week 0

FUNCTIONS OF 2 OR MORE VARIABLES

- function of n variables is function f that assigns real # $f(x_1, \ldots, x_n)$ to each n-tuple $(x_1, \ldots, x_n)$ in domain in $\mathbb{R}^n$
  - ↳ domain is set of all n-tuples for which $f(x_1, \ldots, x_n)$ is defined
  - ↳ range is set of all values $f(x_1, \ldots, x_n)$ for $(x_1, \ldots, x_n)$ is domain
- graph of function f of 2 variables consists of all points $(a, b, f(a,b))$ in $\mathbb{R}^3$ for (a,b) in domain $\mathcal{D}$ of f
  - ↳ assuming f is continuous, graph is surface whose height above/below xy-plane at (a,b) is value $f(a,b)$
- vertical traces are obtained when we freeze either the x-or-y-coordinate so resulting curve is graph intersected w/ plane parallel to vertical coordinate plane
  - ↳ in plane $x=a$: intersection of graph w/ vertical plane $x=a$, consisting of all points $(a, y, f(a,y))$
  - ↳ in plane $y=b$: intersection of graph w/ vertical plane $y=b$, consisting of all points $(x, b, f(x,b))$

- horizontal trace at height c: intersection of graph w/ horizontal plane $z=c$, consisting of points $(x, y, f(x,y))$ st $f(x,y)=c$
  - ↳ level curve: curve $f(x,y)=c$ in xy-plane
    - i.e. horizontal trace projected onto xy-plane
- contour map is plot in domain in xy-plane that shows level curves $f(x,y)=c$ for equally spaced values of c
  - ↳ contour interval m is interval btwn values of c
    - altitude doesn't change when hiking along level curve
    - when hiking from one level curve to next, altitude changes by $\pm m$
  - ↳ level curves often called contour lines
  - ↳ level curves are close tgt if graph is steep & further apart when graph is flatter
  - ↳ contour map of linear function consists of equally spaced parallel lines
- AROC from P to Q = $\frac{\Delta \text{altitude}}{\Delta \text{horizontal}}$
  - ↳ P & Q are points on contour map
  - ↳ no single ROC b/c change in $f(x,y)$ depends on direction
- path of steepest ascent is path that begins at point P & everywhere along the way, points in steepest dir
  - ↳ apx it by drawing sequence of segments that move as directly as possible from one level curve to next
- not possible to draw function of more than 2 variables so use level surfaces for function of 3 variables
  - ↳ e.g. when function rep temp, level surfaces are called isotherms

LIMITS AND CONTINUITY IN SEVERAL VARIABLES

- assume $f(x,y)$ is defined near $P=(a,b)$, then $\lim_{(x,y)\to P} f(x,y) = L$
  - ↳ for any $\epsilon > 0$, there exists $\delta > 0$ st if (x,y) satisfies $0 < d((x,y),(a,b)) < \delta$, then $|f(x,y) - L| < \epsilon$
    - $d((x,y),(a,b)) = \sqrt{(x-a)^2 + (y-b)^2}$
  - ↳ in multivariable limit, $f(x,y)$ must tend to L as (x,y) approaches P from infinitely many diff dir
- function f is continuous at $P=(a,b)$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
  - ↳ f is continuous if it's cts at each point (a,b) in its domain
    - polynomials cts everywhere
    - rational functions cts on domain
- if f is product $f(x,y) = h(x)g(y)$ where $h(x)$ & $g(y)$ are cts, then $\lim_{(x,y)\to(a,b)} f(x,y) = \left(\lim_{x\to a} h(x)\right)\left(\lim_{y\to b} g(y)\right)$
- if $f(x,y)$ is cts at (a,b) & $G(z)$ is continuous at $c=f(a,b)$, then composite function $G(f(x,y))$ is continuous at (a,b)

- to show limit DNE, show limits obtained along 2 diff paths aren't equal some different paths are:

- along x-axis ($y=0$)
- along y-axis ($x=0$)
- along line $y=mx$
- along parabola $y=mx^2$

· can show limit exists by converting to polar coordinates

- $r=\sqrt{x^2+y^2}$ & $\tan\theta=\frac{y}{x}$
- $x=r\cos\theta$
- $y=r\sin\theta$
- $(x,y)\to(0,0) \Rightarrow r\to 0^+$
- e.g. evaluate $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

SOLUTION

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$$
$$= \lim_{r\to0^+} \frac{3(r\cos\theta)^2(r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2}$$
$$= \lim_{r\to0^+} \frac{3r^3\cos^2\theta\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)}$$
$$= \lim_{r\to0^+} 3r\cos^2\theta\sin\theta \longrightarrow \cos^2\theta\sin\theta \text{ is bounded}$$
$$= 0$$

WARNING

· only use this trick when $(x,y)\to(0,0)$

· make sure θ expression is bounded

- e.g. $\frac{\cos^2\theta}{\sin\theta} \to \infty$ when $\sin\theta\to0$

PARTIAL DERIVATIVES

· partial derivatives are ROC wrt each variable separately

· $f(x,y)$ has 2 partial derivatives:

- $f_x(a,b) = \frac{\partial f}{\partial x}\Big|_{(a,b)} = \lim_{h\to0}\frac{f(a+h,b)-f(a,b)}{h}$
- $f_y(a,b) = \frac{\partial f}{\partial y}\Big|_{(a,b)} = \lim_{k\to0}\frac{f(a,b+k)-f(a,b)}{k}$

· to compute, treat other variable as constant

· $f_x(a,b)$ is slope at $x=a$ of tangent line to trace curve $z=f(x,b)$ where b is constant

· $f_y(a,b)$ is slope at $y=b$ of tangent line to trace curve $z=f(a,y)$ where a is constant

· to apx partial derivatives, for small Δx & Δy:

- $f_x(a,b) \simeq \frac{\Delta f}{\Delta x} \simeq \frac{f(a+\Delta x,b)-f(a,b)}{\Delta x}$
- $f_y(a,b) \simeq \frac{\Delta f}{\Delta y} \simeq \frac{f(a,b+\Delta y)-f(a,b)}{\Delta y}$

· second-order partial derivatives are $\frac{\partial^2 f}{\partial x^2}=f_{xx}$, $\frac{\partial^2 f}{\partial y^2}=f_{yy}$, $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=f_{xy}$, & $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=f_{yx}$

Clairaut's Theorem: if f_x, f_y, & f_{xy} exist near (a,b) & f_{xy} is continuous at (a,b), then $f_{yx}(a,b)$ exists & $f_{yx}(a,b)=f_{xy}(a,b)$

