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ENUMERATION basics

- # ways of choosing 1 thing from set A AND 1 thing from set B is Cartesian product of A & B: $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$
  - ↳ for finite sets, cardinalities satisfy $|A \times B| = |A| \cdot |B|$
- # ways of choosing 1 thing from set A OR 1 thing from set B is union of A & B: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
  - ↳ for finite disjoint (no intersection) sets, cardinalities satisfy $|A \cup B| = |A| + |B|$
  - ↳ since sets must have unique elements, intersections btwn sets must be subtracted
    - for finite sets, $|A \cup B| = |A| + |B| - |A \cap B|$

LISTS, PERMUTATIONS, AND SUBSETS

- list of set S is a list of all the elmts of S exactly once, in some order
- permutation is list of set $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$
- to construct list of S, choose any elmt $v \in S$ to be 1st elmt of list, then append list of set $S \setminus \{v\}$
  - ↳ if p_n is # lists of n-elmt set S for $n \in \mathbb{N}$, then $p_n = n \cdot p_{n-1}$
- theorem 1: for every $n \geq 1$, # lists of n-elmt set of S is $n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$, denoted as $n!$
- subset of S is collection of some (or none) of elmts of S, at most 1 each & in no order
- to specify subset X of S, for each elmt of $v \in S$, decide whether it's in or out of X (i.e. choose elmt from {in, out} n times
  - ↳ $\underbrace{|\{in, out\}| \cdot \dots \cdot |\{in, out\}|}_{n} = \underbrace{2 \cdot \ \cdot 2}_{n} = 2^n$
    - e.g. if $S = \{a, b, c\}$, then subset $X = \{a, b\}$ corresponds to list (in, in, out)
  - ↳ # subsets of n-elmt set S is 2^n
- partial list of set S is list of a subset of S
- to make k-tuples $(s_1, \dots, s_k)$ of distinct elmts of S (set w/n elmts), there's n choices for s_1 & n-1 choices for s_2 & ... & $(n-k+1)$ choices for s_k
- theorem 2: for $n, k \geq 0$, # partial lists of length k of an n-elmt set is $n(n-1) \dots (n-k+2)(n-k+1)$
  - ↳ if $k > n$, then one of factors will be 0 so whole product is 0 (i.e. no length-k partial lists)
  - ↳ when $0 \leq k \leq n$, # partial lists of length k of an n-elmt set is $\frac{n!}{(n-k)!}$
    - same formula as permutation
- theorem 3: for $0 \leq k \leq n$, # k-elmt subsets of n-elmt set S is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
  - ↳ same formula as combination
  - ↳ proof:

    Let S be set of size n. Another way to construct partial list of S length k is to choose k-elmt subset X of S AND list of X.

    # length-k partial lists of S
    $= (\text{\# k-elmt subsets of S}) \cdot (\text{\# lists of k-elmt set})$
    $= \binom{n}{k} k!$

    As such, $\frac{n!}{(n-k)!} = \binom{n}{k} k!$

    $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
    - example of combinatorial proof, where we prove identity by counting size of same set in 2 diff ways
- e.g. for $n \geq 0$, $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

  Proof:

  RHS, 2^n, is # subsets of n-elmt set

  For LHS, for $0 \leq k \leq n$, $\binom{n}{k}$ is # k-elmt subsets of S. An addition is OR so LHS is # ways of choosing a single subset of S: choosing 1 of the $\binom{n}{0}$ 0-elmt subsets or

  $\binom{n}{1}$ 1-elmt subsets or

  ⋮

  $\binom{n}{n}$ n-elmt subsets
  - ↳ LHS covers all possible subsets of an n-elmt set
- powerset of set S, denoted P(S), is set of all subsets of S, including empty set
  - ↳ if $|S| = \ell$, then $|P(S)| = 2^{\ell}$

- e.g. for $k, n \geq 1$, prove $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

  $\binom{n}{k}$: # k-elmt subsets of n-elmt set

  $\binom{n-1}{k}$: # k-elmt subsets of (n-1)-elmt set

  $\binom{n-1}{k-1}$: # (k-1)-elmt subsets of (n-1)-elmt set

  Let $S = \{1, \dots, n\}$.
  The LHS, $\binom{n}{k}$, is # k-elmt subsets of S.
  Note $\binom{n-1}{k}$ is # k-elmt subset of $S' = \{1, 2, \dots, n-1\}$.
  Any k-elmt subset X of S either contains n or doesn't contain n. If $n \notin X$, then X is k-elmt subset of (n-1)-elmt set. If $n \in X$, then $X \setminus \{n\}$ is (k-1)-elmt subset of (n-1)-elmt set. Thus, LHS = RHS.
  ↳ allows us to recursively compute $\binom{n}{k}$, leading to Pascal's triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
=
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

