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A reintroduction to proofs

Plan

1. Logic, constructively

2. $\forall : \Pi :: \exists : \Sigma$

3. Peano's axioms, revisited

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Logic, constructively

Conjunction and disjunction

Forget truth tables! Instead, define the logical operators "and" $\wedge$ and "or" $\vee$ by:

Conjunction $\wedge$ is the logical operator defined by the rules:

- $^\wedge$intro: If p is true and q is true, then $p \wedge q$ is true.
- $^\wedge$elim$_1$: If $p \wedge q$ is true, then p is true.
- $^\wedge$elim$_2$: If $p \wedge q$ is true, then q is true.

Disjunction $\vee$ is the logical operator defined by the rules:

- $^\vee$intro$_1$: If p is true, then $p \vee q$ is true.
- $^\vee$intro$_2$: If q is true, then $p \vee q$ is true.
- $^\vee$elim: If $p \vee q$ is true, and if r can be derived from p and from q, then r is true.

Introduction rules explain how to prove a proposition involving a particular connective, while elimination rules explain how to use a hypothesis involving a particular connective.

Implication

Implication $\Rightarrow$ is the logical operator defined by the rules:

- $^{\Rightarrow}$intro: If q can be derived from the assumption that p is true, then $p \Rightarrow q$ is true.
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Theorem. For any propositions p, q, and r, $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$.

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Proof:

givens: p, q, r

goal: $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$

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Proof: By $^{\Rightarrow}$intro, assume that $(p \Rightarrow q) \wedge (q \Rightarrow r)$ is true; our goal is to prove $p \Rightarrow r$.

givens: p, q, r

$(p \Rightarrow q) \wedge (q \Rightarrow r)$

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goal: $p \Rightarrow r$

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givens:

p, q, r

$(p \Rightarrow q) \wedge (q \Rightarrow r)$

$p \Rightarrow q$

$q \Rightarrow r$

goal: $p \Rightarrow r$

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givens:	p, q, r
	$(p \Rightarrow q) \wedge (q \Rightarrow r)$
	$p \Rightarrow q$
	$q \Rightarrow r$
	p
goal:	r

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givens:

p, q, r

$(p \Rightarrow q) \wedge (q \Rightarrow r)$

$p \Rightarrow q$

$q \Rightarrow r$

p

q

goal: r

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Proof: By ${}^{\Rightarrow}$intro, assume that $(p \Rightarrow q) \wedge (q \Rightarrow r)$ is true; our goal is to prove $p \Rightarrow r$. By ${}^{\wedge}$elim$_1$ and ${}^{\wedge}$elim$_2$ it follows that $p \Rightarrow q$ and $q \Rightarrow r$ are true. By ${}^{\Rightarrow}$intro again, also assume p is true; now our goal is just to prove r. By ${}^{\Rightarrow}$elim, from p and $p \Rightarrow q$, we may conclude that q is true. By ${}^{\Rightarrow}$elim again, from q and $q \Rightarrow r$, we may conclude r is true as desired. $\square$

givens:	p, q, r
	$(p \Rightarrow q) \wedge (q \Rightarrow r)$
	$p \Rightarrow q$
	$q \Rightarrow r$
	p
	q
	r
goal:	r

Type theory

Type theory is a formal system for mathematical statements and proofs that has two primitive notions: types A, B and terms $a : A$, $b : B$.

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Mathematics in type theory:

- To state a conjecture, one forms a type that encodes its statement.
- To prove the theorem, one constructs a term in that type.

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Mathematics in type theory:

- To state a conjecture, one forms a type that encodes its statement.
- To prove the theorem, one constructs a term in that type.

Given any types A and B, one may form the

product type $A \times B$, coproduct type $A + B$, function type $A \to B$

whose terms are governed by introduction and elimination (and computation) rules which extend the rules for conjunction, disjunction, and implication.

Conjunction and Products

Conjunction $\wedge$ is the logical operator defined by the rules:

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- $^{\wedge}$elim$_1$: If $p \wedge q$ is true, then p is true.
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Given types A and B, the product type $A \times B$ is governed by the rules:

- $^{\times}$intro: given terms $a : A$ and $b : B$ there is a term $(a, b) : A \times B$
- $^{\times}$elim$_1$: given a term $p : A \times B$ there is a term $\pi_1 p : A$
- $^{\times}$elim$_2$: given a term $p : A \times B$ there is a term $\pi_2 p : B$

plus computation rules that relate pairings and projections.

