Towards a sustainable pay-as-you-go pension system: the Italian case study

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February 28, 2022

Abstract

This project aims to address the issue of the sustainability in the long term of the Italian pension system, therefore we divided our analysis into two main parts: the demographic evolution of the population and the economic model of the Italian pension system. In both parts we identified several relevant control variables that can affect the future trend of the pension system. In conclusion we find that a situation with no immigration whatsoever is not likely to be sustainable in the long term compared to other more favourable scenarios with immigration. We also found that the sustainability of the pension system can be affected by socioeconomic factors, such as an increase of the retirement age or an increase of the contribution rates.

1 Introduction

The issue we want to address with this work is the sustainability in the long term of the Italian pension system, which is currently implemented on the pay-as-you-go basis. Current demographic trends make a clear statement about the ageing of the population in the developed countries, caused by the decline of fertility rates and the rising of life expectations. This drift has several critical implications. We will mainly focus on its consequences on a macroeconomic level, studying the ratio between the pension volume and INPS (National Institution of Welfare) contributions. The dynamics of the model and the possibility of an equilibrium to be reached are studied in detail, considering a number of relevant variables, such as migration flows and economic parameters. In particular, we will consider how manipulating several parameters such as working rates, retirement age, contribution rates and average pension amount can impact on pension burden. We will divide our analysis in two main parts: the demographic evolution of the population and the economic model of the Italian pension system. In this context, immigration and the economic parameters are variables that can be adjusted for the two distinct parts. Therefore we will present simulated scenarios for both.

2 Demographic model

The approach we followed to simulate the demographic evolution consists in using a Leslie Matrix, which is a standard mathematical tool used in Ecology to describe the growth of a population. In this model we introduced as variables the age-specific birth rates $\bar{\alpha}_j \geq 0$, which represent the estimate of live births in one year per age group j per person, and the survival rates $0 < \omega_j < 1$, which represent the normalized probability of a person of age group j of surviving the time period of one year.

2.1 Leslie Matrix

The Leslie Matrix is a useful tool to condense the information about a one-year evolution of a population without migration flows.

For starters, if we consider a population without sex structure with N age groups (from 0 to $N-1$), the number of newborns $x_0(t+1)$ at time $t+1$ (here $x_j(t)$ represents the number of people of the age group j at time t) is the sum of contributions given by the people of all age groups $x_j(t)$ at time t controlled by the age-specific birth rates $\bar{\alpha}_j$:

$$x_0(t+1) = \sum_{j=0}^{N-1} \bar{\alpha}_j \; x_j(t)$$

The people from other age groups $x_j(t+1)$ $(j > 0)$ instead evolve with a certain survival probability ω_{j-1} from the people of the previous age group $x_{j-1}(t)$ at time t:

$$x_j(t+1) = \omega_{j-1} \; x_{j-1}(t), \quad j > 0$$

In order to condense this information, the Leslie Matrix for a population without sex structure is defined as such:

$$L := \begin{bmatrix} \bar{\alpha}_0 & \bar{\alpha}_1 & \cdots & \bar{\alpha}_{N-2} & \bar{\alpha}_{N-1} \\ \omega_0 & 0 & \cdots & 0 & 0 \\ 0 & \omega_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \omega_{N-2} & 0 \end{bmatrix}$$

the information of the population distribution is condensed in the vector

$$x(t) := [x_0(t), \; x_1(t), \; ..., \; x_{N-1}(t)]^T$$

and the dynamics of the population is therefore given by:

$$x(t+1) = L \; x(t)$$

In order to consider a more realistic population with a sex structure, we need to redefine the population vector as

$$x(t) := [x_0^F(t), \; x_1^F(t), \; ..., \; x_{N-1}^F(t), \; x_0^M(t), \; x_1^M(t), \; ..., \; x_{N-1}^M(t)]^T$$

where we divided the male and female population, and we can consider at time $t = 0$

$$\varphi = \frac{x_0^F(0)}{x_0(0)}$$

the fraction of females at birth. The dynamics of the female subpopulation is identical to the dynamics of a population without sex structure with age-specific birth rates $\varphi \, \bar{\alpha}_j$ and survival rates ω_j^F (in this context $\bar{\alpha}_j$ must be interpreted as the estimate of live births of both sexes in one year per age group j per woman). The dynamics of the male subpopulation instead is identical only for age groups $j > 0$ (with survival rates ω_j^M), while the number of newborn males $x_0^M(t+1)$ at time $t+1$ is the sum of contributions given by the people of all female age groups $x_j^F(t)$ at time t controlled by the birth rates $(1 - \varphi) \, \bar{\alpha}_j$:

$$x_0^M(t+1) = \sum_{j=0}^{N-1} (1 - \varphi) \, \bar{\alpha}_j \; x_j^F(t)$$

The dynamics of the population is therefore condensed in the notation

$$x(t+1) = \Lambda \; x(t)$$

where Λ is a $2N \times 2N$ matrix and it is the equivalent of a Leslie Matrix for a sex-divided population, and is defined as such:

$$\Lambda := \begin{bmatrix} L^F & 0 \\ d & 0 \\ 0 & L^M \end{bmatrix}$$

where L^F is a $N \times N$ matrix, d is a $1 \times N$ matrix and L^M is $(N-1) \times N$ matrix, and they are defined as such:

$$L^F := \begin{bmatrix} 0 & 0 & \cdots & \varphi\,\bar{\alpha}_{17} & \cdots & \varphi\,\bar{\alpha}_{50} & \cdots & 0 & 0 \\ \omega_0^F & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \omega_1^F & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & \omega_{N-2}^F & 0 \end{bmatrix}$$

$$d := \begin{bmatrix} 0 & 0 & \cdots & (1-\varphi)\,\bar{\alpha}_{17} & \cdots & (1-\varphi)\,\bar{\alpha}_{50} & \cdots & 0 & 0 \end{bmatrix}$$

$$L^M := \begin{bmatrix} \omega_0^M & 0 & \cdots & 0 & 0 \\ 0 & \omega_1^M & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \omega_{N-2}^M & 0 \end{bmatrix}$$

where we considered only the age-specific birth rates $\bar{\alpha}_j$ with $17 \leq j \leq 50$ to be relevant. In our case, we further set $N = 100$.

