trajectory $x(t)$, $y(t)$, $z(t)$ is shown. The initial point is indicated by a circle. (Drawing by Wm. Dickerson.)

A three-dimensional plot of the trajectory $(x(t), y(t), z(t))$ is shown in Figure 11.5. The famous Lorenz "butterfly" shape is seen.

The trajectories do not cross themselves in the x-y-z space, although they appear to in two-dimensional cross sections. They tend to reside on two bent surfaces (the wings of a "butterfly"), and therefore one says that their "volume" is zero (since a surface has zero volume). The trajectories appear to be attracted to the two "attractors" near P_2 and P_3. A trajectory may wind around P_2 a few times and then be repelled to the neighborhood of P_3, then wind around it a few more times before being sent back to the neighborhood of P_2, and so on. The attractors are called "strange attractors."

It can be shown that there is no limit cycle (periodic solutions) for $r > r_c$ in the Lorenz system (see exercise 5). Its solution is aperiodic (chaotic) for any value of r greater than r_c. The other possibility, that for $r > r_c$ the linearly unstable perturbations grow to infinity, can be ruled out by showing that the solution to the Lorenz system has to be finite (Lorenz, 1963). (See exercise 6.)

The nonlinear behavior of the solution for $r < r_c$ is more complicated. Although there exist two linearly stable equilibria, P_2 and P_3

(called *attracting fixed points*), there also exists *transient chaos* at finite perturbations away from these points for $r \gtrsim 13.926$. A solution not close enough to P_2, say, will be repelled towards the other fixed point, P_3, and, if it does not get close enough to P_3, gets repelled back to P_2. This would happen several times until it approaches one of the fixed points close enough to be attracted into it. On the other hand, it is easy to show (see exercise 8) that $(x^*, y^*, z^*) = (0, 0, 0)$ is nonlinearly stable ("globally attracting") for $r < 1$.

11.5 Comments on Lorenz Equations as a Model of Convection

You probably have noticed that we did not derive the Lorenz equations as a model of a physical system. This is for good reason.

Lorenz's three equations are *not* a good model of the Rayleigh–Bénard convection, although it is based on a set of two partial differential equations describing the fluid convection in two dimensions, with heating below and cooling above (Saltzman, 1962). See Figure 11.3.

Saltzman expressed the two unknowns, the streamfunction and the temperature, in the form of a double Fourier series in the two spatial coordinates, with the coefficients of the series a function of t only. These coefficients satisfy an infinite dynamical system. Lorenz (1963) took this infinite system and truncated it to only three components, $x(t)$, $y(t)$, and $z(t)$, in effect retaining only a single Fourier harmonic in each direction. Since Lord Rayleigh in 1916 found that for r slightly supercritical, i.e., slightly larger than 1, the onset of convection rolls takes the form of a single harmonic in each of the two dimensions, Lorenz felt that his three equations "may give realistic results when the Rayleigh number is slightly supercritical" but cautioned that "their solution cannot be expected to resemble those of [the original partial differential equations] when strong convection occurs, in view of the extreme truncation." Note that interest in the Lorenz system is usually in the parameter regime that gives rise to chaos, namely at very large supercritical r, when the truncated system likely fails!

When r becomes large, the other terms dropped by Lorenz—those representing smaller spatial scales and their interactions with the large scales—become important and cannot be ignored. Curry et al. (1984) looked into the problem of what happens when more and more of the harmonics of the original infinite set are retained. Chaos *disappears* when a sufficient number of terms are kept! In fact, the original partial differential equations describing convection in two dimensions do not possess chaotic behavior.

However, Lorenz's original intention was not to model convection but to look for a set of simple equations that possess the properties he

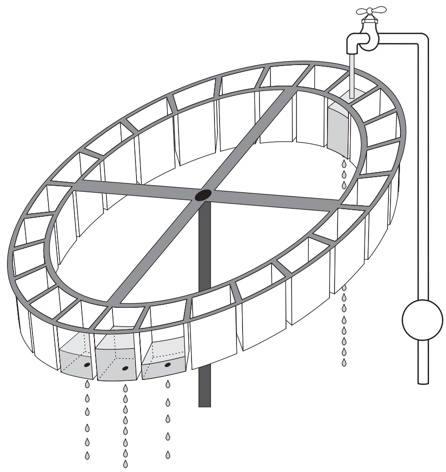

Figure 11.6. Chaotic waterwheel. (Drawing by Wm. Dickerson.)

saw in more complicated systems: aperiodicity and extreme sensitivity to initial conditions. We now know many simple *physical* systems that have the mathematical properties Lorenz sought. An example is the driven pendulum (see exercises 2 and 3 at the end of this chapter).

11.6 Chaotic Waterwheel

Another example is the leaky waterwheel of Malkus and Howard, which is discussed in Strogatz (1994). See Figure 11.6. In a "reverse" modeling effort, applied mathematicians Willem Malkus and Lou Howard at MIT constructed a mechanical contraption that possesses the chaotic properties of the Lorenz system. It is a wheel tilted from the horizontal. On its rim are attached plastic cups hanging vertically, each with a small hole of equal size at its bottom. A faucet is located at the topmost point of the wheel, dripping water into the cup there. Think of the water input rate as our parameter r. For small input rates compared to the rate at which water leaks out of the cup being filled, nothing happens. The wheel remains stationary. This corresponds to the trivial solution P_1. When the fill rate is higher than the leak rate ($r > 1$), the top cup fills up and, being top heavy, the wheel turns, either to the right or to the left, initiating either a clockwise rotation (P_2) or an anticlockwise rotation (P_3). Either rotation is stable for fixed r until r increases beyond a certain point when some of the cups get too full and are unable to make it back up to the top. The wheel slows down and may reverse direction. The wheel may spin in one direction a few times and then change direction erratically.