MULTISETS

- suppose I have bag containing 8 marbles, each of which is either red, green, or blue; how many diff bag contents are possible?
  $\{R, R, R, G, G, G, B, B\} = \{R, G, B\}$ since sets don't have duplicates
  $(R, R, R, G, G, G, B, B) \neq (B, B, G, G, G, R, R, R)$ since order matters in tuples
  $(3, 3, 2)$
  ↑ #R ↑ #G ↑ #B
- let $n \geq 0$, $t \geq 1$ be ints; a multiset of size n w/elmts of t types is sequence of non-ve ints $(m_1, \dots, m_t)$ st $m_1 + \dots + m_t = n$
- theorem 4: for any $n \geq 0$, $t \geq 1$, # n-elmt multisets w/elmts of t types is $\binom{n+t-1}{t-1}$
  ↳ proof: We know $\binom{n+t-1}{t-1}$ is # (t-1)-elmt subsets of (n+t-1)-elmt set. We'll show how to translate btwn (t-1)-elmt subsets of (n+t-1)-elmt set & n-elmt multisets w/t types.
  $n = 8$, $t = 3$
  Row of $n+t-1$ circles:

  $\underbrace{\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ}_{n+t-1}$

  Choose (t-1)-elmt subset & cross out:

  $\circ\circ\otimes\circ\circ\circ\otimes\circ\circ\circ$

  Row now has n circles grouped into t segments, each containing 0 or more consecutive circles.
  Let m_i be length of i^{th} segment of consecutive circles. Thus, $m_1 + \dots + m_t = n$, so $(m_1, \dots, m_t)$ is n-elmt multiset w/t types.
  ↳ going the other way:
  Let $(m_1, \dots, m_n)$ be n-elmt multiset w/t types. Write seq of m_1 circles followed by X, then m_2 circles followed by X, & so on finishing w/X & m_t circles. We'll have written down $n+t-1$ symbols, w/ $t-1$ Xs indicating (t-1)-elmt subset

  ○○X○○○X○○○
  ↳ example of bijection

BIJECTIONS

let A & B be sets & let $f: A \to B$
↳ f is surjective (onto) if for every $b \in B$, there exists an $a \in A$ st $f(a) = b$
↳ f is injective (one-to-one) if for every $a, a' \in A$, if $f(a) = f(a')$ then $a = a'$
↳ f is bijective if it's both surjective & injective

- proposition 1: a function $f: A \to B$ is bijection iff f has an inverse function $g: B \to A$ st $g(f(a)) = a$ for all $a \in A$ & $f(g(b)) = b$ for all $b \in B$
- if there exists a bijection btwn 2 sets A & B, then $A \rightleftharpoons B$
- corollary 1: if $A \rightleftharpoons B$ & at least A or B is finite, then both are finite & $|A| = |B|$
- e.g. for $k, n \geq 1$: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

  Let $S = \{1, \dots, n\}$ & $S' = \{1, \dots, n-1\}$

  Let $X = \{k\text{-elmt subsets of } S\}$ & $Y = \{(k-1)\text{-elmt or } k\text{-elmt subsets of } S'\}$

  By defn, $|X| = \binom{n}{k}$ & $|Y| = \binom{n-1}{k} + \binom{n-1}{k-1}$

  To prove $X \rightleftharpoons Y$:

  Define $f: X \to Y$ by $\overset{u \in X}{f(u)} = \begin{cases} u & \text{, if } n \notin u \\ u \setminus \{n\} & \text{, if } n \in u \end{cases}$

  Define $g: Y \to X$ by $g(v) = \begin{cases} v & \text{, if } |v| = k \\ v \cup \{n\} & \text{, if } |v| = k-1 \end{cases}$

  Is g inverse of f?

  $\forall u \in X$, is $g(f(u)) = u$?

  ↳ yes (check both cases of f, then apply g to get back to where we started)

  $\forall v \in Y$, is $f(g(v)) = v$?

  ↳ yes (check both cases of f, then apply g to get back to where we started)

  Thus, $g = f^{-1}$ & f is a bijection, so $A \rightleftharpoons B$ & $|A| = |B|$

GENERATING SERIES

- formal power series is expression of form $G(x) = \sum_{n=0}^{\infty} g_n x^n$
  - ↳ coeffs $(g_0, g_1, g_2, \dots)$ are sequence of ints (each coeff must be finite)
  - ↳ x is used as indeterminate for which we don't normally sub any particular value
    - don't care if series converges
  - ↳ e.g. $G(x) = 0 - 1x + 2x - 3x^3 + 4x^4 - \dots = \sum_{n=0}^{\infty} (-1)^n n x^n$
- can manipulate formal power series like polynomials
  - ↳ add them term by term
  - ↳ multiply them collecting like powers
  - ↳ e.g. $F(x) = \sum_{n=0} f_n x^n = f_0 + f_1 x + f_2 x^2 + \dots$

$$G(x) = \sum_{n=0} g_n x^n = g_0 + g_1 x + g_2 x^2 + \dots$$

$$(F+G)(x) = \sum_{n=0} (f_n + g_n) x^n$$
$$= (f_0 + g_0) + (f_1 + g_1)x + (f_2 + g_2)x^2 + \dots$$

$$(F \cdot G)(x) = \sum_{n=0} \left(\sum_{k=0}^{n} f_k g_{n-k} \right) x^n$$
$$= f_0 g_0 + (f_0 g_1 + f_1 g_0) x + \dots$$

