Extension of Howe's Method to Lazy Pairs

L Denney
Computer Science
McGill University
Montreal, Canada
261042191

Emily Martin
Computer Science and Biology
McGill University
Montreal, Canada
261019878

Alexander Lin-Hu
Computer Science
McGill University
Montreal, Canada
260344911

***Abstract*—In this project, we extend the proof that similarity in the call-by-name lambda calculus is a pre-congruence using Howe's method. In particular, we extend this proof to cover the addition of a new data type, lazy pairs, to the language. We prove all necessary lemmas and the main theorem on paper and extend the mechanization of the proof in Beluga.**

I. INTRODUCTION

Induction is a well-understood and trusted form of argumentation that is taught in introductory computer science and mathematics courses [3]. In terms of functional programming, finite (inductive) data is defined using constructors and analyzed using familiar pattern matching. Infinite (coinductive) data is defined by the observations that we can make about it using copattern matching [1]. Inductive proofs are functions about finite data that cover all possible cases and are terminating [1]. Similarly, coinductive proofs are functions defined on infinite data that cover all cases and remain productive [1].

However, coinduction is a less widely taught, understood, and trusted proof strategy. As explained in [3], many proofs in the literature that rely on coinduction still end up essentially reasserting the principle every time it is used." In order to provide a resource for understanding coinductive proofs and their computational interpretations, Momigliano, Pientka & Thibodeau [1] created a case-study in coinductive proofs exploring the proof that similarity in the call-by-name lambda calculus with lists is a pre-congruence using Howe's method.

In this project, we explore coinductive structures and reasoning by extending this case study to cover the addition of a new data type, lazy pairs, to the language. In addition, we extend the corresponding mechanization in **Beluga** provided by Momigliano, Pientka & Thibodeau to cover the addition of lazy pairs.

II. BISIMULATION AND HOWE'S METHOD

A. Similarity and Bisimilarity

The proof that we explore here deals with bisimilarity, a notion of program equivalence adapted from concurrency theory [1]. Two closed terms are bisimilar if whenever the first term evaluates to an observable then so does the second, and all subprograms of those are also bisimilar [1]. Essentially, this asserts that the programs are indistinguishable from each other in terms of the observations that can be made about them as they are executed.

As in the case study by Momigliano, Pientka, and Thibodeau [1], we concentrate on similarity, from which bisimilarity is obtained by symmetry.

We first extend the language PCFL defined in [1] to include lazy pairs. Typing and evaluation rules for lazy pairs are borrowed from [2].

In this proof, we deal with a particular form of similarity called *applicative similarity* in which we define similarity of functions not by comparing them directly but rather by asserting that they behave identically on all possible inputs.

An *applicative simulation* is a family of typed relations R_τ on closed terms, one for each constructor in the language. With the addition of lazy pairs, we add a new relation to the family at the type $\tau \times \tau'$:

If $m \ \mathcal{R}_{\tau\times\tau'} \ n$ then $m \Downarrow \langle m_f, m_s \rangle$ entails there are terms m'_f, m'_s s.t. $n \Downarrow \left\langle m'_f, m'_s \right\rangle$ for which $m_f \ \mathcal{R}_\tau \ m'_f$ and $m_s \ \mathcal{R}_{\tau'} \ m'_s$

We call the greatest such relation (which exists by the Knaster-Tarski fixpoint theorem) *applicative similarity*.

We then extend the proof that applicative similarity is reflexive to cover the new cases.

We also extend the definition of a compatible relation with the following cases for pairs (numbering follows from [1],

(C7) $\Gamma \vdash m_1 \ \mathcal{R}_\tau \ m_2$ and $\Gamma \vdash n_1 \ \mathcal{R}_{\tau'} \ n_2$ entails $\Gamma \vdash \langle m_1, n_1 \rangle \ \mathcal{R}_{\tau\times\tau'} \ \langle m_2, n_2 \rangle$

(C8) $\Gamma \vdash m_1 \ \mathcal{R}_{\tau\times\tau'} \ m_2$ entails $\Gamma \vdash \text{fst}(m_1) \ \mathcal{R}_\tau \ \text{fst}(m_2)$

(C9) $\Gamma \vdash m_1 \ \mathcal{R}_{\tau\times\tau'} \ m_2$ entails $\Gamma \vdash \text{snd}(m_1) \ \mathcal{R}_{\tau'} \ \text{snd}(m_2)$

Next, since applicative similarity is only defined for closed terms, we extend it to open terms with substitution, and call this relation *open similarity*.

