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## Arrow induction and the dependent Yoneda lemma

joint with Dom Verity and Mike Shulman



UCLA Distinguished Lecture Series

# What are $\infty$ -categories and what are they for?

*It frames a possible template for any mathematical theory: the theory should have nouns and verbs, i.e., objects, and morphisms, and there should be an explicit notion of composition related to the morphisms; the theory should, in brief, be packaged by a category.*

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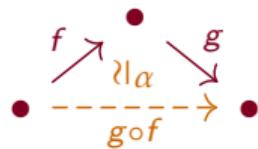
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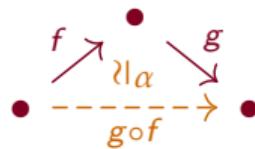
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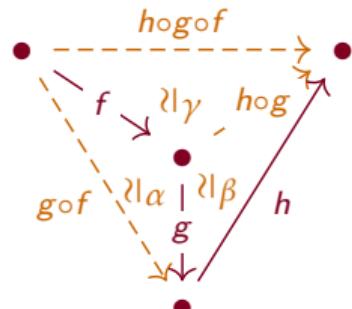
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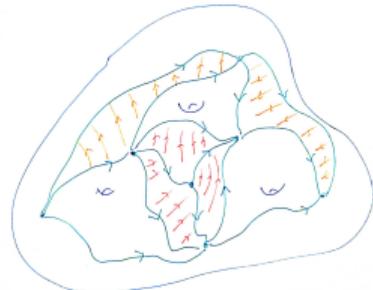
with these witnesses coherent up to invertible morphisms all the way up.



## Examples of $\infty$ -categories

The full homotopy type of a topological space is captured by its **fundamental  $\infty$ -groupoid** whose

- objects are **points**, 1-morphisms are **paths**,
- 2-morphisms are **homotopies** between paths,
- 3-morphisms are **homotopies between homotopies**, ...



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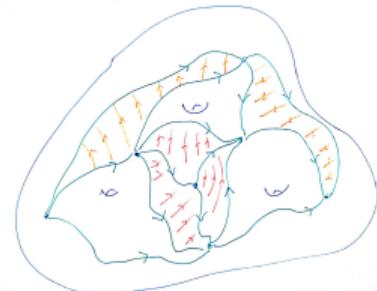
- The **derived category** of a ring is the homotopy category of the  $\infty$ -category of chain complexes.
- The category of closed  $n$ -manifolds and **diffeomorphism classes** of cobordisms is the homotopy category of the  $\infty$ -category of closed  $n$ -manifolds and cobordisms.



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Here “ **$\infty$ -category**” is a nickname for the  $n = 1$  special case of an  **$(\infty, n)$ -category**, a weak infinite dimensional category in which all morphisms above dimension  $n$  are invertible (for fixed  $0 \leq n \leq \infty$ ).

## Curiosity 1: a postponed definition of an $\infty$ -category



In the Introductory Workshop for the Derived Algebraic Geometry and Birational Geometry and Moduli Spaces programs at MSRI in February 2019, Carlos Simpson gave a beautiful three-hour lecture course “ $\infty$ -categories and why they are useful”:

*Abstract: In this series, we'll introduce  $\infty$ -categories and explain their relationships with triangulated categories, dg-categories, and Quillen model categories. We'll explain how the  $\infty$ -categorical language makes it possible to create a moduli framework for objects that have some kind of up-to-homotopy aspect: stacks, complexes, as well as higher categories themselves. The main concepts from usual category theory generalize very naturally. Emphasis will be given to how these techniques apply in algebraic geometry. In the last talk we'll discuss current work related to mirror symmetry and symplectic geometry via the notion of stability condition.*

What's curious is that a **definition** of an  $\infty$ -category doesn't appear until the second half of the second talk.

## Curiosity 2: competing models of $\infty$ -categories



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These competing definitions are referred to as **models** of  $\infty$ -categories.

## Curiosity 3: the necessity of repetition?



Considerable work has gone into defining the key notions for and proving the fundamental results about  $\infty$ -categories, but sometimes this work is later redeveloped starting from a different model.