- can apply to even higher order partial derivatives
  - e.g. $f_{xyyz}=f_{yxzy}$ if f is function of x, y, & z

DIFFERENTIABILITY, TANGENT PLANES, AND LINEAR APPROXIMATION

· tangent lines for f_x & f_y can determine a plane that's possibly tangent to graph

· $f(x,y)$ is differentiable at (a,b) if $f_x(a,b)$ & $f_y(a,b)$ exist & $\lim_{(x,y)\to(a,b)}\frac{f(x,y)-L(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}=0$

- $L(x,y)$ is linear apx of $f(x,y)$
- theorem: if $f_x(x,y)$ & $f_y(x,y)$ exist & are continuous on open disk D, then $f(x,y)$ is differentiable on D

· if $f(x,y)$ is differentiable at (a,b), equation of tangent plane to $z=f(x,y)$ at (a,b) is $z=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$

· if $f(x,y)$ is differentiable at (a,b), linearization of f centred at (a,b) is $L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$

- linear apx for $f(x,y)$ w/ (x,y) being close to (a,b) is $f(x,y)\simeq L(x,y)$

· linear apx for a very small Δx & Δy is $f(a+\Delta x, b+\Delta y) \simeq f(a,b)+f_x(a,b)\Delta x + f_y(a,b)\Delta y$

↳ $\Delta f \approx f(a+\Delta x,\ b+\Delta y) - f(a,b)$

$\Delta f \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$

· differential form of linear apx where $df \approx \Delta f$, $dx \approx \Delta x$, & $dy \approx \Delta y$: $df = f_x(x,y)\,dx + f_y(x,y)\,dy$

$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$

PARAMETRIC EQUATIONS

· parametric curve $c(t) = (x(t),\ y(t))$ describes path of particle moving along curve as function of parameter t

↳ can also be expressed as vector function $\vec{r}(t) = \langle x(t), y(t) \rangle$

↳ parametrizations aren't unique & every curve can be parametrized in infinitely many ways

- $c(t)$ may traverse all / part of curve more than once

· standard parametrizations:

↳ line of slope $m = \frac{s}{r}$ through point (a,b): $c(t) = (a + rt,\ b + st)$

↳ circle of radius R centred at point (a,b): $c(t) = (a + R\cos t,\ b + R\sin t)$

↳ cycloid generated by circle of radius R: $c(t) = (R(t - \sin t),\ R(1 - \cos t))$

↳ graph of $y = f(x)$: $c(t) = (t,\ f(t))$

derivative of $\vec{r}(t) = \langle x(t), y(t) \rangle$ is $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$

↳ tangent line at $t = t_0$ is $\vec{r}(t_0) + s\vec{r}\,'(t_0)$, $s \in \mathbb{R}$

- slope of tangent is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ (only if $x'(t) \neq 0$)

· area under parametric curve $c(t) = (x(t),\ y(t))$ that doesn't go below x-axis & traces graph of function once is $A = \int_{t_0}^{t_1} y(t)\,x'(t)\,dt$

· can use vector functions to describe velocity, speed, & acceleration

· $\vec{r}(t) = \langle x(t),\ y(t) \rangle$

↳ position at time t

· $\vec{r}\,'(t) = v(t) = \langle x'(t),\ y'(t) \rangle$

↳ velocity at time t

↳ $\|\vec{r}\,'(t)\|$ is speed at time t

- $\|\vec{r}\,'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$

$\vec{r}\,''(t) = a(t) = \langle x''(t),\ y''(t) \rangle$

↳ acceleration at time t

↳ $\|\vec{r}\,''(t)\|$ is magnitude of acceleration at time t

week 1

ARC LENGTH AND SPEED

- let $c(t) = (x(t), y(t))$ be parametrization that directly traverses curve $\mathcal{C}$ for $a \le t \le b$; assuming $x'(t)$ & $y'(t)$ exist & are both continuous, then arc length s of $\mathcal{C}$ is $s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$
  - ↳ directly traversing means path traces $\mathcal{C}$ w/o changing dir along the way
  - ↳ b/c of square root in integral, s can't be evaluated explicitly (unless special cases) so we apx it numerically
- distance travelled along path $c(t)$ (regardless if it's direct traversing or not) for $a \le t \le b$ is $\int_a^b \sqrt{x'(t)^2 + y'(t)}\, dt$
- displacement of $c(t)$ over $a \le t \le b$ is distance from starting point $c(a)$ to endpoint $c(b)$
  - ↳ displacement ≤ distance
- distance travelled, starting at t_0, is $s(t) = \int_{t_0}^{t} \sqrt{x'(u)^2 + y'(u)^2}\, du$
- speed at time t is $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$
- let $c(t) = (x(t), y(t))$ & assume $x'(t)$ & $y'(t)$ are continuous ; surface area of surface obtained by rotating $c(t)$ abt x-axis for $a \le t \le b$ is $S = 2\pi \int_a^b y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$
  - ↳ assume $y(t) \ge 0$ so $c(t)$ lies above x-axis
  - ↳ assume $x(t)$ is inc so curve doesn't reverse dir

THE GRADIENT AND DIRECTIONAL DERIVATIVES

- gradient of function $f(x, y)$ at point $P = (a, b)$ is vector $\nabla f_P = \langle f_x(a,b), f_y(a,b) \rangle$
  - ↳ gradient of function of n variables is vector $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$
  - ↳ omitting ref to P, $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$
    - gradient ∇f assigns vector ∇f_P to each point in domain of f
- properties of gradient where $f(x,y,z)$ & $g(x,y,z)$ are differentiable & c is a constant:
  - ↳ $\nabla(f + g) = \nabla f + \nabla g$
  - ↳ $\nabla(cf) = c\nabla f$
  - ↳ $\nabla(fg) = f\nabla g + g\nabla f$
    - product rule
  - ↳ if $F(t)$ is differentiable function of one variable, then $\nabla(F(f(x,y,z))) = F'(f(x,y,z))\nabla f$
    - chain rule

- given function f along parametric path given by $x(t)$ & $y(t)$ in plane, let $r(t)$ rep both vector $\langle x(t), y(t) \rangle$ & point $(x(t), y(t))$
  - ↳ path is traced out by tips of vectors or points
- Chain Rule for Paths: if f & $r(t)$ are differentiable, then $\frac{d}{dt} f(r(t)) = \nabla f_{r(t)} \cdot r'(t)$
  - ↳ e.g. for 2 variables, $\frac{d}{dt} f(r(t)) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t) \rangle$
    $= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

