5

Modeling in the Physical Sciences, Kepler, Newton, and Calculus

Mathematics required:
calculus, vectors, Cartesian and polar coordinates

5.1 Introduction

Nowadays classical mechanics is a deductive science: we know the equations of motion and we use them to *deduce* results. We no longer "model" the orbits of the planets, for example. We *model* biological, psychological, ecological, and social problems because in these fields the governing equations are not yet known.

Astronomy, in particular the prediction of planetary orbits, was an empirical exercise in Johannes Kepler's time (1571–1630), just as biology is to us in modern times. Kepler did not know the law of gravitation. Calculus had not even been invented. Both had to await Isaac Newton (1642–1727). Newton's law of universal gravitation has been regarded as one of the greatest achievements of human thought. But, as Newton freely admitted, he was standing on the shoulders of those before him. If we take *you* back in time, to Kepler's time, but equip you with the mathematics of calculus, can you *model* the empirical data Kepler had in his possession at the time and beat Newton to arrive at the theory of gravitation? This is modeling at its best: to come up with an equation to explain the observational data, to generalize to other situations, to make new predictions, and to verify the new theory with additional experiments.

Johannes Kepler (Figure 5.1) was born near Stuttgart, Germany, and raised mostly by his mother. An infection from smallpox at age 4 left his eyesight much impaired. He was educated in theology and mathematics at the Lutheran seminary at the University of Türbingen, and in 1594 accepted a lectureship at the University of Graz in Austria. In 1600 he became the assistant to the famous Danish-Swedish astronomer Tycho Brahe, the court astronomer to Kaiser Rudolph II at Prague. Prior to this time, Kepler's work had been more metaphysical—his first treatise was entitled *The Cosmic Mystery*. Among other things Greek, Kepler

Figure 5.1. Johannes Kepler (1571–1630).

thought the Golden Ratio was a fundamental tool of God in creating the universe.

Johannes Kepler was a mathematician rather than an observer; he viewed his study of astronomy as fulfilling his Christian duty to understand God's creation, the universe. Kepler, however, had the great fortune to have inherited, upon Brahe's sudden death in 1601, both his master's post as the imperial mathematician and his large collection of accurate data on the motion of Mars. Based on these data and after toiling for two decades, Kepler formed the following three laws on the orbits of planets:

I. The planets move about the sun in elliptical orbits with the sun at one focus.
II. The areas swept over by the radius vector drawn from the sun to a planet in equal times are equal.
III. The squares of the times of describing the orbits (periods) are proportional to the cubes of the major axes.

These three laws (Figures 5.2 and 5.3) represent a distillation of volumes of empirical data compiled by Brahe, which was the most time-consuming part of the whole modeling process. Kepler had been driven and sustained by the conviction that: "By the study of the orbit of Mars, we must either arrive at the secrets of astronomy or forever remain in ignorance of them." Kepler had first tried the circle, believed by the Greeks to be the perfect orbit that heavenly bodies must follow, before settling on the ellipse. It is a tribute to the accuracy of Brahe's

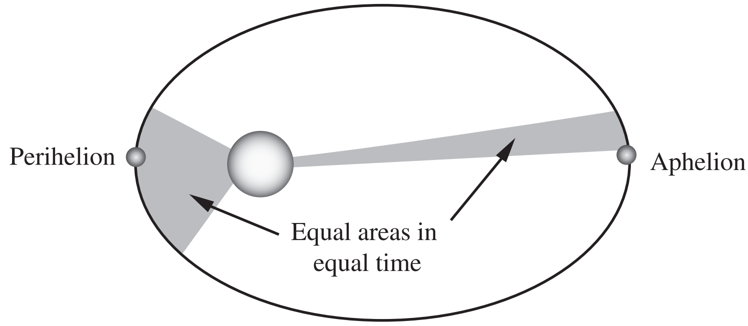

Figure 5.2. Kepler's first and second laws. The sun sits in one of the foci of an ellipse, which forms a planet's orbit. The planet moves fastest when it is closest to the sun (in the orbit's perihelion) and slowest when it is farthest away (in the orbit's aphelion).