One can demonstrate that the matrix Λ has precisely one real positive eigenvalue λ_0, which corresponds to a one-dimensional eigenspace, from which we can choose the normalized eigenvector z_0 to have all real positive components. Furthermore, all the other eigenvalues are smaller in modulus than λ_0. (see Perron-Frobenius Theory for the Λ Matrix appendix for further reference)

2.2 Migration flows

In the context of the Italian population, due to the very low values for the age-specific birth rates $\bar{\alpha}_j$ and the very high values of the survival rates ω_j^F and ω_j^M, as we will show the Simulation framework section, the system is bound to tend towards a very old and small population, which inevitably results in a collapse of the pension system. A possible way to avoid this trend is to work with factors that are external to the simple and bare evolution with the Leslie Matrix: migration flows, for which we will consider in general slightly higher values for the age-specific birth rates α'_j.

In order to introduce immigration and emigration as variables in our model of demography, we define the vectors

$$y^i(t) := [y_0^{i,F}(t),\ y_1^{i,F}(t),\ \ldots,\ y_{N-1}^{i,F}(t),\ y_0^{i,M}(t),\ y_1^{i,M}(t),\ \ldots,\ y_{N-1}^{i,M}(t)]^T$$

$$y^e(t) := [y_0^{e,F}(t),\ y_1^{e,F}(t),\ \ldots,\ y_{N-1}^{e,F}(t),\ y_0^{e,M}(t),\ y_1^{e,M}(t),\ \ldots,\ y_{N-1}^{e,M}(t)]^T$$

to condense the information about the number of immigrants and emigrants per age group for both sexes at time t. In this scenario we consider only the immigrants to have age-specific birth rates α'_j different from those of the Italian citizens.

We are therefore demanded to reconsider at each timestamp t the value of $\bar{\alpha}_j$, which results in a time dependence of the birth rates $\bar{\alpha}_j(t)$, and hence a time dependence also of the evolution matrix $\Lambda(t)$. The evolution equation now becomes:

$$x(t+1) = \Lambda(t)\ x(t) + y^i(t) - y^e(t)$$

where the $\bar{\alpha}_j(t)$ are set to evolve as such:

$$\bar{\alpha}_j(t+1) = \left[\frac{\omega^F_{j-1}\ x^F_{j-1}(t) - y^{e,F}_j(t)}{\omega^F_{j-1}\ x^F_{j-1}(t) - y^{e,F}_j(t) + y^{i,F}_j(t)} \right] \bar{\alpha}_j(t) + \left[\frac{y^{i,F}_j(t)}{\omega^F_{j-1}\ x^F_{j-1}(t) - y^{e,F}_j(t) + y^{i,F}_j(t)} \right] \alpha'_j$$

This evolution equation encodes the fact that the new immigrants come to Italy with different age-specific birth rates, and therefore the existing ones should be updated accordingly, using a weighted average.

2.3 Stationary population

In the context of our model, we consider three different regimes, based on different migration flows.

The first regime is the absence of migration flows: the $\bar{\alpha}_j$ are constant, and so is the matrix Λ. The evolution of the population therefore simply becomes:

$$x(t) = \Lambda^t\ x(0)$$

where one can easily see that, at large t, the evolution is dominated by the eigenvalue λ_0 and the population tends towards the stationary population $u\, z_0$ (u is a positive real constant). This population is stationary in the sense that the age structure, in terms of the relative numbers of individuals in the different age groups, is constant, while the absolute number of individuals is modulated by the value of λ_0, whether it is greater than or less than 1.

The second regime consists in taking the vectors y^i and y^e to have constant values, inferred from real data, throughout the simulation. In contrast to the previous scenario, the $\Lambda(t)$ matrix is time-dependent, since $\bar{\alpha}_j(t)$ are time-dependent, but one can show that the limit $\Lambda^\infty := \lim_{t\to\infty} \Lambda(t)$ is well defined, so there exists a limiting stationary population $u\, z_0^\infty$ defined by:

$$\Lambda^\infty\ z_0^\infty = \lambda_0^\infty\ z_0^\infty$$

where λ_0^∞ is the real positive eigenvalue of Λ^∞. However, the system is not bound to tend to this distribution, since the evolution equation is

$$x(t+1) = \Lambda(t)\ x(t) + y^i - y^e$$

and the y^i and y^e vectors are external to the mathematical machinery of the Leslie Matrix.

In the third regime we follow the treatment in Angrisani et al[1], which consists in using the immigration to reach the stationary population more easily, but unlike them we work with a non-null and constant emigration flux y^e. Much alike to the second regime, the $\Lambda(t)$ matrix is time-dependent, with a well defined limit Λ^∞, with eigenvalue λ_0^∞ and eigenvector z_0^∞, but in this case we define the immigration vector $y^i(t)$ as such:

$$y^i(t) := \left(1 - \mu^{t+1}\right) \left[u^{(t)}\ z_0^{(t)} - \Lambda(t)\ x(t) + y^e\right]$$

where $z_0^{(t)}$ is the normalized eigenvector of $\Lambda(t)$ with positive real eigenvalue $\lambda_0^{(t)}$, where $\mu \in [0, 1[$ in a convergence parameter and where $u^{(t)}$ is defined as

$$u^{(t)} = \max_k \left[\frac{(\Lambda(t)\ x(t) - y^e)_k}{z^{(t)}_{0,k}} \right]$$

The reason behind these definitions is that one uses the immigration to reach at each time t the smallest stationary population $u^{(t)}\ z_0^{(t)}$ of the $\Lambda(t)$ Matrix. Actually, this would be true only if $\mu = 0$, but in limit as $t \to \infty$, one can demonstrate that in this way, the system reaches a stationary population $u\, z_0^\infty$ for the Λ^∞ Matrix, where $z_0^\infty = \lim_{t\to\infty} z_0^{(t)}$ and $\lambda_0^\infty = \lim_{t\to\infty} \lambda_0^{(t)}$ also hold. (see Convergence to the stationary population appendix for further reference)

3 Pension System model

The pension system we decided to implement is based on a pay-as-you-go or PAYGO basis. Expenditures are thus financed with funds that are currently available rather than borrowed, which means that contribution spent by workers at present time is used to pay present day pensions. This implies that the more income is produced one given year, the more sustainable the pension burden will be for the same year.