- consider line through point $P = (a, b)$ in dir of unit vector $\vec{u} = \langle h, k \rangle$, denoted by $r(t) = \langle a + th, b + tk \rangle$
  - ↳ directional derivative of $f(r(t))$ at $P = (a,b)$ is $D_{\vec{u}} f(P) = D_{\vec{u}} f(a,b) = \lim_{t \to 0} \frac{f(a+th, b+tk) - f(a,b)}{t}$
  - ↳ partial derivatives are directional derivatives wrt standard unit vectors $i = \langle 1, 0 \rangle$ & $j = \langle 0, 1 \rangle$
    - $f_x(a,b) = D_i f(a,b)$
    - $f_y(a,b) = D_j f(a,b)$
- if f is differentiable at P & $\vec{u}$ is unit vector, directional derivative in dir of $\vec{u}$ is $D_{\vec{u}} f(P) = \nabla f_P \cdot \vec{u}$
  - ↳ i.e. $D_{\vec{u}} f(a,b) = f_x(a,b)h + f_y(a,b)k$

↳ to get unit vector of some vector $\vec{v}$, $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$

- assume $\nabla f_P \neq 0$ & let $\vec{u}$ be unit vector making angle θ w/ ∇f_P ; $D_{\vec{u}} f(P) = \|\nabla f_P\| \cos\theta$
  - ↳ ∇f_P points in dir of fastest rate of inc of f at P since $\cos 0 = 1$
    - rate is $\|\nabla f_P\|$
    - max value of $D_{\vec{u}} f(P)$ is $\|\nabla f_P\|$
  - ↳ $-\nabla f_P$ points in dir of fastest rate of dec of f at P since $\cos \pi = -1$
    - rate is $-\|\nabla f_P\|$
    - min value of $D_{\vec{u}} f(P)$ is $-\|\nabla f_P\|$
  - ↳ ∇f_P is normal to level curve/surface of f at P since $\cos\frac{\pi}{2} = 0$
- equation of tangent plane to level surface $F(x,y,z) = k$ at $P(a,b,c)$ is $\nabla F_P \cdot \langle x-a, y-b, z-c\rangle = 0$
  - ↳ i.e. $F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$

MULTIVARIABLE CALCULUS AND CHAIN RULES

- if $f(x,y,z)$ is function of x, y, z & if x, y, z depend on 2 other variables s & t, then $f(x,y,z) = f(x(s,t), y(s,t), z(s,t))$ is composite function of s & t
  - ↳ s & t are independent variables
  - ↳ chain rule expresses partial derivatives wrt independent variables in terms of primary derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, & $\frac{\partial f}{\partial z}$
    - $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$
    - $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$
- General Chain Rule: if $f_1(x_1, \dots, x_n)$ is function of n variables & if $x_1, \dots, x_n$ depend on independent variables $t_1, \dots, t_m$, then $\frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_k}$ for any $1 \leq k \leq m$
  - ↳ expressed as dot product: $\frac{\partial f}{\partial t_k} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial x_1}{\partial t_k}, \frac{\partial x_2}{\partial t_k}, \dots, \frac{\partial x_n}{\partial t_k} \right\rangle$

    $= \Delta f \cdot \left\langle \frac{\partial x_1}{\partial t_k}, \frac{\partial x_2}{\partial t_k}, \dots, \frac{\partial x_n}{\partial t_k} \right\rangle$
- when z is defined implicitly by equation $F(x,y,z) = 0$ (i.e. $z = z(x,y)$ is function of x & y), use implicit differentiation to find partial derivatives
  - ↳ $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$
  - ↳ $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

week 2

OPTIMIZATION IN SEVERAL VARIABLES

- function $f(x,y)$ has local extremum at $P=(a,b)$ if there exists an open disk $D(P,r)$ st:
  - ↳ local max: $f(x,y) \le f(a,b)$ for all $(x,y) \in D(P,r)$
  - ↳ local min: $f(x,y) \ge f(a,b)$ for all $(x,y) \in D(P,r)$
- point $P=(a,b)$ in domain of $f(x,y)$ is critical point if $f_x(a,b)=0$ or DNE & $f_y(a,b)=0$ or DNE
  - ↳ more generally, $(a_1,\dots,a_n)$ is critical point of $f(x_1,\dots,x_n)$ if each partial derivative satisfies $f_{x_j}(a_1,\dots,a_n)=0$ or DNE
- saddle point: critical point that's neither local min/max
- discriminant of $f(x,y)$ is $D = f_{xx}f_{yy} - (f_{xy})^2$
- second derivative test for $f(x,y)$: let $P=(a,b)$ be critical point of $f(x,y)$ & assume f_{xx}, f_{yy}, & f_{xy} are continuous near P
  - ↳ if $D>0$ & $f_{xx}(a,b)>0$, then $f(a,b)$ is local min
  - ↳ if $D>0$ & $f_{xx}(a,b)<0$, then $f(a,b)$ is local max
  - ↳ if $D<0$, then $f(a,b)$ is saddle point
  - ↳ if $D=0$, then test is inconclusive
- let $f(x,y)$ be continuous function on closed, bounded domain D in $\mathbb{R}^2$, then:
  - ↳ $f(x,y)$ takes on both global min & max in D
  - ↳ extreme values occur at either critical points in interior of D or at points on boundary of D

to determine extreme values:

1) find critical points in interior of D
2) find candidates for min/max values of f on boundary
3) compare all values of f

LAGRANGE MULTIPLIERS: OPTIMIZING WITH A CONSTRAINT

- global max/min of $f(x,y)$ that's subject to constraint $g(x,y)=k$ occur when level curve of f is tangent to constraint curve
- local extreme values of $f(x,y)$ subject to constraint $g(x,y)=k$ occur at critical points P that satisfy Lagrange condition: $\nabla f_P = \lambda \nabla g_P$
  - ↳ λ is scalar called Lagrange multiplier
  - ↳ assuming $\nabla g_P \ne \vec{0}$
  - ↳ if $g(x,y)$ is bounded, then global max/min of $f(x,y)$ exist

method of Lagrange:

1) find all (x,y) st $\nabla g(x,y) = \vec{0}$ & $g(x,y)=k$
2) find all (x,y) st $\nabla f = \lambda \nabla g$ & $g(x,y)=k$
3) plug all points from 1) & 2) into $f(x,y)$