— e.g., David Kazhdan, Yakov Varshavsky's [Yoneda Lemma for Complete Segal Spaces](#) begins:

*In recent years  $\infty$ -categories or, more formally,  $(\infty, 1)$ -categories appear in various areas of mathematics. For example, they became a necessary ingredient in the geometric Langlands problem. In his books [Lu1, Lu2] Lurie developed a theory of  $\infty$ -categories in the language of quasi-categories and extended many results of the ordinary category theory to this setting.*

*In his work [Re1] Rezk introduced another model of  $\infty$ -categories, which he called complete Segal spaces. This model has certain advantages. For example, it has a generalization to  $(\infty, n)$ -categories (see [Re2]).*

*It is natural to extend results of the ordinary category theory to the setting of complete Segal spaces. In this note we do this for the Yoneda lemma.*

## Curiosity 4: avoiding a precise definition at all



The precursor to Jacob Lurie's [Higher Topos Theory](#) is a 2003 preprint [On  \$\infty\$ -Topoi](#), which avoids selecting a model of  $\infty$ -categories at all:

*We will begin in §1 with an informal review of the theory of  $\infty$ -categories. There are many approaches to the foundation of this subject, each having its own particular merits and demerits. Rather than single out one of those foundations here, we shall attempt to explain the ideas involved and how to work with them. The hope is that this will render this paper readable to a wider audience, while experts will be able to fill in the details missing from our exposition in whatever framework they happen to prefer.*

# Reimagining the foundations of $\infty$ -category theory



A main theme from a new book [Elements of  \$\infty\$ -Category Theory](#) is that the theory of  $\infty$ -categories is model independent.

[elements-book.github.io/elements.pdf](https://elements-book.github.io/elements.pdf)

In more detail:

- Much of the theory of  $\infty$ -categories can be developed **model independently**, in an axiomatic setting we call an  **$\infty$ -cosmos**.
- Change-of-model functors define **biequivalences** of  $\infty$ -cosmoi, which **preserve**, **reflect**, and **create**  $\infty$ -categorical structures.
- Consequently theorems proven both “**synthetically**” and “**analytically**” transfer between models.
- Moreover there is a **formal language** for expressing properties about  $\infty$ -categories that are independent of a choice of model.

# Plan



1. Model-independent foundations of  $\infty$ -category theory
2. The fibrational Yoneda lemma
3. Arrow induction as directed path induction