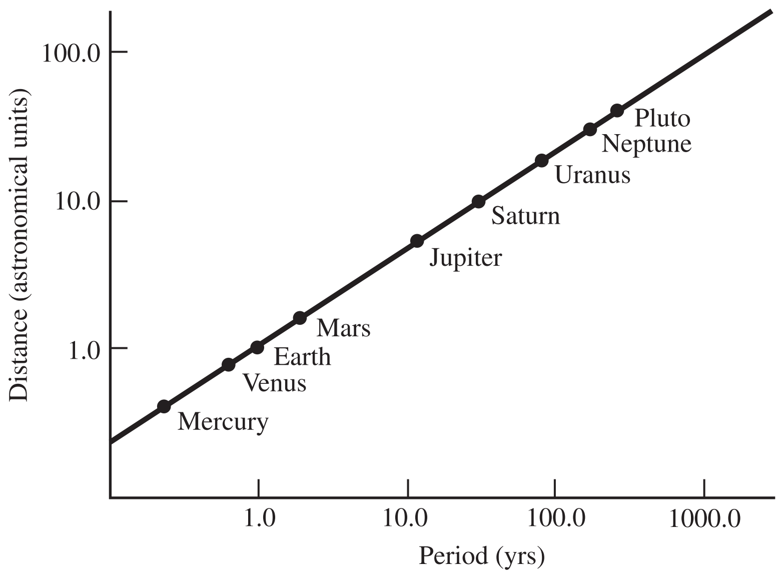

Figure 5.3. A schematic log-log plot of the semimajor axis of a planet and its period around the sun, from Kepler's third law. (Courtesy of Vik Dillion, University of Sheffield.)

data that Kepler could distinguish a circle or an oval from an ellipse in each's fit to the observed orbit of Mars. The first and the second laws (see Figure 5.2) were published in 1609. The third law (Figure 5.3) was not found until 10 years later. Some thought that the invention of logarithms by John Napier (1614) played some role in helping Kepler visualize the 1.5 exponent present in his third law.

Figure 5.4. Sir Isaac Newton (1642–1727).

It turns out that Kepler's three laws already contained the information needed to arrive at the law of gravitation; it needed an additional step of distillation and mathematical modeling, and the latter required a knowledge of calculus. Since you know calculus, let us see if you can model Kepler's three laws by a differential equation and possibly arrive at Newton's law of gravitation. The presentation here is the reverse of what you may find in other texts, where Newton's law of gravitation is assumed axiomatically and Kepler's laws are then *derived*.

5.2 Calculus, Newton, and Leibniz

Calculus is probably your first introduction to college mathematics. Instead of dealing with finite numbers, you encounter "infinitesimals," such as dx and dy, nonintuitive quantities that are infinitely small and yet nonzero. Using it, you are able to find the slope of a curve $y = f(x)$ at any point x as $\frac{dy}{dx}$, and the instantaneous velocity $u(t)$ of a particle whose displacement is $x(t)$ as the derivative $u(t) = \frac{d}{dt}x(t)$. This concept then becomes indispensable in physics, where you need to calculate instantaneous velocity $u(t)$ and acceleration $a(t) = \frac{d}{dt}u(t)$. Using integration calculus you are also able to calculate the area or volume of curved objects.

The foundation of calculus was laid in the 17th century by Isaac Newton, the English scientist better known for his work in physics (Figure 5.4), and the German mathematician Gottfried Leibniz (1646–1716). It appears that the two developed calculus independently. Newton probably started work on what he called the "method of fluxions" earlier, in 1665–1666, and even wrote up the paper in 1671, but failed to get it published until 1736. Leibniz published details of his differential calculus in 1684 and integral calculus in 1686. We owe our modern notation of dy/dx and $\int dx$ to Leibniz in these two publications. Although Leibniz published his results first, he had to defend himself in the last years of his life against charges of plagiarism by various members of the Royal Society of London. It was one of the longest and bitterest priority disputes in the history of mathematics and science and was fought mainly along nationalistic lines. In response to Leibniz's protests, the Royal Society of London set up a committee to investigate and decide on the priority. The committee report found in favor of Newton, who, as the president of the Royal Society, actually wrote the report.

