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Arrow induction and the dependent Yoneda lemma

joint with Dominic Verity and Mike Shulman

 

UCLA Distinguished Lecture Series

What are ∞-categories and what are they for?

> *It frames a possible template for any mathematical theory: the theory should have nouns and verbs, i.e., objects, and morphisms, and there should be an explicit notion of composition related to the morphisms; the theory should, in brief, be packaged by a category.* —*Barry Mazur, "When is one thing equal to some other thing?"*

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$$\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "\wr\|\alpha"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, dashed, "g\circ f"'] & {} & \bullet \end{tikzcd}$$

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- associativity

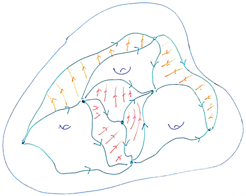
$$\begin{tikzcd} \bullet \arrow[rr, dashed, "h\circ g\circ f"] \arrow[dr, "f"] \arrow[ddr, dashed, "g\circ f"'] & \arrow[d, phantom, "\wr\|\gamma"] & \bullet \\ & \bullet \arrow[ur, dashed, "h\circ g"'] \arrow[d, "g"] \arrow[dl, phantom, "\wr\|\alpha"] \arrow[dr, phantom, "\wr\|\beta"] & \\ & \bullet \arrow[uur, "h"'] & \end{tikzcd}$$

witnessed by invertible 3-morphisms

with these witnesses coherent up to invertible morphisms all the way up.

Examples of ∞-categories

The full homotopy type of a topological space is captured by its fundamental ∞-groupoid whose



- objects are points, 1-morphisms are paths,
- 2-morphisms are homotopies between paths,
- 3-morphisms are homotopies between homotopies, ...

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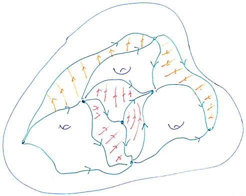

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- The derived category of a ring is the homotopy category of the ∞-category of chain complexes.
- The category of closed n-manifolds and diffeomorphism classes of cobordisms is the homotopy category of the ∞-category of closed n-manifolds and cobordisms.

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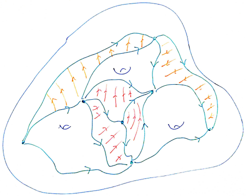

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- The category of closed n-manifolds and diffeomorphism classes of cobordisms is the homotopy category of the ∞-category of closed n-manifolds and cobordisms.

Here "∞-category" is a nickname for the $n = 1$ special case of an (∞, n)-category, a weak infinite dimensional category in which all morphisms above dimension n are invertible (for fixed $0 \leq n \leq \infty$).

Curiosity 1: a postponed definition of an ∞-category

In the Introductory Workshop for the Derived Algebraic Geometry and Birational Geometry and Moduli Spaces programs at MSRI in February 2019, Carlos Simpson gave a beautiful three-hour lecture course "∞-categories and why they are useful":

> *Abstract: In this series, we'll introduce ∞-categories and explain their relationships with triangulated categories, dg-categories, and Quillen model categories. We'll explain how the ∞-categorical language makes it possible to create a moduli framework for objects that have some kind of up-to-homotopy aspect: stacks, complexes, as well as higher categories themselves. The main concepts from usual category theory generalize very naturally. Emphasis will be given to how these techniques apply in algebraic geometry. In the last talk we'll discuss current work related to mirror symmetry and symplectic geometry via the notion of stability condition.*

What's curious is that a definition of an ∞-category doesn't appear until the second half of the second talk.

Curiosity 2: competing models of ∞-categories

That definition of ∞-categories is used in

- André Hirschowitz, Carlos Simpson — Descente pour les n-champs, 1998.

However a different definition appears in

- Pedro Boavida de Brito, Michael Weiss — Spaces of smooth embeddings and configuration categories, 2018.

yet another definition appears in

- Andrew Blumberg, David Gepner, Gonçalo Tabuada — A universal characterization of higher algebraic K-theory, 2013

and still another definition is used at various points in

- Jacob Lurie — Higher Topos Theory, 2009.

These competing definitions are referred to as models of ∞-categories.

Curiosity 3: the necessity of repetition?

Considerable work has gone into defining the key notions for and proving the fundamental results about ∞-categories, but sometimes this work is later redeveloped starting from a different model.