Implication and functions

Implication $\Rightarrow$ is the logical operator defined by the rules:

- $^{\Rightarrow}$intro: If q can be derived from the assumption that p is true, then $p \Rightarrow q$ is true.
- $^{\Rightarrow}$elim: If $p \Rightarrow q$ is true and p is true, then q is true.

Given types A and B, the function type $A \to B$ is governed by the rules:

- $^{\to}$intro: if given any term $x : A$ there is a term $b_x : B$,

  then there is a term $\lambda x.b_x : A \to B$
- $^{\to}$elim: given terms $f : A \to B$ and $a : A$, there is a term $f(a) : B$

plus computation rules that relate λ-abstractions and evaluations.

A proof/construction in type theory

The proof of transitivity of implication constructs the composition function:

Theorem. For any propositions p, q, and r, $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$.

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Theorem. For any types P, Q, and R, $((P \to Q) \times (Q \to R)) \to (P \to R)$.

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Construction:

givens: P, Q, R

---

goal: $(P \to Q) \times (Q \to R) \to (P \to R)$

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Theorem. For any types P, Q, and R, $((P \to Q) \times (Q \to R)) \to (P \to R)$.

Construction: By $^{\to}$intro, suppose given $h : (P \to Q) \times (Q \to R)$; our goal is a term of type $P \to R$.

givens: P, Q, R

$h : (P \to Q) \times (Q \to R)$

goal: $P \to R$

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Theorem. For any types P, Q, and R, $((P \to Q) \times (Q \to R)) \to (P \to R)$.

Construction: By ${}^{\to}$intro, suppose given $h : (P \to Q) \times (Q \to R)$; our goal is a term of type $P \to R$. By ${}^{\times}\text{elim}_1$ and ${}^{\times}\text{elim}_2$, we have $\pi_1 h : P \to Q$ and $\pi_2 h : Q \to R$.

givens: P, Q, R

$h : (P \to Q) \times (Q \to R)$

$\pi_1 h : P \to Q$

$\pi_2 h : Q \to R$

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goal: $P \to R$

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givens:

$$P, Q, R$$
$$h : (P \to Q) \times (Q \to R)$$
$$\pi_1 h : P \to Q$$
$$\pi_2 h : Q \to R$$
$$p : P$$

goal: R

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Construction: By ${}^{\to}$intro, suppose given $h : (P \to Q) \times (Q \to R)$; our goal is a term of type $P \to R$. By ${}^{\times}\text{elim}_1$ and ${}^{\times}\text{elim}_2$, we have $\pi_1 h : P \to Q$ and $\pi_2 h : Q \to R$. By ${}^{\to}$intro again, suppose given $p : P$; now our goal is a term of type R. By ${}^{\to}$elim, from $p : P$ and $\pi_1 h : P \to Q$, we obtain $\pi_1 h(p) : Q$.

givens:

$$P, Q, R$$
$$h : (P \to Q) \times (Q \to R)$$
$$\pi_1 h : P \to Q$$
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$$p : P$$
$$\pi_1 h(p) : Q$$

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Construction: By ${}^{\to}$intro, suppose given $h : (P \to Q) \times (Q \to R)$; our goal is a term of type $P \to R$. By ${}^{\times}\mathrm{elim}_1$ and ${}^{\times}\mathrm{elim}_2$, we have $\pi_1 h : P \to Q$ and $\pi_2 h : Q \to R$. By ${}^{\to}$intro again, suppose given $p : P$; now our goal is a term of type R. By ${}^{\to}$elim, from $p : P$ and $\pi_1 h : P \to Q$, we obtain $\pi_1 h(p) : Q$. By ${}^{\to}$elim again, from $\pi_1 h(p) : Q$ and $\pi_2 h : Q \to R$, we obtain $\pi_2 h(\pi_1 h(p)) : R$ as desired. $\square$

$$
\begin{array}{ll}
\text{givens:} & P, Q, R \\
 & h : (P \to Q) \times (Q \to R) \\
 & \pi_1 h : P \to Q \\
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givens:

P, Q, R

$h : (P \to Q) \times (Q \to R)$

$\pi_1 h : P \to Q$

$\pi_2 h : Q \to R$

$p : P$

$\pi_1 h(p) : Q$

$\pi_2 h(\pi_1 h(p)) : R$

---

goal: R

This constructs a term $\lambda h.\lambda p.\pi_2 h(\pi_1 h(p)) : ((P \to Q) \times (Q \to R)) \to (P \to R)$.