- Lagrange condition for function of 3 variables $f(x,y,z)$ subject to 2 constraints $g(x,y,z)=k$ & $h(x,y,z)=m$ is $\nabla f = \lambda \nabla g + \mu \nabla h$

week 3

INTEGRATION IN 2 VARIABLES

- double integral is integral of function of 2 variables $f(x,y)$: $\iint_D f(x,y)\,dA$
  - ↳ rep signed volume where +ve contributions are from regions above xy-plane & -ve contributions are from below
  - ↳ domain D is plane region whose boundary can be made up of diff curves & segments
    - will first focus on when D is rectangle
- Riemann sum for $f(x,y)$ on rectangle $R = [a,b] \times [c,d]$ is sum of form $S_{N,M} = \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij}) \Delta x_i y_j$
  - ↳ corresponding to partitions of $x \in [a,b]$ & $y \in [c,d]$
  - ↳ P_{ij} are sample points in subrectangle R_{ij}
  - ↳ easier to use regular partitions (i.e. $\Delta x_i = \Delta x$ & $\Delta y_j = \Delta y$ for all i & j)
    - $\Delta x = \frac{b-a}{N}$
    - $\Delta y = \frac{d-c}{M}$
- double integral of $f(x,y)$ over rectangle R is $\iint_R f(x,y)\,dA = \lim_{\|P\| \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij}) \Delta A_{ij}$
  - ↳ $\|P\|$ is max width of Δx_i & Δy_j
  - ↳ as $N, M \to \infty$, $\|P\| \to 0$
- if function f of 2 variables is continuous on rectangle R, then $f(x,y)$ is integrable over R
- assume $f(x,y)$ & $g(x,y)$ are integrable over R, then:
  - ↳ $\iint_R (f(x,y) + g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$
  - ↳ for any constant C, $\iint_R C f(x,y)\,dA = C \iint_R f(x,y)\,dA$
    - $\iint_R C\,dA = C \cdot \text{area}(R)$
- Fubini's Theorem: double integral of continuous function $f(x,y)$ over rectangle $R = [a,b] \times [c,d]$ is equal to iterated integral (i.e. order doesn't matter)
  - ↳ $\iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx = \int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\,dx\,dy$
- if integrating over rectangle & x & y terms can be completely separated, then $\int_a^b \int_c^d g(x) \cdot h(y)\,dy\,dx = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$

DOUBLE INTEGRALS OVER MORE GENERAL REGIONS

- if $f(x,y)$ is continuous on closed domain D whose boundary is simple closed piecewise smooth curve, then $\iint_D f(x,y)$ exists
- assume D is closed, bounded domain whose boundary is simple closed curve or has finite # of corners ; double integral is $\iint_D f(x,y)\,dA = \iint_R \tilde{f}(x,y)\,dA$
  - ↳ R is rectangle containing D
  - ↳ $\tilde{f}(x,y) = \begin{cases} f(x,y) & , (x,y) \in D \\ 0 & , (x,y) \notin D \end{cases}$
    - value of integral doesn't depend on choice/size of R
- for any constant C, $\iint_D C\,dA = C \cdot \text{area}(D)$
- if D is vertically simple w/ $a \leq x \leq b$ & $g_1(x) \leq y \leq g_2(x)$, then $\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$
- if D is horizontally simple w/ $c \leq y \leq d$ & $h_1(y) \leq x \leq h_2(y)$, then $\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$
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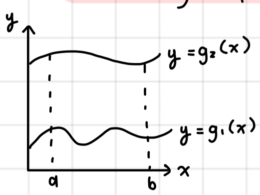
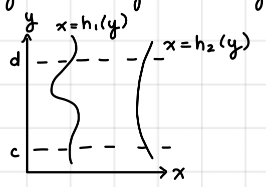


vertically simple



horizontally simple

- let $f(x,y)$ & $g(x,y)$ be integrable functions on D:
  - ↳ if $f(x,y) \le g(x,y)$ for all $(x,y) \in D$, then $\iint_D f(x,y)\,dA \le \iint_D g(x,y)\,dA$
    - i.e. larger functions have larger integrals
  - ↳ if $m \le f(x,y) \le M$ for all $(x,y) \in D$, then $m \cdot \text{area}(D) \le \iint_D f(x,y)\,dA \le M \cdot \text{area}(D)$
- average/mean value of function $f(x,y)$ is $\bar{f} = \frac{1}{\text{area}(D)} \iint_D f(x,y)\,dA = \frac{\iint_D f(x,y)\,dA}{\iint_D dA}$
  - ↳ $\iint_D f(x,y)\,dA = \bar{f} \cdot \text{area}(D)$
- Mean Value Theorem for Double Integrals: if $f(x,y)$ is continuous & D is closed, bounded, & connected, then there exists a point $P \in D$ st $\iint_D f(x,y)\,dA = f(P)\,\text{area}(D)$
  - ↳ i.e. $f(P) = \bar{f}$
- if D is union of domains $D_1, D_2, \ldots, D_N$ that don't overlap (except possibly on boundary curves), then $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \ldots + \iint_{D_N} f(x,y)\,dA$
- if $f(x,y)$ is continuous function on small domain D, then $\iint_D f(x,y)\,dA \simeq f(P)\,\text{area}(D)$
  - ↳ if domains $D_1, \ldots, D_N$ are small & P_j is sample point in D_j, then $\iint_D f(x,y)\,dA \simeq \sum_{j=1}^{N} f(P_j)\,\text{area}(D_j)$

INTEGRATION IN POLAR, CYLINDRICAL, AND SPHERICAL COORDINATES

- polar coordinates are convenient when domain of integration is angular sector / polar rectangle
  - ↳ $R: \theta_1 \le \theta \le \theta_2,\ r_1 \le r \le r_2$
  - ↳ function $f(x,y)$ in polar coordinates is $f(r\cos\theta, r\sin\theta)$
- cylindrical coordinates are useful when domain has axial symmetry
  - ↳ function $f(x,y,z)$ in cylindrical coordinates is $f(r\cos\theta, r\sin\theta, z)$
- function $f(x,y,z)$ in spherical coordinates is $f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$
- area & volume elements:
  - ↳ $dA = r\,dr\,d\theta$
  - ↳ $dV = r\,dz\,dr\,d\theta$
    - cylindrical
  - ↳ $dV = \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$
    - spherical
- double integral in polar coordinates: $\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$
- triple integral $\iiint_W f(x,y,z)\,dV$ conversions:
  - ↳ for cylindrical, $\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$
  - ↳ for spherical, $\int_{\theta_1}^{\theta_2} \int_{\varphi_1}^{\varphi_2} \int_{\rho_1(\theta,\varphi)}^{\rho_2(\theta,\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