By opting for this system we made somewhat of a strong approximation of the complex and diverse Italian pensions panorama, that witnessed a considerable number of temporary laws and reforms in the last few years only. The simplification can be justified considering that our model aims to produce simulations of future scenarios, hence the history of the Italian pension system only relatively impacts on the model. Moreover, some approximations are required when the aim is to build a simple model, and this led us to search for a compromise.

We considered pension stats that matched current data about the Italian pension system, therefore implicitly leaving the laws currently in force unaltered. To be specific, instead of considering a specific set of laws to refer to, we studied datasets containing information about retirement probability at different ages and average pensions, and applied the same values to future scenarios as well, forcing them to remain constant. This simplification thus allowed us to base our simulations on realistic numbers, without having too much bureaucracy complexity to handle.

To implement the working-pension system we had to take some aspects into considerations. We divided contributors into three different categories, each one contributing to the payment of pensions with a distinct share γ_i, depending on professional profile (employed, self-employed and retired). The values we used for these contribution rates correspond to real data, collected from the INPS website[6]. A simplified summary is provided below.

Worker type	Contribution rate γ_i
Employed	$\gamma_E = 33\%$
Self-employed	$\gamma_S = 25\%$
Retired	$\gamma_R = 24\%$

As far as average amount of incomes and pensions are concerned, we extracted relevant data from Italian Finances Department [5] and ISTAT datasets [4], as well as the rates of employment and retirement per age group. We assumed the per capita income to stay constant through the years, thus having the total working income to vary with time as a function of the number of working people only. The overall income and pension values considered act as initial conditions for the model, which will vary accordingly to the evolution of the population during the time span considered for our simulations.

The same kind of assumptions are valid for average pension income and employment/retirement rates. For the latter issue, we defined a retirement ratio for male and female population as

$$r_j^M = \frac{R_j^M}{N_j^M}, \quad r_j^F = \frac{R_j^F}{N_j^F}$$

where R_j^M and R_j^F indicate the number of retired people per age group j for males and females, and N_j^M, N_j^F the total population in that age group, for males and females. The ratios r_j^M and r_j^F hence serve as probabilities of being retired per age group and sex.

We assumed the retirement ratio to be a fixed number in time. The evolution of the system only impacts on the total number of pensioners per age group, and not on the ratio just defined.

We considered pensions to be of four different kinds, as stated by the INPS. Specifically, we took into account longevity pay, disability pay, survivor's pension and pension due to occupational injury. Having considered not only old-age pensions means that r_i is non zero for every age group, although its module becomes significantly higher for people over 60 years old.

Following the same kind of reasoning, the working ratios are defined as

$$w_j^M = \frac{W_j^M}{N_j^M}, \quad w_j^F = \frac{W_j^F}{N_j^F}$$

where W_j^M and W_j^F stand for the number of working people in the age group j per gender. The ratio will naturally be higher in age groups from 15 years old to 55 years old and almost null for people in the retirement age groups. We set w_i to be zero for people younger than 15 years old. Working ratios are assumed to be fixed numbers in time, as described for retirement ratios.

By using real data as initial values, we decided to consider some important aspects of Italian society, focusing mainly on disparities that have a marked impact on demographic and economic trends. We mainly considered the male-female retribution and employment gap.
To take into account the substantial disparity registered in work retribution between the male and female population, we calculated an average ratio between average female income and average male income, using real income data collected from Italian Finances Department. We will refer to this parameter as *gender pay ratio* (GPR) in the following. We assumed the gender pay ratio to remain constant with time, and made an approximation by assuming it has the same percentage impact, there are no GPR variations with age. Although the GPR probably impacts in a more complex way in reality, we considered this approximation fair enough for the purpose of this work.

4 Sustainability indicators

To understand the effect that population evolution and economic parameters have on simulation outcomes we introduced some useful parameters.

The first is the *Inverse Dependency Ratio* (IDR), which is given by the ratio of total population in the 16 - 65 age group (expected working population) and total population over 65 years old (expected retired population) at a given time t.

$$IDR = \frac{expected\, workers}{expected\, pensioners}$$

This indicator gives a first idea of how sustainable the system is. The higher the IDR is at time t, the more affordable it will be to pay time t pensions.

The aim of the simulations is to see under what circumstances the equilibrium between incomes and outcomes is obtained[3].
Incomes are represented by pension contributions regularly deposited by workers, while outcomes are pensions distributed to retired people. To study the equilibrium of the system and its sustainability, we introduced an indicator c which is given by the ratio of total pensions volume distributed at time t and contributions deposited at time t. In the following we will also refer to this parameter as *Pension-Contribution Ratio* (PCR).

Specifically, the contributions volume is composed of three different terms: shares deposited by employed workers, self-employed workers and retired people, as already introduced at the beginning of the paragraph. Each term is obtained by multiplying the number of workers/retirees by their average annual income and by the contribution rate associated to the designated working category.
Initial values for these parameters have been collected from official government data, however, their magnitude has been considered variable with time, as we will explain further down this paragraph.
Pensions volume has been computed as the product: *probability of being retired per age group* · *average pension amount per age group*, summed for each age group and kind of pension, as previously stated.

The ratio then takes the following shape:

$$c = \frac{I_R}{I_E \cdot \gamma_E + I_S \cdot \gamma_S + I_R \cdot \gamma_R}$$

where I_E and I_S stand for average incomes of Employed workers and Self-Employed workers, I_R

represents the average income of Retirees (essentially as a retirement salary), and γ factors are the associated contribution rates. The pensions-contributions ratio needs to be $c \lesssim 1$ for the system to be sustainable. The trend of this indicator has been the main focus of our simulations.

The variation of the previously introduced equilibrium parameters depends on a definite set of variables. We considered three distinct scenarios, which have already been presented in the Demographic model Section.

Additionally, we studied the trends by also varying some economic parameters. Specifically, we considered several possible economic maneuvers that can be implemented to affect the system equilibrium, and thus modify the value of equilibrium parameter c. A list of the considered variables is provided below:

1) average pension amount
2) working ratio
3) contribution rates
4) retirement ratio

As we will show more broadly in the Simulation framework section, in the case of average pension, contribution ratios and working ratio, we simply changed initial values by a fixed percentage amount. In the case of retirement ratio, we implemented a slightly more complex mechanism. We considered current data for retirement probability per age, and gradually set probabilities to zero, with a time span of one year for each update of the probabilities. This operation was repeated until reaching the maximum age, which we set to be at 65. This process can be seen as an increase of minimum retirement age. For example, considering the year 2022, we set to zero the probability of being retired at the age of 40 (which appeared to be the lowest retirement age for 2021). For the simulated year 2023, we set to zero the probability of being retired at 41, and so on until reaching a minimum retirement age of 65.