B. Towards the Goal

Now we want to progress towards our goal of proving (and mechanizing this proof in Beluga) that open similarity is a pre-congruence with our extension of lazy pairs. As described in [1] a direct attempt to prove open similarity is a pre-congruence breaks down, and this issue does not go away with our extension

C. On Howe's Method

The solution Howe offered to this problem in 1996 [4] was to introduce his eponymous Howe relation, a candidate relation which contains open similarity and can be proven to be a pre-congruence by a showing it is substitutive. The key to his idea is that the Howe relation can be shown to coincide with open similarity. If we can prove this coincidence, we have found a way to prove that open similarity is a pre-congruence without the trouble of attempting to prove substitutivity for open similarity directly.

D. Proofs and Mechanization

The proof that howe's relation coincides with open simulation, when extended to lazy pairs, follows the same structure given in [1]. We extend relevant definitions (applicative simulation, and compatible relation) and the proofs of eight lemmata before being equipped to tackle the main goal. The appendix covers the proofs of all lemmata in detail.

We start by adding three additional cases to the definition of Howe's relation to cover pair, fst, and snd.
Proposed Howe's Inference Rules

$$\frac{\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m' \qquad \Gamma \vdash \text{ fst } m' \preccurlyeq^{\circ}_{\tau} n}{\Gamma \vdash \text{ fst}(m) \preccurlyeq^{\mathcal{H}}_{\tau} n}\text{hfst}$$

$$\frac{\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m' \qquad \Gamma \vdash \text{ snd } m' \preccurlyeq^{\circ}_{\tau'} n}{\Gamma \vdash \text{ snd}(m) \preccurlyeq^{\mathcal{H}}_{\tau'} n}\text{hsnd}$$

$$\frac{\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_1' \qquad \Gamma \vdash m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} m_2' \qquad \Gamma \vdash \langle m_1', m_2'\rangle \preccurlyeq^{\circ}_{\tau\times\tau'} n}{\Gamma \vdash \langle m_1, m_2\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n}\text{hpair}$$

It is imperative that we define these rules correctly, otherwise our proofs will fail down the line. Since we have completed all proofs without issue, we are confident that our definitions are correct.

Now we move to working through extending the 8 lemmata required to prove our extended Howe relation coincides with the extended open similarity (appendix, Theorem 3).

Lemma 1 Semi-transitivity of Howe's relation:
$(m \preccurlyeq^{\mathcal{H}}_{\tau} n') \wedge (n' \preccurlyeq^{\circ}_{\tau} m'') \implies m \preccurlyeq^{\mathcal{H}}_{\tau} m''$
The proof of this lemma is straightforward: by cases on the Howe derivation using transitivity of open similarity, then using the Howe inference rules to reconstruct.

Lemma 2 Reflexivity of Howe's relation: if $\Gamma \vdash m : \tau$ then $\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau} m$.
The proof proceeds by induction on the typing derivation, reflexivity of open similarity, and using the Howe inference rules to reconstruct.

Lemma 3 Howe relation is compatible.
This involves proving Howe' relation satisfies (C7)-(C9). Proof is nearly immediate using reflexivity of open similarity and Howe inference rules.

Lemma 4 Open similarity is contained in Howe: if $\Gamma \vdash m \preccurlyeq^{\circ}_{\tau} n$ then $\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau} n$.
This proof follows immediately from lemmas (1) and (2).

Lemma 5 Substitutivity of the Howe relation. if $\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_2$ and $\Psi \vdash \sigma_1 \preccurlyeq^{\mathcal{H}}_{\Gamma} \sigma_2;$ then $\Psi \vdash [\sigma_1]m_1 \preccurlyeq^{\mathcal{H}}_{\tau} [\sigma_2]m_2$.
This lemma is where open similarity initially got stuck. In preparation, we extended the definition of substitution over pairs in the expected way. We find that this proof is much easier on Howe's relation than on open similarity, where it is by induction on the derivation of $\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_2$. Each case generally looks at the shape of the Howe's relation definition, appeals to the induction hypothesis, and refers to the definition of substitution.