Newton, as a physicist, was interested in finding the instantaneous velocity of a particle tracing a path whose coordinate is $(x(t), y(t))$ as time t changes. He denoted the velocity as $(\dot{x}(t), \dot{y}(t))$, which Newton called the "fluxions" of $(x(t), y(t))$ associated with the flux of time. The "flowing quantities" themselves, $(x(t), y(t))$, are called "fluents." To find the tangent to the curve traced by $(x(t), y(t))$, Newton used the ratio of the finite quantities $\dot{y}$ and $\dot{x}$, as $\dot{y}/\dot{x}$. Leibniz, on the other hand, introduced dx and dy as infinitesimal differences in x and y, and obtained the tangent dy/dx as the finite ratio of two infinitesimals. For integration, Newton's method involves finding the fluent given a fluxion, while Leibniz's treats integration as a sum. The way calculus is taught nowadays more closely resembles the concepts of Leibniz than Newton, although modern textbooks still vacillate between treating integration as antidifferentiation (a Newtonian concept) and as a Riemann sum (similar to Leibniz's idea).

5.3 Vector Calculus Needed

Let

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

be the position vector, where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors in the x and y directions, respectively. They are constant (Cartesian) vectors.

Let θ be the angle $\mathbf{r}$ makes relative to the x-axis and r be the magnitude of $\mathbf{r}$. Then (see Figure 5.5):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Let $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ be the unit vectors in the r and θ directions, respectively. They can be expressed in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ as

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}.$$

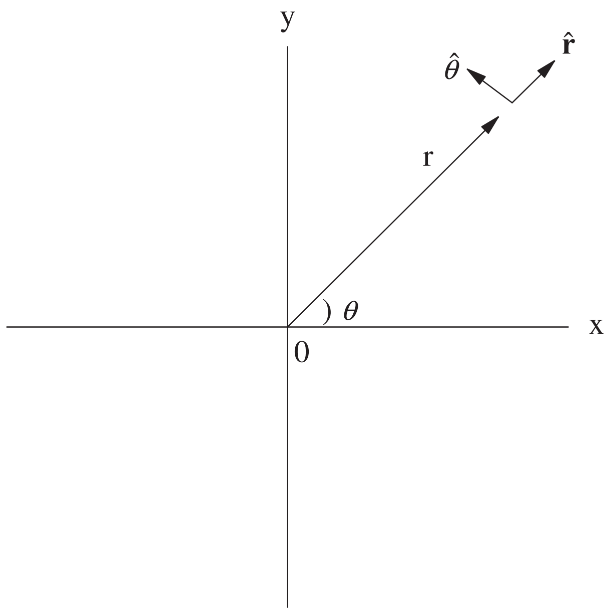

Figure 5.5. Cartesian and polar coordinates.

These (non-Cartesian) vectors may change direction in time although their magnitudes are defined to be always 1. In terms of the polar coordinates, the position vector can be written as

$$\begin{aligned}\mathbf{r} &= r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}} = r(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})\\ &= r\hat{\mathbf{r}}.\end{aligned}$$

If $\mathbf{r}$ is the position of a particle, then its velocity, $\mathbf{v}$, is given by $d\mathbf{r}/dt$, the rate of change of $\mathbf{r}$:

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d}{dt}\hat{\mathbf{r}}.$$

Since

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{r}} &= \frac{d}{dt}(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}) = -\sin\theta\frac{d\theta}{dt}\hat{\mathbf{i}} + \cos\theta\frac{d\theta}{dt}\hat{\mathbf{j}}\\ &= \frac{d\theta}{dt}(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\end{aligned}$$

(and, for later use, $\frac{d}{dt}\hat{\boldsymbol{\theta}} = \frac{d}{dt}(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) = (-\cos\theta\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{j}})\frac{d\theta}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}}$), we have

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}},$$

where we have used an overhead dot to denote d/dt, following Newton. The d/dt notation was due to Leibniz.

Since acceleration is the rate of change of velocity, we have

$$\begin{aligned}\mathbf{a} &= \frac{d}{dt}\mathbf{v} = \frac{d}{dt}(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\frac{d}{dt}\hat{\mathbf{r}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \\ &\equiv a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}},\end{aligned}$$

where we have defined

$$a_r \equiv \ddot{r} - r\dot{\theta}^2 \tag{5.1}$$

as the component of acceleration in the $\hat{\mathbf{r}}$ direction, and

$$a_\theta \equiv r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{5.2}$$

as the component of acceleration in the $\hat{\boldsymbol{\theta}}$ direction.

5.4 Rewriting Kepler's Laws Mathematically

Kepler's three laws are mostly about geometry. Only the second law hints at some rates of change of angle.