— e.g., David Kazhdan, Yakov Varshavsky's Yoneda Lemma for Complete Segal Spaces begins:

> *In recent years ∞-categories or, more formally, $(\infty,1)$-categories appear in various areas of mathematics. For example, they became a necessary ingredient in the geometric Langlands problem. In his books [Lu1, Lu2] Lurie developed a theory of ∞-categories in the language of quasi-categories and extended many results of the ordinary category theory to this setting.*
>
> *In his work [Re1] Rezk introduced another model of ∞-categories, which he called complete Segal spaces. This model has certain advantages. For example, it has a generalization to (∞, n)-categories (see [Re2]).*
>
> *It is natural to extend results of the ordinary category theory to the setting of complete Segal spaces. In this note we do this for the Yoneda lemma.*

Curiosity 4: avoiding a precise definition at all

The precursor to Jacob Lurie's Higher Topos Theory is a 2003 preprint On ∞-Topoi, which avoids selecting a model of ∞-categories at all:

> *We will begin in §1 with an informal review of the theory of ∞-categories. There are many approaches to the foundation of this subject, each having its own particular merits and demerits. Rather than single out one of those foundations here, we shall attempt to explain the ideas involved and how to work with them. The hope is that this will render this paper readable to a wider audience, while experts will be able to fill in the details missing from our exposition in whatever framework they happen to prefer.*

Reimagining the foundations of ∞-category theory

A main theme from a new book Elements of ∞-Category Theory is that the theory of ∞-categories is model independent.

`elements-book.github.io/elements.pdf`

In more detail:

- Much of the theory of ∞-categories can be developed model independently, in an axiomatic setting we call an ∞-cosmos.
- Change-of-model functors define biequivalences of ∞-cosmoi, which preserve, reflect, and create ∞-categorical structures.
- Consequently theorems proven both "synthetically" and "analytically" transfer between models.
- Moreover there is a formal language for expressing properties about ∞-categories that are independent of a choice of model.

Plan

1. Model-independent foundations of ∞-category theory

2. The fibrational Yoneda lemma

3. Arrow induction as directed path induction

1

Model-independent foundations of ∞-category theory

Models of ∞-categories

The meaning of the term ∞-category is made precise by several models, connected by "change-of-model" functors.

$$
\begin{tikzcd}
& \mathcal{R}\mathsf{ezk} \arrow[r] \arrow[dr, shift left] & \mathcal{S}\mathsf{egal} \arrow[d, shift left] & \\
\mathcal{R}\mathsf{el}\mathcal{C}\mathsf{at} \arrow[ur] \arrow[dr] \arrow[drr] & & & \mathcal{T}\mathsf{op}\text{-}\mathcal{C}\mathsf{at} \arrow[dl] \\
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- topological categories and relative categories are the simplest to define but the maps between them are too strict
- $\begin{cases} \text{quasi-categories (née weak Kan complexes),} \\ \text{Rezk spaces (née complete Segal spaces),} \\ \text{Segal categories, and} \\ \text{(saturated 1-trivial weak) 1-complicial sets} \end{cases}$
  each have the correct maps and also an internal hom, and in fact any of these categories can be enriched over any of the others

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Two strategies:

- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

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Our method: introduce an ∞-cosmos to axiomatize the common features of the categories $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ of ∞-categories.

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Idea: An ∞-cosmos is an infinite-dimensional category whose objects are ∞-categories: an "$(\infty, 2)$-category with $(\infty, 2)$-categorical limits."

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An ∞-cosmos is a category that

- is enriched over quasi-categories, i.e., functors $f : A \to B$ between ∞-categories define the points of a quasi-category $\mathrm{Fun}(A, B)$,
- has a class of isofibrations $p : E \twoheadrightarrow B$ with familiar closure properties,
- and has the expected limits of diagrams of ∞-categories and isofibrations, which satisfy simplicially-enriched universal properties.

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Theorem. $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, and $1\text{-}\mathcal{C}omp$ define ∞-cosmoi, and so do certain models of (∞, n)-categories for $0 \leq n \leq \infty$, fibered versions of all of the above, and many more things besides (ordinary categories, Kan complexes, stable ∞-categories, ...).

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Henceforth ∞-category and ∞-functor are technical terms that refer to the objects and morphisms of some ∞-cosmos.