Lemma 6 Howe's relation mimics simulation conditions: if $\langle m_1, m_2\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n$ then $n \Downarrow \langle p_1, p_2\rangle$
s.t. $m_1 \preccurlyeq^{\mathcal{H}}_{\tau} p_1,\ m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} p_2$
Proof proceeds by cases on the Howe relation, generally using inversion on the Howe derivation and Lemma (1). In our extension only pairs needs to be covered.

Lemma 7 Downward closure: if $m \preccurlyeq^{\mathcal{H}}_{\tau} n$ and $m \Downarrow v$ then $v \preccurlyeq^{\mathcal{H}}_{\tau} n$.
Proof is by induction of the derivation of the evaluation ($\mathcal{E}$) and proceeds by cases. This is the key lemma for the proof of the main theorem (Theorem 3 in Appendix) and was significantly more challenging than the previous lemmata. In the appendix, the case where m is a pair isn't included because pairs are values and thus the proof is immediate since values evaluate to themselves meaning $m = v$.

Lemma 8 $p \preccurlyeq^{\mathcal{H}}_{\tau} q$ entails $p \preccurlyeq_{\tau} q$.
At this step, we have almost reached the main theorem of Howe coinciding with open sim, and we see it approaching because this lemma tells us Howe coincides with simulation for the closed terms. It is a proof by co-induction on the derivation on the Howe relation where we consider cases of the derivation of the Howe relation where all terms are closed. This means we add a case for pairs that uses downwards closure (lemma 7), the proof that Howe mimics the simulation conditions (lemma 6), and the co-inductive hypothesis.

At this point, we are prepared to finally prove that the Howe relation coincides with open similarity. It turns out no extension is required for this second-to-last step. The right to left is lemma 4 which we already extended and left to right is by induction on Γ, where the base case is lemma 8, and the step case is closure under substitution which deals with an arbitrary instance of Howe's relation. So cases specifying anything about pairs or fst and snd don't appear!

Finally, we can conclude with our ultimate goal by a corollary of the above: open similarity, with extension to lazy pairs, is a pre-congruence.

III. Conclusion

This project has shown open similarity in the call-by-name lambda calculus remains a pre-congruence with the extension of lazy pairs by following the form of the case study on Howe's method presented in [1]. All proofs have been mechanized in Beluga, where built-in support for coinductive reasoning and simultaneous substitutions permits elegant implementations of these proofs.

References

[1] A. MOMIGLIANO, B. PIENTKA, and D. THIBODEAU, "A case study in programming coinductive proofs: Howe's method," Mathematical Structures in Computer Science, vol. 29, no. 8, pp. 1309–1343, Oct. 2018, doi: https://doi.org/10.1017/s0960129518000415.

[2] A. Pitts, "Operationally-Based Theories of Program Equivalence," in Semantics and Logics of Computation, A. M. Pitts and P. Dybjer, Eds. Cambridge: Cambridge University Press, 1997, pp. 241–298

[3] D. KOZEN and A. SILVA, "Practical Coinduction," Mathematical Structures in Computer Science, vol. 27, special issue 7, pp. 1132-1152.
[4] D. J. Howe, "Proving Congruence of Bisimulation in Functional Programming Languages," Information & Computation, vol. 124, no. 2, pp. 103–112, Feb. 1996, doi: https://doi.org/10.1006/inco.1996.0008.

Appendix

Rules of Inference

$$\frac{\Gamma \vdash M_1 : \sigma \qquad \Gamma \vdash M_2 : \sigma'}{\Gamma \vdash \langle M_1, M_2 \rangle : \sigma \times \sigma'} \vdash \text{ pair}$$

$$\frac{\Gamma \vdash P : \sigma \times \sigma'}{\Gamma \vdash \text{ fst}(P) : \sigma} \vdash \text{ fst} \qquad \frac{\Gamma \vdash P : \sigma \times \sigma'}{\Gamma \vdash \text{ snd}(P) : \sigma'} \vdash \text{ snd}$$

$$\frac{P \Downarrow \langle M_1, M_2 \rangle \qquad M_1 \Downarrow c}{\text{fst}(P) \Downarrow c} \Downarrow \text{ fst} \qquad \frac{P \Downarrow \langle M_1, M_2 \rangle \qquad M_2 \Downarrow c}{\text{snd}(P) \Downarrow c} \Downarrow \text{ snd}$$