Disjunction and coproducts

Disjunction $\vee$ is the logical operator defined by the rules:

- ${}^{\vee}\mathrm{intro}_1$: If p is true, then $p \vee q$ is true.
- ${}^{\vee}\mathrm{intro}_2$: If q is true, then $p \vee q$ is true.
- ${}^{\vee}\mathrm{elim}$: If $p \vee q$ is true, and if r can be derived from p and from q, then r is true.

Given types A and B, the coproduct type $A + B$ is governed by the rules:

- ${}^{+}\mathrm{intro}_1$: given a term $a : A$, there is a term $\iota_1 a : A + B$
- ${}^{+}\mathrm{intro}_2$: given a term $b : B$, there is a term $\iota_2 b : A + B$
- ${}^{+}\mathrm{elim}$: given a types C and terms $c_a, d_b : C$ for each $a : A$ and $b : B$ respectively, there is a term ${}^{+}\mathrm{ind}(c, d)(x) : C$ for each $x : A + B$

plus computation rules that relate the inclusions and the elimination.

Another proof/construction in type theory

Theorem. For any types A, B, and C, $((A+B) \to C) \to ((A \to C) \times (B \to C))$.

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Theorem. For any types A, B, and C, $((A+B) \to C) \to ((A \to C) \times (B \to C))$.

Construction: By $\to$intro, suppose given $h : (A+B) \to C$; our goal is a term of type $(A \to C) \times (B \to C)$.

- $\to$intro: if given any term $x : A$ there is a term $b_x : B$, there is a term $\lambda x.b_x : A \to B$

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Theorem. For any types A, B, and C, $((A+B) \to C) \to ((A \to C) \times (B \to C))$.

Construction: By ${}^{\to}$intro, suppose given $h : (A+B) \to C$; our goal is a term of type $(A \to C) \times (B \to C)$. By ${}^{\times}$intro, it suffices to define terms of type $A \to C$ and type $B \to C$.

- ${}^{\to}$intro: if given any term $x : A$ there is a term $b_x : B$, there is a term $\lambda x.b_x : A \to B$
- ${}^{\times}$intro: given terms $a : A$ and $b : B$ there is a term $(a, b) : A \times B$

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Theorem. For any types A, B, and C, $((A + B) \to C) \to ((A \to C) \times (B \to C))$.

Construction: By ${}^{\to}$intro, suppose given $h : (A + B) \to C$; our goal is a term of type $(A \to C) \times (B \to C)$. By ${}^{\times}$intro, it suffices to define terms of type $A \to C$ and type $B \to C$. By ${}^{\to}$intro, to define a term of type $A \to C$ it suffices to assume a term $a : A$ and define a term of type C.

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- ${}^{\to}$intro: if given any term $x : A$ there is a term $b_x : B$, there is a term $\lambda x.b_x : A \to B$
- ${}^{\times}$intro: given terms $a : A$ and $b : B$ there is a term $(a, b) : A \times B$
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- ${}^{\to}$intro: if given any term $x : A$ there is a term $b_x : B$, there is a term $\lambda x.b_x : A \to B$
- ${}^{\times}$intro: given terms $a : A$ and $b : B$ there is a term $(a, b) : A \times B$
- ${}^{+}$intro$_1$: given a term $a : A$, there is a term $\iota_1 a : A + B$
- ${}^{\to}$elim: given terms $f : A \to B$ and $a : A$, there is a term $f(a) : B$

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Theorem. For any types A, B, and C, $((A + B) \to C) \to ((A \to C) \times (B \to C))$.