The homotopy 2-category

The homotopy 2-category of an ∞-cosmos is a strict 2-category whose:

- objects are the ∞-categories A, B in the ∞-cosmos
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- 2-cells we call ∞-natural transformations $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$ which are defined to be homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$

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Theorem (RV). Equivalences in the homotopy 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\Downarrow\cong"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, equal] \arrow[r, bend right, "fg"'] \arrow[r, phantom, "\Downarrow\cong"] & B \end{tikzcd}$$

coincide with equivalences in the ∞-cosmos.

Thus, non-evil 2-categorical definitions are "homotopically correct."

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A: The more delicate task is to prove that these synthetic definitions coincide with the previously-established definitions when interpreted in one of the models (but they do).

2

The fibrational Yoneda lemma

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- The Yoneda embeddings (valued in the ∞-category of spaces $\mathcal{S}$)

$$\begin{array}{ccc} A & \overset{よ}{\hookrightarrow} & \mathcal{S}^{A^{\mathrm{op}}} \\ y & \longmapsto & \mathrm{Hom}_A(A, y) \end{array} \qquad \begin{array}{ccc} A^{\mathrm{op}} & \overset{よ}{\hookrightarrow} & \mathcal{S}^{A} \\ x & \longmapsto & \mathrm{Hom}_A(x, A) \end{array}$$

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The arrow ∞-category

The hom ∞-groupoids are defined as fibers of a two-sided discrete fibration:

$$\begin{tikzcd}
\mathrm{Hom}_A(x,y) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{2} \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"] \\
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Summary: The hom-bifunctor $A^{\mathrm{op}} \times A \xrightarrow{\mathrm{Hom}_A} \mathcal{S}$ is encoded by

$$\begin{tikzcd}
A^{\mathbb{2}} \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"] \\
A \times A
\end{tikzcd}$$

Fibrations of ∞-categories

Consider an isofibration $p\colon E \twoheadrightarrow B$ and perform an analogous construction, defining the comma ∞-category over p in the homotopy 2-category:

$$\begin{tikzcd} \mathrm{Hom}_B(B,p) \arrow[r, "\ulcorner\phi\urcorner"] \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B^{\mathbb{2}} \arrow[d, two heads, "(\mathrm{cod},\mathrm{dom})"] \\ E \times B \arrow[r, "p\times\mathrm{id}"'] & B \times B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathrm{Hom}_B(B,p) \arrow[r, bend left, two heads, "\mathrm{cod}"] \arrow[dr, bend right, two heads, "\mathrm{dom}"'] & E \arrow[d, two heads, "p"] \arrow[dl, phantom, "\Uparrow\phi"] \\ & B \end{tikzcd}$$

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Definition (RV). An isofibration $p\colon E \twoheadrightarrow B$ is a cartesian fibration just when the comma cone ϕ admits a p-cartesian lift χ:

$$\begin{tikzcd} \mathrm{Hom}_B(B,p) \arrow[r, bend left, two heads, "\mathrm{cod}"] \arrow[dr, bend right, two heads, "\mathrm{dom}"'] & E \arrow[d, two heads, "p"] \arrow[dl, phantom, "\Uparrow\phi"] \\ & B \end{tikzcd} = \begin{tikzcd} \mathrm{Hom}_B(B,p) \arrow[r, bend left, two heads, "\mathrm{cod}"] \arrow[r, bend right, dashed, "r"'] \arrow[r, phantom, "\Uparrow\chi"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

A Chevalley criterion for cartesian fibrations

Street's "Chevalley criterion" characterizing cartesian fibrations extends to ∞-categories:

Theorem (RV). For an isofibration $p\colon E \twoheadrightarrow B$ the following are equivalent:

(i) $p\colon E \twoheadrightarrow B$ defines a cartesian fibration.