Proposed Howe's Inference Rules

$$\frac{\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m' \qquad \Gamma \vdash \text{ fst } m' \preccurlyeq^{\circ}_{\tau} n}{\Gamma \vdash \text{ fst}(m) \preccurlyeq^{\mathcal{H}}_{\tau} n} \text{hfst} \qquad \frac{\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m' \qquad \Gamma \vdash \text{ snd } m' \preccurlyeq^{\circ}_{\tau'} n}{\Gamma \vdash \text{ snd}(m) \preccurlyeq^{\mathcal{H}}_{\tau'} n} \text{hsnd}$$

$$\frac{\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_1' \qquad \Gamma \vdash m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} m_2' \qquad \Gamma \vdash \langle m_1', m_2' \rangle \preccurlyeq^{\circ}_{\tau\times\tau'} n}{\Gamma \vdash \langle m_1, m_2 \rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n} \text{hpair}$$

Objective. Prove the following theorems on our extension.

- Extend Definition 1 Applicative Simulation to pairs
- Prove Theorem 1 (Reflexivity of Applicative Similarity, $\forall m : \tau, m \preccurlyeq_{\tau} m$) holds for our extension
- Extend Definition 2 Compatible relation to pairs.
- Prove Lemma 6 (Substitutivity of the Howe Relation) holds for our extension
- Prove Theorem 3 (Howe's Relation Coincide with Similarity, $\Gamma \vdash p \preccurlyeq^{\mathcal{H}}_{\tau} q \iff \Gamma \vdash p \preccurlyeq^{\circ}_{\tau} q$). Achieved by first extending the lemmata proofs to pairs, then concluding the theorem.
  - Semi-transitivity; $(m \preccurlyeq^{\mathcal{H}}_{\tau} m') \wedge (m' \preccurlyeq^{\circ}_{\tau} m'') \implies m \preccurlyeq^{\mathcal{H}}_{\tau} m''$
  - Reflexive;
  - Compatibility; Our extension of definition 2 holds
  - Open similarity is contained in Howe; $\Gamma \vdash m \preccurlyeq^{\circ}_{\tau} n \implies \Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau} n$
  - Howe relation is substitutive (immediate from lemma 6)
  - Howe relation mimics the Simulation condition, extend for pairs. If $\langle m_1, m_2 \rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n$ then $n \Downarrow \langle p_1, p_2 \rangle$ with $m_1 \preccurlyeq^{\mathcal{H}}_{\tau} p_1$ and $m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} p_2$
  - Downward closure, $(p \preccurlyeq^{\mathcal{H}}_{\tau} q) \wedge (p \Downarrow v) \implies v \preccurlyeq^{\mathcal{H}}_{\tau} q$
  - $p \preccurlyeq^{\mathcal{H}}_{\tau} q \implies p \preccurlyeq_{\tau} q$

For proofs where fst, snd are similar, we use fst as the representative case.

Definition 1. Applicative Simulation We extend to pairs by adding the following,

- If $m \mathrel{\mathcal{R}_{\tau\times\tau'}} n$ then $m \Downarrow \langle m_f, m_s \rangle$ entails there are terms m_f', m_s' s.t. $n \Downarrow \left\langle m_f', m_s' \right\rangle$ for which $m_f \mathrel{\mathcal{R}_{\tau}} m_f'$ and $m_s \mathrel{\mathcal{R}_{\tau'}} m_s'$

Theorem 1. Reflexitivity of Applicative Similarity We show $\forall m : \tau, m \preccurlyeq_{\tau} m$ holds with our extension of applicative simulation by adding a case for pairs. We note S_{τ} to be the family $\left\{(m, m) : \dot{\vdash} m : \tau\right\}$. Assume $m \mathrel{\mathcal{S}_{\tau\times\tau'}} m$ and $m \Downarrow \langle m_f, m_s \rangle$. Pick m_f', m_s' to be m_f, m_s. By definition of simulation, $m_f \mathrel{\mathcal{S}_{\tau}} m_f$ and $m_s \mathrel{\mathcal{S}_{\tau'}} m_s$ □

