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Contractibility as Uniqueness

Hardy Lecture Tour, Manchester

An analogy

contractibility :: uniqueness

1. Contractibility as Uniqueness
2. Categorifying Uniqueness
3. ∞-Categorifying Uniqueness

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Contractibility as Uniqueness

The algebra of paths

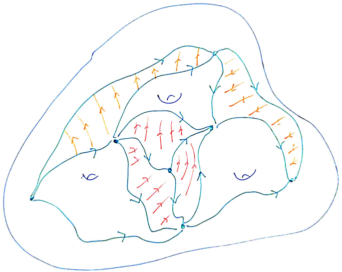
The standard technique used to distinguish your favorite space A from other spaces is to compute an algebraic invariant of the space.

The "algebra of paths" of a space is described in increasing precision by:

- the fundamental group $\pi_1(A, x)$ of loops in A based at x up to homotopy
- the fundamental groupoid $\pi_1 A$ of paths in A up to homotopy
- the fundamental ∞-groupoid $\pi_\infty A$ of paths in A

$\pi_\infty A$ has:

- points of A as objects
- paths of A as 1-arrows
- paths between paths in A as 2-arrows
- paths between paths between paths in A as 3-arrows, and so on ...



Witnesses to composition

Q: How do we define the composite of two paths in $\pi_\infty A$? A: We don't!

Instead of a composition operation, composites of paths are witnessed by higher paths.

$$\begin{tikzcd}
& \bullet \arrow[dr, squiggly, "g"] & \\
\bullet \arrow[ur, squiggly, "f"] \arrow[rr, squiggly, "k"'] & \arrow[u, phantom, "\Downarrow\alpha"] & \bullet
\end{tikzcd}$$

Q: How unique is path composition in $\pi_\infty A$? Partial A: Unique enough for associativity.

Given composable paths f, g, h and specified higher paths α, β, γ witnessing composition relations, these higher paths compose. More precisely, a 3-arrow expresses a coherence between compositions witnessed by 2-arrows.

$$\begin{tikzcd}
\bullet \arrow[rr, squiggly, bend left, "(h\circ g)\circ f"] \arrow[dr, squiggly, "f"] \arrow[ddr, squiggly, "g\circ f"'] & \arrow[d, phantom, "\Uparrow\gamma"] & \bullet \\
{} & \bullet \arrow[ur, squiggly, "h\circ g"'] \arrow[d, squiggly, "g"] \arrow[l, phantom, "\Swarrow\alpha"] \arrow[r, phantom, "\Nearrow\beta"] & {} \\
& \bullet \arrow[uur, squiggly, "h"'] &
\end{tikzcd}$$

Homotopical uniqueness of path composition

Theorem. The space of composites of two paths f and g in A is contractible.

Proof: The space of composites of paths f and g in A is defined by the pullback:

$$
\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, hook] & \mathrm{Comp}(f,g) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta} \arrow[d, two heads, "\text{restrict}"] \\
D^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed, bend right] & * \arrow[r, "f \wedge g"'] & A^{\Lambda}
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
S^{n-1} \times \Delta \cup_{S^{n-1}\times\Lambda} D^n \times \Lambda \arrow[r] \arrow[d, hook] & A \\
D^n \times \Delta \arrow[u, dotted, bend right] \arrow[ur, dashed] &
\end{tikzcd}
$$

A space is contractible just when any sphere S^{n-1} can be filled to a disk D^n for $n \geq 0$. This filling problem transposes to an extension problem, and the extension exists since the inclusion admits a continuous deformation retract. □

Summary

- In a group(oid), composable arrows have a unique composite.
- In a ∞-group(oid), composable arrows have a contractible space of composites.

The analogy

$$\begin{array}{ccc} \text{set-based mathematics} & :: & \text{“higher” mathematics} \\ \text{uniqueness} & :: & \text{contractibility} \end{array}$$

can be made even tighter.

Aim: Express the classical notion of uniqueness more categorically.

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Categorifying Uniqueness

Uniqueness

To say C has a unique element means

$$\exists x \in C, \forall y \in C, x = y$$

Here "$x = y$" is a predicate — a mathematical statement that is either true or false, depending on two free variables $x, y \in C$.

In proof-relevant mathematics, we interpret "$x = y$" as the set of all proofs that x equals y (which is empty if x and y are not equal).

Then we can form the set $\sum_{x \in C} \prod_{y \in C} x = y$

inspired by a notational analogy with the sentence $\exists x \in C, \forall y \in C, x = y$.

The set $\sum_{x \in C} \prod_{y \in C} x = y$ is also a set of proofs — but proofs of what?

Proofs of uniqueness

The set $\sum_{x \in C} \prod_{y \in C} x = y$ is also a set of proofs — but proofs of what?

