

Johns Hopkins University

Path induction and the indiscernibility of identicals

### Plan



- 1. Induction over the natural numbers
- 2. Dependent type theory
- 3. Identity types
- 4. Path induction
- 5. Epilogue: univalent foundations



Induction over the natural numbers

## Peano's postulates



In Dedekind's 1888 book "Was sind und was sollen die Zahlen" and Peano's 1889 paper "Arithmetices principia, nova methodo exposita," the natural numbers  $\mathbb N$  are characterized by:

- There is a natural number  $0 \in \mathbb{N}$ .
- Every natural number  $n \in \mathbb{N}$  has a successor  $sucn \in \mathbb{N}$ .
- 0 is not the successor of any natural number.
- No two natural numbers have the same successor.
- The principle of mathematical induction:

$$\forall P, P(0) \rightarrow (\forall k \in \mathbb{N}, P(k) \rightarrow P(\operatorname{suc} k)) \rightarrow (\forall n \in \mathbb{N}, P(n))$$

By Dedekind's categoricity theorem, all triples given by a set  $\mathbb{N}$ , an element  $0 \in \mathbb{N}$ , and a function  $suc : \mathbb{N} \to \mathbb{N}$  satisfying the Peano postulates are isomorphic.

#### Natural numbers induction



In the statement of the principle of mathematical induction:

$$\forall P, P(0) \to (\forall k \in \mathbb{N}, P(k) \to P(\mathsf{suc}k)) \to (\forall n \in \mathbb{N}, P(n))$$

the variable P is a predicate over the natural numbers.

A predicate over the natural numbers is a function

$$P \colon \mathbb{N} \to \{\top, \bot\}$$

that associates a truth value  $\top$  or  $\bot$  to each  $n \in \mathbb{N}$ .

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Thus, to prove a sentence of the form  $\forall n \in \mathbb{N}, P(n)$  it suffices to:

- prove the base case, showing that P(0) is true, and
- prove the inductive step, showing for each  $k \in \mathbb{N}$  that P(k) implies  $P(\operatorname{suc} k)$ .

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- In the base case, when n = 0,  $0^2 + 0 = 2 \times 0$ , which is even.
- For the inductive step, assume for  $k \in \mathbb{N}$  that  $k^2 + k = 2 \times m$  is even. Then

$$(k+1)^{2} + (k+1) = (k^{2} + k) + ((2 \times k) + 2)$$

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#### By the principle of mathematical induction

$$\forall P, P(0) \rightarrow (\forall k \in \mathbb{N}, P(k) \rightarrow P(\mathsf{suc}k)) \rightarrow (\forall n \in \mathbb{N}, P(n))$$

this proves that  $n^2 + n$  is even for all  $n \in \mathbb{N}$ .

The induction proof not only demonstrates for all  $n \in \mathbb{N}$  that  $n^2 + n$  is even but also defines a function  $m : \mathbb{N} \to \mathbb{N}$  so that  $n^2 + n = 2 \times m(n)$ .

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By the principle of mathematical recursion, this defines a function  $m: \mathbb{N} \to \mathbb{N}$  so that  $n^2 + n = 2 \times m(n)$  for all  $n \in \mathbb{N}$ .

#### Induction and recursion

Recursion can be thought of as the constructive form of induction

$$\forall P, P(0) \rightarrow (\forall k \in \mathbb{N}, P(k) \rightarrow P(\mathsf{suc}k)) \rightarrow (\forall n \in \mathbb{N}, P(n))$$

in which the predicate

$$P \colon \mathbb{N} \to \{\top, \bot\}$$
 such as  $P(n) := \exists m \in \mathbb{N}, n^2 + n = 2 \times m$ 

is replaced by an arbitrary family of sets

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The output of a recursive construction is a dependent function  $p \in \prod_{n \in \mathbb{N}} P(n)$  which specifies a value  $p(n) \in P(n)$  for each  $n \in \mathbb{N}$ .

$$\forall P, (p_0 \in P(0)) \to (p_s \in \prod_{k \in \mathbb{N}} P(k) \to P(\operatorname{suc} k)) \to (p \in \prod_{n \in \mathbb{N}} P(n))$$

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The recursive function  $p \in \prod_{n \in \mathbb{N}} P(n)$  satisfies computation rules:

$$p(0) := p_0$$
  $p(\operatorname{suc} n) := p_s(n, p(n)).$ 





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- There is a type N.
- There is a term  $0: \mathbb{N}$  and a function  $suc: \mathbb{N} \to \mathbb{N}$ .
- For any family of types  $P : \mathbb{N} \to \mathsf{Type}$  there is a term

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Note the final two postulates — that 0 is not a successor and suc is injective — are missing because they are provable.