Definition 2. Compatible relation We extend the definition of a compatible relation with the following,

(C7) $\Gamma \vdash m_1\ \mathcal{R}_\tau\ m_2$ and $\Gamma \vdash n_1\ \mathcal{R}_{\tau'}\ n_2$ entails $\Gamma \vdash \langle m_1, n_1\rangle\ \mathcal{R}_{\tau\times\tau'}\ \langle m_2, n_2\rangle$

(C8) $\Gamma \vdash m_1\ \mathcal{R}_{\tau\times\tau'}\ m_2$ entails $\Gamma \vdash \mathrm{fst}(m_1)\ \mathcal{R}_\tau\ \mathrm{fst}(m_2)$

(C9) $\Gamma \vdash m_1\ \mathcal{R}_{\tau\times\tau'}\ m_2$ entails $\Gamma \vdash \mathrm{snd}(m_1)\ \mathcal{R}_{\tau'}\ \mathrm{snd}(m_2)$

Theorem 3

Lemma 1 Semi-transitivity of Howe's relation. $(m \preccurlyeq^{\mathcal{H}}_\tau n') \wedge (n' \preccurlyeq^{\circ}_\tau m'') \implies m \preccurlyeq^{\mathcal{H}}_\tau m''$

We define $m = \langle m_1, m_2\rangle, n = \langle n_1, n_2\rangle, n' = \langle n_1', n_2'\rangle$.

Case. With assumption $n' \preccurlyeq^{\circ}_{\tau\times\tau'} m''$

$$\mathcal{D} = \quad \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_\tau n_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} n_2\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ \Gamma \vdash \langle n_1, n_2\rangle \preccurlyeq^{\circ}_{\tau\times\tau'} n'\end{array}}{\Gamma \vdash \langle m_1, m_2\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n'}\ \text{hpair}$$

$$\begin{array}{rl} n \preccurlyeq^{\circ}_{\tau\times\tau'} m'' & \text{By Transitivity of } \preccurlyeq^{\circ}_{\tau\times\tau'} \text{ on } \mathcal{D}_3 \text{ and Assumption}\\ \Gamma \vdash \langle m_1, m_2\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m'' & \text{By hpair on } \mathcal{D}_1, \mathcal{D}_2, \text{ and above}\end{array}$$

Case. With assumption $\mathrm{fst}(n') \preccurlyeq^{\circ}_\tau \mathrm{fst}(m'')$

$$\mathcal{D} = \quad \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash \mathrm{fst}(n) \preccurlyeq^{\circ}_\tau \mathrm{fst}(n')\end{array}}{\Gamma \vdash \mathrm{fst}(m) \preccurlyeq^{\mathcal{H}}_\tau \mathrm{fst}(n')}\ \text{hfst}$$

$$\begin{array}{rl} \mathrm{fst}(n) \preccurlyeq^{\circ}_\tau \mathrm{fst}(m'') & \text{By Transitivity of } \preccurlyeq^{\circ}_\tau \text{ on } \mathcal{D}_2 \text{ and assumption}\\ \Gamma \vdash \mathrm{fst}(m) \preccurlyeq^{\mathcal{H}}_\tau \mathrm{fst}(m'') & \text{By hfst on } \mathcal{D}_1 \text{ and above}\end{array}$$

□

Lemma 2 Reflexivity of Howe's relation. If $\Gamma \vdash m : \tau$ then $\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_\tau m$. Proof by induction on typing,

Case.