5 Simulation framework

The simulations implemented for this work strongly rely on initial conditions. As already mentioned, we collected data for initial conditions mostly from ISTAT and other reliable sources[4][5][6]. In the following, we will give some insight on how raw data has been extracted and processed in a Python development environment.

5.1 Demographic analysis

For our model, we considered maximum age to be set at 99 years old, as data shows that the amount of people older than 100 years is negligible.

Taking into account the relevant impact of the COVID-19 pandemic on demographic trends, we decided to consider birth rates and survival rates extracted from 2019 data instead of more recent ones, so not to risk considering oddly small values caused by decreased fertility and increased mortality in the last few years. For the birth rates analysis, we computed them as the ratio between the estimated number of live births for each age group and mother origin and the female population in the same group. The needed data for the Italian population was computed as the difference between the total resident population and the immigrant population. We also aggregated birth rates for people less than 17 years old in the 17 years old slot and data for women older than 50 in the 50 years old slot. This operation is justified by the extremely low birth rates observed for the very young and the over-fifty females. As we can see in Figure 1, the birth rates have different distributions and cumulative values for native and immigrant population, for the most part the immigrant birth rates being slightly higher than those of the Italian people. In fact, the cumulative value of this indicator reaches 1.14 for native people, while it is much higher, approximately 1.85, for immigrant population.

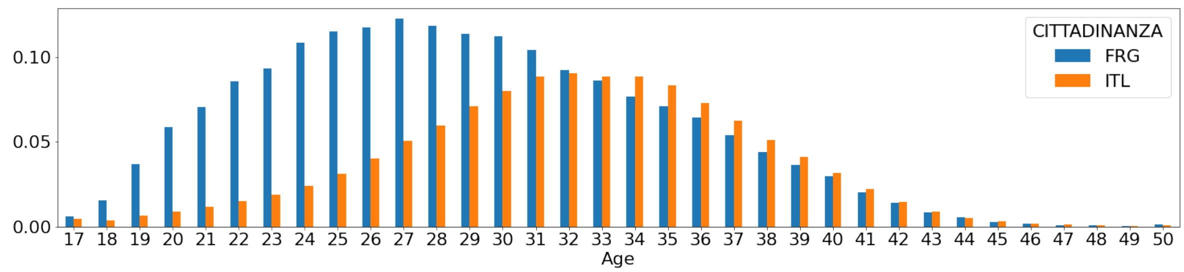

Figure 1: Birth rates for Italian (ITL) and foreign (FRG) population.

As far as the immigrant arrivals and emigrant departures each year are concerned, civil registration data are used, but since ISTAT only provides data aggregated in four age groups, we had to use the averages as values for each single age.

Finally, the demographic evolution from 2021 until 2071 - a 50 years span[1] - has been simulated converting relevant data - stored as Pandas dataframes initially - into Numpy arrays and applying the Leslie matrix with an iterative approach.

For the scenario "with current migration", the same number of individuals are added and subtracted each year to model constant immigration and emigration.

The scenario "with controlled immigration" differs from the previous one as the new immigrant population is computed by the optimization algorithm, with parameter of convergence set to 0.9. Finally, since the goal is to counteract the ageing of the population, a sigmoid-like function was used to modulate the flow of immigrants according to age:

$$sig(z) = \frac{1}{1 + \exp\left(\frac{z-M}{T}\right)}$$

where z are ages between 0 and 99, T is a parameter used to modulate the steepness of the curve (set to 4 in our simulation) and M is used to translate the curve (set to 32 in our simulation). This modulation acts as an admission filter: immigrants younger than 32 are more likely to be admitted.

5.2 Economic analysis

In order to compute the sustainability indices, data regarding pensions and incomes was needed. We extracted average pension data from ISTAT datasets, which contained relevant information up to the year 2019 for each type of pension: longevity, survival, injury and disability. Pension amounts are subject to an automatic revaluation according to measured inflation for a certain year, so we took into account inflation rates measured in 2020 and 2021[2] to reevaluate average pensions data and obtain relevant values for the year 2021, which we set to be the starting point of our simulations.

As far as working probabilities are concerned, we set w_i to be zero for people under 15. We aggregated working probability data of 65+ population on 65.
As previously stated, working population has been divided into employed and self-employed. The self-employed population has been computed as the difference between the total working population and the employed population. The proportion between members of these two working categories has been kept constant with time.

Data regarding Italian population income has been extracted from aggregated IRPEF data from 2019 available on the Finances Department website. As stated in their Methodology Report, the self-employed income amounts are reported before pension contributions, unlike the employed ones, which we had to estimate from the net amount.

Since the available datasets provided disaggregated data for each age group but not for each sex, we computed for each work profile (employed and self-employed) the GPR to estimate the incomes for

[1] A 50 years time span has been chosen since this is ISTAT's span for demographic forecasts, justified by the greater unpredictability of distant events.

[2] Specifically, we considered a revaluation rate of 0.5% in 2020 (which means that the average pension was 0.5% greater than in 2019), and 0% in 2021.

each age group and sex using the following equations:

$$W_j I_j = W_j^F I_j^F + W_j^M I_j^M, \quad GPR = \frac{I_j^F}{I_j^M}$$

where W_j and I_j are the working population and the average income in the age group j.

Finally, the economic measures to make the system more sustainable have been implemented in the following ways:

1) 10% pension amount decrease
2) 10% employment rate increase
3) 40% contribution rate for employed workers and 30% contribution rate for self-employed workers and pensioners, instead of 33%, 25% and 24%, respectively
4) annual longevity-pension age increase starting from the youngest pensioner age (40 years) up until 65

The last measure entailed shifting probabilities to take into account not only the reduced retired population but also the increased working population. A balance equation similar to the one used to estimate male and female incomes was applied to estimate the new employed and self-employed ratios of the threshold age at each time.

6 Results

A summary of our results concerning the demographic evolution is presented in Figures 2 and 3.