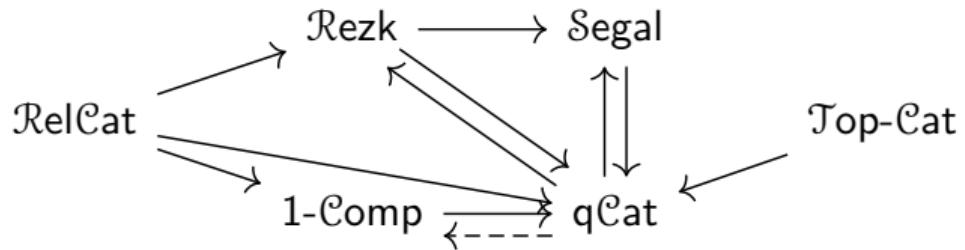


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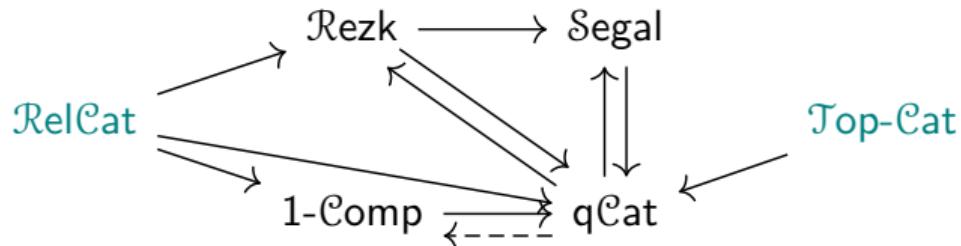
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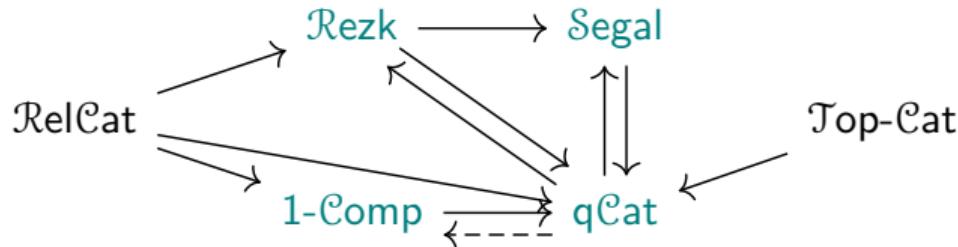


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- topological categories and relative categories are the simplest to define but the maps between them are too strict
  - quasi-categories (née weak Kan complexes),
- - Rezk spaces (née complete Segal spaces),
  - Segal categories, and
  - (saturated 1-trivial weak) 1-complicial setseach have the correct maps and also an internal hom, and in fact any of these categories can be enriched over any of the others

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Two strategies:

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Our method: introduce an  **$\infty$ -cosmos** to axiomatize the common features of the categories [qCat](#), [Rezk](#), [Segal](#), [1-Comp](#) of  $\infty$ -categories.



## $\infty$ -cosmoi of $\infty$ -categories

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**Definition (RV).** An  $\infty$ -cosmos is a category that

- is enriched over quasi-categories, i.e., functors  $f : A \rightarrow B$  between  $\infty$ -categories define the points of a quasi-category  $\text{Fun}(A, B)$ ,
- has a class of isofibrations  $p : E \twoheadrightarrow B$  with familiar closure properties,
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Henceforth  $\infty$ -category and  $\infty$ -functor are technical terms that refer to the objects and morphisms of some  $\infty$ -cosmos.



## The homotopy 2-category

The **homotopy 2-category** of an  $\infty$ -cosmos is a strict 2-category whose:

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**Theorem (RV).** **Equivalences** in the homotopy 2-category

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \gamma \\ \xleftarrow{g} \end{array} & B \\ & \Downarrow \cong & \\ A & \xrightarrow{\quad gf \quad} & A \\ & \Downarrow \cong & \\ B & \xrightarrow{\quad fg \quad} & B \end{array}$$

coincide with **equivalences** in the  $\infty$ -cosmos.

Thus, non-evil 2-categorical definitions are “homotopically correct.”

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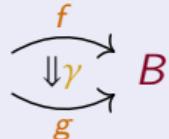
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**A:** The more delicate task is to prove that these synthetic definitions coincide with the previously-established definitions when interpreted in one of the models (but they do).



2

The fibrational Yoneda lemma

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## The arrow $\infty$ -category

The hom  $\infty$ -groupoids are defined as fibers of a two-sided discrete fibration:

$$\begin{array}{ccc} \text{Hom}_A(x,y) & \longrightarrow & A^2 \\ \downarrow & \lrcorner & \downarrow (\text{cod},\text{dom}) \\ \mathbb{1} & \xrightarrow{(\text{y},\text{x})} & A \times A \end{array}$$

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Summary: The hom-bifunctor  $A^{\text{op}} \times A \xrightarrow{\text{Hom}_A} \mathcal{S}$  is encoded by

$$\begin{array}{c} A^2 \\ \downarrow (\text{cod},\text{dom}) \\ A \times A \end{array}$$