If over time Δt, the orbit of a planet describes an arc with an extended angle $\Delta\theta$, the area swept over by the radius vector (measured from the sun to the planet) in Δt is, from geometry,

$$\Delta A = \frac{1}{2}r^2\Delta\theta.$$

This result is obtainable by assuming that for very small $\Delta\theta$, there is no difference between an elliptical sector and a circular sector. For the latter, the area of a sector is $\frac{\Delta\theta}{2\pi}$ of πr^2, which is $\frac{1}{2}r^2\Delta\theta$. Now you are beginning to appreciate the advantages of being able to examine infinitesimal changes afforded by calculus.

Kepler's second law says:

$$\Delta A/\Delta t \ \text{ is a constant.}$$

Thus

$$\frac{1}{2}r^2\Delta\theta/\Delta t \ \text{ is a constant.}$$

Taking the limit $\Delta t \to 0$, we get

$$r^2\dot{\theta} = h, \quad \text{a constant,} \tag{5.3}$$

since $\dot{\theta} = \frac{d}{dt}\theta = \lim \Delta\theta/\Delta t$. Equation (5.3) is a mathematical statement of Kepler's second law. (This is actually a statement of the conservation of angular momentum in Newtonian mechanics, but Kepler was not aware of its general nature.) It can also be stated as

$$\frac{d}{dt}(r^2\dot{\theta}) = \frac{d}{dt}(h) = 0. \tag{5.4}$$

Since the acceleration in the $\hat{\boldsymbol{\theta}}$ direction is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}, \quad \text{then } r a_\theta = \frac{d}{dt}(r^2\dot{\theta}).$$

Equation (5.4) implies that

$$a_\theta = 0, \quad r > 0. \tag{5.5}$$

Stated this way, Kepler's second law says that the acceleration of the planet is purely in the radial direction, likely towards the sun. This is the first hint of a force of attraction between the planet and the sun. Next we shall rewrite the radial acceleration in a mathematical form.

Kepler's first law states that the orbit of a planet can be described by an ellipse. In terms of its radial distance from the sun, the formula for an ellipse is

$$r = \frac{p}{1 + e\cos\theta}, \quad p > 0, \quad 0 < e < 1. \tag{5.6}$$

This is a standard formula for an ellipse whose semimajor axis is given by $p/(1 - e^2)$.

From Eq. (5.1), the acceleration in the radial direction is

$$a_r = \ddot{r} - r\dot{\theta}^2.$$

We next proceed to find a_r by differentiating Eq. (5.6) with respect to time:

$$\dot{r} = \frac{ep\sin\theta\dot{\theta}}{(1 + e\cos\theta)^2} = \frac{e}{p}r^2\dot{\theta}\sin\theta = \frac{eh}{p}\sin\theta,$$

using Eq. (5.3) in the last step. Continuing,

$$\ddot{r} = \frac{eh}{p}\cos\theta\dot{\theta} = \frac{eh^2}{pr^2}\cos\theta;$$

again Eq. (5.3) was used in the last step. Also,

$$r\dot{\theta}^2 = r^4\dot{\theta}^2/r^3 = h^2/r^3.$$

Thus,

$$a_r = \ddot{r} - r\dot{\theta}^2 = -\frac{h^2}{r^2}\left[\frac{1}{r} - \frac{e\cos\theta}{p}\right]$$

$$= -\frac{h^2}{r^2}\left[\frac{1+e\cos\theta}{p} - \frac{e\cos\theta}{p}\right].$$

Finally, with the cancellation of the cosine terms:

$$a_r = -\left(\frac{h^2}{p}\right)\frac{1}{r^2}. \tag{5.7}$$

This is the inverse square law! The acceleration is inversely proportional to the square of the radial distance and it is negative, i.e., towards the sun.

Equation (5.7) is not yet a "universal law" because h and p are characteristics of individual planets. Kepler's third law remedies this situation. We are fortunate that Kepler did not stop at his first two laws but worked an extra 10 years to get his third law in order. (It turns out that Kepler's third law is only approximately correct, and needs a correction if the sun and the planet are of comparable mass. This does not matter much for our purpose, though.)

Let R be the semimajor axis of an ellipse. Then $R = p/(1-e^2)$. Let T be the period, the time it takes for the planet to make a complete revolution around the sun. Kepler's third law states that

$$T^2/R^3 = \text{ the same constant} \tag{5.8}$$

for any planet around the same sun.

