AN INVESTIGATION OF (NEARLY) WEAKLY PRIMES AND RELATED SIEVE FORMULATIONS

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Abstract. A prime number p is called a *weakly prime* in base b if, when any single digit of the base b expansion of p is switched to any other digit $0, \dots, b-1$, the resulting number is composite. Tao proves that for any fixed base b, a positive proportion of primes are weakly in base b. In this paper, using a combination of analysis and computational evidence, we investigate the exact proportion of such primes for a given base b, and how the values of b and $\phi(b)$ affect this proportion. We provide two possible sieve formulations for determining the proportion of *nearly weakly primes*, and demonstrate the connection to the study of gaps between primes. We prove several propositions regarding weakly primes across bases and examine weakly primes in small prime bases. We also include an efficient algorithm for generating weakly primes and the numerical data generated by this algorithm to support our conjectures.

1. Introduction

In 1978, Murray Klamkin asked if there exists a prime number such that if any (base-10) digit is changed to any other digit, the resulting number is always composite. Erdős showed that there are in fact infinitely many such primes, the first of which are $294001, 505447, 584141, 604171, 971767$ [3]. This construction easily generalizes to any base $b \geq 1$, and such primes are called *weakly primes* in base b. An even stronger result holds, that if you fix a base b, a positive proportion of primes are in fact weakly primes base b. Tao [13] proves the following theorem.

Theorem 1.1. *(Tao) Let $K \geq 2$ be an integer. For all sufficiently large integers N, the number of primes p between N and $(1+\frac{1}{K})N$ such that $|kp \pm ja^i|$ is composite for all integers $1 \leq a, j, k \leq K$ and $0 \leq i \leq K \log N$ is at least $c_K \frac{N}{\log N}$ for some constant c_K depending only on K.*

The following corollary is then immediate in combination with the prime number theorem.

Corollary 1.2. *(Tao) Let $b \geq 2$ be a base. Then there is a positive proportion of prime numbers such that if any single digit is changed in their base-b expansion to any other digit, the resulting number is composite.*

An interesting consequence of this theorem is that any prime testing algorithm must read in every digit of a number to determine if it is prime, regardless of the base the number is written in. In the conclusion, Tao suggests other ways this result can be improved, one of which Pollack shows in [9].

Theorem 1.3. *(Pollack) Fix an integer $K \geq 2$. There is a constant c_K such that the following holds for all sufficiently large N : Let $S_N \subset [-KN, KN]$ be an arbitrary set of integers of cardinality at most K. Let K_N be the number of primes $N \leq p \leq (1 + \frac{1}{K})N$ such* *that* $|kp + ja^i + s|$ *is either equal to* p *or composite for all combinations of* a, i, j, k, s, *where* $1 \leq a, |j|, k \leq K,\ 0 \leq i \leq K \log N,\ s \in S_N$. *Then* $K_N \geq c_K \frac{N}{\log N}$.

Then in a similar way, the following corollary is immediate.

Corollary 1.4. *(Pollack) In any fixed base, a positive proportion of prime numbers becomes composite if one modifies any single digit and appends a bounded number of digits to the beginning or end.*

For other work regarding digits and primality, we direct the reader to [1], [4], [6], [10], and [11]. Many of these papers employ covering spaces to show that there are infinitely many of such numbers.

Tao and Pollack use the Selberg upper bound sieve to prove Theorems 1.1 and 1.3, respectively, and both results are much stronger than the corresponding corollaries, in the sense that they place more restrictions on the primes than the definition of weakly primes.

The author is interested in the proportion of weakly primes in a given base b, and in particular, how the proportion of weakly primes varies across bases. Along these lines, we make the following conjecture, with supporting numerical evidence.

Conjecture 1.5. *The proportion of primes that are weakly primes in a given base* b *is asymptotic and depends on* b *and* $\phi(b)$, *where* ϕ *is Euler's totient function.*

The general trend is that the proportion decreases as b increases, and that the proportion decreases as $\phi(b)$ decreases. The latter relationship is intuitive, because if you switch any digit d_m other than the ones digit of prime p with base-b expansion

$$p = d_n b^n + \cdots + d_1 b + d_0, \tag{1.1}$$

the resulting number

$$r = d_n b^n + \cdots \gamma b^m + \cdots + d_1 b + d_0 \tag{1.2}$$

is congruent to p modulo b. That is, $p \equiv r \equiv d_0 \pmod{b}$. Therefore, $\gcd(r, b) = 1$, and r cannot be divisible by any prime factors of b. In this paper, we will give two possible ways to formulate this as a sieve-theoretic problem with supporting numerical evidence.

Given a prime p with base-b expansion (1.1), for p to be weakly prime base b, we need each element (not equal to p) of the following matrix to be composite.

$$\begin{bmatrix} d_n b^n + \cdots + d_1 b + 0 & d_n b^n + \cdots + d_1 b + 1 & \cdots & d_n b^n + \cdots + d_1 b + (b-1) \\ d_n b^n + \cdots + 0 \cdot b + d_0 & d_n b^n + \cdots + 1 \cdot b + d_0 & \cdots & d_n b^n + \cdots + (b-1) \cdot b + d_0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 \cdot b^n + \cdots + d_1 b + d_0 & 1 \cdot b^n + \cdots + d_1 b + d_0 & \cdots & (b-1) b^n + \cdots + d_1 b + d_0 \end{bmatrix}$$

Notice that out of the $(b-1) \cdot (\lfloor \log_b p \rfloor + 1)$ integers, $(b-1) \cdot \lfloor \log_b p \rfloor$ of such integers are congruent to p modulo b. Namely, the second through last row of the above matrix. With this in mind, we define the following indexing sets and a relaxed definition of weakly primes. Let

$$M = \{1, 2, \ldots, n\}, \quad J = \{0, 1, \ldots, b-1\}. \tag{1.3}$$

Definition 1.6. *We say a prime p is a* ***nearly weakly prime*** *base b if p becomes composite when any single digit other than possibly the ones digit is modified in the base-b expansion. In other words, $p = d_n b^n + \cdots + d_0$ is prime and*

$$d_n b^n + \cdots + j b^m + \cdots + d_0$$

is composite for $(m, j) \in M \times J \setminus \{(m, d_m)_{m \in M}\}$.

Since the proportion of primes p such that $|p + \alpha|$ is prime (Theorem 6.3.4 in [2]) equals 0 for any $\alpha \in \mathbb{Z} \setminus \{0\}$, it follows that for any base b, the proportion of primes such that $|p + \alpha|$ is prime for some $\alpha \in \{\pm 1, \ldots, \pm b - 1\}$ is 0.

Proposition 1.7. *In any base b, the proportion of primes that are weakly primes base b is equal to the proportion of primes that are nearly weakly primes base b.*

We use the term **strictly nearly weakly prime** to mean a prime that is a nearly weakly prime in base b but not a weakly prime base b. This means that there exists a prime q such that $|p - q| \leq b - 2$. In bases 2 and 3, a prime $p > 3$ is weakly if and only if it is nearly weakly, thus there are no non-trivial strictly nearly weakly primes. However, this is not true in most bases, and we show examples of such bases in Section 6.

A natural question to ask is whether there are infinitely more nearly weakly primes than weakly primes in a given base, and such a result would have many consequences regarding gaps between primes, such as the following example.