- can sometimes invert them
  - ↳ if $F(x)$ & $G(x)$ are formal power series st $F(x)G(x) = 1$, then $G(x) = F(x)^{-1}$ or $G(x) = \frac{1}{F(x)}$
  - ↳ e.g. Let $G(x) = 1 + x + x^2 + x^3 + \dots = \sum_{n \geq 0} x^n$ (geometric series)

$$xG(x) = x + x^2 + x^3 + x^4 + \dots$$
$$= x \sum_{n \geq 0} x^n$$
$$= \sum_{n \geq 0} x^{n+1}$$

$$G(x) - xG(x) = 1 + x + x^2 + x^3 + \dots - x - x^2 - x^4 - \dots$$
$$= (1-0)x^0 + (1-1)x + (1-1)x^2 + \dots$$
$$= \sum_{n \geq 0} x^n - \sum_{n \geq 1} x^n$$
$$= 1$$
$$(1-x)G(x) = 1$$

So, $G(x) = \frac{1}{1-x}$ & we can see $G(x)$ & $(1-x)$ are inverses of each other

- proposition 2: inverse of $F(x) = \sum_{n \geq 0} f_n x^n$ exists iff $f_0 \neq 0$
  - ↳ if $f_0 = 0$, then it results in division by 0

BINOMIAL THEOREM

theorem 5 (Binomial Theorem): for any $n \in \mathbb{N}$, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k \geq 0} \binom{n}{k} x^k$

↳ combinatorial proof:

Let $P(n)$ be power set of $\{1, \dots, n\}$. We'll use correspondance btwn subsets & indicator vector indicating whether elmt is in subset or not. We have bijection btwn $P(n)$ & $\{0,1\}^n$, where if S is subset of $\{1, \dots, n\}$, its corresponding indicator vector is $\vec{a} = (a_1, \dots, a_n)$ where $a_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$.

Note $|S| = a_1 + a_2 + \dots + a_n$ (i.e. size of subset S is sum of entries in its indicator vector).
Intro indeterminate x. For every subset S & its corresponding vector $\vec{a}$, we have:

$$x^{|S|} = x^{a_1 + \dots + a_n}$$

Due to bijection, summing over all subsets is equivalent to summing overall indicator vectors:

$$\sum_{S \in P(n)} x^{|S|} = \sum_{\vec{a} \in \{0,1\}^n} x^{a_1 + \dots + a_n}$$

Simplify LHS:

$$\sum_{S \in P(n)} x^{|S|} = \sum_{k=0}^{n} \sum_{S \in P(n), |S| = k} x^{|S|}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k$$ ⟵ since there's $\binom{n}{k}$ k-elmt subsets of n-elmt set for each $0 \leq k \leq n$

Simplify RHS:

$$\sum_{\vec{a} \in \{0,1\}^n} x^{a_1 + \dots + a_n} = \sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \dots \sum_{a_n=0}^{n} x^{a_1} x^{a_2} \dots x^{a_n}$$

$$= \left(\sum_{a_1=0}^{1} x^{a_1} \right) \left(\sum_{a_2=0}^{1} x^{a_2} \right) \cdots \left(\sum_{a_n=0}^{1} x^{a_n} \right)$$

$$= \underbrace{(1+x)(1+x) \cdots (1+x)}_{n}$$

$= (1+x)^n$

Thus, we've proven the identity

- theorem 6 (-ve binomial theorem): if $t \geq 1$, then $(1-x)^{-t} = \sum_{n \geq 0} \binom{n+t-1}{t-1} x^n$
  - ↳ i.e. binomial series theorem
  - ↳ combinatorial proof:

    We recognize $\binom{n+t-1}{t-1}$ as # n-elmt multisets w/ t types (denote $|\mathcal{M}(n,t)|$), which are non -ve int seqs $(m_1, m_2, \dots, m_t)$ st $m_1 + m_2 + \dots + m_t = n$. Consider $\mathcal{M}(t)$ to be set of all multisets (w/any #elmts) of t types:

    $\mathcal{M}(t) = \bigcup_{n \geq 0} \mu(n,t)$

    There's bijection btwn $\mathcal{M}(t)$ & $\mathbb{N}^t$: every multiset $\mu \in \mathcal{M}(t)$ corresponds to non -ve int seq $(m_1, m_2, \dots, m_t)$ & vice versa. Note # elmts in μ is $|\mu| = m_1 + m_2 + \dots + m_t$.

$$\sum_{n \geq 0} \binom{n+t-1}{t-1} x^n = \sum_{n \geq 0} |\mathcal{M}(n,t)| x^n$$

$$= \sum_{\mu \in \mathcal{M}(t)} x^{|\mu|} \quad \text{another way of counting multisets}$$

$$= \sum_{(m_1, m_2, \dots, m_t) \in \mathbb{N}^t} x^{m_1 + m_2 + \dots + m_t} \quad \text{bijection}$$

$$= \sum_{m_1 \geq 0} \sum_{m_2 \geq 0} \dots \sum_{m_t \geq 0} x^{m_1 + m_2 + \dots + m_t} \quad \text{Cartesian prod}$$

$$= \sum_{m_1 \geq 0} x^{m_1} \sum_{m_2 \geq 0} x^{m_2} \dots \sum_{m_t \geq 0} x^{m_t} \quad \text{multiplication}$$