Dependent type theory



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- type families, e.g.,  $\mathbb{R}^-:\mathbb{N} \to \mathsf{Type}$  ,  $\mathsf{Mat}_{-\times-}(-):\mathbb{N} \to \mathbb{N} \to \mathsf{Ring} \to \mathsf{Type}$



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all of which can occur in an arbitrary context of variables from previously-defined types.

In a mathematical statement of the form "Let ...be ...then ..." The stuff following the "let" likely declares the names of the variables in the context described after the "be", while the stuff after the "then" most likely describes a type or term in that context.

#### Type constructors



Type constructors build new types from given ones:

- given  $A, B \rightsquigarrow \text{products } A \times B$ , coproducts A + B, function types  $A \rightarrow B$ ,
- given  $P: A \to \mathsf{Type} \leadsto \mathsf{dependent\ pairs\ } \sum_{x:A} P(x), \mathsf{dependent\ functions\ } \prod_{x:A} P(x)$
- given  $A \leadsto \text{identity types } -=_A -: A \to A \to \mathsf{Type}$

#### Type constructors

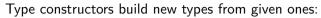
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#### Each type constructor comes with rules:

- (i) formation: a way to construct new types
- (ii) introduction: ways to construct terms of these types
- (iii) elimination: ways to use them to construct other terms
- (iv) computation: the way (ii) and (iii) relate

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#### The rules suggest a logical naming for certain types:



# Product types and function types



### Product types are governed by the rules

- $\times$ -form: given types A and B there is a type  $A \times B$
- $\times$ -intro: given terms a:A and b:B there is a term  $(a,b):A\times B$
- $\times$ -elim: given  $p:A\times B$  there are terms  $\operatorname{pr}_1p:A$  and  $\operatorname{pr}_2p:B$

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#### Function types are governed by the rules

- $\rightarrow$ -form: given types A and B there is a type  $A \rightarrow B$
- $\rightarrow$ -intro: if in the context of a variable x:A there is a term  $b_x:B$

there is a term  $\lambda x.b_x : A \to B$ 

 $^{\rightarrow}$ -elim: given terms  $f:A \to B$  and a:A there is a term f(a):B plus computation rules that relate  $\lambda$ -abstractions and evaluations.

# Mathematics in dependent type theory

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$$\lambda p. \operatorname{pr}_2 p(\operatorname{pr}_1 p) : (A \times (A \to B)) \to B.$$



Identity types

# The traditional view of equality

In first order logic, the binary relation "=" is governed by the following rules:

- Reflexivity:  $\forall x, x = x$ .
- Indiscernibility of Identicals:

 $\forall x, y, x = y$  implies that for all predicates  $P, P(x) \leftrightarrow P(y)$ 

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Symmetry and transitivity of equality can be proven from these rules.

The formation and introduction rules for identity types are:

- =-form: given a type A and terms x, y : A, there is a type  $x =_A y$
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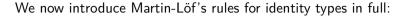
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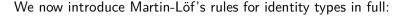
- types are interpreted as "spaces",
- terms are interpreted as points,
- a term  $p: x =_A y$  may be thought of as a path from x to y in A, and
- a term  $h: p =_{x=A^y} q$  is interpreted as a homotopy between paths,

we know that iterated identity types can have interesting higher structure.



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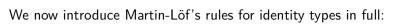
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Now that terms  $p: x =_A y$  are called paths, we re-brand =-elim as:

Path induction: For any type family P(x,y,p) over  $x,y:A,p:x=_A y$ , to prove P(x,y,p) for all x,y,p it suffices to assume y is x and p is  $refl_x$ . That is

$$\mathsf{path}\text{-}\mathsf{ind}: \Bigl(\prod\nolimits_{x:A} P(x,x,\mathsf{refl}_x)\Bigr) \to \Bigl(\prod\nolimits_{x,y:A} \prod\nolimits_{p:x=_{A}y} P(x,y,p)\Bigr).$$



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Path induction: For any type family P(x, y, p) over  $x, y : A, p : x =_A y$ , to prove P(x, y, p) for all x, y, p it suffices to assume y is x and p is refl<sub>x</sub>. That is

$$\mathsf{path}\text{-}\mathsf{ind}: \Big(\prod\nolimits_{x:A} P(x,x,\mathsf{refl}_x)\Big) \to \Big(\prod\nolimits_{x,y:A} \prod\nolimits_{p:x=_A y} P(x,y,p)\Big).$$

A computation rule establishes that the proof of  $P(x, x, refl_x)$  is the given one.



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Proposition. Paths can be reversed:  $(-)^{-1}:\prod_{x,y:A}x=_Ay\to y=_Ax$ .

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## The $\infty$ -groupoid of paths

Identity types can be iterated: given x, y : A and  $p, q : x =_A y$  there is a type  $p =_{x =_A y} q$ .