CHANGE OF VARIABLES

- mapping: function $G: X \to Y$ from set X (domain) to another set Y
  - ↳ for $x \in X$, $G(x)$ belongs to Y & is image of x
    - set of all images $G(x)$ is image/range of G
- when mapping from $D \to \mathbb{R}^2$ (where D is in $\mathbb{R}^2$), use u & v as domain variables & x & y for range
  - ↳ written as $G(u,v) = (x(u,v), y(u,v))$
- map is linear if it has form $G(u,v) = (Au + Cv, Bu + Dv)$
  - ↳ $A, B, C,$ & D are constants
  - ↳ G has following linearity properties:
    - $G(u_1 + u_2, v_1 + v_2) = G(u_1, v_1) + G(u_2, v_2)$
    - for any constant c, $G(cu, cv) = cG(u,v)$
  - ↳ G maps segments joining any 2 points P & Q to segment joining $G(P)$ & $G(P)$
- Jacobian of a map $G(u,v) = (x(u,v), y(u,v))$ is determinant: $\text{Jac}(G) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$
  - ↳ tells how area changes under map of G

- let $G: D_0 \to D$ be C^1 mapping that's one-to-one on interior of D_0; if $f(x,y)$ is continuous, then
  $\iint_D f(x,y)\,dx\,dy = \iint_{D_0} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$
  - ↳ C^1 map means component functions x & y have continuous partial derivatives
- Change of Variables formula in 2 & 3 variables:
  - ↳ $dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$
  - ↳ $dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$
- Change of Variables formula turns xy-integral into uv-integral but map G goes from uv-domain to xy-domain
  - ↳ sometimes more convenient to find map F going in opp dir
  - ↳ if $G = F^{-1}$ & $Jac(F) \neq 0$, then $Jac(G) = Jac(F)^{-1}$
    - i.e. $\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$

week 4

APPLICATIONS OF MULTIPLE INTEGRALS

- total amount of quantity distributed in $\mathbb{R}^2$ / $\mathbb{R}^3$ can be written as double/triple integral: $\iint_D \delta(x,y)\,dA$ or $\iiint_W \delta(x,y,z)\,dV$
  - ↳ δ is density function & has units of amount per unit area/volume
- can use double integrals to find area of region R: $A = \iint_R 1\,dA$
  - ↳ volume is area of $R \times$ height
- moments of lamina (i.e. thin plate on plane) D are $M_x = \iint_D x\,\delta(x,y)\,dA$ & $M_y = \iint_D y\,\delta(x,y)\,dA$
- centre of mass (COM) is $P_{CM} = (x_{CM}, y_{CM})$
  - ↳ $x_{CM} = \frac{M_y}{M}$
  - ↳ $y_{CM} = \frac{M_x}{M}$
- if δ is constant, then COM coincides w/ centroid, whose coordinates are avg values over domain $(\bar{x}, \bar{y}, \bar{z})$
  - ↳ for $\mathbb{R}^2$, $\bar{x} = \frac{1}{A}\iint_D x\,dA$ & $\bar{y} = \frac{1}{A}\iint_D y\,dA$
    - $A = \iint_D 1\,dA$

	In $\mathbf{R}^2$	In $\mathbf{R}^3$
Total mass	$M = \iint_D \delta(x,y)\,dA$	$M = \iiint_W \delta(x,y,z)\,dV$
Moments	$M_x = \iint_D y\delta(x,y)\,dA$ $M_y = \iint_D x\delta(x,y)\,dA$	$M_{yz} = \iiint_W x\delta(x,y,z)\,dV$ $M_{xz} = \iiint_W y\delta(x,y,z)\,dV$ $M_{xy} = \iiint_W z\delta(x,y,z)\,dV$
Center of mass	$x_{CM} = \frac{M_y}{M}, \quad y_{CM} = \frac{M_x}{M}$	$x_{CM} = \frac{M_{yz}}{M}, \quad y_{CM} = \frac{M_{xz}}{M}, \quad z_{CM} = \frac{M_{xy}}{M}$
Moments of inertia	$I_x = \iint_D y^2\delta(x,y)\,dA$ $I_y = \iint_D x^2\delta(x,y)\,dA$ $I_0 = \iint_D (x^2+y^2)\delta(x,y)\,dA$ $(I_0 = I_x + I_y)$	$I_x = \iiint_W (y^2+z^2)\delta(x,y,z)\,dV$ $I_y = \iiint_W (x^2+z^2)\delta(x,y,z)\,dV$ $I_z = \iiint_W (x^2+y^2)\delta(x,y,z)\,dV$

  - ↳ I_0 is polar moment of inertia & it's relative to z-axis
  - ↳ radius of gyration: $r_g = \sqrt{I/M}$
- random variables X & Y have joint probability density function $p(x,y)$ if $P(a \le X \le b;\ c \le Y \le d) = \int_{x=a}^{b}\int_{y=c}^{d} p(x,y)\,dy\,dx$
- 2 conditions for joint density functions:
  - ↳ $p(x,y) \ge 0$ for all x & y b/c probabilities can't be -ve
  - ↳ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dy\,dx = 1$
    - normalization condition

TRIPLE INTEGRALS

triple integral over box $B = [a,b] \times [c,d] \times [p,q]$ is equal to iterated integral $\iiint_B f(x,y,z)\,dV$

$= \int_{x=a}^{b} \int_{y=c}^{d} \int_{z=p}^{q} f(x,y,z)\, dz\, dy\, dz$

↳ may be evaluated in any order when it's a rectangular box (6 diff orders)

- z-simple region W is $\mathbb{R}^3$ consists of points (x,y,z) btwn 2 surfaces $z = z_1(x,y)$ & $z = z_2(x,y)$, where $z_1(x,y) \leq z_2(x,y)$
  - ↳ lies over domain D in xy-plane (i.e. projection of W onto xy-plane)
  - ↳ $W = \{(x,y,z) : (x,y) \in D$ & $z_1(x,y) \leq z \leq z_2(x,y)\}$
  - ↳ x-simples & y-simples also exist
- triple integral over z-simple region W is $\iiint_W f(x,y,z) = \iint_D \left(\int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz \right) dA$
- graph of $f(x,y,z)$ lives in 4D space but triple integrals can still be used to compute diff quantities in 3D settings
  - ↳ volume of region W is $V = \iiint_W 1\, dV$
- avg value of $f(x,y,z)$ on region W of volume V is $\bar{f} = \frac{1}{V} \iiint_W f(x,y,z)\, dV$

week 5

NEWTON'S METHOD

- procedure for finding numerical apx to zeroes of functions
- Newton's Method to apx root of $f(x)=0$:
  1) choose initial guess x_0 (close to desired root if possible)
  2) generate successive apx $x_1, x_2, \dots,$ where $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- if x_n & x_{n+1} agree to m decimal places, apx x_n is correct to these m places
- sometimes, Newton's Method doesn't give correct root
  - ↳ may converge to another root
  - ↳ iterations may diverge to ∞
- try to choose x_0 to be as close as possible to root & consult graph