Construction: By ${}^{\to}$intro, suppose given $h : (A + B) \to C$; our goal is a term of type $(A \to C) \times (B \to C)$. By ${}^{\times}$intro, it suffices to define terms of type $A \to C$ and type $B \to C$. By ${}^{\to}$intro, to define a term of type $A \to C$ it suffices to assume a term $a : A$ and define a term of type C. By ${}^{+}$intro$_1$, we then have a term $\iota_1 a : A + B$. Then by ${}^{\to}$elim we obtain a term $h(\iota_1 a) : C$. Similarly, by ${}^{\to}$intro, ${}^{+}$intro$_2$, and ${}^{\to}$elim we have $\lambda b.h(\iota_2 b) : B \to C$. □

- ${}^{\to}$intro: if given any term $x : A$ there is a term $b_x : B$, there is a term $\lambda x.b_x : A \to B$
- ${}^{\times}$intro: given terms $a : A$ and $b : B$ there is a term $(a, b) : A \times B$
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Construction: By ${}^{\to}$intro, suppose given $h : (A+B) \to C$; our goal is a term of type $(A \to C) \times (B \to C)$. By ${}^{\times}$intro, it suffices to define terms of type $A \to C$ and type $B \to C$. By ${}^{\to}$intro, to define a term of type $A \to C$ it suffices to assume a term $a : A$ and define a term of type C. By ${}^{+}\text{intro}_1$, we then have a term $\iota_1 a : A + B$. Then by ${}^{\to}$elim we obtain a term $h(\iota_1 a) : C$. Similarly, by ${}^{\to}$intro, ${}^{+}\text{intro}_2$, and ${}^{\to}$elim we have $\lambda b.h(\iota_2 b) : B \to C$. $\square$

- ${}^{\to}$intro: if given any term $x : A$ there is a term $b_x : B$, there is a term $\lambda x.b_x : A \to B$
- ${}^{\times}$intro: given terms $a : A$ and $b : B$ there is a term $(a, b) : A \times B$
- ${}^{+}\text{intro}_1$: given a term $a : A$, there is a term $\iota_1 a : A + B$
- ${}^{\to}$elim: given terms $f : A \to B$ and $a : A$, there is a term $f(a) : B$

This constructs $\lambda h.(\lambda a.h(\iota_1 a), \lambda b.h(\iota_2 b)) : ((A+B) \to C) \to ((A \to C) \times (B \to C))$.

2

$$\forall : \Pi :: \exists : \Sigma$$

Universal and existential quantification

Let $p : X \to \{\bot, \top\}$ be an X-indexed family of propositions, a predicate $p(x)$ on $x \in X$. For example:

- "$2^{2^n} - 1$ is prime" is a predicate on $n \in \mathbb{N}$
- "$z^2 = -1$" is a predicate on $z \in \mathbb{C}$

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For example:

- "$2^{2^n} - 1$ is prime" is a predicate on $n \in \mathbb{N}$
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Universal quantification $\forall x \in X, p(x)$ is the logical formula defined by the rules:

- $^{\forall}$intro: If $p(x)$ can be derived from the assumption that x is an arbitrary element of X, then $\forall x \in X, p(x)$ is true.
- $^{\forall}$elim: If $\forall x \in X, p(x)$ is true and $a \in X$, then $p(a)$ is true.

Existential quantification $\exists x \in X, p(x)$ is the logical formula defined by the rules:

- $^{\exists}$intro: If $a \in X$ and $p(a)$ is true, then $\exists x \in X, p(x)$ is true.
- $^{\exists}$elim: If $\exists x \in X, p(x)$ is true and q can be derived from the assumption that $p(a)$ is true for some $a \in X$, then q is true.

Exchanging quantifiers

$^{\forall}$-intro: If $p(x)$ for any $x \in X$, then $\forall x \in X, p(x)$.
$^{\forall}$elim: If $\forall x \in X, p(x)$ and $a \in X$, then $p(a)$.