$$= \left(\frac{1}{1-x}\right)^t \quad \text{geometric series}$$

GENERATING SERIES

- e.g. write $\left(\frac{x^2}{1+3x}\right)^4$ as formal power series

$$\left(\frac{x^2}{1+3x}\right)^4 = x^8 \left(\frac{1}{1+3x}\right)^4$$

$$= x^8 \left(\frac{1}{1-(-3x)}\right)^4$$

$$= x^8 (1-(-3x))^{-4}$$

$$= x^8 \cdot \sum_{n \geq 0} \binom{n+4-1}{4-1} (-3x)^n \quad \text{-ve binomial theorem}$$

$$= \sum_{n \geq 0} \binom{n+3}{3} (-3)^n x^{n+8}$$

$$= \sum_{n \geq 8} \binom{n-5}{3} (-3)^{n-8} x^n$$

- coeff extraction: let $G(x) = \sum_{n \geq 0} g_n x^n$ be formal power series; for $k \in \mathbb{N}$, we can define $[x^k]G(x) = g_k$
  - ↳ $[x^k]$ is operator applied to $G(x)$ that extracts coeff x^k
  - ↳ e.g. from above example:

$$[x^k]\left(\frac{x^2}{1+3x}\right)^4$$

$$= [x^k] \sum_{n \geq 8} \binom{n-5}{3} (-3)^{n-8} x^n$$

$$= \begin{cases} 0 & \text{, if } 0 \leq k < 8 \\ \binom{n-5}{3} (-3)^{n-8} & \text{, if } k \geq 8 \end{cases}$$

- rules abt coeff extraction (let $a, b \in \mathbb{R}$):
  - ↳ $[x^k](aF(x) + bG(x)) = a[x^k]F(x) + b[x^k]G(x)$
  - ↳ $[x^k](x^\ell F(x)) = [x^{k-\ell}]F(x)$
  - ↳ $[x^k](F(x)G(x)) = \sum_{\ell=0}^{k} ([x^\ell]F(x))([x^{k-\ell}]G(x))$
- we'll use formal power series to encode counting info abt set, so then it's called a generating series
- e.g. Let $\mathcal{M} = \{$January, February, March, ..., December$\}$.

  Let $\mathcal{M}_n = \{\alpha \in \mathcal{M} : \alpha$ has n days (no leap years)$\}$.
  - ↳ $\mathcal{M}_{30} = \{$April, June, September, November$\}$
  - ↳ $\mathcal{M}_7 = \{\}$

  So, $\sum_{n \geq 0} |\mathcal{M}_n| x^n = x^{28} + 4x^{30} + 7x^{31}$

  $= \sum_{\alpha \in \mathcal{M}} x^{(\text{\#days in } \alpha)}$

- let S be set; a weight fcn $\omega : S \to \mathbb{N}$ if for every $n \in \mathbb{N}$, # elmts of S of weight n is finite (i.e. $\{\alpha \in S : \omega(\alpha) = n\}$ is finite for all $n \in \mathbb{N}$)
- let S be set & ω be weight fcn on S; generating series of S wrt ω is $\Phi_S^\omega(x) = \Phi_S(x) = \sum_{\alpha \in S} x^{\omega(\alpha)}$
  - ↳ superscript ω is sometimes omitted if unambiguous
  - ↳ e.g. cont from above example:

    $\omega(\alpha)$ counts #days in α & we have generating series of $\mathcal{M}$ wrt α is $\Phi_{\mathcal{M}}(x) = \sum_{\alpha \in \mathcal{M}} x^{\omega(\alpha)} = x^{28} + 4x^{30} + 7x^{31}$
    - $\mathcal{M}$ is set of months
    - $\omega(\alpha)$ is weight fcn counting #days in month α

proposition 3: let ω be weight fcn on set S; then $\Phi_S^{\omega}(x) = \sum_{n\geq 0} \underbrace{|\{\alpha\in S : \omega(\alpha)=n\}|}_{\#\text{elmts in } S \text{ of weight } n} x^n$

↳ i.e. for each $k\geq 0$, $[x^k]\Phi_S^{\omega}(x) = |\{\alpha\in S : \omega(\alpha)=k\}|$

↳ proof:

Let $S_n = \{\alpha\in S : \omega(\alpha)=n\}$. Then, $S = \bigcup_{n\geq 0} S_n$.

$$\begin{aligned}\Phi_S^{\omega}(x) &= \sum_{\alpha\in S} x^{\omega(\alpha)} \\ &= \sum_{n\geq 0}\Big(\sum_{\alpha\in S_n} x^n\Big) \\ &= \sum_{n\geq 0} |S_n| x^n \\ &= \sum_{n\geq 0} |\{\alpha\in S \cdot \omega(\alpha)=n\}| x^n\end{aligned}$$

- e.g. write generating series for set S of all binary strings wrt weight fcn ω which records length of string

$$\begin{aligned}\Phi_S^{\omega}(x) &= \sum_{\alpha\in S} x^{\omega(\alpha)} \\ &= \sum_{n\geq 0} |\{\alpha\in S : \text{length of } \alpha \text{ is } n\}| x^n \\ &= \sum_{n\geq 0} 2^n x^n \\ &= \sum_{n\geq 0} (2x)^n \\ &= \frac{1}{1-2x}\end{aligned}$$

generalization of geometric series

SUM AND PRODUCT LEMMAS

- e.g.