An element of $\sum_{x \in C} \prod_{y \in C} x = y$ is

- the choice of some element $c \in C$
- together with a proof, for all $z \in C$, that c equals z.

Thus $\sum_{x \in C} \prod_{y \in C} x = y$ is the set of proofs of the sentence $\exists x \in C, \forall y \in C, x = y$

asserting that C has a unique element.

It remains to explain the analogy:

logic	$\exists$	$\forall$
sets	$\sum$	$\prod$

Digression: quantifiers as adjoints

A set function $f : S \to T$ induces order-preserving functions between their powersets:

$$\begin{tikzcd} \mathfrak{P}(S) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & \mathfrak{P}(T) \arrow[l, "\Delta_f"'] \end{tikzcd}$$

Δ_f is inverse image: $B \subset T \mapsto \{s \in S \mid f(s) \in B\} \subset S$

$\exists_f$ is direct image: $A \subset S \mapsto \{t \in T \mid \exists s \in S, f(s) = t \wedge s \in A\} \subset T$

$\forall_f$ is pushforward: $A \subset S \mapsto \{t \in T \mid \forall s \in S, f(s)=t \Rightarrow s \in A\} \subset T$

For the unique function $! : S \to *$ these reduce to $\begin{tikzcd} \mathfrak{P}(S) \arrow[r, bend left, "\exists"] \arrow[r, bend right, "\forall"'] & \mathfrak{P}(*) \arrow[l, "\Delta"'] \end{tikzcd} = \{*, \emptyset\}$

The set $\mathfrak{P}(S) = \{A \subset S\}$ can be identified with the set of predicates $p(s)$ with one free variable $s \in S$ — the corresponding subset is $\{s \in S \mid p(s) \text{ is true}\}$. If we interpret the two elements of $\mathfrak{P}(*)$ by $* =:$ "true" and $\emptyset =:$ "false" then

- $\exists$ is the function that sends the predicate $p(s)$ to the sentence $\exists s \in S, p(s)$
- $\forall$ is the function that sends the predicate $p(s)$ to the sentence $\forall s \in S, p(s)$

Digression: locally cartesian closed categories

For any function $f\colon S \to T$ there are functors:

$$\begin{tikzcd} \mathfrak{P}(S) \arrow[r, bend left=50, "\exists_f"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=50, "\forall_f"'] & \mathfrak{P}(T) \arrow[l, "\Delta_f"'] \end{tikzcd}$$

Δ_f is inverse image: $B \subset T \mapsto \{s \in S \mid f(s) \in B\} \subset S$

$\exists_f$ is direct image: $A \subset S \mapsto \{t \in T \mid \exists s \in S, f(s) = t \wedge s \in A\} \subset T$

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In proof-relevant mathematics, it is natural to replace the poset $\mathfrak{P}(S)$ by the category $\mathsf{Set}_{/S}$ of S-indexed sets. An object $\{P(s)\}_{s\in S}$ is a family of sets where $P(s)$ can be thought of as the set of proofs of some predicate $p(s)$ on $s \in S$.

$$\begin{tikzcd} \mathsf{Set}_{/S} \arrow[r, bend left=50, "\Sigma_f"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=50, "\Pi_f"'] & \mathsf{Set}_{/T} \arrow[l, "\Delta_f"'] \end{tikzcd}$$

Δ_f is substitution: $\{Q(t)\}_{t\in T} \mapsto \{Q(f(s))\}_{s\in S}$

$\sum_f$ is sum: $\{P(s)\}_{s\in S} \mapsto \{\sum_{f(s)=t} P(s)\}_{t\in T}$

$\prod_f$ is product: $\{P(s)\}_{s\in S} \mapsto \{\prod_{f(s)=t} P(s)\}_{t\in T}$

Summary

$$\begin{tikzcd}[column sep=huge] \mathsf{Set}_{/S} \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathsf{Set}_{/T} \arrow[l, "\Delta_f" description] \end{tikzcd}$$

Δ_f is substitution: $\{Q(t)\}_{t\in T} \mapsto \{Q(f(s))\}_{s\in S}$

$\sum_f$ is sum: $\{P(s)\}_{s\in S} \mapsto \{\sum_{f(s)=t} P(s)\}_{t\in T}$

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The triple of adjoint functors gives a more formal way to understand the set

$$\sum_{x\in C}\prod_{y\in C} x = y$$

- ⇝ The set of proofs "$x = y$" defines an indexed set $\{x = y\}_{x,y\in C} \in \mathsf{Set}_{/C\times C}$
- ⇝ Product along the projection $\pi_1 : C\times C \to C$ gives $\{\prod_{y\in C} x = y\}_{x\in C} \in \mathsf{Set}_{/C}$
- ⇝ Sum along $! : C \to *$ gives the set $\sum_{x\in C}\prod_{y\in C} x = y \in \mathsf{Set}_{/*} \simeq \mathsf{Set}$

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∞-Categorifying Uniqueness

A convenient category of spaces

Now replace Set by a convenient category Space of spaces and continuous maps that is locally cartesian closed.