Proposition 1.8. *If there are infinitely many more nearly weakly primes than weakly primes in base 5, then the twin prime conjecture holds.*

We prove Proposition 1.8 in Section 5, where we discuss (nearly) weakly primes for small bases b.

Notice that each of the modified numbers is between b^n and b^{n+1}, except for the bottom left number

$$0 \cdot b^n + \cdots + d_1 b + d_0,$$

where $0 \cdot b^n + \cdots + d_1 b + d_0 = p \pmod{b^n}$. With this in mind, let

$$\begin{aligned} I &= M \times J \setminus \{(n, 0)\} \\ I_p &= I \setminus \{(m, d_m)_{m \in M}\} \end{aligned} \tag{1.4}$$

where I_p is defined for prime p with base-b expansion (1.1).

Definition 1.9. *We say that a prime p is* ***nearly weakly-($n, 0$)*** *in base b if that prime becomes composite if any digit $d_m, m \geq 1$, is modified to any other digit except possibly when the largest digit is modified to 0. In other words, $p = d_n b^n + \cdots + d_0$ is prime and*

$$d_n b^n + \cdots + j b^m + \cdots + d_0$$

is composite for $(m, j) \in I_p$.

Let $\pi(b^n; p, b)$ denote the number of primes $q \equiv p \pmod{b}$ less than b^n. Then from Theorem 2.3.1 in [7], we have

$$\pi(b^n; p, b) \leq \frac{2b^n}{\phi(b) \log(b^{n-1})} + O\left(\frac{b^{n-1}}{(\log b^{n-1})^2}\right),$$

and the proportion of primes $q \equiv p \pmod{b}$ goes to 0 as $n \to \infty$. We then propose the following question.

Question 1.10. *In any base b, is the proportion of primes that are weakly primes base b equal to the proportion of primes that are nearly weakly-$(n, 0)$ base b?*

For our sieve formulations, we will consider the proportion of nearly weakly-$(n, 0)$ primes. In this paper we show two ways in which we can formulate the proportion of nearly weakly-$(n, 0)$ primes as a sieve theory problem, the first of which leads to the following conjecture regarding the proportion of weakly primes.

Conjecture 1.11. *Let $S_b(\mathcal{A}, P, n)$ be the set of nearly weakly-$(n, 0)$ primes in base b between $N_1 = b^n$ and $N_2 = b^{n+1}$. Then*

$$\#S_b(\mathcal{A}, P, n) \sim \pi(N_1, N_2)W(n)$$

where

$$W(n) = \prod_{p\in\mathcal{P}} (1 - \delta_p)$$

and $\delta_p = \frac{n}{\phi(b)\cdot b^{n-1}}$. In other words,

$$W(n) = \left(1 - \frac{n}{\phi(b) \cdot b^{n-1}}\right)^{\pi(N_1,N_2)}.$$

The second formulation leads us to the following conjecture.

Conjecture 1.12. *Let I be the indexing set as defined in (1.4) and let $S_b(\mathcal{A}, I, n)$ be the set of nearly weakly-$(n, 0)$ primes in base b between $N_1 = b^n$ and $N_2 = b^{n+1}$. Then*

$$\#S_b(\mathcal{A}, I, n) \sim \pi(N_1, N_2)V(n)$$

where

$$V(n) = \prod_{(m,j)\in I} (1 - \delta_{m,j})$$

and $\delta_{m,j} = \frac{b}{\phi(b)} \frac{1}{\log N_2}$. In other words,

$$V(n) = \left(1 - \frac{b}{\phi(b)} \frac{1}{\log N_2}\right)^{n(b-1)-1}.$$

This paper is structured as follows: in Section 2, we give a brief introduction to sieve theory and include three important sieves, namely the Sieve of Eratosthenes-Legendre, Brun's Sieve, and the Turán Sieve. In Section 3, we describe two possible formulations for nearly weakly primes excluding the $(n, 0)$ case. In Section 4, we prove several short but interesting propositions about weakly primes. In Section 5, we describe a different approach for small bases that lead to larger and more accurate proportions of (nearly) weakly primes, and we present more propositions and conjectures. We then include an algorithm used to generate weakly primes, and tables of (nearly) weakly primes in Section 6. We conclude with Section

7 and talk about the next steps in this research.

2. Sieve Methods

Many methods in sieve theory focus on determining the proportion of a certain type of number by eliminating or sieving numbers that do not fit the pertinent criteria.. The classic example is prime numbers. If you begin with a set of positive integers $\mathcal{A}$, such that $\sqrt{z} < a < z$ for all $a \in A$, you can then delete every multiple of two, and then from the remaining set delete every multiple of three, and so on. If you repeat this process for all primes up to $\sqrt{z}$, the resulting set is the set of prime elements $p \in \mathcal{A}$. The resulting set is denoted $S(\mathcal{A}, \mathcal{P}(\sqrt{z}))$, where $\mathcal{P}(\sqrt{z})$ is the set of primes less than $\sqrt{z}$.

Let $\mathcal{A}$ denote some finite set of positive integers with largest integer z, $\mathbb{P}$ denote the set of all primes, $\mathcal{P} \subset \mathbb{P}$ denote some specified subset of primes, and

$$P = \prod_{p \in \mathcal{P}} p.$$

For each $p \mid P$, we can associate a subset of $\mathcal{A}$, denoted $\mathcal{A}_p$, where

$$\mathcal{A}_p = \{a \in \mathcal{A} : a \equiv 0 \pmod{p}\}$$

and

$$\mathcal{A}_d = \bigcap_{p \mid d} \mathcal{A}_p$$

for some squarefree d. It follows that

$$S(\mathcal{A}, \mathcal{P}) = \mathcal{A} \setminus \bigcup_{d \mid P} \mathcal{A}_d.$$

Then the inclusion-exclusion principle states

$$\#S(\mathcal{A}, \mathcal{P}) = \sum_{d \mid P} (-1)^{\nu(d)} \#\mathcal{A}_d, \tag{2.1}$$

where $\nu(d)$ is the number of divisors of d. That is,

$$\#S(\mathcal{A}, \mathcal{P}) = \#\mathcal{A} - \sum_{p_1 \in \mathcal{P}} \#\mathcal{A}_{p_1} + \sum_{\substack{p_1, p_2 \in \mathcal{P} \\ p_1 < p_2}} \#\mathcal{A}_{p_1 p_2} - \sum_{\substack{p_1, p_2, p_3 \in \mathcal{P} \\ p_1 < p_2 < p_3}} \#\mathcal{A}_{p_1 p_2 p_3} \pm \cdots$$

Many sieves, including the sieve of Eratosthenes-Legendre and Brun's Sieve, are formed by cleverly approximating this alternating series under specified conditions. To see this, we let

$$\#\mathcal{A}_d = \#\mathcal{A} \cdot \delta_d + r_d, \tag{2.2}$$

where

$$\delta_d = \prod_{p \mid d} \delta_p$$

with $0 \leq \delta_p < 1$ for all $p \in \mathcal{P}$. We can think of δ_d as the approximate proportion of elements in $\mathcal{A}$ that are in $\mathcal{A}_d$, and r_d as the error in that approximation that we want to minimize. In particular, we want the following to hold for any d.

$$|r_d(x)| \leq \delta_d d. \tag{2.3}$$

For most sieves, we will also require the following condition, or some stronger form of it.

$$\sum_{p \leq y, p \in \mathcal{P}} \delta_p \log p = \kappa \log y + O(1). \tag{2.4}$$

Another necessary condition is

$$\sum_{p \in \mathcal{P}} \delta_p^2 \log p < \infty. \tag{2.5}$$

Finally, we state the the classic first sieve example as stated in [5] as Theorem 4.2.