(ii) The functor $i_1\widehat{\pitchfork}p\colon E^{\mathbb{2}} \twoheadrightarrow \mathrm{Hom}_B(B,p)$ admits a right adjoint right inverse:

$$\begin{tikzcd} E^{\mathbb{2}} \arrow[r, two heads, bend left, "i_1\widehat{\pitchfork}p"] \arrow[r, phantom, "\perp"] & \mathrm{Hom}_B(B,p) \arrow[l, dashed, bend left, "\ulcorner\chi\urcorner"] \end{tikzcd}$$

where

$$\begin{tikzcd} E^{\mathbb{2}} \arrow[dr, dashed, "i_1\widehat{\pitchfork}p"] \arrow[drr, two heads, bend left, "p^{\mathbb{2}}"] \arrow[ddr, bend right, "\mathrm{cod}"'] & & \\ & \mathrm{Hom}_B(B,p) \arrow[r, two heads, "\ulcorner\phi\urcorner"] \arrow[d, two heads, "\mathrm{cod}"'] \arrow[dr, phantom, "\lrcorner"] & B^{\mathbb{2}} \arrow[d, two heads, "\mathrm{cod}"] \\ & E \arrow[r, two heads, "p"'] & B \end{tikzcd}$$

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The right adjoint right inverse adjunction is fibered over $\mathrm{Hom}_B(B, p)$, which allows us to interpret $\ulcorner \chi \urcorner \colon \mathrm{Hom}_B(B, p) \to E^{\mathbb{2}}$ as a terminal element in $E^{\mathbb{2}}$ over $\mathrm{Hom}_B(B, p)$.

A Chevalley criterion for cartesian arrows

A relative form of the Chevalley criterion also characterizes p-cartesian arrows:

Theorem (RV). For $p \colon E \twoheadrightarrow B$ and an arrow
$$\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\psi"] & E \end{tikzcd}$$
the following are equivalent:

(i) ψ is p-cartesian.

(ii) The commutative triangle of functors defines an absolute right lifting diagram:

$$\begin{tikzcd} & E^{\mathbb{2}} \arrow[d, two heads, "i_1 \hat{\pitchfork} p"] \\ X \arrow[ur, "\ulcorner \psi \urcorner"] \arrow[r, "\ulcorner p\psi \urcorner"'] \arrow[ur, phantom, "\parallel"] & \mathrm{Hom}_B(B, p) \end{tikzcd}$$

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A relative form of the Chevalley criterion also characterizes p-cartesian arrows:

Theorem (RV). For $p \colon E \twoheadrightarrow B$ and an arrow $X \underset{e}{\overset{e'}{\rightrightarrows}} E$ with $\Downarrow\psi$ the following are equivalent:

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An absolute right lifting diagram is a relative adjunction: here the right adjoint right inverse $\ulcorner \psi \urcorner$ is only defined after restricting along $\ulcorner p\psi \urcorner \colon X \to \mathrm{Hom}_B(B, p)$.

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When applied to the universal p-cartesian lift χ, $\ulcorner p\chi\urcorner = \ulcorner\phi\urcorner = \mathrm{id}$ and this theorem specializes to the Chevalley criterion for cartesian fibrations.

The domain fibration

Example. Consider the domain fibration $\mathrm{dom}\colon A^{\mathbb{2}} \twoheadrightarrow A$. Up to equivalence, the canonical functor below-left is defined by cotensoring with the 1-categories below-right:

$$
\begin{tikzcd}
(A^{\mathbb{2}})^{\mathbb{2}} \arrow[dr, dashed, two heads, "i_1 \hat{\pitchfork} \mathrm{dom}"] \arrow[drr, bend left, two heads, "\mathrm{dom}^{\mathbb{2}}"] \arrow[ddr, bend right, two heads, "\mathrm{cod}"'] & & \\
& \mathrm{Hom}_A(A, \mathrm{dom}) \arrow[r, two heads] \arrow[d, two heads, "\mathrm{cod}"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "\mathrm{cod}"] \\
& A^{\mathbb{2}} \arrow[r, two heads, "\mathrm{dom}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{2} \times \mathbb{2} & & \\
& \mathbb{3} \arrow[ul, dashed, "k"] & \mathbb{2} \arrow[l, hook'] \arrow[ull, bend right, hook', "i_0 \times \mathbb{2}"'] \\
& \mathbb{2} \arrow[u, hook] \arrow[uul, bend left, hook, "\mathbb{2} \times i_1"] & \mathbb{1} \arrow[l, hook', "i_0"] \arrow[u, hook, "i_1"']
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Since the inclusion $k\colon \mathbb{3} \hookrightarrow \mathbb{2} \times \mathbb{2}$ has a left adjoint left inverse, $i_1 \widehat{\pitchfork} \mathrm{dom}$ has a right adjoint right inverse; hence, $\mathrm{dom}\colon A^{\mathbb{2}} \twoheadrightarrow A$ is a cartesian fibration.

