Cantor's Theorem: Infinite Infinities

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1 Introduction

In set theory, Cantor's Theorem states that the cardinality of a set must be less than the cardinality of its power set such that for a finite set A:

$$|A| < |P(A)|$$

This is fairly obvious with finite sets, as their power sets have 2^n elements (where n is the number of elements in the original set) (Hosch, 2016).

For example, take $A = \{1, 2, 3\}$:

$$P(A) = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{2\}, \{2, 3\}, \{3\}, \{1, 2, 3\}\}$$

$$|A| = 3; |P(A)| = 8 = 2^3$$

However, this theorem was not proven with infinite sets in mind until 1891, when Georg Cantor used his Diagonalization Argument to show that there are different levels of infinity ("4.10 Cantor's Theorem").

In this paper, a proof will be given to explain why the Diagonalization Argument works and how to prove the Cantor-Schröder-Bernstein Theorem.

2 The Diagonalization Argument

Before discussing diagonalization, it would be beneficial to go over some concepts integral to the proof.

2.1 Bijections and Countability

For a set A to form a bijection with a set B, there must be a relation R that is both one-to-one and onto. In other words, every element of A must correspond to one element of B.

A set is countably infinite iff the set can form a bijection with a set containing the elements of $\mathbb{N}$. Not every infinite set can claim to be countably infinite (Martin).

2.2 Uncountable Reals

Now, this is where the Diagonalization Argument comes in. Instead of overcomplicating this and attempting to list every real number in existence, instead, say that you are working with an interval I from 0 to 1 inclusive. Try to find a bijective relation R such that $x\mathrm{R}y$ where $x \in \mathbb{N}$ and $y \in \mathbb{R}$.

To start, list m real numbers that fall in this interval where $m \in \mathbb{N}$. Making a list with m random elements of $\mathbb{R}$ in I, each with m digits after the decimal point, you could get a list like this:

$$0.10000000, 0.33333333, 0.25837481, 0.99931733,$$

$$0.53238492, 0.77789482, 0.59390173, 0.44282749$$

Here, $m = 8$. Now, after removing the "0." in front of every entry, put every number in a square grid:

Table 1: Grid With Real Numbers Between 0 and 1

1	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3
2	5	8	3	7	4	8	1
9	9	9	3	1	7	3	3
5	3	2	3	8	4	9	2
7	7	7	8	9	4	8	2
5	9	3	9	0	1	7	3
4	4	2	8	2	7	4	9

Taking the diagonal starting from the top left in Table 1, you'd get:
13838479
Then add 1 to each digit (note: any 9 will loop to a 0 rather than becoming a 10):
24949580
Finally, re-add the "0." to the front of the number:
0.24949580

Looking at the table, there is no number of the form 0.24949580. No matter what numbers were chosen, there is no possible way to perform this set of actions while having a real number along the diagonal that is identical to a real number in the table (Bazett, 2018).

This can be easily proven via contradiction. Assume that the real number from the diagonal, d, is indeed equal to one of the real numbers in one of the rows, r. This means that every decimal place is equal. However, this cannot be true, since the one decimal value of r on d has to be equal to itself plus 1. This is, of course, impossible. The relation $x\mathrm{R}y$ from $\mathbb{N}$ to $\mathbb{R}$ is not surjective (Weisstein).

This shows how there is no bijective function from $\mathbb{N}$ to $\mathbb{R}$, and therefore, the set $\mathbb{R}$ is uncountably infinite. $\square$

3 The Cantor-Schröder-Bernstein Theorem

The Cantor-Schröder-Bernstein theorem is another important theorem in Set Theory. It is usually visualized by creating a bijection map which contains set A and B. Cantor's earlier proof effectively relied on the axiom of choice, by deriving the inference of the well-ordered theorem. This proof is only using the the classical Zermelo-Fraenkel axioms. This theory implies that if two cardinalities are "less than or equal to" each other, then they are equal.

3.1 Definition

In other words, the theorem states that if there exists an injection f: $A \to B$ and there is an injection h: $B \to A$, then there is a bijection j: $A \to B$.

To summarize the above conditions: if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. This is a very useful feature in the ordering of cardinalities (Whitman).