Let $S_1 = \{$omelette, waffles, pancakes, eggs, cereal$\}$ ← entrees

weight fcn (cost) $\omega_1 : S_1 \to \mathbb{N}$: omelette → 10, waffles → 10, pancakes → 8, eggs → 8, cereal → 5

↳ generating series for breakfast entrees wrt cost:

$$\begin{aligned}\Phi_{S_1}^{\omega_1}(x) &= \sum_{\alpha\in S_1} x^{\omega_1(\alpha)} \\ &= \sum_{\alpha\in S_1} |\{\alpha\in S : \omega_1(\alpha)=n\}| x^n \\ &= x^5 + 2x^8 + 2x^{10}\end{aligned}$$

Let $S_2 = \{$bacon, hashbrowns, toast$\}$ ← sides

weight fcn (cost) $\omega_2 : S_2 \to \mathbb{N}$: bacon → 5, hashbrowns → 4, toast → 3

$$\begin{aligned}\Phi_{S_1}^{\omega_2}(x) &= \sum_{\alpha\in S_2} x^{\omega_2(\alpha)} \\ &= x^3 + x^4 + x^5\end{aligned}$$

Consider $S_1 \cup S_2$.

$\omega : S_1 \cup S_2 \to \mathbb{N}$

$$\omega(\alpha) = \begin{cases}\omega_1(\alpha) & \text{if } \alpha\in S_1 \\ \omega_2(\alpha) & \text{if } \alpha\in S_2\end{cases}$$

$$\begin{aligned}\Phi_{S_1}^{\omega_1}(x) + \Phi_{S_2}^{\omega_2}(x) &= x^3 + x^4 + 2x^5 + 2x^8 + 2x^{10} \\ &= \sum_{\alpha\in S_1\cup S_2} x^{\omega(\alpha)} \\ &= \Phi_{S_1\cup S_2}^{\omega}(x)\end{aligned}$$

← generating series for each # items of each price

$$\begin{aligned}\Phi_{S_1}^{\omega_1}(x)\cdot\Phi_{S_2}^{\omega_1}(x) &= (x^5 + 2x^8 + 2x^{10})(x^3 + x^4 + x^5) \\ &= x^8 + x^9 + x^{10} + 2x^{11} + 2x^{12} + 4x^{13} + 2x^{14} + 2x^{15} \\ &= \Phi_{S_1\times S_2}^{\omega}(x)\end{aligned}$$

← generating series for # combos of each price

↳ joint weight fcn is $\omega(\alpha_1,\alpha_2) = \omega_1(\alpha_1) + \omega_2(\alpha_2)$

- **lemma 1 (sum lemma)**: let S_1, S_2 be disjoint sets; let ω be weight fcn on $S_1\cup S_2$; then $\Phi_{S_1}(x) + \Phi_{S_2}(x) = \Phi_{S_1\cup S_2}(x)$

↳ proof:

$$\begin{aligned}\Phi_{S_1}(x) + \Phi_{S_2}(x) &= \sum_{\alpha\in S_1} x^{\omega(\alpha)} + \sum_{\alpha\in S_2} x^{\omega(\alpha)} \\ &= \sum_{\alpha\in S_1\cup S_2} x^{\omega(\alpha)} \quad \text{since sets are disjoint} \\ &= \Phi_{S_1\cup S_2}(x)\end{aligned}$$

- **lemma 2 (infinite sum lemma)**: let $S_0, S_1, S_2, \ldots$ be disjoint sets w/union S & let ω be weight fcn on S; then $\Phi_S(x) = \sum_{n\geq 0}\Phi_{S_n}(x)$
- **lemma 3 (product lemma)**: let S_1, S_2 be sets & let ω_1 & ω_2 be weight fcns on S_1 & S_2 respectively; then $\Phi_{S_1}^{\omega_1}(x)\,\Phi_{S_2}^{\omega_2}(x) = \Phi_{S_1\times S_2}^{\omega}(x)$

↳ ω is weight fcn on $S_1\times S_2$ defined by $\omega(\alpha_1,\alpha_2) = \omega_1(\alpha_1) + \omega_2(\alpha_2)$

↳ sets don't have to be disjoint

- e.g.

Let $S=\{2,4,6,8,\ldots\}$. How many pairs (a,b) are there st $a,b\in S$ & $a+b=50$?