## Fibrations of $\infty$ -categories

Consider an isofibration  $p: E \twoheadrightarrow B$  and perform an analogous construction, defining the **comma  $\infty$ -category** over  $p$  in the homotopy 2-category:

$$\begin{array}{ccc} \text{Hom}_B(B, p) & \xrightarrow{[\phi]} & B^2 \\ (\text{cod}, \text{dom}) \downarrow & \lrcorner & \downarrow (\text{cod}, \text{dom}) \rightsquigarrow \\ E \times B & \xrightarrow{p \times \text{id}} & B \times B \end{array}$$
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**Definition (RV).** An isofibration  $p: E \twoheadrightarrow B$  is a **cartesian fibration** just when the comma cone  $\phi$  admits a  **$p$ -cartesian lift**  $\chi$ :

$$\begin{array}{ccc} \text{Hom}_B(B, p) & \xrightarrow{\text{cod}} & E \\ \uparrow \phi & \nearrow \text{dom} & \downarrow p \\ & = & \\ \text{Hom}_B(B, p) & \xrightarrow{\text{cod}} & E \\ & \uparrow \chi & \downarrow p \\ & r & \Rightarrow B \end{array}$$

# A Chevalley criterion for cartesian fibrations



Street's "Chevalley criterion" characterizing cartesian fibrations extends to  $\infty$ -categories:

**Theorem (RV).** For an isofibration  $p: E \twoheadrightarrow B$  the following are equivalent:

- (i)  $p: E \twoheadrightarrow B$  defines a cartesian fibration.
- (ii) The functor  $i_1 \hat{\pitchfork} p: E^2 \rightarrow \text{Hom}_B(B, p)$  admits a right adjoint right inverse:

$$E^2 \begin{array}{c} \xrightarrow{i_1 \hat{\pitchfork} p} \\ \perp \\ \xleftarrow{\lceil x \rceil} \end{array} \text{Hom}_B(B, p)$$

where

$$\begin{array}{ccccc} E^2 & \xrightarrow{p^2} & & & B^2 \\ \dashv & \nearrow i_1 \hat{\pitchfork} p & \searrow & \nearrow [\phi] & \downarrow \text{cod} \\ & \text{Hom}_B(B, p) & \xrightarrow{\lceil \phi \rceil} & B^2 & \downarrow \text{cod} \\ \text{cod} \downarrow & & \lrcorner & & \downarrow \text{cod} \\ E & \xrightarrow{p} & B & & \end{array}$$

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The right adjoint right inverse adjunction is fibered over  $\text{Hom}_B(B, p)$ , which allows us to interpret  $\lceil x \rceil: \text{Hom}_B(B, p) \rightarrow E^2$  as a terminal element in  $E^2$  over  $\text{Hom}_B(B, p)$ .

# A Chevalley criterion for cartesian arrows



A relative form of the Chevalley criterion also characterizes  $p$ -cartesian arrows:

Theorem (RV). For  $p: E \rightarrow B$  and an arrow  $X \xrightarrow{\psi} E$  the following are equivalent:

- (i)  $\psi$  is  $p$ -cartesian.
- (ii) The commutative triangle of functors defines an absolute right lifting diagram:

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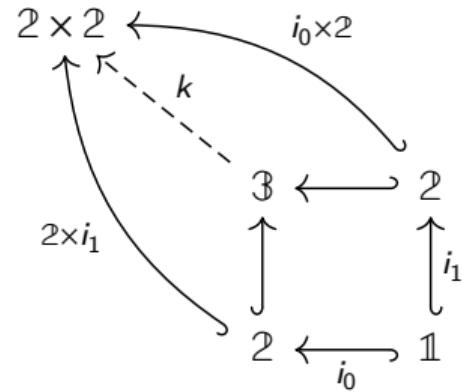
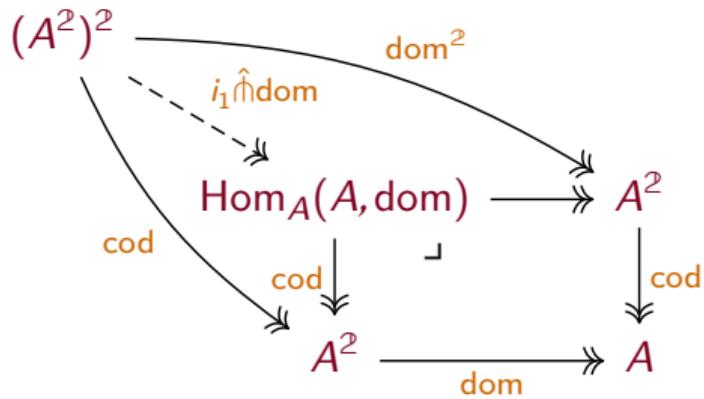
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When applied to the **universal  $p$ -cartesian lift**  $\chi$ ,  $[p\chi] = [\phi] = \text{id}$  and this theorem specializes to the Chevalley criterion for cartesian fibrations.