11.7 Exercises

1. Damped pendulum

A simple pendulum, with a point mass m at one end of a rigid weightless rod of fixed length L that is hinged at the other end, satisfies the following equation (Newton's second law: mass times acceleration = force):

$$m\frac{d^2}{dt^2}(L\theta) = -mg\sin\theta - m\gamma\frac{d}{dt}(L\theta).$$

Here θ is the angle the rod makes with the vertical, $L\theta$ is the displacement (arc length) from the vertical, $mg\sin\theta$ is the projection of gravitational force in the direction of motion (the angular direction), and $m\gamma\frac{d}{dt}(L\theta)$ is the air resistance, which is proportional to the velocity $\frac{d}{dt}(L\theta)$. γ is a positive constant of proportionality. This system does not possess chaotic behavior because it is a dynamical system of only second degree.

a. Let $x = \theta$, $y = \frac{d}{dt}\theta$. Write the above equation in the form of a dynamical system of the form

$$\frac{dx}{dt} = f(x, y),$$

$$\frac{dy}{dt} = g(x, y).$$

Denote g/L by ω^2.

b. Find the equilibria of the system in (a).

c. Determine the linear stability of the equilibria found in (b).

2. Forced, damped pendulum

Adding a forcing term to the equation in exercise 3, we have the following equation for the case of a forced damped pendulum:

$$\frac{d^2}{dt^2}\theta + \omega^2\sin\theta + \gamma\frac{d}{dt}\theta = D\cos(\omega_D t),$$

where $\omega = \sqrt{g/L}$, and ω_D is the (specified) frequency of the forcing.

Write this equation in the form of a dynamical system:

$$\frac{dx}{dt} = f(x, y, z),$$

$$\frac{dy}{dt} = g(x, y, z),$$

$$\frac{dz}{dt} = h(x, y, z),$$

thus showing that you need a system of three equations. (Define z appropriately. Note that f, g, h do not contain t explicitly.)

3. Use a MATLAB program to plot $y \equiv \frac{d}{dt}\theta$, the angular velocity, as a function of nondimensional time ωt. Vary D and γ until you get chaotic-looking behavior.

4. Let $\mathbf{F} = (f, g, h)$, where f, g, h are functions of x, y, and z. In addition, they are the right-hand sides of the dynamical system:

$$\frac{dx}{dt} = f(x, y, z), \frac{dy}{dt} = g(x, y, z), \frac{dz}{dt} = h(x, y, z).$$

a. Find $\nabla \cdot \mathbf{F} \equiv \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g + \frac{\partial}{\partial z} h$ for the Lorenz system. Show that it is a negative constant.

b. Let D_0 be the region in the x-y-z space where the trajectories of $x(t)$, $y(t)$, and $z(t)$ reside at $t = 0$ and let $D(t)$ be the region at time t. Let $V(t)$ be the "volume" of this region $D(t)$:

$$V(t) = \int_{D(t)} dx\, dy\, dz.$$

We wish to find the time rate of change of this "moving volume." Show that in general we have (provided that $\nabla \cdot \mathbf{F}$ exists) (*hint*: Liouville's theorem):

$$\frac{d}{dt} V(t) = \int_{D(t)} \nabla \cdot \mathbf{F}\, dx\, dy\, dz.$$

c. For the Lorenz system, show that

$$V(t) = e^{-(\sigma+b+1)t} V(0).$$

d. What can you say about the boundedness of solutions to the Lorenz system? Does $V(t) \to 0$ as $t \to \infty$ necessarily mean that the trajectories will tend to the origin $(x, y, z) = (0, 0, 0)$?

5. Show that there are no periodic solutions of the Lorenz equations. (*Hint*: Suppose there are periodic (or quasi-periodic) solutions, whose trajectories $(x(t), y(t), z(t))$ reside on the surface of a volume $D(t)$. Then the volume should not shrink as time increases. Contrast this with the result from exercise 6.)

6. Show that for $r < 1$, the equilibrium $(x^*, y^*, z^*) = (0, 0, 0)$ is globally (nonlinearly) stable for the Lorenz system. That is, any $(x(t), y(t), z(t))$ would eventually approach $(0, 0, 0)$ as $t \to \infty$.

Consider the "volume"

$$V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2.$$

a. Show that, using the Lorenz equations,

$$\begin{aligned}\frac{dV}{dt} &= \frac{2}{\sigma}x\frac{d}{dt}x + 2y\frac{d}{dt}y + 2z\frac{d}{dt}z \\ &= -2\left[x - \frac{r+1}{2}y\right]^2 - 2\left[1 - \left(\frac{r+1}{2}\right)^2\right]y^2 - 2bz^2.\end{aligned}$$

b. Show that $\frac{d}{dt}V$ is strictly negative unless one reaches $(x, y, z) = (0, 0, 0)$. Thus argue that the point $(0, 0, 0)$ is the final destination of all trajectories (x, y, z) for $r < 1$.