Sol:

Let $w(\alpha)=\alpha$ for each $\alpha\in S$. Then # pairs $(a,b)\in S\times S$ w/ $w(\alpha)+w(b)=50$ is

$$
\begin{aligned}
[x^{50}]\Phi_{S\times S}(x) &= [x^{50}](\Phi_S(x)\times\Phi_S(x)) && \text{by Product Lemma}\\
&= [x^{50}](x^2+x^4+x^6+\ldots)^2 && \text{sub generating series for } \Phi_S(x)\\
&= [x^{50}](x^2(1+x^2+x^4+\ldots))^2\\
&= [x^{50}]\left(\frac{x^2}{1-x^2}\right)^2 && \text{geometric series}\\
&= [x^{50}]\frac{x^4}{(1-x)^2}\\
&= [x^{46}](1-x^2)^{-2} && \text{coeff extraction rules}\\
&= [x^{46}]\sum_{n\geq 0}\binom{n+2-1}{2-1}(x^2)^n && \text{-ve binomial theorem}\\
&= \binom{23+2-1}{2-1} && \text{take } n=23 \text{ to get } x^{46}\\
&= 24
\end{aligned}
$$

- star operator: let A be set, then define $A^* = \bigcup_{k\geq 0} A^k = \{$ all tuples of elmts of $A\}$
  - ↳ e.g. $\{0,1\}^* = \{\underbrace{\overset{\varepsilon}{()}}_{A^0}, \underbrace{\overset{0}{(0)}, \overset{1}{(1)}}_{A^1}, \underbrace{\overset{00}{(0,0)}, \overset{01}{(0,1)}, \overset{10}{(1,0)}, \overset{11}{(1,1)}}_{A^2}, \underbrace{\overset{000}{(0,0,0)}}_{A^3}, \ldots\}$

- given weight fcn w on A, define w^* on A^* st for $(a_1,a_2,\ldots,a_k)\in A^*$, $w^*((a_1,a_2,\ldots,a_k)) = \sum_{i=1}^{k} w(a_i)$
  - ↳ well-defined when no elmts of A have weight 0
  - ↳ e.g.

    Let $S=\{0,1\}$ & w count length.

    $w^*((1,1,0,1)) = w(1)+w(1)+w(0)+w(1) = 1+1+1+1 = 4$

    $w^*((1)) = w(1) = 1$

    $w^*(()) = 0$

    - w^* also counts length
    - how are $\Phi_S(x)$ & $\Phi_{S^*}(x)$ related?

      $\Phi_S(x) = \sum_{\alpha\in S} x^{w(\alpha)} = x^{w(0)} + x^{w(1)} = 2x$

      $$
      \begin{aligned}
      \Phi_{S^*}(x) &= \Phi_{\{(),(0),(1),(0,0),\ldots\}}(x)\\
      &= 1x^0 + 2x^1 + 4x^2 + \ldots\\
      &= \sum_{n\geq 0}(2x)^n\\
      &= \frac{1}{1-2x} && \text{geometric series}\\
      &= \frac{1}{1-\Phi_S(x)}
      \end{aligned}
      $$

  - lemma 4 (string lemma): let A be set w/ weight fcn w st no elmts of A have weight 0, then $\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)}$
    - ↳ proof:

      $$
      \begin{aligned}
      \Phi_{A^*}(x) &= \Phi_{A_0\cup A_1\cup A_2\cup\ldots}(x)\\
      &= \sum_{n\geq 0}\Phi_{A_n}(x) && \text{infinite sum lemma}\\
      &= \sum_{n\geq 0}(\Phi_A(x))^n && \text{product lemma}\\
      &= \frac{1}{1-\Phi_A(x)} && \text{geometric series}
      \end{aligned}
      $$

COMPOSITIONS

- composition is finite seq of +ve ints $\gamma = (c_1, c_2, \ldots, c_k)$
  - ↳ each $c_i \in \mathbb{Z}_{>0}$ is called part
  - ↳ length is # parts
    - $\ell(\gamma) = k$
  - ↳ size is sum of parts
    - $|\gamma| = c_1 + c_2 + \ldots + c_k$
  - ↳ if s is size of γ, then γ is composition of s

- e.g.

Composition	Size	Length

$()$	0	0 parts
$(1,2,3,4)$	10	4 parts
$(1,1,2)$	4	3 parts

- **theorem 7**: generating series for all int compositions wrt size is $\Phi_{\text{compositions}}(x) = \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x}$
  - ↳ proof:

$$\begin{aligned}
\Phi_{\text{compositions}}(x) &= \Phi_{(\mathbb{Z}>0)^*}(x) \\
&= \frac{1}{1-\Phi_{\mathbb{Z}>0}(x)} && \text{string lemma} \\
&= \frac{1}{1-\sum_{\alpha\in\mathbb{Z}>0} x^{\text{size}(\alpha)}} \\
&= \frac{1}{1-\sum_{\alpha>0} x^{\alpha}} \\
&= \frac{1}{1-x\sum_{\alpha\geq 0} x^{\alpha}} \\
&= \frac{1}{1-x(\frac{1}{1-x})} && \text{geometric series} \\
&= \frac{1-x}{(1-x)-x} \\
&= \frac{1-x}{1-2x}
\end{aligned}$$

  - ↳ e.g.

How many int compositions are there of 123?

$$\begin{aligned}
\#\text{compositions of } 123 &= [x^{123}]\,\Phi_{\text{compositions}}(x) \\
&= [x^{123}]\frac{1-x}{1-2x} \\
&= [x^{123}](1-x)\left(\frac{1}{1-2x}\right) \\
&= [x^{123}](1-x)\sum_{n\geq 0}(2x)^n \\
&= [x^{123}]\sum_{n\geq 0}(2x)^n - [x^{123}]\,x\sum_{n\geq 0}(2x)^n \\
&= [x^{123}]\sum_{n\geq 0}(2x)^n - [x^{122}]\sum_{n\geq 0}(2x)^n \\
&= 2^{123} - 2^{122} \\
&= 2^{122}(2-1) \\
&= 2^{122}
\end{aligned}$$

- **theorem 8**: for $n \in \mathbb{N}$, # compositions of size n is $\begin{cases} 1, & \text{if } n=0 \\ 2^{n-1}, & \text{if } n\geq 1 \end{cases}$
- e.g.