3.2 Proof

One way to understand the the Cantor-Schröder-Bernstein proof is by comparing it to Hilbert's Infinite Hotel. Hilbert's Hotel is a thought experiment that illustrates a hotel with a countably infinite number of rooms that can accommodate a countably infinite number of new guests. Each room is occupied by one guest. To accommodate a countable infinite number of guests, move the guest in room n to room 2n and put the new guests in room 2n-1. Therefore, the old guests are in the infinitely even rooms and the new guests are in the odd rooms. Through iteration you can accommodate infinitely many sets of new guests. By using the Cantor-Schröder-Bernstein Theorem, one can show that each room is occupied and each guest is matched to a distinct room (Kragh).

To start off, assume there are injections f: $A \to B$ and $h : B \to A$
Let $A_0 = A$ h[B] and $A_{n+1} = hf[A_n]$, therefore $A_\infty = \bigcup_{n=0}^{\infty} A_n$
A_0 are the guests that don't have rooms, A_{n+1} are the guests occupying the rooms where the guests of A_0 should be (Lobo, 2021).

Define $j : A \to B$

$$j(x) = \begin{cases} f(x)\ if\ X \in A_\infty \\ h^{-1}(x) \text{ otherwise} \end{cases}$$

Claim: j is a bijection. Therefore, prove that j is injective and surjective.

To prove j is injective, one can use proof by cases to show $j(x) = j(y)$ implies $x = y$.
Case 1:
x, $y \in A_\infty$,
$f(x) = j(x) = f(y) = j(y)$, thus $x = y$ since f is injective.

Case 2:
$x, y \notin A_\infty$
By the proposed injectivity of h, the injectivity of h implies the injectivity of h inverse.

Case 3:
$x \in A_\infty$ and $y \notin A_\infty$ then $x \in A_n$ for some n.
By definition, $f(x) = j(x) = j(y) = h^{-1}$ (x) such that $h(f(x)) = y$. Consider $x \in A_m$ for $m \geq 1$. This means that $y = h(f(x))$ for $\in h(f(A_m)) = A_{m+1}$. Which implies that $y \in A_\infty$ which is a contradiction. Therefore, j is injective. In terms of the Hilbert Hotel, this means that two guests are not mapped to the same room. Instead, each guest is matched to a different room with no overlap (Blargoner, 2016).

The next step is to show that j is surjective. This implies that for each $r \in B$, there is $X \in A$ such that $j(x) = z$. Proving for surjectivity ensures that each room has a guest.

Consider $x = h(z)$:
Suppose $X \notin A_\infty$ then $j(x) = h^{-1}(x) = z$ Suppose $x \in A_\infty$ then $X \in A_n$ for some n. Therefore $n > 0$ because $X \in h[B]$. So $X \in hf[A_{n-1}]$. This means that $x = hf(x^{'})$ for some $X' \in A_{n-1}$.

$$z = h^{-1}(x) = f(x^{'}) = j(x^{'})$$

In terms of Hilbert's Hotel, x is a guest that has been moved, but you will get a different guest that will occupy x's room. Therefore, j is surjective and it is also injective, as proved earlier. This means that j is bijective; j: $A \rightarrow B$ (Blargoner, 2016). $\square$

4 Conclusion

To sum up, Cantor has come up with many different approaches to conceptualizing infinity. Specifically, he discovered whether or not infinite sets are 'countable' or 'uncountable'. The Diagonal Argument perfectly depicts the impossible bijection from the set containing all the elements of $\mathbb{N}$ to the set containing all the elements of $\mathbb{R}$, proving that $|\mathbb{N}| \neq |\mathbb{R}|$. This, in turn, would mean that there are infinite infinities. With the help of Hilbert's Hotel, the Cantor-Schröder-Bernstein Theorem was proven to show that if two sets have cardinalities that are "less than or equal to" each other, then the two sets have equal cardinalities.

Cantor would go on to prove many more interesting aspects of infinity, such as founding the idea of transfinite numbers (the different levels of infinity), stating $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = \aleph_1$. Essentially, for all $n \in \mathbb{N}$ where $n \geq 0$, $2^{\aleph_n} = \aleph_{n+1}$.

5

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