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Theorem. For any predicate $p(x, y)$ on $x \in X$ and $y \in Y$,

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Exchanging quantifiers

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Proof:

givens: p

goal: $\exists y \in Y, \forall x \in X, p(x, y) \Rightarrow \forall x' \in X, \exists y' \in Y, p(x', y')$

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Proof: By $^{\Rightarrow}$intro, we may assume $\exists y \in Y, \forall x \in X, p(x, y)$; our goal is to prove $\forall x' \in X, \exists y' \in Y, p(x', y')$.

givens: p

$\exists y \in Y, \forall x \in X, p(x, y)$

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goal: $\forall x' \in X, \exists y' \in Y, p(x', y')$

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givens:

p

$\exists y \in Y, \forall x \in X, p(x, y)$

y_0

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givens:
p
$\exists y \in Y, \forall x \in X, p(x, y)$
y_0
$\forall x \in X, p(x, y_0)$
x'

goal: $\exists y' \in Y, p(x', y')$

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givens:	p
	$\exists y \in Y, \forall x \in X, p(x, y)$
	y_0
	$\forall x \in X, p(x, y_0)$
	x'
	$p(x', y_0)$
goal:	$\exists y' \in Y, p(x', y')$

Exchanging quantifiers

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Proof: By $^{\Rightarrow}$intro, we may assume $\exists y \in Y, \forall x \in X, p(x, y)$; our goal is to prove $\forall x' \in X, \exists y' \in Y, p(x', y')$. By $^{\exists}$elim, we may assume $y_0 \in Y$ makes $\forall x \in X, p(x, y_0)$ true. By $^{\forall}$intro, we may fix $x' \in X$; our goal is to prove that $\exists y' \in Y, p(x', y')$. But by $^{\forall}$elim, we know that $p(x', y_0)$ is true. So by $^{\exists}$intro, it follows that $\exists y' \in Y, p(x', y')$ is true. □

givens:
p
$\exists y \in Y, \forall x \in X, p(x, y)$
y_0
$\forall x \in X, p(x, y_0)$
x'
$p(x', y_0)$
$\exists y' \in Y, p(x', y')$

goal: $\exists y' \in Y, p(x', y')$

Dependent type theory

Dependent type theory is a formal system for mathematical statements and proofs that, in addition to the types A, B and terms $a : A$, $b : B$, also has primitive notions of type families and term families that are indexed by previously-defined types.

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Type families $B : A \to \mathrm{Type}$ are analogous to predicates and also to indexed families of sets, e.g.,

$$\text{is-prime} : \mathbb{N} \to \mathrm{Type},\ =_A : A \to A \to \mathrm{Type},\ \mathbb{R}^{\bullet} : \mathbb{N} \to \mathrm{Type},\ \mathrm{Mat}_{\bullet \times \bullet} : \mathbb{N} \to \mathbb{N} \to \mathrm{Type}$$

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Term families $f : \prod_{x:A} B(x)$ are analogous to universal proofs or indexed families of elements and define dependent functions, e.g.,

$$\vec{0}^\bullet : \prod_{n:\mathbb{N}} \mathbb{R}^n\ ,\ \ I_\bullet : \prod_{n:\mathbb{N}} \mathrm{Mat}_{n,n}\ ,\ \ S_\bullet : \prod_{n:\mathbb{N}} \mathrm{Group}$$

Universal quantification and dependent functions

For any predicate $p : X \to \{\bot, \top\}$, the universal quantification $\forall x \in X, p(x)$ is the logical formula defined by the rules:

- $^{\forall}$intro: If $p(x)$ can be derived from the assumption that x is an arbitrary element of X, then $\forall x \in X, p(x)$ is true.
- $^{\forall}$elim: If $\forall x \in X, p(x)$ is true and $a \in X$, then $p(a)$ is true.

For any family of types $B : A \to \mathrm{Type}$, the dependent function type $\prod_{x:A} B(x)$ is governed by the rules:

- $^{\Pi}$intro: if given any $x : A$ there is a term $b_x : B(x)$

  there is a term $\lambda x.b_x : \prod_{x:A} B(x)$
- $^{\Pi}$elim: given terms $f : \prod_{x:A} B(x)$ and $a : A$ there is a term $f(a) : B(a)$

plus computation rules that relate λ-abstractions and evaluations.