# The domain fibration

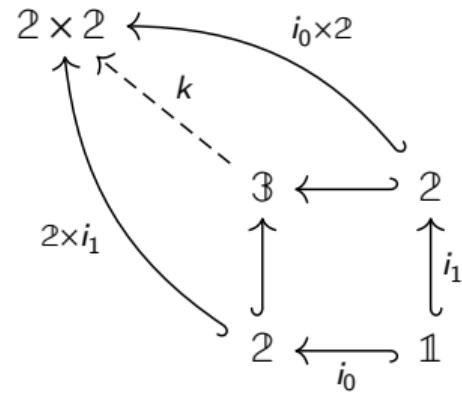
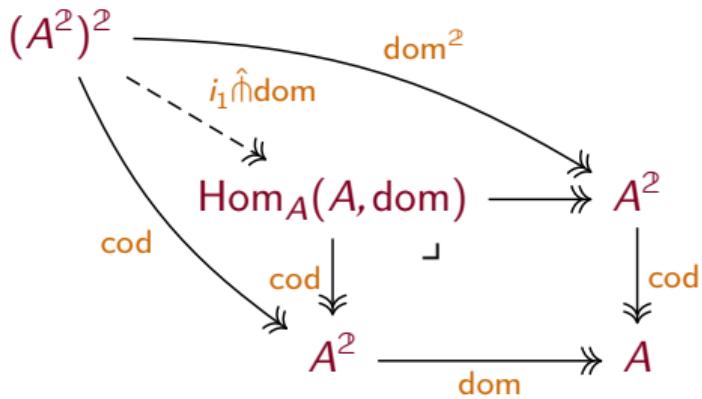
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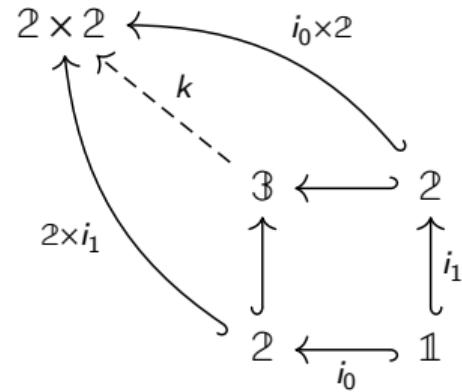
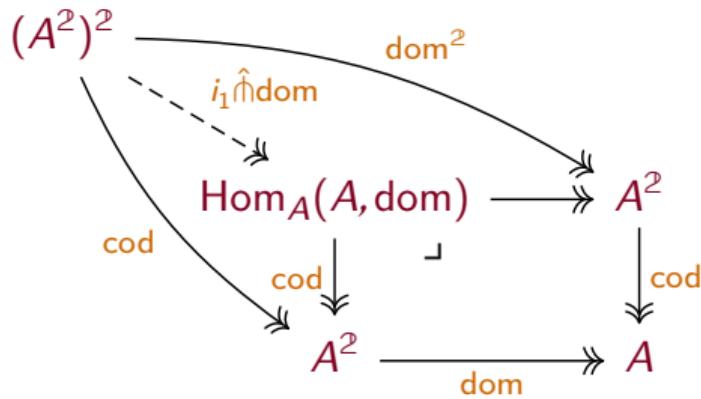


Since the inclusion  $k: 3 \hookrightarrow 2 \times 2$  has a left adjoint left inverse,  $i_1 \hat{\pitchfork} \text{dom}$  has a right adjoint right inverse; hence,  $\text{dom}: A^2 \twoheadrightarrow A$  is a cartesian fibration.