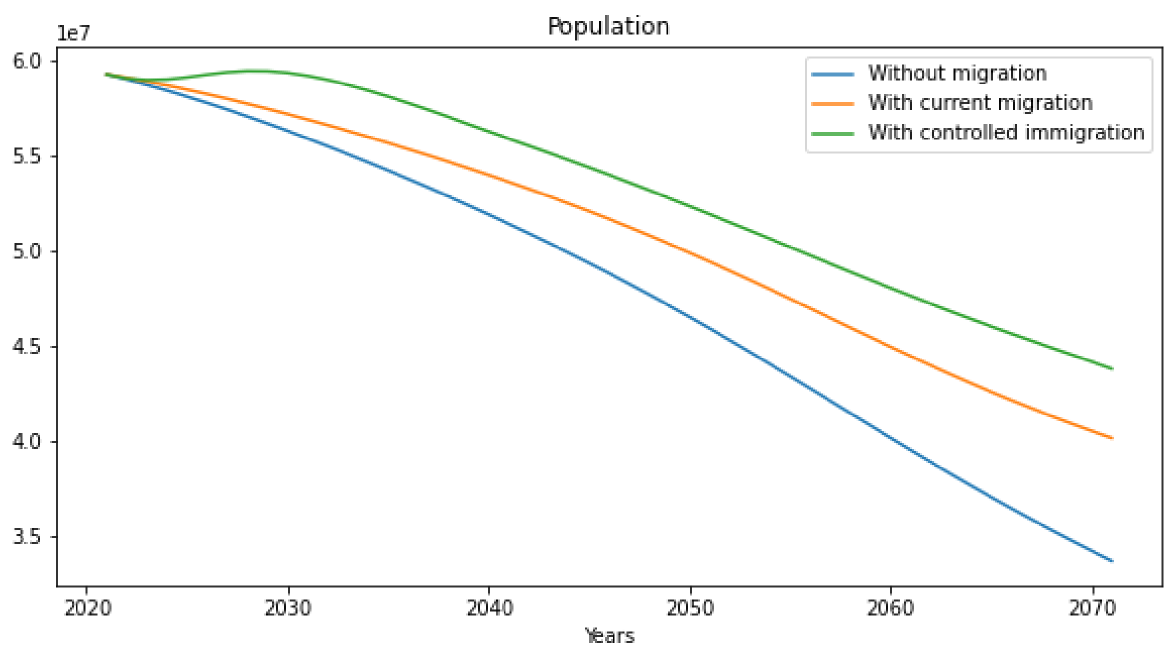

Figure 2: Total population evolution. The three immigration scenarios are considered in a time span of 50 years.

In Figure 2, for all scenarios, the Italian population is subject to a remarkable and steady fall. The worst scenario is the “no immigration” one, that foresees a drop of more than 25 million people in 50 years only. The other two cases, by including immigration flows, considerably mitigate the magnitude of the negative trend.

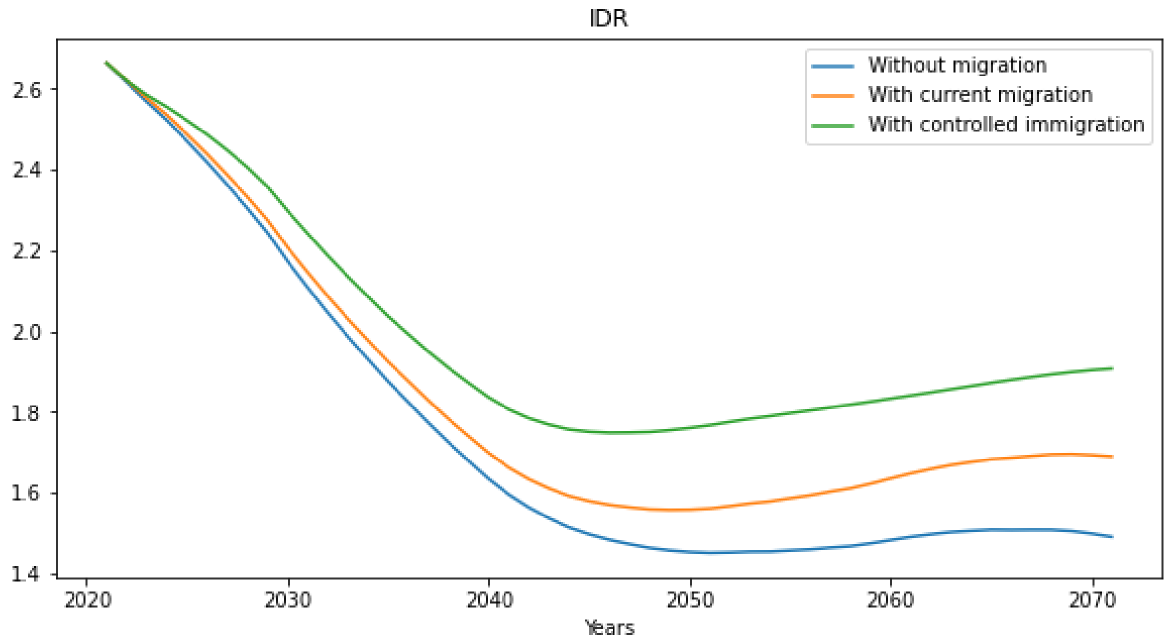

Figure 3: IDR evolution. The three immigration scenarios are considered in a time span of 50 years.

The considerations made about demographic evolution have a concrete impact on the IDR trend, as one can see in Figure 3. In all migration scenarios, the indicator witnesses a steep descent in the first 20 years of the simulation, and then a relative stabilization in the following years, where the system is in proximity of the stationary population, and therefore the relative number of individuals is the various age groups is approximately constant. The very high values for the survival rates considered in our model demonstrate here that the Italian population is bound to have a social structure oriented towards old people.