Theorem 2.1. *(Sieve of Eratosthenes-Legendre) Let $\mathcal{A} = (a_n)$ be a sequence of non-negative real numbers and $\mathcal{P}$ a set of primes. Assume that the density function δ_p satisfies* (2.4) *and* (2.5)*, and the remainder terms satisfy* (2.3) *for every $d \leq x$. Let $s \geq 1$. Then for $z = x^{\frac{1}{s}}$ we have*

$$\#S(\mathcal{A}, z) = XW(z)\{h(s) + O((\log)^{2\kappa-1})\} + O(x(\log x)^{\kappa-1}),$$

where

$$W(z) = \prod_{p|P(z)} (1 - \delta_p),$$

with $X = \#\mathcal{A}$ and the implied constants depend on κ and s.

The function $h(s)$ is the continuous solution to a differential-difference equation, which we will not describe in this paper (see 4.16 in [5]). Although for many problems this sieve has been replaced with stronger sieves such as Brun's Sieve, it still provides a good asymptotic formula for $\#S(\mathcal{A}, z)$ if $X >> x$.

We now introduce the conditions required for Brun's Sieve. First, there exists a constant $C_1 \geq 1$ such that for all $p \in \mathcal{P}$,

$$0 \leq \delta_p \leq 1 - \frac{1}{C_1}. \tag{2.6}$$

Second, we need a stronger version of (2.4), which states that there exist constants $\kappa > 0$, $C_2 \geq 1$ such that

$$\sum_{w \leq p < z} \delta_p \log p \leq \kappa \log \frac{z}{w} + C_2 \tag{2.7}$$

if $2 \leq w \leq z$.

Theorem 2.2. *(Brun's Sieve) Let $\mathcal{A}, \mathcal{A}_d, \delta_d, r_d$, and $S(\mathcal{A}, \mathcal{P})$ be as defined above, and assume that* (2.3)*,* (2.6)*, and* (2.7) *are satisfied. Then*

$$XW(z)\{1 - E_1\} + O(z^{e_1}) \leq \#S(\mathcal{A}, \mathcal{P}) \leq XW(z)\{1 + E_2\} + O(z^{e_2}).$$

Ideally e_1, e_2 are less than 1, but depend on many constants. See Theorem 6.2.5 in [2] for details regarding the error terms.

Although both of these sieves (as well as many sieves) use a set of primes $\mathcal{P}$ to form $\mathcal{A}_d$, it is possible to sieve over indexing sets. We begin with a set $\mathcal{S}$ and indexing set I. For each $i \in I$, let $\Omega(i)$ denote some specified condition. Let

$$S_i = \{s \in \mathcal{S} : s \text{ satisfies } \Omega(i)\}$$

and

$$\pi_s(I) = \#\{i \in I : s \text{ satisfies } \Omega(i)\}.$$

Then we can write

$$\frac{\#\mathcal{S}_i}{\#\mathcal{S}} = \delta_i + r_i,$$

where r_i is an error term. Additionally, for $i \neq j \in I$, we have

$$\frac{\#(\mathcal{S}_i \cap \mathcal{S}_j)}{\#\mathcal{S}} = \delta_i \delta_j + r_{i,j},$$

where $r_{i,j}$ is the error term, and we say $\mathcal{S}_i, \mathcal{S}_j$ are quasi-independent. We can also state the inclusion-exclusion principle in a more general form.

$$\#S(\mathcal{S}, I) = \sum_{D \subset I} (-1)^{\#D} \#\mathcal{S}_D, \tag{2.8}$$

where

$$\mathcal{S}_D = \bigcap_{i \in D} \mathcal{S}_i.$$

The Turán sieve is an upper bound sieve, which is not as useful in this context as an asymptotic sieve. Many asymptotic sieves, including Theorems 2.1, 2.2, cannot easily be generalized in this way, but the Turán sieve can be and so we state it here (Corollary 1.2 in [12]).

Theorem 2.3. *(Turán Sieve) Let $v = \sum_{i \in I} \delta_i$, and let $S(\mathcal{S}, I) = \{s \in \mathcal{S} : \pi_s(I) = 0\}$, that is, $S(\mathcal{S}, I)$ is the number of elements of $\mathcal{S}$ that do not satisfy $\Omega(i)$ for any $i \in I$. Then*

$$\#S(\mathcal{S}, I) \leq \frac{\#\mathcal{S}}{v} + \frac{\#\mathcal{S}}{v^2} + \sum_{i,j \in I} |r_{i,j}| + \frac{2\#\mathcal{S}}{v} \sum_{i \in I} |r_i|.$$

3. Formulations

We provide two ways to formulate the proportion of nearly weakly-$(n, 0)$ primes for a fixed base b. For simplicity, we may refer to such primes simply as nearly weakly for this section. In the first formulation, we construct our sieve by removing sets $\mathcal{A}_p$ that contain primes that p causes not to be weakly. This allows us to sieve over a subset of $\mathbb{P}$, the set of primes. The resulting formulation leads us to Conjecture 1.11. In the second formulation, we use a less conventional index set to sieve pairs (m, j). The resulting formulation leads us to Conjecture 1.12. Let $\mathcal{A}$ be the set of primes between $N_1 = b^n$ and $N_2 = b^{n+1}$ and $P = P(N_1, N_2)$ be the product of all such primes. That is, let

$$\mathcal{A} = \{N_1 < p < N_2 : p \text{ prime}\}, \tag{3.1}$$

and

$$P = \prod_{p \in \mathcal{A}} p. \tag{3.2}$$

We let $\pi(N_1, N_2) = \#\mathcal{A}$ denote the number of primes between N_1 and N_2. For each prime $p \in \mathcal{A}$ with base b expansion

$$p = d_n b^n + \cdots + d_m b^m + \cdots + d_1 b + d_0, \tag{3.3}$$

we associate the set $\{a_{p,m}\}_{m=1}^n$ where

$$\begin{aligned} a_{p,m} &= d_n b^n + \cdots + 0 \cdot b^m + \cdots + d_1 b + d_0 \\ &= p - (p \mod b^m) + (p \mod b^{m-1}) \end{aligned} \tag{3.4}$$

and $(p \mod b^m)$, $(p \mod b^{m-1})$ denote the least residue of p modulo b^m, b^{m-1}, respectively. Let

$$Q_p := \{a_{p,m} + jb^m : (m, j) \in I_p\}, \tag{3.5}$$

where $\#Q_p = \#I_p = n(b-1) - 1$. Note that any element of Q_p is necessarily between N_1 and N_2. Consider

$$\begin{aligned} \mathcal{A}_p &:= \{a_{p,m} + jb^m \neq p \text{ prime } : (m, j) \in I\} \\ &= \mathcal{A} \cap Q_p \end{aligned} \tag{3.6}$$

.