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For a constant type family $B : A \to \text{Type}$, the dependent function type recovers $A \to B$

Existential quantification and dependent sums

For any predicate $p : X \to \{\bot, \top\}$, the existential quantification $\exists x \in X, p(x)$ is the logical formula defined by the rules:

- $^{\exists}$intro: If $a \in X$ and $p(a)$ is true, then $\exists x \in X, p(x)$ is true.
- $^{\exists}$elim: If $\exists x \in X, p(x)$ is true and q can be derived from the assumption that $p(a)$ is true for some $a \in X$, then q is true.

For any family of types $B : A \to \mathrm{Type}$, the dependent sum type $\sum_{x:A} B(x)$ is governed by the rules:

- $^{\Sigma}$intro: if there are terms $a : A$ and $b : B(a)$, there is a term $(a, b) : \sum_{x:A} B(x)$
- $^{\Sigma}$elim: given a term $p : \sum_{x:A} B(x)$ there are terms $\pi_1 p : A$ and $\pi_2 p : B(\pi_1 p)$

plus computation rules that relate pairings and projections.

For a constant type family $B : A \to \mathrm{Type}$, the dependent sum type recovers $A \times B$.

Exchanging quantifiers, revisited

Theorem. For any $p(x, y)$, $\exists y \in Y, \forall x \in X, p(x, y) \Rightarrow \forall x' \in X, \exists y' \in Y, p(x', y')$.

Theorem. For any $P : X \to Y \to \mathrm{Type}$, $\Sigma_{y:Y} \Pi_{x:X} P(x, y) \to \Pi_{x':X} \Sigma_{y':Y}, P(x', y')$.

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Proof:

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Proof: By ${}^{\to}$intro, we may assume $h : \Sigma_{y:Y}\Pi_{x:X} P(x,y)$; our goal is of type $\Pi_{x':X}\Sigma_{y':Y} P(x',y')$.

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The constructs $\lambda h.\lambda x'.(\pi_1 h, \pi_2 h(x')) : \Sigma_{y:Y}\Pi_{x:X}P(x,y) \to \Pi_{x':X}\Sigma_{y':Y}, P(x',y')$.

3

Peano's axioms, revisited

The natural numbers

Dedekind's Categoricity Theorem. The natural numbers $\mathbb{N}$ are characterized by Peano's postulates:

- There is a natural number $0 \in \mathbb{N}$.
- Every natural number $n \in \mathbb{N}$ has a successor $\mathsf{suc}\, n \in \mathbb{N}$.
- 0 is not the successor of any natural number.
- No two natural numbers have the same successor.
- The principle of mathematical induction: for all predicates $P : \mathbb{N} \to \{\bot, \top\}$

$$P(0) \Rightarrow (\forall k \in \mathbb{N}, P(k) \Rightarrow P(\mathsf{suc}\, k)) \Rightarrow (\forall n \in \mathbb{N}, P(n))$$

A proof by induction

Theorem. For any $n \in \mathbb{N}$, $n^2 + n$ is even.

Proof: By induction on $n \in \mathbb{N}$:

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A proof by induction

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Proof: By induction on $n \in \mathbb{N}$:

- In the base case, when $n = 0$, $0^2 + 0 = 2 \times 0$, which is even.
- For the inductive step, assume for $k \in \mathbb{N}$ that $k^2 + k = 2 \times m$ is even. Then

$$\begin{aligned}(k+1)^2 + (k+1) &= (k^2 + k) + ((2 \times k) + 2) \\ &= (2 \times m) + (2 \times (k+1)) \\ &= 2 \times (m + k + 1) \qquad \text{is even.}\end{aligned}$$

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By the principle of mathematical induction

$$\forall P, P(0) \Rightarrow (\forall k \in \mathbb{N}, P(k) \Rightarrow P(\mathrm{suc}k)) \Rightarrow (\forall n \in \mathbb{N}, P(n))$$

this proves that $n^2 + n$ is even for all $n \in \mathbb{N}$. $\square$

A construction by induction

The inductive proof not only demonstrates for all $n \in \mathbb{N}$ that $n^2 + n$ is even but also defines a function $m : \mathbb{N} \to \mathbb{N}$ so that $n^2 + n = 2 \times m(n)$.

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Theorem. There is a function $m\colon \mathbb{N} \to \mathbb{N}$ so that $n^2 + n = 2 \times m(n)$ for all $n \in \mathbb{N}$.