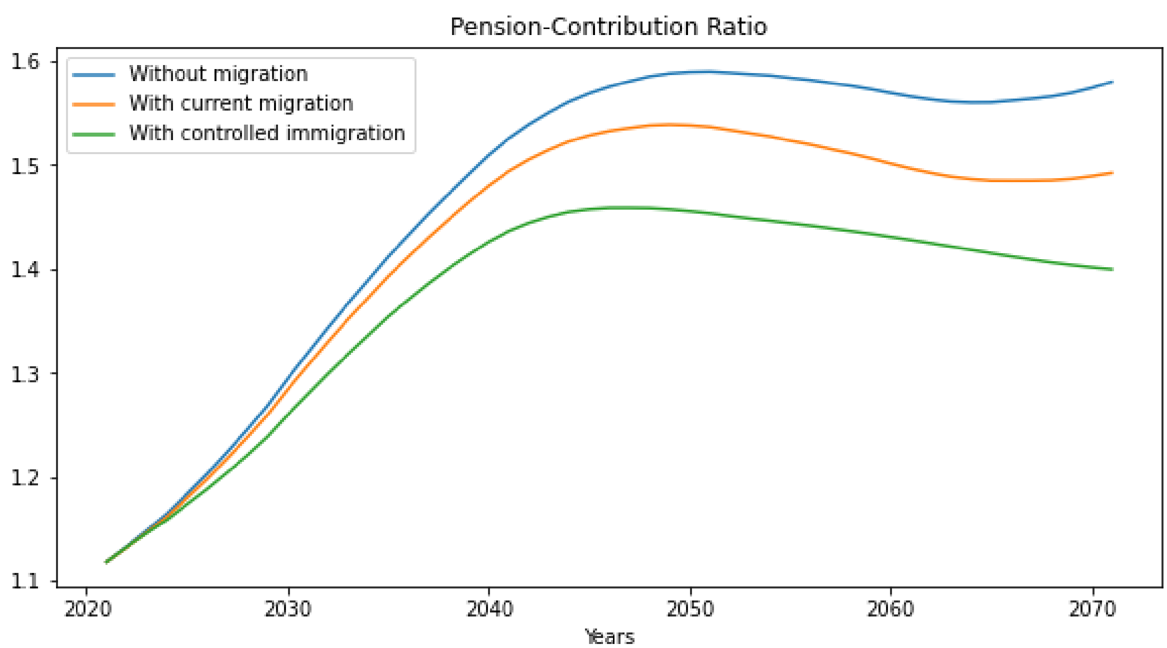

Figure 4: PCR with no economic measures applied. The three immigration scenarios are considered in a time span of 50 years.

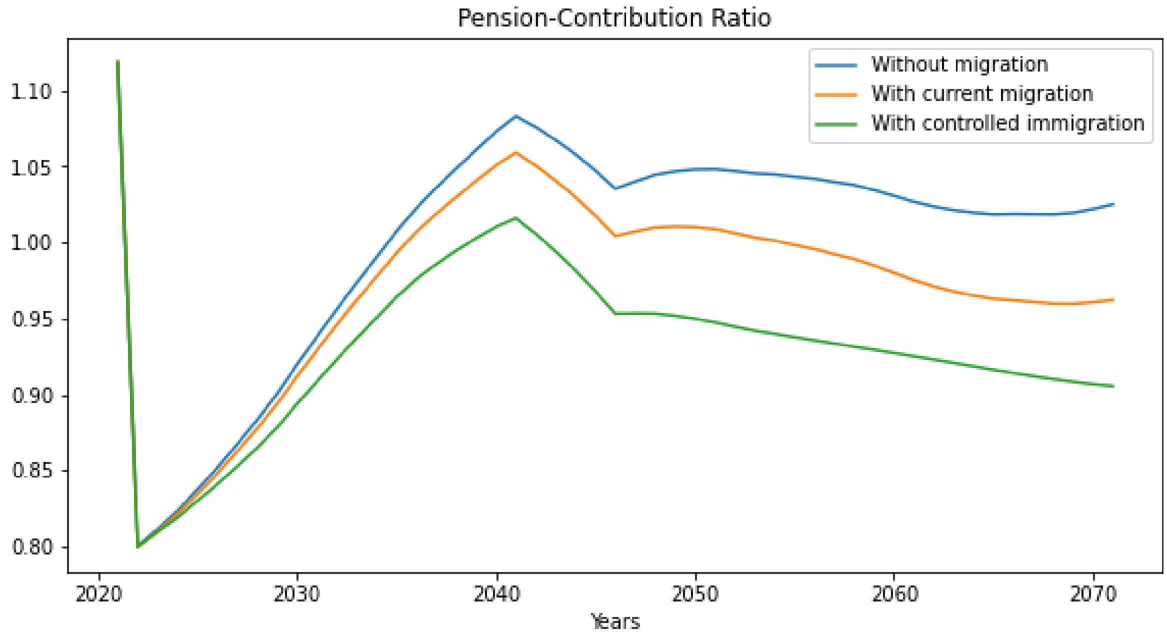

Figure 5: PCR with all presented economic measures applied. The three immigration scenarios are considered in a time span of 50 years.

The comparison between Figures 4 and 5 clearly shows that the considered economic parameters have a sizeable impact on the trend of the PCR. The measures 1), 2) and 3), being implemented in the second year of our simulation, determine the sharp decrease of the PCR visible in Figure 5, while the measure 4) is much more gradual. Both in the cases of controlled immigration and current immigration, the ratio goes below the unit after reaching a peak around the year 2040, while the scenario without migration flows remains unsustainable in the long run (PCR > 1). Instead, as one can witness from Figure 4, the situation where no economic measures are applied is far from sustainable in every possible demographic scenario, the controlled immigration always being the most favourable one between the three.

7 Conclusions

We have addressed the Italian pension sustainability issue by analyzing the main demographic and economic factors and implementing simulations based on real data.

It has been proven that among all demographic scenarios the most favourable is the controlled immigration one, followed by the current scenario and the scenario without migration, even if in each one of these situations the future Italian population is inevitably bound to decrease and shift towards the elder age classes (low Inverse Dependency Ratio). Nevertheless, the pension system is not meant to be completely unsustainable, as long as relevant measures are applied, such as pension amount reduction, contribution rates increase, employment rates increase and pension age increase, which may result in a sustainable Pension-Contribution Ratio (smaller than 1) in the long run.

However, it is worth to be noted that the results presented here are to be seen as qualitative rather than quantitative, since the model has strong assumptions and there are other factors that have to be taken into account in a real decision-making context, such as inflation and socio-political aspects.