POLYNOMIAL INTERPOLATION

- given set of data points, can find polynomial passing through them
  - ↳ if given k points, can use $(k-1)^{th}$ order polynomial
- n^{th} order polynomial passing through points $(0, y_0), (1, y_1), \dots, (n, y_n)$ is given by

$$y = y_0 + x\Delta y_0 + x(x-1)\frac{\Delta^2 y_0}{2!} + x(x-1)(x-2)\frac{\Delta^3 y_0}{3!} + \dots + x(x-1)(x-2)\cdots(x-(n-1))\frac{\Delta^n y_0}{n!}$$

  - ↳ only applies when x-values are 0, 1, 2, ...
- to find fwd finite diffs, use y-values from given points
  - ↳ e.g. given (x_0, y_0), (x_1, y_1), & (x_2, y_2):

$$\begin{array}{ccc} y_0 & & \\ & \Delta y_0 & \\ y_1 & & \Delta^2 y_0 \\ & \Delta y_1 & \\ y_2 & & \end{array}$$

  - $\Delta y_0 = y_1 - y_0$
  - $\Delta y_1 = y_2 - y_1$
  - $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$
- for more general interpolating polynomial: given $n+1$ equidistant nodes x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_n = x_0 + nh$ & their corresponding y-values, n^{th} order polynomial passing through all points is given by $y = y_0 + \frac{(x-x_0)}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 y_0 + \dots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{n!\,h^n}\Delta^n y_0$
  - ↳ h is distance btwn each x-value

TAYLOR POLYNOMIALS

- Taylor polynomials are useful for apx functions
- n^{th} Taylor polynomial centred at $x=a$ for function f is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

  - ↳ in summation notation, $P_{n,a} = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j$
  - ↳ f is zeroth derivative so $f^{(0)} = f$
- Maclaurin's Theorem: if $P(x) = a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)^n$ & $P^{(k)}(x_0) = f^{(k)}(x_0)$ for all $k = 0, 1, 2, \dots, n$, then $P(x) = P_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$
- Maclaurin polynomial: n^{th} order Taylor polynomial centred at $a=0$
  - ↳ if $P(x) = n^{th}$ deg Maclaurin polynomial for $f(x)$, $P(kx^m) = (mn)^{th}$ deg Maclaurin polynomial for $f(kx^m)$
    - k is some constant
- let $P(x) = n^{th}$ order Taylor polynomial for $f(x)$ centred at x_0 st $P(x_0) = f(x_0)$, $P'(x_0) = f'(x_0), \dots, P^{(n)}(x_0) = f^{(n)}(x_0)$:
  - ↳ $P'(x) = P_{n-1,x_0}(x)$ for $f'(x)$
  - ↳ $\int P(x)\,dx = P_{n+1,x_0}(x)$ for $\int f(x)\,dx$
    - to find C, set $P_{n,x_0}(x_0) = f(x_0)$ (i.e. point of tangency)

- if $P(x)$ denotes n^{th} order Maclaurin polynomial for $f(x)$, then for all integers $m \geq 0$, $x^m P(x)$ is $(m+n)^{th}$ order Maclaurin polynomial for $x^m f(x)$
- when apx f w/ T_n, size of error depends on size of $(n+1)^{st}$ derivative
  - ↳ if $f^{(n+1)}$ exists & is continuous, then Error Bound is $|T_n(x) - f(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}$
    - K is # st $|f^{(n+1)}(u)| \leq K$ for all u btwn a & x

week 6

TAYLOR'S INEQUALITY

- Taylor's Remainder Theorem: if $f(x)$ has $n+1$ derivatives at x_0, then $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!} + R_n(x)$
  - ↳ 1^{st} term is $P_{n,x_0}(x)$
  - ↳ remainder / error is $R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$
- Taylor's Inequality: if $R_n(x)$ denotes error in apx $f(x)$ w/ its n^{th} order Taylor polynomial $P_{n,x_0}(x)$, then $|R_n(x)| \leq \frac{K \cdot |x-x_0|^{n+1}}{(n+1)!}$
  - ↳ K is upper bound for $|f^{(n+1)}(z)|$ for all z btwn x_0 & x
    - i.e. $|f^{(n+1)}(z)| \leq K$
  - ↳ when using $R_n(x)$ to find error bounds for $f(x)$, always round $R_n(x)$ up
- e.g. estimate $\int_0^1 e^{x^2}\,dx$

SOLUTION

Use substitution $u = x^2$ & find $P_{2,0}$

$$e^u = 1 + u + \frac{u^2}{2} + R_2(u)$$

To find $R_2(u)$:

$$|R_2(u)| \leq \frac{K\,|u-0|^3}{3!} \longrightarrow \frac{d}{du}(e^u) = e^u \text{, max value } K \text{ in interval } [0,1] \text{ is } e$$

$$\leq \frac{e\,|u|^3}{3!}$$

$$\leq \frac{3\,|u|^3}{3!} = \frac{|u|^3}{2}$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + R_2(x^2)$$

$$= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{2}$$

This means $1 + x^2 + \frac{x^4}{2} - \frac{x^6}{2} \leq e^{x^2} \leq 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{2}$

$$\int_0^1 1 + x^2 + \frac{x^4}{2} - \frac{x^6}{2}\,dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{2}\,dx$$

$$\int_0^1 1 + x^2 + \frac{x^4}{2} - \int_0^1 \frac{x^6}{2}\,dx \leq \int_0^1 e^{x^2}\,dx \leq \int_0^1 1 + x^2 + \frac{x^4}{2} + \int_0^1 \frac{x^6}{2}\,dx$$

$$\frac{43}{30} - \frac{1}{14} \leq \int_0^1 e^{x^2}\,dx \leq \frac{43}{30} + \frac{1}{14}$$

week 7

TAYLOR SERIES

- if f is infinitely differentiable at $x=c$, then Taylor series for $f(x)$ centred at c is power series:

$$T(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

  - ↳ partial sum $T_k(x)$ is k^{th} Taylor polynomial
  - ↳ Maclaurin series where $c=0$ is $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$
- a function f is rep by its Taylor series iff remainder $R_k(x) = f(x) - T_k(x)$ tends to 0 as $k \to \infty$
  - ↳ i.e. Taylor series converges to $f(x)$ iff $\lim_{k \to \infty} R_k(x) = 0$
- let $I = (c-R, c+R)$ w/ $R > 0$ & assume f is infinitely differentiable on I; suppose there exists $K > 0$ st $|f^{(k)}(x)| \leq K$ for all $k \geq 0$ & $x \in I$
  - ↳ then, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ for all $x \in I$
    - i.e. f is rep by its Taylor series on I
- to find Taylor series of function, start w/ known Taylor series & apply differentiation, integration, multiplication, or substitution
  - ↳