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& \mathrm{Hom}_A(A, \mathrm{dom}) \arrow[r, two heads] \arrow[d, two heads, "\mathrm{cod}"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^{\mathbb{2}} \arrow[d, two heads, "\mathrm{cod}"] \\
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Representable fibrations

Fix an ∞-category A and an element $a \colon \mathbb{1} \to A$. The cartesian fibration $\mathrm{dom} \colon \mathrm{Hom}_A(A, a) \twoheadrightarrow A$ behaves similarly to $\mathrm{dom} \colon A^{\mathbb{2}} \twoheadrightarrow A$ except the right adjoint right inverse adjunction provided by the Chevalley criterion is an adjoint equivalence:

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$\rightsquigarrow$ $\mathrm{dom} \colon \mathrm{Hom}_A(A, a) \twoheadrightarrow A$ is a discrete cartesian fibration aka a right fibration.

Lemma (RV). The canonical element $\mathrm{id}_a : \mathbb{1} \to \mathrm{Hom}_A(A, a)$ in the ∞-category $\mathrm{Hom}_A(A, a)$ is terminal:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "a"] \arrow[dr, dashed, "\mathrm{id}_a"] \arrow[ddr, bend right] & A \arrow[dr, "\Delta"] & & & & \\
& \mathrm{Hom}_A(A,a) \arrow[r, hook] \arrow[d, two heads, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbb{2}} \arrow[d, "\mathrm{cod}"] & \rightsquigarrow & \mathrm{Hom}_A(A,a) \arrow[r, bend left, two heads, "!"] \arrow[r, phantom, "\perp"] & \mathbb{1} \arrow[l, bend left, dashed, "\mathrm{id}_a"] \\
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\end{tikzcd}$$

The fibrational Yoneda lemma

The Yoneda lemma characterizes natural transformations out of a representable functor, here encoded by a discrete cartesian fibration $\mathrm{dom}\colon \mathrm{Hom}_A(A, a) \twoheadrightarrow A$.

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Yoneda Lemma (RV). For any discrete cartesian fibration $p\colon E \twoheadrightarrow A$ and element $a\colon \mathbb{1} \to A$, evaluation at id_a defines an equivalence of Kan complexes:

$$\mathrm{Fun}_A\left(\begin{array}{c} \mathrm{Hom}_A(A,a) \\ \Big\downarrow{\scriptstyle \mathrm{dom}} \\ A \end{array} , \begin{array}{c} E \\ \Big\downarrow{\scriptstyle p} \\ A \end{array} \right) \xrightarrow{\mathrm{ev}_{\mathrm{id}_a}} \mathrm{Fun}(\mathbb{1}, E_a)$$

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Proof Sketch: The terminal element $! \dashv \mathrm{id}_a$ induces an adjoint equivalence upon mapping into a discrete cartesian fibration. □

The dependent Yoneda lemma

The fibrational Yoneda lemma follows easily from:

Proposition (RV). For any terminal element $t\colon \mathbb{1} \to B$ and cartesian fibration $q\colon F \twoheadrightarrow B$, evaluation at t admits a right adjoint that defines an adjoint equivalence of Kan complexes in when q is discrete.

$$\begin{tikzcd} \mathrm{Fun}_B(B, F) \arrow[r, bend left, "\mathrm{ev}_t"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(1, B_t) \arrow[l, bend left, dashed, "よ"] \end{tikzcd}$$

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As a special case:

Dependent Yoneda lemma (RV). For any $a \colon \mathbb{1} \to A$ and discrete cartesian fibration $p \colon F \twoheadrightarrow \mathrm{Hom}_A(A, a)$, evaluation at id_a defines an equivalence of Kan complexes:

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Q: Where did these statements come from?

3

Arrow induction as directed path induction

∞-categories in homotopy type theory

The identity type family gives each type the structure of an ∞-groupoid: each type A has a family of identity types $x =_A y$ over $x, y : A$ whose terms $p : x =_A y$ are called paths.

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each type A also has a family of hom types $\mathrm{Hom}_A(x, y)$ over $x, y : A$ whose terms $f : \mathrm{Hom}_A(x, y)$ are called arrows.