Lemma 3.1. *Let nearly weakly refer to a prime being nearly weakly - $(n, 0)$ in a fixed base b as in (1.9). Then the following statements hold.*

(1) p is a nearly weakly prime if and only if $\#\mathcal{A}_p = 0$.
(2) $q \in \mathcal{A}_p$ if and only if $p \in \mathcal{A}_q$.
(3) $q \in \mathcal{A}_p$ if and only if there exists a unique m such that $a_{p,m} = a_{q,m}$.
(4) A prime p is not nearly weakly if and only if $p \in \mathcal{A}_q$ for some $q \in \mathcal{A}$.

Proof. Let $p \in \mathcal{A}$ have the base-b expansion in (3.3). By definition, if p is nearly weakly prime base b, then $\#\mathcal{A}_p = 0$. Alternatively, if p is not a nearly weakly prime in base b, there exists a prime $q = a_{p,m} + jb^m \in \mathcal{A}_p$. Therefore p is a nearly weakly prime if and only if $\#\mathcal{A}_p = 0$, proving (1)

Without loss of generality, suppose $q \in \mathcal{A}_p$, and we will show $p \in \mathcal{A}_q$. We know $q \neq p$ is prime and $q = a_{p,m} + jb^m$ for some $(m, j) \in I_p$. Since $(m, j) \neq (n, 0)$ and $m \in \{1, \ldots, n\}$, q is a prime between N_1 and N_2 thus $q \in \mathcal{A}$ as stated in (3.6). Then q has the same expansion as p except with d_m replaced with j, that is,

$$q = d_n b^n + \cdots + jb^m + \cdots + d_0.$$

Therefore $a_{p,m} = a_{q,m}$, and it follows that $p = a_{q,m} + d_m b^m$ for $(m, d_m) \in I$. Since $p \neq q$, we have $(m, d_m) \neq (m, j)$ and so $(m, d_m) \in I_q$. Therefore $p \in \mathcal{A}_q$, proving (2). This m is clearly unique, otherwise $p = q$, a contradiction as $p \notin \mathcal{A}_p$, proving the forward direction of (3). The converse of (3) follows almost immediately from the definition: if $a_{p,m} = a_{q,m}$ for some unique m, then $p \neq q$ and $q = a_{p,m} + jb^m$ for some $j \neq d_m$, hence $q \in \mathcal{A}_p$.

For (4), suppose p is not nearly weakly. Then there exists $q \in \mathcal{A}_p$, and by (2) this implies $p \in \mathcal{A}_q$ for some $q \in \mathcal{A}$. Conversely, if $p \in \mathcal{A}_q$ for some $q \in \mathcal{A}$, again by (2) we have $q \in \mathcal{A}_p$, and so p is not nearly weakly base b. □

3.1. *Formulation 1*

Lemma 3.1 implies that we can sieve nearly weakly primes by each other given this formulation. If we define,

$$\mathcal{A}_d = \bigcap_{p|d} \mathcal{A}_p \tag{3.7}$$

for a squarefree integer d, then the following theorem holds.

Theorem 3.2. *Let $S_b(\mathcal{A}, P, b)$ be the set of nearly weakly -$(n, 0)$ primes in base b between $N_1 = b^n$ and $N_2 = b^{n+1}$. Then*

$$S_b(\mathcal{A}, P, n) = \mathcal{A} \setminus \bigcup_{d|P} \mathcal{A}_d. \tag{3.8}$$

Following the inclusion-exclusion principle, we have

$$\begin{aligned} \#\mathcal{S}_b(\mathcal{A}, P, n) &= \#\mathcal{A} - \sum_{p_1} \#\mathcal{A}_{p_1} + \sum_{\substack{p_1p_2, \\ p_1<p_2}} \#\mathcal{A}_{p_1p_2} - \sum_{\substack{p_1p_2p_3, \\ p_1<p_2<p_3}} \#\mathcal{A}_{p_1p_2p_3} + \cdots \\ &= \sum_{d} (-1)^{\nu(d)} \#\mathcal{A}_d, \end{aligned} \tag{3.9}$$

where $p_1, p_2, \ldots$ are primes.

Proof. Suppose a prime $p \in \mathcal{A} \setminus \bigcup_{d|P} \mathcal{A}_d$. Then p is a prime, $N_1 < p < N_2$ such that $p \notin \mathcal{A}_d$ for any $d \mid P$. Then by Lemma 3.1, $q \notin \mathcal{A}_p$ for any $q \in \mathcal{A}$, and so $p \in \mathcal{S}_b(\mathcal{A}, P, n)$. Conversely, suppose $p \in S_b(\mathcal{A}, P, n)$. Then by Lemma 3.1, $\#\mathcal{A}_p = 0$, which implies $p \notin \mathcal{A}_q$ for any $q \in \mathcal{A}$, and thus $p \notin \mathcal{A}_d$ for any $d \mid P$. Thus $p \notin \bigcup_{d|P} \mathcal{A}_d$, and the equality holds. □

Corollary 3.3. *Let $S_b(\mathcal{A}, P, b)$ be the set of nearly weakly -$(n, 0)$ primes in base b between $N_1 = b^n$ and $N_2 = b^{n+1}$. Then*

$$\#S_b(\mathcal{A}, P, n) = \sum_{p \in \mathcal{A}} \sum_{d|(\#\mathcal{A}_p+1)} \mu(d), \tag{3.10}$$

where μ is the Möbius function.

Proof. We first note that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases},$$

hence if $\#\mathcal{A}_p = 0$, then p is a nearly weakly prime and $\#\mathcal{A}_p + 1 = 1$, and if p is not a nearly weakly prime, then $\#\mathcal{A}_p > 0$ and $\#\mathcal{A}_p + 1 > 1$. Therefore the summation in (3.10) correctly counts the number of nearly weakly primes base b in $\mathcal{A}$. □

We now discuss how we can formulate this as a sieve problem. Suppose q is prime. Then q is in one of $\phi(b^{n+1}) - \phi(b^n)$ residue classes, and $\mathcal{A}_p$ is the set of primes $q \in \mathcal{A}$ that belong to exactly one of $(n(b-1)-1) \approx n(b-1)$ fixed residue classes out of $\phi(b^{n+1}) - \phi(b^n)$ possible residue classes modulo b^{n+1}. Therefore we estimate the probability that a prime $q \in \mathcal{A}$ is in $\mathcal{A}_p$ as

$$\delta_p = \frac{n(b-1)}{\phi(b^{n+1}) - \phi(b^n)}. \tag{3.11}$$

Let $b^* = \frac{b}{\phi(b)} > 1$. Simplify

$$\delta_p = \frac{n(b-1)}{\phi(b^{n+1}) - \phi(b^n)} = \frac{n(b-1)b^*}{b^{n+1} - b^n} = \frac{nb^*}{b^n} = \frac{n}{\phi(b) \cdot b^{n-1}}. \tag{3.12}$$

Note that δ_p does not depend on p, because

$$\frac{1}{\log x^n} - \frac{1}{\log x^{n+1}} = \frac{1}{n(n+1)\log x}$$

goes to 0 as $n \to \infty$, so by the prime number theorem, we assume any arbitrary $q \in (b^n, b^{n+1})$ has the same probability $\frac{1}{\log N_2}$ of being prime. Consider

$$\#\mathcal{A}_p = X\delta_p + R_p = \pi(N_1, N_2)\frac{n}{\phi(b) \cdot b^{n-1}} + R_p \tag{3.13}$$

where $\pi(N_1, N_2)$ is the number of primes between N_1 and N_2. Then $0 \leq \#\mathcal{A}_p \leq n(b-1)$, and $0 < \delta_p < 1$, so $0 \leq R_p \leq (b-1) \cdot \log_b N_1$.