$$\mathcal{D} = \quad \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash M_1 : \sigma\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash M_2 : \sigma'\end{array}}{\Gamma \vdash \langle M_1, M_2\rangle : \sigma\times\sigma'}\ \vdash \text{pair}$$

$$\begin{array}{lr} \Gamma \vdash M_1 \preccurlyeq^{\mathcal{H}}_\sigma M_1 & \text{By IH on } \mathcal{D}_1 \text{ (denote as } \mathcal{F}_1)\\ \Gamma \vdash M_2 \preccurlyeq^{\mathcal{H}}_\sigma M_2 & \text{By IH on } \mathcal{D}_2 \text{ (denote as } \mathcal{F}_2)\\ \Gamma \vdash \langle M_1, M_2\rangle \preccurlyeq^{\circ}_{\sigma\times\sigma'} \langle M_1, M_2\rangle & \text{By reflexivity of } \preccurlyeq^{\circ}_{\sigma\times\sigma'} \text{ (denote as } \mathcal{F}_3)\\ \Gamma \vdash \langle M_1, M_2\rangle \preccurlyeq^{\mathcal{H}}_{\sigma\times\sigma'} \langle M_1, M_2\rangle & \text{By hpair on } \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\end{array}$$

Case.

$$\mathcal{D} = \quad \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash P : \sigma\times\sigma'\end{array}}{\Gamma \vdash \mathrm{fst}(P) : \sigma}\ \vdash \text{fst}$$

$$\begin{array}{lr} \Gamma \vdash P \preccurlyeq^{\mathcal{H}}_{\sigma\times\sigma'} P & \text{By IH on } \mathcal{D}_1 \text{ (denote as } \mathcal{F}_1)\\ \Gamma \vdash \mathrm{fst}(P) \preccurlyeq^{\circ}_\sigma \mathrm{fst}(P) & \text{By reflexivity of } \preccurlyeq^{\circ}_\sigma \text{ (denote as } \mathcal{F}_2)\\ \Gamma \vdash \mathrm{fst}(P) \preccurlyeq^{\mathcal{H}}_\sigma \mathrm{fst}(P) & \text{By hfst on } \mathcal{F}_1, \mathcal{F}_2\end{array}$$

□

Lemma 3 Howe relation is compatible. Extend proof for C7, C8, C9.

C7. Observe,

$$\dfrac{\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_2 \qquad \Gamma \vdash n_1 \preccurlyeq^{\mathcal{H}}_{\tau'} n_2 \qquad \dfrac{}{\Gamma \vdash \langle m_2, n_2\rangle \preccurlyeq^{\circ}_{\tau\times\tau'} \langle m_2, n_2\rangle}\text{Reflex-Osim}}{\Gamma \vdash \langle m_1, n_1\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} \langle m_2, n_2\rangle}\text{hpair}$$

C8. Observe (C9 is very similar),

$$\dfrac{\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m_2 \qquad \dfrac{}{\Gamma \vdash \mathsf{fst}(m_2) \preccurlyeq^{\circ}_{\tau} \mathsf{fst}(m_2)}\text{Reflex-Osim}}{\Gamma \vdash \mathsf{fst}(m_1) \preccurlyeq^{\mathcal{H}}_{\tau} \mathsf{fst}(m_2)}\text{hfst}$$

□

Lemma 4. Open similarity is contained in Howe. Assume $\Gamma \vdash m \preccurlyeq^{\circ}_{\tau} n$, we show $\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau} n$

$\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau} m$	By howe reflexivity
$\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau} n$	By semi-transitivity on above and assumption

□

Lemma 5. Substitutivity of the Howe Relation, suppose we have $\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_2$ and $\Psi \vdash \sigma_1 \preccurlyeq^{\mathcal{H}}_{\Gamma} \sigma_2$; then $\Psi \vdash [\sigma_1]m_1 \preccurlyeq^{\mathcal{H}}_{\tau} [\sigma_2]m_2$. Proof by induction on the derivations of $\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_2$. We extend this to the derivations of the Howe relation for pairs.