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Definition (RS). A type A is an ∞-category if:

- Every pair of arrows $f : \mathrm{Hom}_A(x, y)$ and $g : \mathrm{Hom}_A(y, z)$ has a unique composite, defining a term $g \circ f : \mathrm{Hom}_A(x, z)$.

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- Every pair of arrows $f : \mathrm{Hom}_A(x, y)$ and $g : \mathrm{Hom}_A(y, z)$ has a unique composite, defining a term $g \circ f : \mathrm{Hom}_A(x, z)$.
- Paths in A are equivalent to isomorphisms in A.

Covariant type families

Fix $a : A$ where A is an ∞-category. The family of types $\mathrm{Hom}_A(a, x)$ over $x : A$ varies covariantly over the arrows of A in the following sense:

Definition (RS). A family of types $E(x)$ over $x : A$ is covariant if for every $f : \mathrm{Hom}_A(x, y)$ and $u : E(x)$ there is a unique lift of f with domain u, defining $f_* u : B(y)$.

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Proposition (RS). For $u : E(x)$, $f : \mathrm{Hom}_A(x, y)$, and $g : \mathrm{Hom}_A(y, z)$,

$$g_*(f_* u) = (g \circ f)_* u \qquad \text{and} \qquad (\mathrm{id}_x)_* u = u.$$

Proposition (RS). Fix $a : A$. The type family $\mathrm{Hom}_A(a, x)$ over $x : A$ is covariant.

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Yoneda lemma (RS). The type family $\mathrm{Hom}_A(a, x)$ over $x : A$ is freely generated by $\mathrm{id}_a : \mathrm{Hom}_A(a, a)$ in the sense that the maps

$$\text{ev-id} := \lambda\phi.\phi(a, \mathrm{id}_a) : \left(\prod_{x:A} \mathrm{Hom}_A(a, x) \to E(x)\right) \to E(a)$$

and

$$\mathrm{yon} := \lambda u.\lambda x.\lambda f.f_* u : E(a) \to \left(\prod_{x:A} \mathrm{Hom}_A(a, x) \to E(x)\right)$$

are inverse equivalences for any covariant family $E(x)$ over $x : A$.

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are inverse equivalences for any covariant family $E(x)$ over $x : A$.

For example, taking $E(x) :\equiv \mathrm{Hom}_A(b, x)$ for some fixed $b : A$, the Yoneda lemma provides an equivalence between

$$\mathrm{Hom}_A(b, a) \qquad \text{and} \qquad \prod_{x:A} \mathrm{Hom}_A(a, x) \to \mathrm{Hom}_A(b, x)$$

The dependent Yoneda lemma

Yoneda lemma (RS). If A is an ∞-category, $a : A$, and $E(x)$ is a covariant family over $x : A$, then evaluation at (a, id_a) defines an equivalence

$$\text{ev-id} : \left(\prod_{x:A} \mathrm{Hom}_A(a,x) \to E(x)\right) \to E(a)$$

The Yoneda lemma has a dependently-typed generalization analogous to the full induction principle for identity types.

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Dependent Yoneda lemma (RS). If A is an ∞-category, $a : A$, and $F(x, f)$ is a covariant family over $x : A$ and $f : \mathrm{Hom}_A(a,x)$, then evaluation at (a, id_a) defines an equivalence

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This is useful for proving equivalences between various types of coherent or incoherent adjunction data.

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Path induction: If $P(x,p)$ is a type family dependent on $x : A$ and $p : a =_A x$, then to prove $P(x,p)$ it suffices to assume x is a and p is refl_a.

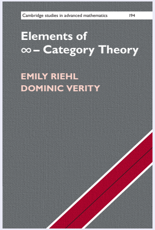
$$\text{path-ind}_a : P(a, \mathrm{refl}_a) \to \left(\prod_{x:A} \prod_{p:a=_A x} P(x,p)\right).$$

Arrow induction: If $F(x,f)$ is a covariant family dependent on $x : A$ and $f : \mathrm{Hom}_A(a,x)$ and A is an ∞-category, then to prove $F(x,f)$ it suffices to assume x is a and f is id_a.

$$\text{arrow-ind}_a : F(a, \mathrm{id}_a) \to \left(\prod_{x:A} \prod_{f:\mathrm{Hom}_A(a,x)} F(x,f)\right).$$

References

For more on the synthetic theory of ∞-categories see:



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Thank you!