From this formulation we propose Conjecture 1.11. When we apply the Eratosthenes-Legendre Sieve (Theorem 2 in [7]) to this formulation, we obtain too large an error. Additionally, we cannot apply Brun's Sieve (Theorem 6.2.5 in [2]) because we cannot satisfy (2.7) where $z = N_2$. In any formulation, we will need δ_p to depend on $\log_b p$, which contradicts (2.7), and so it is likely that the Brun's Sieve cannot be applied in this context.

3.2. *Formulation 2*

We begin with a similar construction as in Section 3.1. Rather than indexing by primes, we index by $i = (m, j) \in I$, where I is as defined in (1.4). Let

$$\mathcal{A}_{m,j} = \{q = a_{p,m} + jb^m \in \mathcal{A} \text{ for some prime } p \neq q\}. \tag{3.14}$$

and for $D \subset I$, let

$$\mathcal{A}_D = \bigcap_{(m,j)\in D} \mathcal{A}_{m,j}. \tag{3.15}$$

Theorem 3.4. *Let $S_b(\mathcal{A}, P, b)$ be the set of nearly weakly -$(n, 0)$ primes in base b between $N_1 = b^n$ and $N_2 = b^{n+1}$. Then*

$$S_b(\mathcal{A}, P, n) = \mathcal{A} \setminus \bigcup_{(m,j)\in I} \mathcal{A}_{m,j}, \tag{3.16}$$

and following the inclusion-exclusion principle in (2.8)*, we have*

$$\#S(\mathcal{A}, I) = \sum_{D \subset I} (-1)^{\#D} \#\mathcal{A}_D. \tag{3.17}$$

Proof. Suppose $p \in S_b(\mathcal{A}, P, n)$. Then clearly $p \in \mathcal{A}$, and we want to show that there does not exists $(m, j) \in I$ such that $p \in \mathcal{A}_{m,j}$. Assume for a contradiction that such an $(m, j) \in I$ exists. Then $p = a_{q,m} + jb^m$ for some prime $p \neq q \in \mathcal{A}$. Since $p \neq q$, we have $(m, j) \in I_q$, and so $p \in \mathcal{A}_q$. By Lemma 3.1, p is not a nearly weakly prime, a contradiction. Thus no such $(m, j) \in I$ exists and $p \in \mathcal{A} \setminus \bigcup_{(m,j)\in I} \mathcal{A}_{m,j}$.

Suppose instead that $p \in \mathcal{A} \setminus \bigcup_{(m,j)\in I} \mathcal{A}_{m,j}$. Then for any $(m, j) \in I$, $p \neq a_{q,m} + jb^m$ for any prime $p \neq q \in \mathcal{A}$. Therefore $p \notin \mathcal{A}_q$ for any q, and so by Lemma 3.1, p is nearly weakly prime. Therefore $p \in S_b(\mathcal{A}, P, n)$, and the equality holds. □

Consider $\mathcal{A}_{m,j}$ for some $(m, j) \in I$. We have $\pi(N_1, N_2)$ elements in $\mathcal{A}$. Any element $q \in \mathcal{A}$ is in exactly one of the $\frac{\phi(b)}{b}(N_2 - N_1)$ possible congruence classes modulo b^{n+1}. We estimate the probability that $q = a_{p,m} + jb^m$ for a given prime $p \in \mathcal{A}$ as

$$\frac{1}{\frac{\phi(b)}{b}(N_2 - N_1)}.$$

The probability $\delta_{m,j}$ that $q \in \mathcal{A}_{m,j}$ is the probability that $q = a_{p,m} + jb^m$ for any prime $p \in \mathcal{A}$, where $\#\mathcal{A} = \pi(N_1, N_2)$. We estimate the probability

$$\begin{aligned} \delta_{m,j} &= \frac{\pi(N_1, N_2)}{\frac{\phi(b)}{b}(N_2 - N_1)} = \frac{b}{\phi(b)} \cdot \frac{\pi(N_1, N_2)}{(N_2 - N_1)} \\ &\sim \frac{b}{\phi(b)} \cdot \frac{N_2}{N_2 \log N_2} = \frac{b}{\phi(b)} \cdot \frac{1}{\log N_2}, \end{aligned} \tag{3.18}$$

of being in $\mathcal{A}_{m,j}$, so we have

$$\begin{aligned} \#\mathcal{A}_{m,j} &= \#\mathcal{A} \cdot \delta_{m,j} + r_{m,j} \\ &= \pi(N_1, N_2) \cdot \frac{b}{\phi(b)} \cdot \frac{\pi(N_1, N_2)}{(N_2 - N_1)} + r_{m,j} \\ &= \frac{b}{\phi(b)} \cdot \frac{N_2}{\log N_2} \cdot \frac{1}{\log N_2} + r'_{m,j} \\ &= \frac{b}{\phi(b)} \cdot \frac{N_2}{(\log N_1)^2} + r'_{m,j}. \end{aligned} \tag{3.19}$$

This formulation leads us to Conjecture 1.12.

4. Propositions

We know from [13] that there is a positive proportion of primes that are weakly in any given base b, and more than that there is a positive proportion of primes that are weakly in bases $b_1, \dots, b_m$ if we choose K large enough in Theorem 1.1. This first proposition is regarding primes that are weakly in base b^k for some $b, k \geq 2$.

Proposition 4.1. *Let p be a prime. If p is a weakly prime base b^k, then p is weakly base b.*

Proof. Let $x = b^k$, and $p = a_n x^n + \cdots + a_1 x + a_0$ be the base x representation of p, that is, $a_i \in \{0, \dots, x-1\}$ for $i = 0, \dots, n$. Then we can write $a_i = c_{i,k-1} b^{k-1} + \cdots + c_{i,1} b + c_{i,0}$, where $c_{i,j} \in \{0, \dots, b-1\}$. Consider

$$\begin{aligned}
p &= a_n x^n + \cdots + a_1 x + a_0 \\
&= (c_{n,k-1} b^{k-1} + \cdots + c_{n,1} b + c_{n,0}) x^n + \cdots + (c_{0,k-1} b^{k-1} + \cdots + c_{0,1} b + c_{0,0}) \\
&= c_{n,k-1} b^{k-1} x^n + \cdots + c_{n,1} b x^n + c_{n,0} x^n + \cdots + c_{0,1} b + c_{0,0} \\
&= c_{n,k-1} b^{k-1} (b^k)^n + \cdots + c_{i,j} b^i (b^j)^k + \cdots + c_{0,1} b + c_{0,0} \\
&= c_{n,k-1} b^{kn+k-1} + \cdots + c_{i,j} b^{ki+j} + \cdots + c_{0,0}.
\end{aligned}$$

Therefore we have a degree $kn+k-1$ representation of p in base b. Assume for a contradiction that p is not a weakly prime base b. Then there exists a $c_{i,j}$ such that for some $d \in \{0, \dots, b-1\}$,

$$p' = c_{n,k-1} b^{kn+k-1} + \cdots + c_{i,j+1} b^{ki+j+1} + d b^{ki+j} + \cdots + c_{0,0}$$

is prime. But then

$$d_i = c_{i,k-1} b^{k-1} + \cdots + c_{i,j+1} b^j + d b^j + \cdots + c_{i,0} \in \{0, \dots, b^k - 1\},$$

and so p' has the following base x representation.

$$p' = a_n x^n + \cdots + d_i x^i + \cdots + a_0.$$

This is a contradiction because $p' \neq p$ and $d_i \neq a_i$. Thus p is a weakly prime in base b as well. ∎

The next proposition is regarding twin primes such that at least one of the pair is weakly. One might ask whether such primes exist, and so we also provide Table 6.3 that includes twin primes such that both are weakly for bases $2, 3, 5$, and 7.