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## Representable fibrations

Fix an  $\infty$ -category  $A$  and an element  $a: \mathbb{1} \rightarrow A$ . The cartesian fibration  $\text{dom}: \text{Hom}_A(A, a) \twoheadrightarrow A$  behaves similarly to  $\text{dom}: A^2 \twoheadrightarrow A$  except the right adjoint right inverse adjunction provided by the Chevalley criterion is an [adjoint equivalence](#):



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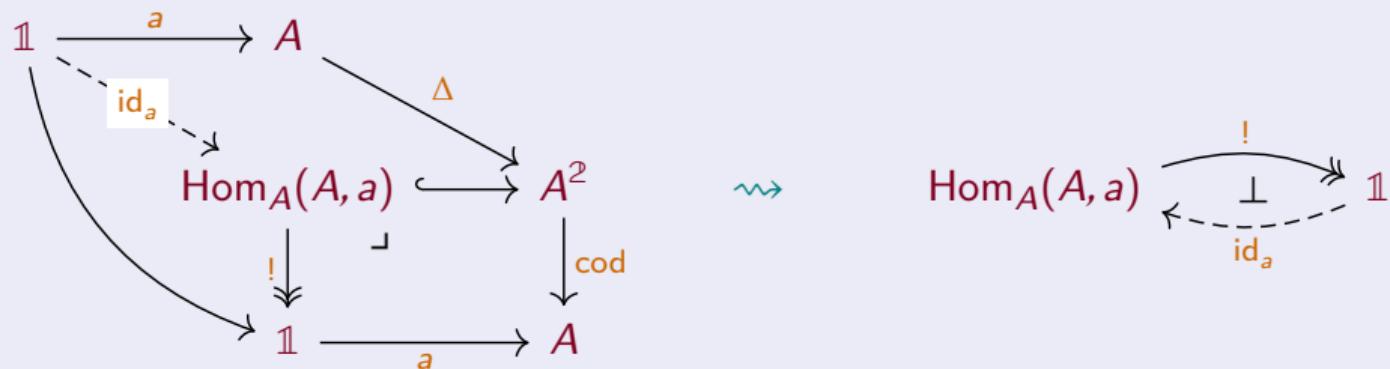


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$\rightsquigarrow \text{dom}: \text{Hom}_A(A, a) \twoheadrightarrow A$  is a **discrete cartesian fibration** aka a **right fibration**.

**Lemma (RV).** The canonical element  $\text{id}_a: \mathbb{1} \rightarrow \text{Hom}_A(A, a)$  in the  $\infty$ -category  $\text{Hom}_A(A, a)$  is **terminal**:





## The fibrational Yoneda lemma

The Yoneda lemma characterizes natural transformations out of a representable functor, here encoded by a discrete cartesian fibration  $\text{dom}: \text{Hom}_A(A, a) \rightarrow\!\!\! \rightarrow A$ .



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**Proof Sketch:** The terminal element  $! \dashv \text{id}_a$  induces an adjoint equivalence upon mapping into a discrete cartesian fibration. □



## The dependent Yoneda lemma

The fibrational Yoneda lemma follows easily from:

**Proposition (RV).** For any terminal element  $t: \mathbb{1} \rightarrow B$  and cartesian fibration  $q: F \twoheadrightarrow B$ , evaluation at  $t$  admits a right adjoint that defines an adjoint equivalence of Kan complexes in when  $q$  is discrete.

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Q: Where did these statements come from?



3

Arrow induction as directed path induction

# $\infty$ -categories in homotopy type theory



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- Paths in  $A$  are equivalent to isomorphisms in  $A$ .

## Contravariant type families



Fix  $a : A$  where  $A$  is an  $\infty$ -category. The family of types  $\text{Hom}_A(x, a)$  over  $x : A$  varies contravariantly over  $A$  in the following sense:

**Definition (RS).** A family of types  $E(x)$  over  $x : A$  is **contravariant** if for every  $f : \text{Hom}_A(x, y)$  and  $u : E(y)$  there is a unique lift of  $f$  with codomain  $u$ .