8 Appendices

8.1 Perron-Frobenius Theory for the Λ Matrix

It is a well known result (Perron-Frobenius Theorem[2]) that an *irreducible*[3] *non-negative*[4] square matrix A always has a positive eigenvalue λ_0 that is equal to its *spectral radius* $\rho(A)$[5], that is associated to a one-dimensional eigenspace, where the corresponding eigenvector z_0 can be chosen to be *positive*. As one can quickly check, the Λ Matrix defined in the Demographic model section, despite being non-negative, is not irreducible, so the Perron-Frobenius Theorem does not directly apply. In order to get around this issue we can re-write Λ as:

$$\Lambda = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$$

where A is a 51×51 matrix defined as such:

$$A := \begin{bmatrix} 0 & 0 & \cdots & \varphi\,\bar{\alpha}_{17} & \cdots & \varphi\,\bar{\alpha}_{49} & \varphi\,\bar{\alpha}_{50} \\ \omega_0^F & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \omega_1^F & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \omega_{49}^F & 0 \end{bmatrix}$$

and where B and C are respectively 149×149 and 149×51 matrices defined specifically to obtain the Λ Matrix. As one can quickly check, A is irreducible, so the Perron-Frobenius Theorem holds, and A admits a positive eigenvalue λ_0^A with positive eigenvector z_0^A. The relevant fact here that saves the Perron-Frobenius result also for the whole Λ Matrix is that the B matrix is in lower triangular form (with 0s on the diagonal). Using the Laplace recursive formula to compute the characteristic polynomial of Λ, one finds that all the last 149 diagonal terms of Λ have 0 above them in the column, hence

$$\det(\lambda I - \Lambda) = \lambda^{149}\ \det(\lambda I - A)$$

So the remaining part of the Λ Matrix contributes only with eigenvalues equal to 0, and one can then conclude that the Λ Matrix admits a positive eigenvalue $\lambda_0 = \lambda_0^A$ that is equal to the spectral radius $\rho(\Lambda)$, associated to a one-dimensional eigenspace where a positive eigenvector z_0 lives (which is an extension to the whole Λ Matrix of z_0^A).

8.2 Convergence to the stationary population

In the context of an evolution without immigration (first regime), that can be summarized as

$$x(t) = \Lambda^t\ x(0)$$

the issues of convergence to the stationary population can be easily resolved if one decomposes the $x(0)$ vector into an eigenvector basis of Λ:

$$x(0) = \sum_{\lambda} c_\lambda\, z_\lambda$$

One can then see that the evolution equation becomes

$$x(t) = \sum_{\lambda} \lambda^t\, c_\lambda\, z_\lambda$$

[3]A square matrix A is said to be irreducible if the directed graph associated to the matrix G_A, obtained by linking the vertex i to the vertex j if $A_{ij} \neq 0$, is *strongly connected*, meaning that for every pair of vertices i, j there exists a path from i to j.

[4]A matrix is non-negative if all its entries are. Similarly, we will talk about positivity, negativity, and so on, for matrices and also for vectors, and we will always mean a property about all of the components of the object.

[5]The spectral radius $\rho(A)$ of a square matrix is defined as $\max\{|\lambda_1|\,,\ \ldots,\ |\lambda_n|\}$ where $\{\lambda_1,\ \ldots,\ \lambda_n\}$ are the eigenvalues of A.

and it is clear that for great values of t, the main contribution is given by the eigenvalue λ_0, which is the greatest in modulus among all other eigenvalues, so the population can be approximated by a multiple of z_0.

In a regime where the $\bar{\alpha}_j(t)$ evolve because of immigration, the sequence of $\Lambda(t)$ is convergent to a matrix Λ^∞, with principal eigenvalue λ_0^∞ and eigenvector z_0^∞ (proof in Angrisani et al[1]). In the second regime, while the sequence of matrices is convergent, due to the fact that the immigration and emigration vectors are constant and arbitrary, the population does not automatically tend towards a stationary population.

In the third regime however, the immigration is specifically used to reach more rapidly the stationary population. We can then slightly modify the proof presented in Angrisani et al[1] to prove the convergence of the sequence of populations to the stationary population of Λ^∞ with constant emigration y^e (which is assumed to never exceed the existing Italian population).

The target is to reach by means of immigration at each time stamp t the minimal stationary population for the $\Lambda(t)$ Matrix:

$$\Lambda(t)\ x(t) - y^e + y^i(t) = u\, z_0^{(t)}, \quad y^i(t) \geq 0$$

In order to obtain the minimal stationary population we therefore need to minimize the function

$$f^{(t)}(u) = \sum_k y_k^i(t) = \sum_k \left(u\, z_0^{(t)} - \Lambda(t)\ x(t) + y^e\right)_k$$

with the constraint that

$$\left(u\, z_0^{(t)} - \Lambda(t)\ x(t) + y^e\right)_k \geq 0, \quad \forall k$$

If we define

$$u^{(t)} = \max_k \left[\frac{(\Lambda(t)\ x(t) - y^e)_k}{z_{0,k}^{(t)}}\right]$$

one can quickly see that the constraint has now become that $u \geq u^{(t)}$, and that therefore the function $f^{(t)}(u)$ is minimized with this constraint at $u = u^{(t)}$. We can now define the optimal immigration as

$$y^i(t) := \left(1 - \mu^{t+1}\right) \left[u^{(t)}\, z_0^{(t)} - \Lambda(t)\ x(t) + y^e\right]$$

At each iteration the population therefore becomes

$$x(t+1) = \Lambda(t)\ x(t) - y^e + y^i(t) = u^{(t)}\, z_0^{(t)} - \mu^{t+1} \left[u^{(t)}\, z_0^{(t)} - \Lambda(t)\ x(t) + y^e\right].$$

From the constraint we find that

$$(\Lambda(t)\ x(t) - y^e)_k \leq u^{(t)}\, z_{0,k}^{(t)}, \quad \forall k$$

which in turn implies that

$$\|\Lambda(t)\ x(t) - y^e\| \leq \left\|u^{(t)}\, z_0^{(t)}\right\| = u^{(t)}$$

If we then calculate the quantity

$$\begin{aligned}
\left\| x(t+1) - u^{(t)}\, z_0^{(t)} \right\| &= \left\| u^{(t)}\, z_0^{(t)} - \mu^{t+1} \left[u^{(t)}\, z_0^{(t)} - \Lambda(t)\; x(t) + y^e \right] - u^{(t)}\, z_0^{(t)} \right\| \\
&= \mu^{t+1} \left\| u^{(t)}\, z_0^{(t)} - \Lambda(t)\; x(t) + y^e \right\| \\
&\leq \mu^{t+1} \left(u^{(t)} \left\| z_0^{(t)} \right\| + \left\| \Lambda(t)\; x(t) - y^e \right\| \right) \\
&\leq 2\mu^{t+1}\, u^{(t)}
\end{aligned}$$

And so we can conclude that

$$\left\| \frac{x(t+1)}{u^{(t)}} - z_0^{(t)} \right\| \leq 2\mu^{t+1} \xrightarrow[t\to\infty]{} 0$$

Which concludes the demonstration that, with such immigration, the normalized population tends as $t \to \infty$ to $z_0^{\infty} = \lim_{t\to\infty} z_0^{(t)}$.