Proposition 4.2. *Let $p, p+2$ be twin primes. Then if either of $p, p+2$ is a weakly prime base b, we have $p \equiv -1 \pmod{b}$.*

Proof. In base 2, this is trivially true as every prime is congruent to -1 modulo 2. In base 3, this is trivially true, since for every pair of twin primes $p, p+2$, we have p is congruent to -1 modulo 3, otherwise 3 divides p or $p+2$.

Now assume $b > 3$. Since $2, 3 < b$, it follows that the smallest weakly prime base b is greater than or equal to b. Bertrand's postulate implies that there exists a prime p such that $b < p \leq 2b - 1$, thus b cannot be a weakly prime base b. Therefore we know p cannot be congruent to 0 or -2 modulo b. Suppose $p \equiv 1, \ldots, b-3 \pmod{b}$, and let $p, p+2$ have the base b representations

$$\begin{aligned} p &= d_n b^n + \cdots + d_1 b + d_0 \\ p &= d_n b^n + \cdots + d_1 b + d_0 + 2, \end{aligned}$$

where $d_0 + 2 \in \{3, \ldots, b-1\}$. This implies that neither p nor $p+2$ are weakly base b, which is a contradiction. Thus $p \equiv -1 \pmod{b}$. □

5. Small Bases

In the previous sections, we do not take into account the specific digits of a prime p in the base b expansion,

$$p = (d_n \ldots d_1 d_0)_b = d_n b^n + \cdots + d_1 b + d_0,$$

where $0 \leq d_i \leq b - 1$. That is, we assume the same probability of a prime being weakly prime regardless of the value of $d_i, i \geq 1$. For base 2 and for large bases, even $b > 10$, this is a fair assumption, however for small bases, in particular, bases $b = 3, 5$, and 7, the value of d_i greatly affects the probability that p will be a weakly prime in base b, and accounting for this variance increases the resulting proportion of weakly primes that is obtained. In particular, this explains why although the trend is that the proportion of weakly primes in prime bases generally decreases as b increases, instead the proportion of weakly primes in base 3 is much higher than that of base 2.

We represent a prime number p in base b as a $(n+1) \times b$ array, where $n = \lfloor \log_b p \rfloor$ table where the red asterisks represent the digits of p base b. For example, the prime $4007 = 14453_7$ is weakly in base 7 and has the following representation.

$$\begin{array}{cccccccc}
7^4 & \bullet & * & \bullet & \bullet & \bullet & \bullet & \bullet \\
7^3 & \bullet & \bullet & \bullet & \bullet & * & \bullet & \bullet \\
7^2 & \bullet & \bullet & \bullet & \bullet & * & \bullet & \bullet \\
7^1 & \bullet & \bullet & \bullet & \bullet & \bullet & * & \bullet \\
7^0 & \bullet & \bullet & \bullet & * & \bullet & \bullet & \bullet \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

When we write a prime in this way (this construction was used in [3] for base 10), we can see that we need to check whether the number obtained by shifting one red asterisk at a time to any of the other black nodes in the row is composite. We can then sieve out certain nodes we know to be composite because of base.

Instead of representing the entire prime, we will consider the possible representations of a single arbitrary row. We denote each possible row representation with a letter, and say that a prime p is weakly in row m (base b) if when the mth digit of p is changed to any other digit base b, the resulting number is composite. For example, consider base 3. Then each $d_m, m \geq 0$, can have one of three possible positions.

$$\begin{array}{lccc}
\text{a:} & {\color{red}*} & \bullet & \bullet \\
\text{b:} & \bullet & {\color{red}*} & \bullet \\
\text{c:} & \bullet & \bullet & {\color{red}*} \\
 & 0 & 1 & 2
\end{array}$$

That is, $d_m = 0, 1,$ or 2 in the base 3 expansion. Note that $d_0 \neq 0$, and can only have positions b, c. Since $2 \nmid 3$ and $2 \nmid p$, we have that $2 \mid p \pm 3^m$ for $m \geq 0$. Then the following blue crosses most be composite.

$$\begin{array}{lcccl}
\text{a:} & {\color{red}*} & {\color{blue}\times} & \bullet & :1 \\
\text{b:} & {\color{blue}\times} & {\color{red}*} & {\color{blue}\times} & :0 \\
\text{c:} & \bullet & {\color{blue}\times} & {\color{red}*} & :1 \\
 & 0 & 1 & 2 &
\end{array}$$

Therefore a prime p with $d_m = 1, m \geq 0$ in its base 3 expansion is automatically weakly in that row.

Proposition 5.1. *Let p be a prime with base 3 expansion $p = (1 \ldots 11)_3$. Then p is a weakly prime in base 3.*

Also, note that p will always be weakly base 3 in the ones place as $3 \mid (d_n \ldots d_1 0)_3$. Therefore to verify that a number is a weakly prime base 3, we only have to check at most $\lfloor \log_3 p \rfloor$ numbers are composite, all of which will be coprime to 6. This shows that out of the $2\lfloor \log_3 p \rfloor - 1$ numbers coprime to 3 that we must verify, at least half of them are necessarily even. Furthermore, if we assume that the primes are distributed evenly over positions $d_m = 0, 1, 2$ for $d_m \geq 1$, then we have that on average we only need to check $\frac{2}{3}\lfloor \log_3 p \rfloor$ numbers are composite.