How many compositions are there of 123 where each part is either 1 or 3?

Let $R = \{1,3\}$. Note $\Phi_R(x) = x^1 + x^3$. Set of compositions w/ each part being either 1 or 3 is R^*.

$$\begin{aligned}
[x^{123}]\,\Phi_{R^*}(x) &= [x^{123}]\frac{1}{1-\Phi_R(x)} && \text{string lemma} \\
&= [x^{123}]\frac{1}{1-x-x^3}
\end{aligned}$$

- e.g.

Find generating series for compositions where each part is odd?

Let $S = \{1,3,5,7,\dots\}$. Note $\Phi_S(x) = x^1 + x^2 + x^5 + \dots = x(1+x^2+x^4+\dots) = \frac{x}{1-x^2}$. Set of compositions w/ only odd parts is S^*.

$$\begin{aligned}
\Phi_{S^*}(x) &= \frac{1}{1-\Phi_S(x)} && \text{string lemma} \\
&= \frac{1}{1-\frac{x}{1-x^2}} \\
&= \frac{1-x^2}{1-x^2-x} \\
&= 1 + \frac{x}{1-x-x^2}
\end{aligned}$$

- e.g.

Find generating series for compositions where each part is 2 or greater?

Let $T = \{2,3,4,5,\dots\}$. Note $\Phi_T(x) = x^2+x^3+x^4+\dots = x^2(1+x+x^2+\dots) = \frac{x^2}{1-x}$. Set of compositions where each part ≥ 2 is T^*.

$$\begin{aligned}
\Phi_{T^*}(x) &= \frac{1}{1-\Phi_T(x)} && \text{string lemma} \\
&= \frac{1}{1-\frac{x^2}{1-x}} \\
&= \frac{1-x}{1-x-x^2} \\
&= 1 + \frac{x^2}{1-x-x^2}
\end{aligned}$$

- note generating series for last 2 examples were similar

- claim 1: # compositions of n where each part is ≥ 2 is equal to # compositions of n-1 where each part is odd
  - proof: $[x^{n-1}]\,\Phi_{S^*}(x) = [x^n]\,\Phi_{T^*}(x)$

$$\begin{aligned}
\text{LHS} &= [x^{n-1}]\,\Phi_{S^*}(x) \\
&= [x^{n-1}]\left(1 + \frac{x}{1-x-x^2}\right) \\
&= [x^{n-1}]1 + [x^{n-1}]\frac{x}{1-x-x^2} \\
&= 0 + [x^{n-2}]\frac{1}{1-x-x^2}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= [x^n]\,\Phi_{T^*}(x) \\
&= [x^n]\left(1 + \frac{x^2}{1-x-x^2}\right) \\
&= [x^n]1 + [x^n]\left(\frac{x^2}{1-x-x^2}\right) \\
&= 0 + [x^{n-2}]\left(\frac{1}{1-x-x^2}\right) \\
&= \text{LHS}
\end{aligned}$$

  - these finite sets are same size so there exists bijection btwn them

Compositions of n=7 w/parts ≥ 2	Compositions of n-1 = 6 w/odd parts
(7)	(1, 5)
(5, 2)	(5, 1)
(2, 5)	(1, 1, 1, 3)
(4, 3)	(1, 1, 3, 1)
(3, 4)	(1, 3, 1, 1)
(3, 2, 2)	(3, 1, 1, 1)
(2, 3, 2)	(3, 3)
(2, 2, 3)	(1, 1, 1, 1, 1, 1, 1)

- one systematic way to convert from one to other:
  1) take composition of 7 w/parts ≥ 2
  2) subtract 1 from last part so it's composition of 6 w/all but last part ≥ 2
  3) for each even part 2k, split into (1, 2k - 1) so it's composition of w/all odd parts