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Construction: By induction on $n \in \mathbb{N}$:

- In the base case, $0^2 + 0 = 2 \times 0$, so we define $m(0) \coloneqq 0$.

A construction by induction

The inductive proof not only demonstrates for all $n \in \mathbb{N}$ that $n^2 + n$ is even but also defines a function $m : \mathbb{N} \to \mathbb{N}$ so that $n^2 + n = 2 \times m(n)$.

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Construction: By induction on $n \in \mathbb{N}$:

- In the base case, $0^2 + 0 = 2 \times 0$, so we define $m(0) \coloneqq 0$.
- For the inductive step, assume for $k \in \mathbb{N}$ that $k^2 + k = 2 \times m(k)$. Then

$$\begin{aligned}(k+1)^2 + (k+1) &= (k^2 + k) + ((2 \times k) + 2) \\ &= (2 \times m(k)) + (2 \times (k+1)) \\ &= 2 \times (m(k) + k + 1)\end{aligned}$$

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By the principle of mathematical recursion, this defines a function $m : \mathbb{N} \to \mathbb{N}$ so that $n^2 + n = 2 \times m(n)$ for all $n \in \mathbb{N}$. $\square$

Induction and recursion

Recursion can be thought of as the constructive form of induction

$$\forall P, P(0) \Rightarrow (\forall k \in \mathbb{N}, P(k) \Rightarrow P(\mathsf{suc}\,k)) \Rightarrow (\forall n \in \mathbb{N}, P(n))$$

in which the predicate

$$P : \mathbb{N} \to \{\top, \bot\} \quad \text{such as} \quad P(n) := \exists m \in \mathbb{N}, n^2 + n = 2 \times m$$

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The recursive function $p \in \prod_{n \in \mathbb{N}} P(n)$ satisfies computation rules:

$$p(0) \coloneqq p_0 \qquad p(\mathrm{suc} n) \coloneqq p_s(n, p(n)).$$

The natural numbers in dependent type theory

The natural numbers type $\mathbb{N}$ is governed by the rules:

- ${}^{\mathbb{N}}$intro: there is a term $0 : \mathbb{N}$ and for any term $n : \mathbb{N}$ there is a term $\mathsf{suc}\, n : \mathbb{N}$

The elimination rule strengthens the principle of mathematical induction by replacing the predicate $P : \mathbb{N} \to \{\perp, \top\}$ by an arbitrary family of types $P : \mathbb{N} \to \mathsf{Type}$.

- ${}^{\mathbb{N}}$elim: for any type family $P : \mathbb{N} \to \mathsf{Type}$, to prove $p : \prod_{n:\mathbb{N}} P(n)$ it suffices to prove $p_0 : P(0)$ and $p_s : \prod_{k:\mathbb{N}} P(k) \to P(\mathsf{suc}\, k)$. That is

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Note the other two Peano postulates are missing because they are provable!

$$\infty$$

$$=$$

Identity types

The following rules for identity types were developed by Martin-Löf:

Given a type A and terms $x, y : A$, the identity type $x =_A y$ is governed by the rules:

- $^{=}$intro: given a type A and term $x : A$ there is a term $\mathrm{refl}_x : x =_A x$

The elimination rule for the identity type defines an induction principle analogous to recursion over the natural numbers: it provides sufficient conditions for which to define a dependent function out of the identity type family.

- $^{=}$elim: for any type family $P(x, y, p)$ over $x, y : A$ and $p : x =_A y$, to prove $P(x, y, p)$ for all x, y, p it suffices to assume y is x and p is refl_x. That is

$$^{=}\mathrm{ind} : \left(\prod_{x:A} P(x, x, \mathrm{refl}_x)\right) \to \left(\prod_{x,y:A} \prod_{p:x=_A y} P(x, y, p)\right)$$

A computation rule establishes that the proof of $P(x, x, \mathrm{refl}_x)$ is the given one.

Symmetry and transitivity of identifications

${}^{=}$elim: For any type family $P(x, y, p)$ over $x, y : A$ and $p : x =_A y$,

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Theorem (symmetry). $(-)^{-1} : \prod_{x,y:A} x =_A y \to y =_A x$.