We can use a similar method for base 5, as represented below.

$$\begin{array}{lccccc}
\text{a:} & {\color{red}*} & \bullet & \bullet & \bullet & \bullet \\
\text{b:} & \bullet & {\color{red}*} & \bullet & \bullet & \bullet \\
\text{c:} & \bullet & \bullet & {\color{red}*} & \bullet & \bullet \\
\text{d:} & \bullet & \bullet & \bullet & {\color{red}*} & \bullet \\
\text{e:} & \bullet & \bullet & \bullet & \bullet & {\color{red}*} \\
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

We first sieve out all the multiples of 2 with the blue $\times$.

$$\begin{array}{lccccc}
\text{a:} & {\color{red}*} & {\color{blue}\times} & \bullet & {\color{blue}\times} & \bullet \\
\text{b:} & {\color{blue}\times} & {\color{red}*} & {\color{blue}\times} & \bullet & {\color{blue}\times} \\
\text{c:} & \bullet & {\color{blue}\times} & {\color{red}*} & {\color{blue}\times} & \bullet \\
\text{d:} & {\color{blue}\times} & \bullet & {\color{blue}\times} & {\color{red}*} & {\color{blue}\times} \\
\text{e:} & \bullet & {\color{blue}\times} & \bullet & {\color{blue}\times} & {\color{red}*} \\
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

Then for each position, there are two possibilities as to which numbers are divisible by 3. We show these below in light blue crosses. Note that we only show a, b, c because d, e are the reflection of b, a, respectively.

$$\begin{array}{lcccccl}
\text{a}_1: & {\color{red}*} & {\color{blue}\times} & {\color{cyan}\times} & {\color{blue}\times} & \bullet & :1 \\
\text{a}_2: & {\color{red}*} & {\color{blue}\times} & \bullet & {\color{blue}\times} & {\color{cyan}\times} & :1
\end{array}$$

$$\begin{array}{rccccccl}
\mathrm{b}_1: & \times & * & \times & \times & \times & :0 \\
\mathrm{b}_2: & \times & * & \times & \bullet & \times & :1 \\
\\
\mathrm{c}_1: & \times & \times & * & \times & \bullet & :1 \\
\mathrm{c}_2: & \bullet & \times & * & \times & \times & :1 \\
 & 0 & 1 & 2 & 3 & 4 &
\end{array}$$

Similarly, if we assume that the mth digit, $m \geq 1$, has the same probability of being $a_i, \ldots, e_i$, $i = 1, 2$, then on average we need to check $\frac{4}{5}\lfloor \log_5 N \rfloor$ numbers, all coprime to 30.

We also note the relationship between nearly weakly and weakly primes in base 5 and twin primes. Recall that a twin prime is a prime p such that $p+2$ or $p-2$ is prime as well. For any base, if a prime number is a strictly nearly weakly prime in base b, then there exists an $1 \leq \alpha < b-1$ such that $p \pm \alpha$ is prime. In particular, if a prime p is strictly nearly weakly prime in base 5, then p is a twin prime. We can see this because the zeroth row representation of such a prime is necessarily in position b, c, d, e with the green $\times$ representing that that integer is divisible by 5.

$$\begin{array}{rccccc}
\mathrm{b}: & \times & * & \times & \bullet & \times \\
\mathrm{c}: & \times & \times & * & \times & \bullet \\
\\
\mathrm{d}: & \times & \bullet & \times & * & \times \\
\mathrm{e}: & \times & \times & \bullet & \times & * \\
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

We see that in each row representation, at most one other number can be prime, and that number is either 2 greater than (b, c) or two less than (d, e) the prime in consideration. Therefore if a number is strictly nearly weakly prime base 5, it is necessarily a twin prime as well. Thus having infinitely many strictly nearly weakly primes in base 5 implies that there are infinitely many twin primes, as stated in Proposition 1.8.

Similar results in prime gaps can be established when considering other small bases. Additionally, the effect that the position of the digit (that is, the letter representation of a digit) has on determining whether a prime is weakly decreases as the base b increases. Therefore these diagrams are less important for large b (say, $b > 11$). However, they are crucial for understanding small bases, and in particular, they explain why numerical evidence supports that there is a much larger proportion of primes that are weakly in base $b = 3$ (almost $\frac{1}{5}$) than base $b = 2$ (almost $\frac{1}{7}$).

In Conjecture 1.11 and Conjecture 1.12, we do not take the above representation into account, which we believe a safe assumption for large bases b. However, for small bases, and in particular, bases $3, 5, 7$, taking this into consideration would give us a different proportion. We end this section with the following conjecture.

Conjecture 5.2. *Let $S_b(\mathcal{A}, N)$ be the set of weakly primes less than N in base b, and let*

$$P_b = \prod_{\substack{p \in \mathbb{P} \\ p \leq b}} p.$$

Then there exists a constant c_b depending only on the base b such that

$$\# S_b(\mathcal{A}, N) \sim \pi(N) \left(1 - \frac{P_b}{\phi(P_b) \log N}\right)^{c_b \cdot \lfloor \log_b N \rfloor}.$$

6. Algorithms and Tables

The algorithms that we were able to find for determining whether an integer is a weakly prime store each prime as an integer array, and then permute each digit individually and test for primality. However, this requires converting the integer array into an integer at each primality check, which occurs up to $\lfloor \log_b(p) \rfloor \cdot (b-1)$ times each time the algorithm is run. In our approach, we avoid using an integer array by using modular arithmetic.

In this section, we present two efficient algorithms for determining whether an integer is a weakly prime in a given base. We then include charts with information concerning (nearly) weakly primes across bases.

6.1. *Algorithms*

Suppose you have a primality test with K time-complexity and S space complexity. Then the following two algorithms have $O(K \log n)$ time complexity and $O(S)$ space complexity for a prime $p = n$. The following algorithm we did not use because it requires converting between floats and integers and the built-in log function, However, we include this algorithm because it follows the procedure used in our sieving set up (note that q in the algorithm is the previously used $a_{p,m}$) and is easier to adjust.

```
isWeaklyAlt(b,p)
---------------------------------------------------------------
INPUT: b, the base (int); p, a weakly prime candidate (int)
OUTPUT: True when p weakly prime base b; False otherwise
---------------------------------------------------------------
def isWeaklyAlt(b,p):
    if not isPrime(p): return False
    n = floor(log(p,x))
    for m in range(0,n+1):
        q = p-(p mod b**m)+(p mod b**(m-1))
        for j in range(0,x-1):
            s = q + j*x**m
            if s != p and isPrime(s):
                return False
    return True
```

Next we present the main algorithm, which we did use to generate weakly, nearly weakly, and nearly-$(n, 0)$ primes. This algorithm again has $O(K \log n)$ time complexity and $O(S)$ space complexity for a prime $p = n$.

```
isWeakly(b,p)
---------------------------------------------------------------
```

```
INPUT: b, the base (int); p, a weakly prime candidate (int)
OUTPUT: True when p weakly prime base b; False otherwise
```

```
def isWeakly(p,b):
    # Check first that p is prime
    if not isPrime(p): return False
    # Below algorithm does not work for base 3, prime 2
    elif b == 3 and p == 2: return True
    else:
        # Initialize m=0, m range across integers between 0 and log_b p
        m = 0
        while p >= b**m:
            # Counter determines range of j for p-jb^m
            counter = 0
            # Initialize j with respect to m
            j = 1
            current = p + j*b**m

            # Check all p+jb^m are composite
            while p mod b**m != current mod (b**(m+1)):
                if isPrime(current): return False
                # Increment counter and j
                counter += 1
                j += 1
                current = p + j*b**m

            # Check all p-jb^m are composite
            for j in range(1,b - counter):
                current = p - j*b**m
                if isPrime(current): return False
            # Increment m
            m += 1

        # Return True if p is a weakly prime base b
        return True
```

The above algorithm can easily be modified to accept nearly weakly primes or nearly-$(n, 0)$ weakly primes rather than only weakly primes. In the first case, initialize $m = 1$. In the second case, also initialize $m = 1$, and change the third to last line to

```
if current >= b**m and isPrime(current): return False
```

so that all the integers checked are between b^n and b^{n+1}, where $n = \lfloor \log_b p \rfloor$.

6.2. *(Nearly) Weakly Primes*

In the upper table we list the first 9 weakly primes for each base $b = 2, \ldots, 11$.