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**Proposition (RS).** For  $u : E(z)$ ,  $f : \text{Hom}_A(x, y)$ , and  $g : \text{Hom}_A(y, z)$ ,

$$f^*(g^* u) = (g \circ f)^* u \quad \text{and} \quad (\text{id}_z)^* u = u.$$

**Proposition (RS).** Fix  $a : A$ . The type family  $\text{Hom}_A(x, a)$  over  $x : A$  is contravariant.

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For example, taking  $E(x) := \text{Hom}_A(x, b)$  for some fixed  $b : A$ , the Yoneda lemma provides an equivalence between

$$\text{Hom}_A(a, b) \quad \text{and} \quad \prod_{x:A} \text{Hom}_A(x, a) \rightarrow \text{Hom}_A(x, b)$$

## The dependent Yoneda lemma



Yoneda lemma (RS). If  $A$  is an  $\infty$ -category,  $a : A$ , and  $E(x)$  is a contravariant family over  $x : A$ , then evaluation at  $(a, \text{id}_a)$  defines an equivalence

$$\text{ev-id} : \left( \prod_{x:A} \text{Hom}_A(x, a) \rightarrow E(x) \right) \rightarrow E(a)$$

The Yoneda lemma has a dependently-typed generalization analogous to the full induction principle for identity types.

# The dependent Yoneda lemma



Yoneda lemma (RS). If  $A$  is an  $\infty$ -category,  $a : A$ , and  $E(x)$  is a contravariant family over  $x : A$ , then evaluation at  $(a, \text{id}_a)$  defines an equivalence

$$\text{ev-id} : \left( \prod_{x:A} \text{Hom}_A(x, a) \rightarrow E(x) \right) \rightarrow E(a)$$

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Dependent Yoneda lemma (RS). If  $A$  is an  $\infty$ -category,  $a : A$ , and  $F(x, f)$  is a contravariant family over  $x : A$  and  $f : \text{Hom}_A(x, a)$ , then evaluation at  $(a, \text{id}_a)$  defines an equivalence

$$\text{ev-id} : \left( \prod_{x:A} \prod_{f:\text{Hom}_A(x, a)} F(x, f) \right) \rightarrow F(a, \text{id}_a)$$

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This is useful for proving equivalences between various types of coherent or incoherent adjunction data.

## Dependent Yoneda is directed path induction



Takeaway: the dependent Yoneda lemma is directed path induction.

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Path induction: If  $P(x, p)$  is a type family over  $x : A$  and  $p : a =_A x$ , then to prove  $P(x, p)$  it suffices to assume  $x$  is  $a$  and  $p$  is  $\text{refl}_a$ .

$$\text{path-ind}_a : P(a, \text{refl}_a) \rightarrow \left( \prod_{x:A} \prod_{p:a=_A x} P(x, p) \right).$$

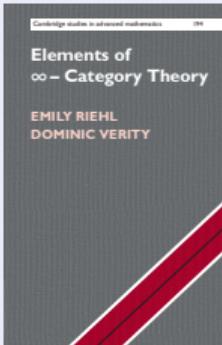
Arrow induction: If  $F(x, f)$  is a contravariant family over  $x : A$  and  $f : \text{Hom}_A(x, a)$  and  $A$  is an  $\infty$ -category, then to prove  $F(x, f)$  it suffices to assume  $x$  is  $a$  and  $f$  is  $\text{id}_a$ .

$$\text{arrow-ind}_a : F(a, \text{id}_a) \rightarrow \left( \prod_{x:A} \prod_{f:\text{Hom}_A(x, a)} F(x, f) \right).$$

# References



For more on the synthetic theory of  $\infty$ -categories see:



Emily Riehl and Dominic Verity

- *Elements of  $\infty$ -Category Theory*

Cambridge University Press 2022

[elements-book.github.io/elements.pdf](https://elements-book.github.io/elements.pdf)

Emily Riehl and Michael Shulman

- *A type theory for synthetic  $\infty$ -categories*

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Thank you!