A family of spaces $\{E_b\}_{b\in B} \in \mathsf{Space}_{/B}$ is a continuous map $\pi\colon E \to B$, where the space E_b is the fiber over a point $b \in B$ in the base space.

There is an adjoint triple:

$$\begin{tikzcd} \mathsf{Space}_{/B} \arrow[rr, bend left, "\Sigma_B"] \arrow[rr, bend right, "\Pi_B"'] & & \mathsf{Space} \arrow[ll, "\Delta_B" description] \end{tikzcd}$$

Δ_B is the constant family: $F \mapsto \{F\}_{b\in B}$

$\sum_B$ is the total space: $\{E_b\}_{b\in B} \mapsto E =: \sum_{b\in B} E_b$

$\prod_B$ is the space of sections: $\{E_b\}_{b\in B} \mapsto \prod_{b\in B} E_b$

A convenient category of spaces

$$\begin{tikzcd} \mathsf{Space}_{/B} \arrow[r, bend left=50, "\Sigma_B"] \arrow[r, bend right=50, "\Pi_B"'] & \mathsf{Space} \arrow[l, "\Delta_B" description] \end{tikzcd}$$

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More generally, any continuous $f : S \to T$ gives rise to an adjoint triple:

$$\begin{tikzcd} \mathsf{Space}_{/S} \arrow[r, bend left=50, "\Sigma_f"] \arrow[r, bend right=50, "\Pi_f"'] & \mathsf{Space}_{/T} \arrow[l, "\Delta_f" description] \end{tikzcd}$$

Δ_f is pullback: $\{F_t\}_{t\in T} \mapsto \{F_{s(t)}\}_{s\in S}$

$\sum_f$ is composition: $\{E_s\}_{s\in S} \mapsto \{\sum_{f(s)=t} E_s\}_{t\in T}$

$\prod_f$ is pushforward: $\{E_s\}_{s\in S} \mapsto \{\prod_{f(s)=t} E_s\}_{t\in T}$

Identifications as paths

Q: For a space C, how to interpret the family of spaces $\{x = y\}_{x,y \in C} \in \mathsf{Space}_{/C \times C}$?

First guess:

$$\begin{tikzcd} C \arrow[d, "\Delta"'] \\ C \times C \end{tikzcd} \in \mathsf{Space}_{/C\times C} \text{ — but a better choice is the path space } \begin{tikzcd} C^I \arrow[d, two heads] \\ C \times C \end{tikzcd} \in \mathsf{Space}_{/C\times C}.$$

New idea:

A point $p \in x = y$ is a path from x to y in C, providing a proof that x equals y.

A space of proofs

What is a point in the space $\sum_{x \in C} \prod_{y \in C} x = y$?

The functor $\sum_B : \mathsf{Space}_{/B} \to \mathsf{Space}$ takes $\{E_b\}_{b \in B}$ to the total space $\sum_{b \in B} E_b$.

$\rightsquigarrow$ a point in $\sum_{b \in B} E_b$ is a pair (a, e_a) of a point $a \in B$ and a point $e_a \in E_a$

The functor $\prod_B : \mathsf{Space}_{/B} \to \mathsf{Space}$ takes $\{E_b\}_{b \in B}$ to the space of sections $\prod_{b \in B} E_b$.

$\rightsquigarrow$ a point in $\prod_{b \in B} E_b$ is section $s : B \to \sum_{b \in B} E_b$ of the projection to B

- So a point in $\sum_{x \in C} \prod_{y \in C} x = y$ is a pair (c, h) where $c \in C$ and $h \in \prod_{y \in C} c = y$.
- The point $h \in \prod_{y \in C} c = y$ is a section $h : C \to \sum_{y \in C} c = y$ to the projection.

Together $(c, h) \in \sum_{x \in C} \prod_{y \in C} x = y$ defines:

- a center of contraction c and
- a contracting homotopy h,

proving that the space C is contractible!

Contractibility as uniqueness

In summary, a point in the set

$$\sum_{x \in C} \prod_{y \in C} x = y$$

is a proof that C is unique, while a point in the space

$$\sum_{x \in C} \prod_{y \in C} x = y$$

is a proof that C is contractible.

The point: this gives a glimpse of how uniqueness in homotopy type theory in fact expresses a contractibility condition — thus, uniqueness is “homotopical uniqueness.”

Thank you!