$f(x)$	Maclaurin series	Converges to $f(x)$ for
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	All x
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$	$\lvert x\rvert < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots$	$\lvert x\rvert < 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\lvert x\rvert < 1$ and $x = 1$
$\tan^{-1} x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\lvert x\rvert \leq 1$
$(1+x)^a$	$\sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$	$\lvert x\rvert < 1$

- binomial series: for any exponent a & $|x| < 1$, $(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \ldots + \binom{a}{n}x^n + \ldots$

SUMMING AN INFINITE SERIES

- an infinite series is expression $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$
  - ↳ a_n is general term
  - ↳ can begin at $n=k$ for any integer k
- N^{th} partial sum is finite sum of terms up to & including a_N
  - ↳ $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \ldots + a_N$
- an infinite series $\sum_{n=k}^{\infty} a_n$ converges to sum S if sequence of its partial sums $\{S_N\}$ converges to S: $\lim_{N \to \infty} S_N = S$
  - ↳ i.e. $S = \sum_{n=k}^{\infty} a_n$
  - ↳ if limit DNE, infinite series diverges
  - ↳ if limit is ∞, infinite series diverges to ∞
- if an infinite series converges, then it's linear; if $\sum a_n$ & $\sum b_n$ converge, then the following all converge:
  - ↳ $\sum (a_n + b_n) = \sum a_n + \sum b_n$
  - ↳ $\sum (a_n - b_n) = \sum a_n - \sum b_n$
  - ↳ $\sum c\, a_n = c \sum a_n$

- c is any constant

- for any geometric series $\sum_{n=0}^{\infty} cr^n$ w/ $r \neq 1$, then partial sum is $S_N = c + cr + cr^2 + \ldots + cr^N = \frac{c(1 + r^{N+1})}{1-r}$
- let $c \neq 0$, if $|r| < 1$, then sum of geometric series is $\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + \ldots = \frac{c}{1-r}$
  - ↳ if $|r| \geq 1$, then geometric series diverges
    - i.e. harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
- n^{th} term divergence test: if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges
  - ↳ even if $\lim_{n\to\infty} a_n = 0$, series might not converge

CONVERGENCE OF SERIES WITH POSITIVE TERMS

- in a +ve series Σa_n (where $a_n > 0$ for all n), visualize terms of +ve series as rectangles of width 1 & height a_n
  - ↳ partial sum $S_N = a_1 + a_2 + \ldots + a_N$ is area of 1st N rectangles
    - form increasing sequence (i.e. $S_N < S_{N+1}$)
- partial sum theorem for +ve series: if $\sum_{n=1}^{\infty} a_n$ is +ve series, then either S_N are bounded above so $\sum_{n=1}^{\infty} a_n$ converges or S_N aren't bounded above so $\sum_{n=1}^{\infty} a_n$ diverges
  - ↳ remains true if $a_n \geq 0$
  - ↳ n doesn't have to be 1 b/c convergence / divergence of series isn't affected by 1st finite terms
- Integral Test: let $a_n = f(n)$, where f is +ve, decreasing, & continuous function of x for $x \geq 1$
  - ↳ if $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
  - ↳ if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges
- convergence of p-series: infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ & diverges otherwise
- Direct Comparison Test: suppose Σa_n & Σb_n are series of +ve terms & $a_n \leq b_n$ for all n
  - ↳ if Σa_n diverges, then Σb_n diverges
  - ↳ if Σb_n converges, then Σa_n converges
- Limit Comparison Test: let $\{a_n\}$ & $\{b_n\}$ be +ve sequences & assume $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists
  - ↳ if $L > 0$, then either both Σa_n & Σb_n converge or both diverge
  - ↳ if $L = \infty$ & Σa_n converges, then Σb_n converges
  - ↳ if $L = 0$ & Σb_n converges, then Σa_n converges

week 8

ALTERNATING SERIES TEST

- AST: consider series $\sum_{k=0}^{\infty} (-1)^k a_k = a_0 - a_1 + a_2 - a_3 + \ldots$ where $a_k > 0$ for every k
  - ↳ if $\lim_{k\to\infty} a_k = 0$ & sequence $\{a_k\}$ is decreasing, then series converges
    - $\{a_k\}$ can be eventually dec meaning $a_{k+1} < a_k$ for all k greater than some int
  - ↳ although test failure doesn't explicitly prove divergence, if $\lim_{k\to\infty} a_k \neq 0$, then can use n^{th} term divergence test
- Alternating Series Estimation Theorem (ASET): consider convergent alternating series $\sum (-1)^k a_k$; if n^{th} partial sum S_n is used to estimate sum S, then error is $|S - S_n| \leq a_{n+1}$.
  - ↳ truncation error is at most the 1^{st} term not included in S_n
  - ↳ if last term added was +ve, S_n is overestimation
  - ↳ if last term added was -ve, S_n is underestimation

- a series $\sum a_k$ is absolutely convergent if $\sum |a_k|$ is convergent
  - ↳ e.g. $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$ is absolutely convergent since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent (using p-series test)
  - ↳ sum will not change if terms are rearranged
- a series $\sum a_k$ is conditionally convergent if it's convergent but $\sum |a_k|$ is divergent
  - ↳ e.g. $\sum (-1)^k \frac{1}{k}$ is convergent (by AST) but harmonic series $\sum \frac{1}{k}$ is divergent
    - can force terms to add up to any sum by selecting right order