From Eq. (5.3) we have

$$r^2\frac{d\theta}{dt} = h \quad \text{or} \quad dt = r^2 d\theta/h.$$

We can integrate both sides to yield the period $T \equiv \int_0^T dt$:

$$T = \frac{1}{h}\int_0^{2\pi} r^2 d\theta = \frac{p^2}{h}\int_0^{2\pi} \frac{d\theta}{(1+e\cos\theta)^2} = \frac{p^2}{h}\cdot\frac{4\pi}{(1-e^2)^{3/2}},$$

using tables of integrals to do the last trigonometric integral. A better way is to use complex variables, but some readers may not have learned that yet.

Now, since

$$T^2 = \frac{(4\pi)^2 p^4}{h^2(1-e^2)^3} \text{ and } R^3 = \frac{p^3}{(1-e^2)^3},$$

$$T^2/R^3 = (4\pi)^2 p/h^2.$$

Kepler's third law, Eq. (5.8), then says that $K \equiv h^2/p$ must be the same constant regardless of which planet we are considering.

The inverse square law, Eq. (5.7), becomes

$$\boxed{a_r = -\frac{K}{r^2},} \qquad (5.9)$$

where K is *independent of the planet's properties*. This is almost a "universal" law!

5.5 Generalizations

Using calculus, we distilled from Kepler's three laws the following results: the acceleration of the planet is purely in the radial direction towards the sun and is given by Eq. (5.9). The constant K is independent of the properties of the planet under consideration, but may depend on the properties of the sun. After all, all the observations of Brahe and Kepler were for planets in our solar system. To find out what K is, we need to go beyond Kepler. The process of reasoning is not difficult at all.

It is reasonable to assume that

a. the sun is the source of this "force of attraction"—the cause for the radial acceleration of the planet towards it; and
b. this attraction should be stronger for a sun that is more massive.

1. Let us start with the "working hypothesis" that K is proportional to the mass of the sun M, i.e.,

$$K = GM,$$

where G is the proportionality constant. G should not depend on the properties of the sun, nor should it depend on the properties of the planet. So G should be a "universal" constant. Thus

$$\boxed{a_r = -\frac{GM}{r^2}.} \tag{5.10}$$

Equation (5.10) is, essentially, the Universal Law of Gravitation, and we have just derived it! (The form you probably have seen is $F_r = -\frac{GmM}{r^2}$, for the force on a planet of mass m. It is equivalent to Eq. (5.10), since $F_r = ma_r$.)

2. How do we determine G? Let's wait until later. Newton didn't know the value of G either and never figured it out, except that he knew it must be very small.

3. Newton later called this "force of attraction" *gravity* and postulated that it exists not just between a planet and the sun but between the moon and the earth as well. (The nature of such a force was not clarified until Einstein, three centuries later.) Galileo (1564–1642) had observed the four brightest moons of Jupiter in orbit around Jupiter just as the planets revolve around the sun. It was known in Kepler's time that our moon revolves around the earth in an elliptical orbit with a very small eccentricity. Applying Eq. (5.10) to the moon, we get

$$a_r = -\frac{GM_{\text{Earth}}}{r^2}, \quad \text{towards the earth.}$$

4. This gravitational pull by the earth should apply to "falling apples" and "cannonballs" as well as the moon. In the case of falling objects near the surface of the earth, it was determined that they accelerate downward (towards the center of the earth) with magnitude

$$|a_r| = g = 980\,\text{cm/s}^2.$$

So, letting $r = a + z$, where a is the radius of the earth ($a \cong 6{,}400$ km) and z is the height above the surface of the earth, we have

$$g = \frac{GM_{\text{Earth}}}{r^2} = \frac{GM_{\text{Earth}}}{(a+z)^2} \cong \frac{GM_{\text{Earth}}}{a^2}, \qquad \text{for } z/a \ll 1.$$

This way, G is determined from

$$GM_{\text{Earth}} = a^2 g = 4 \times 10^{20}\,\text{cm}^3\,\text{s}^{-2}.$$

5. If we can somehow "weigh" the earth and find that $M_{\text{Earth}} \cong 6 \times 10^{27}$ gm, we will eventually get

$$G = 0.67 \times 10^{-7}\,\text{cm}^3 \cdot \text{s}^{-2} \cdot \text{gm}^{-1}.$$

The experiment to "weigh the earth" was not carried out until Henry Cavendish (1731–1810) did it in 1797–1798, several decades after Newton's death. However, even without knowing G explicitly, many interesting results can be obtained, as in the first three of the exercises at the end of this chapter.