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The  $\infty$ -groupoid structure of A has

- terms x : A as objects
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The required structures are proven from the path induction principle:

- constant paths (reflexivity) refl<sub>x</sub>: x = x
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and furthermore concatenation is associative and unital, the associators are coherent ...



Path induction proves the (higher) coherences in the  $\infty$ -groupoid of paths:

Proposition. For any type A and terms w, x, y, z : A

assoc : 
$$\prod_{p:w=_{A^X}} \prod_{q:x=_{A^Y}} \prod_{r:y=_{A^Z}} (p*q)*r =_{w=_{A^Z}} p*(q*r).$$



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By the computation rules for path induction  $refl_w * -$  is the identity function. Thus, we must show

$$\prod_{q:w=_{A}y} \prod_{r:y=_{A}z} q * r =_{w=_{A}z} q * r,$$

for which we have the proof  $refl_{q*r}: q*r =_{w=AZ} q*r$ .



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Construction: By path induction, it suffices to assume y is x and p is  $refl_x$ , in which case we have the identity equivalence.



Epilogue: univalent foundations

# The homotopy type theoretic Rosetta stone



type theory	logic	set theory	homotopy theory
A	proposition	set	space
<b>x</b> : <b>A</b>	proof	element	point
$\emptyset, 1$	$\perp, \top$	$\emptyset, \{\emptyset\}$	$\emptyset, *$
$A \times B$	A and $B$	set of pairs	product space
A + B	A or B	disjoint union	coproduct
${\sf A}  o {\sf B}$	A implies B	set of functions	function space
$P\colon A o Type$	predicate	family of sets	fibration
$f:\prod_{x:A}P(x)$	conditional proof	family of elements	section
$\prod_{x:A} P(x)$	$\forall x.P(x)$	product	space of sections
$\sum_{x:A} P(x)$	$\exists x. P(x)$	disjoint union	total space
$p: x =_A y$	proof of equality	x = y	path from $x$ to $y$
$\sum_{x,y:A} x =_A y$	equality relation	diagonal	path space for $A$

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A type A is contractible if it comes with a term of type

$$is-contr(A) := \sum_{a:A} \prod_{x:A} a =_A x$$

By  $^{\Sigma}$ -elim a proof of contractibility provides:

- a term c: A called the center of contraction and
- a dependent function  $h: \prod_{x:A} c =_A x$  called the contracting homotopy, which can be thought of as a continuous choice of paths  $h(x): c =_A x$  for each x:A.



Contractible types, those types *A* for which the type

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has a term, form the bottom level of Voevodsky's hierarchy of types.



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• a succ(n)-type for  $n : \mathbb{N}$  if

$$is-succ(n)-type(A) := \prod_{x \in A} is-n-type(x =_A y)$$



Similarly, homotopy theory suggests definitions of when two types A and B are equivalent or when a function  $f:A\to B$  is an equivalence:

An equivalence between types A and B is a term of type:

$$A \simeq B := \sum_{f:A \to B} \left( \sum_{g:B \to A} \prod_{a:A} g(f(a)) =_A a \right) \times \left( \sum_{h:B \to A} \prod_{b:B} f(h(b)) =_B b \right)$$

A term of type  $A \simeq B$  provides:

- functions  $f: A \rightarrow B$  and  $g, h: B \rightarrow A$  and
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This type is not a proposition and may have non-trivial higher structure.

Another notion of sameness between types is provided by the universe  $\mathcal{U}$  of types, which has (small) types A, B as its terms  $\longrightarrow$  A, B:  $\mathcal{U}$ .

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- The structure-identity principle, which specializes to the statement that for set-based structures (monoids, groups, rings) isomorphic structures are identical.
- Function extensionality: for any  $f, g: A \to B$ , the canonical function defines an equivalence between the identity type and the type of homotopies:

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Via path induction, Voevodsky's univalence axiom — which is justified by the homotopical model of type theory — captures the common mathematical practice of applying results proven about one object to any other object that is equivalent to it!

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By univalence, the equivalence  $A \simeq \left(\sum_{x,y:A} x =_A y\right)$  gives rise to an equivalence

$$\left(\prod_{x:A} P(x,x,\mathsf{refl}_x)\right) \simeq \left(\prod_{(x,y,p):\sum_{x,y:A} x = _A y} P(x,y,p)\right) \simeq \left(\prod_{x,y:A} \prod_{p:x = _A y} P(x,y,p)\right).$$

#### References

Homotopy Type Theory: Univalent Foundations of Mathematics

homotopytypetheory.org/book/

Egbert Rijke, Introduction to Homotopy Type Theory

arXiv:2212.11082

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discord.gg/tkhJ9zCGs9

Thank you!