Case

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_1'} \qquad \overset{\mathcal{D}_2}{\Gamma \vdash m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} m_2'} \qquad \overset{\mathcal{D}_3}{\Gamma \vdash \langle m_1', m_2'\rangle \preccurlyeq^{\circ}_{\tau\times\tau'} n}}{\Gamma \vdash \langle m_1, m_2\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n}\text{hpair}$$

WTS: $\Psi \vdash [\sigma_1]\langle m_1, m_2\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} [\sigma_2]n$

$\Psi \vdash [\sigma_1]m_1 \preccurlyeq^{\mathcal{H}}_{\tau} [\sigma_2]m_1'$	by IH on $\mathcal{D}_1$
$\Psi \vdash [\sigma_1]m_2 \preccurlyeq^{\mathcal{H}}_{\tau} [\sigma_2]m_2'$	by IH on $\mathcal{D}_2$
$\Psi \vdash [\sigma_2]\langle m_1', m_2'\rangle \preccurlyeq^{\circ}_{\tau\times\tau'} [\sigma_2]n$	by cus on $\mathcal{D}_3$
$[\sigma_2]\langle m_1', m_2'\rangle \;=\; \langle [\sigma_2]m_1', [\sigma_2]m_2'\rangle$	by def. of substitution

Construct:

$$\dfrac{\Psi \vdash [\sigma_1]m_1 \preccurlyeq^{\mathcal{H}}_{\tau} [\sigma_2]m_1' \qquad \Psi \vdash [\sigma_1]m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} [\sigma_2]m_2' \qquad \Psi \vdash \langle [\sigma_2]m_1', [\sigma_2]m_2'\rangle \preccurlyeq^{\circ}_{\tau\times\tau'} [\sigma_2]n}{\Psi \vdash \langle [\sigma_1]m_1, [\sigma_1]m_2\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} [\sigma_2]n}\text{hpair}$$

By the conclusion of the constructed derivation, $\Psi \vdash [\sigma_1]\langle m_1, m_2\rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} [\sigma_2]n$.

Case

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m'} \qquad \overset{\mathcal{D}_2}{\Gamma \vdash \mathsf{fst}(m') \preccurlyeq^{\circ}_{\tau} n}}{\Gamma \vdash \mathsf{fst}(m) \preccurlyeq^{\mathcal{H}}_{\tau} n}\text{hfst}$$

$\Psi \vdash [\sigma_1]m \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} [\sigma_2]m'$	By IH on $\mathcal{D}_1$ (denote as $\mathcal{F}_1$)
$\Psi \vdash [\sigma_2]\mathsf{fst}(m') \preccurlyeq^{\circ}_{\tau} [\sigma_2]n$	By (cus) on $\mathcal{D}_2$
$\Psi \vdash \mathsf{fst}([\sigma_2]m') \preccurlyeq^{\circ}_{\tau} [\sigma_2]n$	By Substitution (denote as $\mathcal{F}_2$)
$\Psi \vdash \mathsf{fst}([\sigma_1]m) \preccurlyeq^{\mathcal{H}}_{\tau} [\sigma_2]n$	By hfst on $\mathcal{F}_1, \mathcal{F}_2$
$\Psi \vdash [\sigma_1]\mathsf{fst}(m) \preccurlyeq^{\mathcal{H}}_{\tau} [\sigma_2]n$	By Substitution

□

Lemma 6. Howe's relation mimics simulation conditions, extended to pairs. Given

$$\langle m_1, m_2 \rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n$$

Show

$$n \Downarrow \langle p_1, p_2 \rangle \quad \text{s.t.} \quad m_1 \preccurlyeq^{\mathcal{H}}_{\tau} p_1,\ m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} p_2$$

By inversion of Howe's relation,

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash m_1 \preccurlyeq^{\mathcal{H}}_{\tau} m_1'\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash m_2 \preccurlyeq^{\mathcal{H}}_{\tau'} m_2'\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ \Gamma \vdash \langle m_1', m_2' \rangle \preccurlyeq^{\circ}_{\tau\times\tau'} n\end{array}}{\langle m_1, m_2 \rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} n}\ \text{hpair}$$

We note $\langle m_1', m_2' \rangle \Downarrow \langle m_1', m_2' \rangle$ by pairs evaluate to themselves, and by similarity on $\mathcal{D}_3$ we have $\exists p_1, p_2$ s.t. $n \Downarrow \langle p_1, p_2 \rangle$ where,

$$H : m_1' \preccurlyeq^{\circ}_{\tau} p_1 \qquad G : m_2' \preccurlyeq^{\circ}_{\tau'} p_2$$

Then,

$m_1 \preccurlyeq^{\mathcal{H}}_{\tau} p_1$	By semi-transitivity on $\mathcal{D}_1$ and H
$m_2 \preccurlyeq^{\mathcal{H}}_{\tau} p_2$	By semi-transitivity on $\mathcal{D}_2$ and G

Lemma 7. Downward closure: if $m \preccurlyeq^{\mathcal{H}}_{\tau} n$ and $m \Downarrow v$ then $v \preccurlyeq^{\mathcal{H}}_{\tau} n$.
Proof is by induction of the derivation $\mathcal{E}$ of the evaluation and proceeds by cases. We note case pair is covered by val since pairs can only evaluate to themselves. Thus there is no extension proof required for them.