8.3 Relevant figures

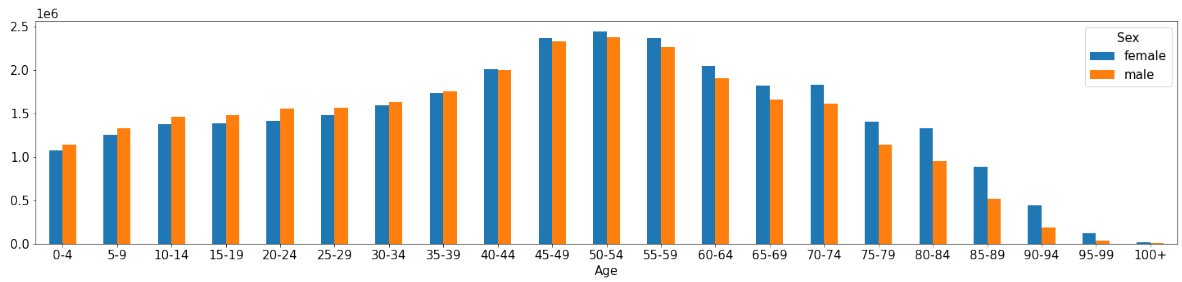

Figure 6: Age distribution for female and male population in 2021 (ISTAT data).

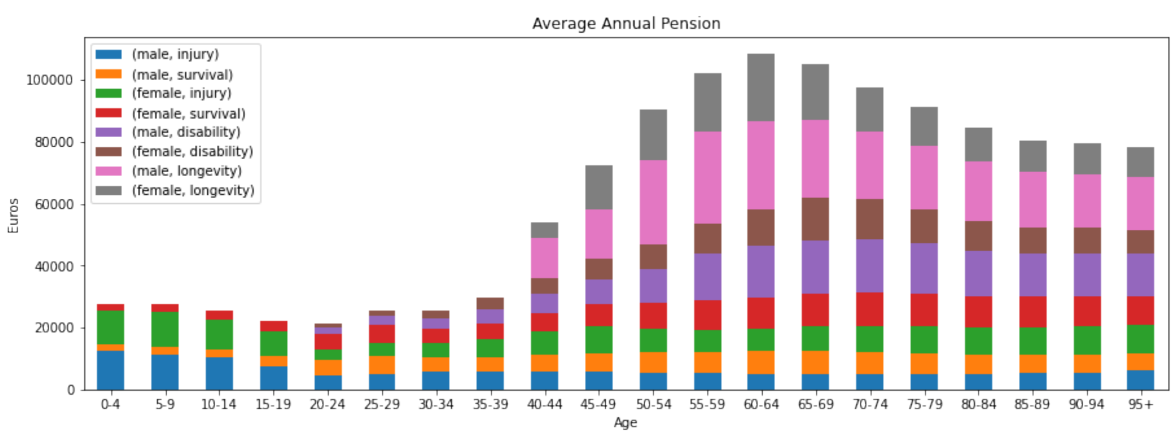

Figure 7: Average annual pension distribution for male and female population. Injury, survival, disability ad longevity pensions have been considered.

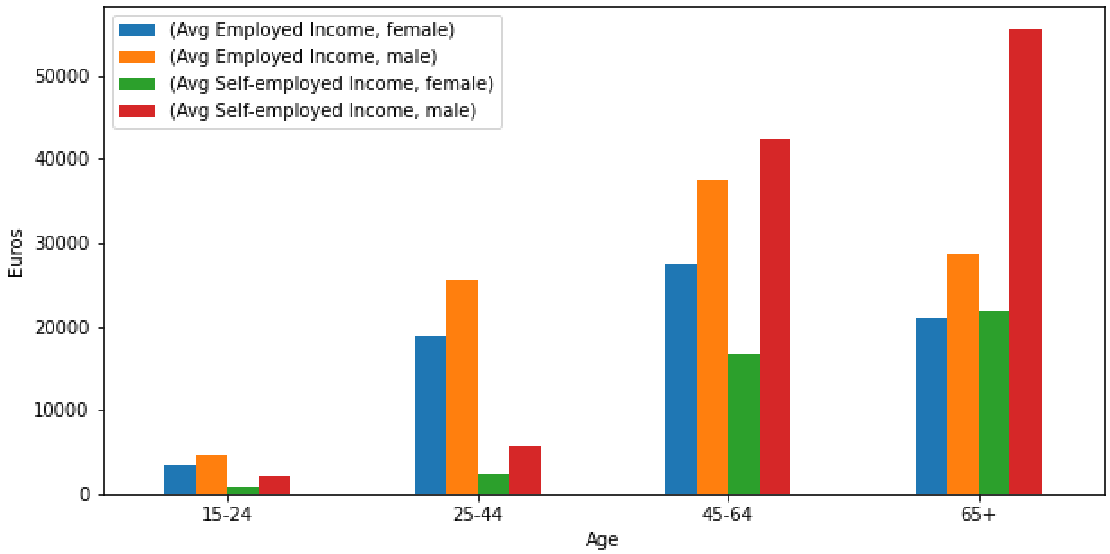

Figure 8: Average income distribution for male and female population. Working population has been divided in employed and self-employed.

References

[1] M. Angrisani, A. Attias, S. Bianchi, Z. Varga (2011) *Sustainability of a pay-as-you-go pension system by dynamic immigration control.* Applied Mathematics and Computation. 219. 10.1016/j.amc.2012.08.080.

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[3] F. G. Alonso (2009) *Can the rising pension burden in Europe be mitigated by immigration? Modelling the effects of selected demographic and socio-economic factors on ageing in the European Union, 2008-2050.* Vienna Yearbook of Population Research Vol. 7, Impact of migration on demographic change and composition in Europe (2009), pp. 123-147.

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