2	127	173	191	223	233	239	251	257	277
3	2	7	13	19	31	41	149	239	283
4	373	2333	2917	2999	3779	6211	6323	6379	7043
5	83	233	277	397	487	509	593	647	739
6	28151	82913	153887	437771	632987	676297	685169	873359	903781
7	223	409	491	587	701	1051	1163	1237	1361
8	6211	57803	62213	83477	130769	132589	145289	153259	161869
9	2789	4027	6421	8963	20521	20719	23143	23473	27631
10	294001	505447	584141	604171	971767	1062599	1282529	1524181	2017963
11	3347	3761	5939	6481	8831	9257	9749	10487	11411

In the lower table, we list the first 9 strictly nearly weakly primes greater than b, for each base b, that is, primes which are nearly weakly but not weakly. Note that any prime less than b is trivially nearly weakly base b. Also note that there are no non-trivial strictly nearly weakly primes in bases $2, 3$.

4	2803	6451	9929	12823	14249	17959	20443	20983	24917
5	11	19	43	101	197	463	827	1033	1093
6	10597	1202629	1327009	1626259	3254393	3703373	4055897	4509073	4585127
7	11	71	79	107	281	461	827	1213	1217
8	2803	24917	49547	53549	68447	76421	101377	104801	126337
9	1753	5779	6091	10601	10847	11353	12011	19843	23603
10	976559	991607	1743739	1770331	2343881	4556911	5476061	6037007	6147133
11	659	727	1877	2003	2381	2531	3037	3851	4801

6.3. *Twin Weakly Primes*

We give the first 5 pairs or twin primes that are both weakly in bases $2, 3, 5, 7$.

2	1451, 1453	35531, 35533	68819, 68821	85931, 85933	98639, 98641
3	2711, 2713	3359, 3361	10067, 10069	10091, 10093	10331, 10333
5	9239, 9241	11489, 11491	51419, 51421	51719, 51721	52289, 52291
7	160481, 160483	175391, 175393	175937, 175939	183917, 183919	528821, 528823

6.4. *(Nearly) Weakly Prime Count*

In this table, we show the number of weakly primes, nearly weakly primes, and nearly-$(n, 0)$ primes in bases $b = 2, \dots, 21$ that are less than one million, ten million, and a hundred million. Notice the general trend that the number of all such primes decreases as b increases, and that the number is larger for larger $\phi(b)$. Note that **W**, **NW**, and $(n, 0)$ stand for the number of weakly primes, nearly weakly primes, and nearly weakly-$(n, 0)$ primes, respectively.

		$p < 10^6$			$p < 10^7$			$p < 10^8$		
b	$\phi(b)$	**W**	**NW**	$(n, 0)$	**WP**	**NW**	$(n, 0)$	**WP**	**NW**	$(n, 0)$
2	1	10338	10340	11755	89486	89488	99216	794760	794762	864812
3	2	15556	15557	16816	130900	130901	139896	1130147	1130148	1208152
4	2	1880	2081	2467	17173	18859	21388	159471	172283	192984
5	4	10272	11970	13072	93243	105540	114546	836536	930104	968342
6	2	13	17	22	146	175	241	1204	1416	1785
7	6	4785	6487	7206	44019	56828	60537	402391	496075	524830
8	4	76	124	154	959	1363	1582	9796	13086	14967
9	6	407	666	764	4304	6409	6998	41678	58606	63609
10	4	5	11	13	35	58	69	334	509	615
11	10	1095	2147	2362	11057	18555	19859	105978	165158	175442
12	4	0	5	5	0	5	5	3	10	12
13	12	490	1129	1260	5332	10612	11321	57311	102779	108766
14	6	0	7	7	2	12	15	51	104	116
15	8	1	18	21	29	78	94	299	578	666
16	8	1	8	9	5	21	26	85	157	182
17	16	164	495	564	1536	3986	4405	14990	34134	36096
18	6	0	7	7	0	7	7	0	7	7
19	18	66	277	310	686	2260	2414	9149	24322	25640
20	8	0	8	8	0	8	8	0	8	8
21	12	0	8	8	2	21	22	42	145	154

6.5. *(Nearly) Weakly Prime Proportion and Conjecture Comparison*

In the table on the following page, we show the proportion of primes less than b^n that are weakly, nearly weakly, and nearly-$(n, 0)$ in bases $b = 2, \ldots, 19$, where n is specified for each base b. We also show the corresponding proportion of nearly-$(n, 0)$ primes estimated by conjectures 1.11 and 1.12. Notice that these estimated proportions are less than the actual proportion, as we do not take into account the digit position.

b	$\phi(b)$	n	**Weakly**	**Nearly Weakly**	**Nearly-$(n,0)$**	**Conj 1.11**	**Conj 1.12**
2	1	27	.139077	.139077	.151101	.002762	.052945
3	2	17	.194484	.194484	.212534	.016774	.068620
4	2	13	.027950	.030241	.034359	.003477	.014484
5	4	11	.144003	.160762	.172816	.023363	.053455
6	2	10	.000229	.000272	.000349	.000067	.000262
7	6	9	.070288	.087951	.095781	.018818	.036550
8	4	9	.001778	.002374	.002742	.000709	.001784
9	6	8	.007698	.011001	.012316	.003268	.006771
10	4	7	.000053	.000087	.000104	.000047	.000112
11	10	7	.018466	.030054	.032738	.009853	.017132
12	4	7	.000001	.000005	.000005	.000002	.000004
13	12	7	.010241	.018637	.020248	.006716	.011324
14	6	7	.000009	.000018	.000020	.000013	.000027
15	8	6	.000045	.000116	.000140	.000095	.000187
16	8	6	.000015	.000036	.000046	.000034	.000067
17	16	6	.002891	.007099	.007787	.003451	.005723
18	6	6	.000000	.000003	.000003	.000000	.000000
19	18	6	.001579	.004407	.004815	.002379	.003908

7. Conclusion

The study of (nearly) weakly and nearly primes relates to interesting number theoretic concepts, such as gaps in primes and primality testing. A possible direction for future research would be to prove whether there is a finite or infinite number of strictly weakly primes in a given base b. Proving there are infinitely many strictly nearly weakly primes in base 5, for example, implies the twin prime conjecture. Additionally, one could study the prevalence of twin weakly primes in a given base.

A direction of further research is to pursue proving Conjectures 1.11 and 1.12 about the proportion of (nearly) weakly-$(n, 0)$ primes using sieve-theoretic tools. Numerical evidence supports the conjecture that the proportion of weakly primes in a given base b depends on the values of b and $\phi(b)$, and that this proportion of weakly primes less than N converges to some absolute proportion.

Another potential direction is finding a polynomial analogue for weakly primes, such as irreducible polynomials in a finite field that become reducible if any any coefficient is modified. Alternatively, one could consider polynomials that are reducible over a finite field $\mathbb{F}_p$ but irreducible over $\mathbb{F}_q$ for $q \neq p$. This relates to work done by Guralnick, Schacher, and Sonn in [8], who show that for any positive integer n, there exist polynomials $f(x) \in \mathbb{Z}[x]$ of degree n which are irreducible over $\mathbb{Q}$ and reducible over $\mathbb{Q}_p$ for all primes p, if and only if n is composite.

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