5.6 Newton and the Elliptical Orbit

In August 1684, the English astronomer Edmond Halley (1656–1742) visited Isaac Newton in Cambridge and asked him if he knew the shape of the orbit of a planet subjected to a force that is proportional to the reciprocal of the square of the distance from the sun. Newton was able to say in reply that he had solved this problem and it was an ellipse, though the result had not yet been published. At Halley's urging and expense, Newton published his celebrated *Principia Mathematica* in 1687, where the three laws of mechanical motion were stated as axioms along with the law of universal gravitation. If you take the axioms as given, the elliptical orbits will follow inevitably. The solution is outlined below. Note that the *solution* procedure is the reverse of the *modeling* procedure.

From the second law of motion, the acceleration of an object is equal to the force (per unit mass) acting on it. So in the case of gravitational force, we have, for the planet

$$a_r = -\frac{GM}{r^2}, \qquad (5.11)$$

$$a_\theta = 0,$$

where M is the mass of the sun and r is the distance of the planet from it. From Eq. (5.1), we know that $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$. Thus

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}, \qquad (5.12)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \qquad (5.13)$$

From Eq. (5.13) we know that $r^2\dot{\theta} = h$ is constant, which is specified by the initial velocity. Substituting this result into Eq. (5.12), we obtain an equation involving $r(t)$ exclusively:

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{GM}{r^2}. \tag{5.14}$$

To obtain the shape of the orbit (i.e., r as a function of θ), we make use of $d\theta = \frac{h}{r^2}dt$, so that

$$\ddot{r} = \frac{d^2}{dt^2}r = \frac{d}{dt}\frac{d\theta}{dt}\frac{dr}{d\theta} = \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{d}{d\theta}r\right) = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right).$$

Consequently, Eq. (5.14) becomes

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{GM}{h^2}. \tag{5.15}$$

This is a linear second-order ordinary differential equation for $(\frac{1}{r})$ with constant forcing. The solution is

$$\frac{1}{r} = A\cos(\theta - \theta_0) + \frac{GM}{h^2}. \tag{5.16}$$

The two constants, A and θ_0, can be determined by the initial conditions on the position. The solution in Eq. (5.16) is in the form of an ellipse, (5.6).

5.7 Exercises

1. How far is the moon?

The moon's orbit around the earth is nearly circular, with a period of 28 days. Determine how far the moon is from (the center) of the earth using only this and the following information, which was available to Newton:

The gravitational acceleration of the moon caused by the gravitational pull of the earth is

$$GM_{\text{Earth}}/r^2,$$

where M_{Earth} is the mass of the earth (but the values of G or M_{Earth} were not known separately).

The orbit of the moon (or any satellite, for that matter) is a balance between the gravitational acceleration and the centrifugal force v^2/r. That is,

$$GM_{\text{Earth}}/r^2 = v^2/r,$$

where $v = r\frac{d\theta}{dt}$ is the velocity in the angular direction.

The gravitational acceleration for an object near the earth's surface ($r \cong a = 6{,}400$ km) is known to be $g = 980\,\text{cm/s}^2$.

2. ***The mass of the sun***

Determine the mass of the sun in units of the earth mass (i.e., find $\overline{M} = M_{\text{sun}}/M_{\text{Earth}}$) using only the information provided in exercise 1 and the following information:

a. the period of earth's orbit is 1 year.

b. the sun–earth distance is, on average, about 1.5×10^8 km.

3. ***Geosynchronous satellite***

If you want to put a satellite in a geosynchronous orbit (so that the satellite will always appear to be above the same spot on earth), how high (measured from the center of the earth) must it be placed? You are given $GM_E = a^2 g = 4 \times 10^{20}\,\text{cm}^3\,\text{s}^{-2}$.

4. ***Weighing a planet***

Newton's law of gravitation and the requirement that the centrifugal acceleration of a body revolving around a planet should be equal to the gravitational pull of the planet suggest a way to determine the mass M of any planet with a satellite or moon. For this exercise you need to know $G = 0.67 \times 10^{-7}\,\text{cm}^3\,\text{s}^{-2}\,\text{gm}^{-1}$.

Determine the mass of a planet when you know only that its moon revolves around it with a nearly circular orbit with $r = 4 \times 10^5$ km once every30 days.

5. ***Weighing Jupiter***

Find the mass of Jupiter, given that its moon Callisto has a mean orbital radius of 1.88×10^6 km and an orbital period of 16 days and 16.54 hours. For this exercise you need to know G.