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Theorem (transitivity). $* : \prod_{x,y,z:A} x =_A y \to (y =_A z \to x =_A z)$.

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Functions preserve identifications

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In set theory, a function $f : X \to Y$ is well-defined: if $x = x'$ then $f(x) = f(x')$.

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Theorem. For any $f : A \to B$ and $a, a' : A$, there is a term

$$ \mathrm{ap}_f : (a =_A a') \to (f(a) =_B f(a')). $$

Construction: Let $f : A \to B$. By ${}^{=}$elim applied to the family $P(x, y, p) :\equiv f(x) =_B f(y)$, to define $\mathrm{ap}_f : \prod_{a,a':A}(a =_A a') \to (f(a) =_B f(a'))$ we may reduce to the case $\prod_{a:A} f(a) =_B f(a)$, for which we have $\lambda a.\mathrm{refl}_{f(a)} : \prod_{a:A} f(a) =_B f(a)$. $\square$

Inductive constructions over the natural numbers

$^{\mathbb{N}}$elim: For any type family $P(n)$ over $n : \mathbb{N}$,

$$^{\mathbb{N}}\mathsf{ind} : P(0) \to \left(\prod_{k \in \mathbb{N}} P(k) \to P(\mathsf{suc}\, k)\right) \to \left(\prod_{n \in \mathbb{N}} P(n)\right)$$

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- $+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ is defined by $\begin{cases} m + 0 := m \\ m + \mathrm{suc}\,k := \mathrm{suc}(m + k) \end{cases}$
- $\mathrm{dist}_{2\times} : \prod_{m:\mathbb{N}} \prod_{n:\mathbb{N}} 2 \times m + 2 \times n =_{\mathbb{N}} 2 \times (m + n)$ is defined by

$$\begin{cases} \mathrm{dist}_{2\times}(m, 0) := \mathrm{refl}_{2\times m} \\ \mathrm{dist}_{2\times}(m, \mathrm{suc}\,k) := \mathrm{ap}_{\mathrm{suc}\circ\mathrm{suc}}(\mathrm{dist}_{2\times}(m, n)) \end{cases}$$

A constructive proof revisited

We proved for any $n \in \mathbb{N}$, that $n^2 + n$ is even by induction and by recursively defining $m : \mathbb{N} \to \mathbb{N}$ so that $n^2 + n = 2 \times m(n)$.

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Theorem. For $\text{square+self} : \mathbb{N} \to \mathbb{N}$ given by $\begin{cases} \text{square+self}(0) := 0 \\ \text{square+self}(\text{suc}\,k) := \\ \quad \text{square+self}(k) + 2 \times \text{suc}\,k \end{cases}$

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- For $\mathsf{succ}\,k : \mathbb{N}$, from $m(k) : \mathbb{N}$ and $p(k) : \mathsf{square{+}self}(k) =_{\mathbb{N}} 2 \times m(k)$ we have:

$$\mathsf{ap}_{+2\times\mathsf{succ}\,k}\,p(k) : \mathsf{square{+}self}(k) + 2 \times \mathsf{succ}\,k =_{\mathbb{N}} 2 \times m(k) + 2 \times \mathsf{succ}\,k$$
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Composing these identifications yields the desired term:

$$(m(k) + \mathsf{suc}\,k, \mathsf{ap}_{+2\times\mathsf{suc}\,k}\, p(k) \cdot \mathsf{dist}_{2\times}(m(k), 2 \times \mathsf{suc}\,k)) : \sum_{m:\mathbb{N}} \mathsf{square{+}self}(\mathsf{suc}\,k) =_{\mathbb{N}} 2 \times m \quad \square$$

References

A reintroduction to proofs using introduction and elimination rules:

- Clive Newstead, An Infinite Descent into Pure Mathematics
  `https://infinitedescent.xyz/`

On dependent type theory and identity types (plus much more):

- Egbert Rijke, Introduction to Homotopy Type Theory,
  `arXiv:2212.11082` and forthcoming from *Cambridge University Press*

To explore computer formalization:

- Kevin Buzzard and Mohammad Pedramfar, The natural numbers game,
  `https://adam.math.hhu.de/#/`

Thank you!