RATIO TEST

- suppose $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L$
  - ↳ if $L < 1$, then series $\sum a_k$ is absolutely convergent
  - ↳ if $L > 1$, then series $\sum a_k$ is divergent
  - ↳ if $L = 1$, then test fails
    - can't draw any conclusions
- ratio test works well w/ factorials
- Root Test: suppose $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L$
  - ↳ if $L < 1$, then series $\sum a_k$ is absolutely convergent
  - ↳ if $L > 1$, then series $\sum a_k$ is divergent
  - ↳ if $L = 1$, then test fails
    - can't draw any conclusions
- useful limits for root test:
  - ↳ for $c > 0$, $\lim_{k\to\infty} \sqrt[k]{c} = 1$
  - ↳ for $p > 0$, $\lim_{k\to\infty} \sqrt[k]{k^p} = 1$
  - ↳ $\lim_{k\to\infty} \sqrt[k]{\ln(k)} = 1$
  - ↳ $\lim_{k\to\infty} \sqrt[k]{n!} = \infty$

POWER SERIES

- power series centred at x_0 is any series of form $\sum_{k=0}^{\infty} c_k (x - x_0)^k = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots$
  - ↳ usually call series a Taylor series if obtained from function & power series if using it to define a function
  - ↳ i.e. power series is more general form of Taylor series
- given power series, can figure out for what values of x it converges by applying Ratio

Test

↳ $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{c_{k+1}(x-x_0)^{k+1}}{c_k(x-x_0)^k}\right|$

$= \lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| \cdot |x-x_0|$

$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = |x-x_0| \lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right|$

- series converges absolutely if $|x-x_0| \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| < 1$

$|x-x_0| < \frac{1}{\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|}$

$|x-x_0| < \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$

$R = \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$ is known as radius of convergence

↳ if $R=0$, series converges only at $x=x_0$

↳ if $R=\infty$, series converges for all x

↳ if $0<R<\infty$, series converges for $x \in (x_0-R, x_0+R)$ & diverges for $x \in (-\infty, x_0-R) \cup (x_0+R, +\infty)$

- at 2 endpoints $x=x_0-R$ & $x=x_0+R$, apply other tests to draw conclusions

week 9

MANIPULATION OF POWER SERIES

- if series $\sum c_k (x-x_0)^k$ has radius of convergence R, then R won't change when we:
  - ↳ differentiate it
  - ↳ integrate it
  - ↳ multiply through w/constant
  - ↳ add it to another series of radius of convergence $\geq R$
    - i.e when adding series, use smaller R for result's radius of convergence
- when differentiating a series, might need to change starting index if there's constants being eliminated
- when integrating a series, to find constant of integration, plug in centre value x_0 b/c we know function & Taylor series are always equal there
- interval of convergence may change when applying above methods so check endpoints again
- when subbing in a diff term for x in series, radius & interval of convergence may change
- building blocks for manipulating Maclaurin series:

↳

Maclaurin Series	Radius R	Interval I
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	∞	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$	∞	$(-\infty, \infty)$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \ldots$	∞	$(-\infty, \infty)$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$	1	$(-1, 1)$

- given any $m \in \mathbb{R}$, binomial series for $(1+x)^m$ is:

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n$$

$$= 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots + \frac{m(m-1)\cdots(m-n+1)}{n!} x^n + \ldots$$

  - ↳ radius of convergence is $R = 1$
  - ↳ useful for various root functions

"BIG-O" ORDER SYMBOL

- given 2 functions f & g, can say f is of order g as $x \to x_0$, written as $f(x) = O(g(x))$ as $x \to x_0$, if there exists a constant $A > 0$ st $|f(x)| \leq A|g(x)|$ on some interval around x_0
  - ↳ x_0 may be excluded from interval since we're describing behaviour of in limit as we approach x_0
- when $f(x) = O(1)$ as $x \to 0$, it means that it's bounded by some constant A near 0
  - ↳ i.e. $f(x)$ doesn't have vertical asymptote at $x=0$
- e.g. $\sin x = O(x)$ as $x \to 0$ since $|\sin x| \leq x$ for $x \in \mathbb{R}$
  - ↳ rearranging gives $\left|\frac{\sin x}{x}\right| \leq 1$ for all $x \neq 0$, $x \in \mathbb{R}$ so $\frac{\sin x}{x} = O(1)$ as $x \to 0$
    - although $\frac{\sin x}{x}$ isn't defined at 0, it's bounded nearby
- definition of O symbol assigns no importance to value of constant A
  - ↳ e.g. $\sin x = O(x)$ as $x \to 0$ & $10^{10} \sin x = O(x)$ as $x \to 0$

- since $R_n(x)$, which is error of n^{th}-order Taylor polynomial used to apx $f(x)$, is $|R_n(x)| \leq \frac{K|x-x_0|^{n+1}}{(n+1)!}$, then $R_n(x) = O((x-x_0)^{n+1})$ as $x \to x_0$
- can use expression $f(x) = P_{n,x_0}(x) + O((x-x_0)^{n+1})$ as $x \to x_0$
- treat order as placeholder for all omitted terms in Taylor series

- algebra rules for Big-$\mathcal{O}$.
  - ↳ $k\mathcal{O}(x^n) = \mathcal{O}(x^n)$
    - k is any constant
  - ↳ $\mathcal{O}(x^m) + \mathcal{O}(x^n) = \mathcal{O}(x^q)$
    - $q = \min(m, n)$
  - ↳ $\mathcal{O}(x^m) \cdot \mathcal{O}(x^n) = \mathcal{O}(x^{m+n})$
  - ↳ $(\mathcal{O}(x^n))^m = \mathcal{O}(x^{mn})$
  - ↳ $\dfrac{\mathcal{O}(x^m)}{x^n} = \mathcal{O}(x^{m-n})$
- e.g. evaluate $\lim_{x \to 0} \dfrac{x\cos x - \sin x}{x^3}$ using Taylor polynomials & Big-O

SOLUTION

$$\lim_{x \to 0} \frac{x\cos x - \sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{x\left(1 - \frac{x^2}{2!} + \mathcal{O}(x^4)\right) - \left(x - \frac{x^3}{3!} + \mathcal{O}(x^5)\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{x - \frac{x^3}{2!} + \mathcal{O}(x^5) - x + \frac{x^3}{3!} - \mathcal{O}(x^5)}{x^3}$$

$$= \lim_{x \to 0} \frac{-\frac{x^3}{3} + \mathcal{O}(x^5)}{x^3}$$

$$= \lim_{x \to 0} -\frac{1}{3} + \mathcal{O}(x^2)$$

$$= -\frac{1}{3} + 0$$

$$= -\frac{1}{3}$$