We use case hfst as our representative case.

Case

$$\mathcal{E} = \cfrac{\begin{array}{c}\mathcal{E}_1\\ m \Downarrow \langle m_1, m_2 \rangle\end{array} \qquad \begin{array}{c}\mathcal{E}_2\\ m_1 \Downarrow v\end{array}}{\text{fst}(m) \Downarrow v}\ \Downarrow \text{fst}$$

Given $\text{fst}(m) \preccurlyeq^{\mathcal{H}}_{\tau} n$ and $\text{fst}(m) \Downarrow v$ (denoted as A_1, A_2 respectively), want to show we have $v \preccurlyeq^{\mathcal{H}}_{\tau} n$.

By inversion on the Howe relation assumption we get

$$\cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash m \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m'\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \vdash \text{fst}(m') \preccurlyeq^{\circ}_{\tau} n\end{array}}{\Gamma \vdash \text{fst}(m) \preccurlyeq^{\mathcal{H}}_{\tau} n}\ \text{hfst}$$

Then,

$\langle m_1, m_2 \rangle \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} m'$	By IH on $\mathcal{D}_1, \mathcal{E}_1$
$m_1 \preccurlyeq^{\mathcal{H}}_{\tau} \text{fst}(m')$	By simulation
$v \preccurlyeq^{\mathcal{H}}_{\tau} \text{fst}(m')$	By IH on above and $\mathcal{E}_2$
$v \preccurlyeq^{\mathcal{H}}_{\tau} n$	By howe-osim transitivity on above and $\mathcal{D}_2$

Lemma 8. Howe relation corresponds to applicative similarity (closed terms only). Formally: if $\quad p \preccurlyeq^{\mathcal{H}}_{\tau} q \quad$ then $p \preccurlyeq_{\tau} q$.
Proof is by coinduction. We extend this proof by adding a single case: when the Howe derivation ends in the `hpair` rule

Case

$$\begin{array}{c}\mathcal{D}\\ p \preccurlyeq^{\mathcal{H}}_{\tau\times\tau'} q\end{array} \qquad\qquad \begin{array}{c}\mathcal{E}\\ p \Downarrow \langle p_1, p_2 \rangle\end{array}$$

By downward closure on $\mathcal{D}, \mathcal{E}$ we have,

$$\langle p_1, p_2 \rangle \preccurlyeq^{\mathcal{H}}_{\tau \times \tau'} q$$

By simulation, $\exists\, p_1', p_2'$ s.t. $q \Downarrow \langle p_1', p_2' \rangle$ s.t. $p_1 \preccurlyeq^{\mathcal{H}}_{\tau} p_1'$ and $p_2 \preccurlyeq^{\mathcal{H}}_{\tau} p_2'$. Then,

$p_1 \preccurlyeq_\tau p_1' \quad \text{and} \quad p_2 \preccurlyeq_\tau p_2'$	By co-IH
$\langle p_1, p_2 \rangle \preccurlyeq_{\tau \times \tau'} \langle p_1', p_2' \rangle$	By compatibility (C7)
$p \preccurlyeq_{\tau \times \tau'} q$	Pairs always evaluate to themselves

□

Theorem 3. Recall the main statement,

$$\Gamma \vdash p \preccurlyeq^{\mathcal{H}}_{\tau} q \iff \Gamma \vdash p \preccurlyeq^{\circ}_{\tau} q$$

($\leftarrow$ direction) By lemma 4.

($\rightarrow$ direction) Proof by induction on Γ, using lemma 8 as the base case and closure under substitution for the step case. The step case works on arbitrary howe relation, thus this proof does not need to be extended to work for pairs. □