BINARY STRINGS

- binary string of length $n \geq 0$ is finite seq $\sigma = b_1 b_2 \dots b_n$ where each bit $b_i \in \{0,1\}$
  - ↳ correspond to elmts of set $\{0,1\}^* = \{(), (0), (1), (0,0), (0,1), \dots\}$
- #binary strings of length n is 2^n
  - ↳ $[x^n]\Phi_{\{0,1\}^*}(x) = [x^n]\frac{1}{1-\Phi_{\{0,1\}}(x)}$
    $= [x^n]\frac{1}{1-2x}$
    $= 2^n$
- concatenate binary strings: if $\sigma = a_1 a_2 \cdots a_m$ & $\tau = b_1 b_2 \cdots b_n$ are bin strs, then concatenation $\sigma\tau$ is bin str $a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n$
  - ↳ σ^k denotes k-fold concatenation of σ w/itself: $\sigma^k = \underbrace{\sigma\sigma\cdots\sigma}_{k}$
- σ is substring of $\tau = b_1 \cdots b_n$ if it's empty str or some str $b_i b_{i+1} \cdots b_j$ for $1 \leq i \leq j \leq n$
  - ↳ i.e. there exist bin strs γ_1, γ_2 st $\gamma_1 \sigma \gamma_2 = \tau$
- concatenation product: if S & T are sets of bin strs, then $ST = \{\sigma\tau : \sigma \in S, \tau \in T\}$
  - ↳ note that $S^k = \underbrace{SS\cdots S}_{k}$
  - ↳ e.g.
    $\{0,1\}\{\epsilon, 00, 11\} = \{0, 000, 011, 1, 100, 111\}$
  - ↳ e.g.
    $\{00, 01\}^2 = \{0000, 0001, 0100, 0101\}$
  - ↳ e.g.
    $\{0, 00\}^2 = \{00, \underset{\uparrow}{000}, 0000\}$
    can get this 2 ways but duplicates are removed in sets
  - ↳ $|ST| \leq |S||T|$

REGULAR EXPRESSIONS

- regular expression is defined recursively as any of the following:
  - ↳ ϵ, 0, or 1
  - ↳ $R \smile S$ where R & S are reg exprs
    - $\smile$ is "smile" & it means "or"
  - ↳ RS where R & S are reg exprs w/ R^k for any $k \in \mathbb{N}$
  - ↳ R^* where R is reg expr
- e.g.
  - ↳ 010
  - ↳ $010 \smile 01$
  - ↳ $(010 \smile 01)^*$
  - ↳ $(11)(010 \smile 01)^*(\epsilon \smile 0^*)$
  - ↳ $(00)^5 \smile ((11)(010 \smile 01)^*(\epsilon \smile 0^*))$
- reg expr R produces set $\mathcal{R}$ of bin strs
  - ↳ e.g.
    $R_1 = \epsilon \smile 0 \smile 1$ prod $\mathcal{R}_1 \{\epsilon, 0, 1\}$
    $R_2 = (01)(0 \smile 11)$ prod $\mathcal{R}_2 \{010, 0111\}$
    $R_3 = (00 \smile 11)^*$ prod $\mathcal{R}_3 \{\epsilon, 00, 11, 0000, 0011, 1100, 1111, \dots\}$
- define production recursively:
  - ↳ reg expr ϵ, 0, & 1 prod sets $\{\epsilon\}, \{0\}, \& \{1\}$ respectively
  - ↳ if R prod $\mathcal{R}$ & S prod $\mathcal{S}$, then:
    - $R \smile S$ prod $\mathcal{R} \cup \mathcal{S}$ (set union)
    - RS prod $\mathcal{RS}$ (concatenation product)
    - R^* prod $\mathcal{R}^* = \bigcup_{k \geq 0} \mathcal{R}^k$ (concatenation powers)
- e.g.
  - ↳ 0^* prod $\{\epsilon, 0, 00, \dots\}$
  - ↳ $(0 \smile 1)^*$ prod {all bin strs}

↳ $(11)0^*$ prod $\{11, 110, 1100, 11000, \ldots\}$
↳ $(00 \cup 000)^*$ prod $\{\varepsilon, 00, 000, 0000, 00000, \ldots\}$
↳ $(0 \cup 00)^*$ prod same as 0^*

- if $\mathcal{R}$ is set of strings that can be prod by reg expr R, then $\mathcal{R}$ is rational language
  ↳ e.g. $\{1, 110, 011, \varepsilon, 0\}$ is rational lang b/c it can be prod by $1 \cup 110 \cup 011 \cup \varepsilon \cup 0$
  ↳ e.g. $\{\underbrace{1010\ldots10}_{m}\underbrace{00\ldots0}_{n} : m, n \geq 1\}$ is rational lang b/c it can be prod $10(10)^*00^*$
  ↳ $\{\varepsilon, 01, 0011, 000111, \ldots, \underbrace{00\ldots0}_{n}\underbrace{11\ldots1}_{n}\}$ isn't rational lang
    - context-free lang

UNAMBIGUOUS EXPRESSIONS

- e.g. $(0 \cup 01)(0 \cup 10)$ prod $\{0, 01\}\{0, 10\} = \{00, 010, 010, 0110\} = \{00, 010, 0110\}$
- reg expr R is unambiguous if every string in lang $\mathcal{R}$ prod by R is prod in exactly 1 way by R
  ↳ otherwise, R is ambiguous
  ↳ e.g. 010 is unambiguous
  ↳ e.g. $(0 \cup 1)^*$ is unambiguous
    - prod every bin str exactly once
  ↳ e.g. $(\varepsilon \cup 0 \cup 1)^*$, $(0 \cup 1)^*(0 \cup 1)^*$, & $(0 \cup 1 \cup 01)^*$ are ambiguous
    - prod every bin str but some are prod more than once
- to prove a reg expr is ambiguous, give 1 example str that can be prod in 2 ways
- lemma 5 (unambiguous expr): reg exprs $\varepsilon, 0,$ & 1 are unambiguous exprs; if R & S are unambiguous exprs that prod $\mathcal{R}$ & $\mathcal{S}$ respectively, then:
  ↳ $R \cup S$ is unambiguous iff $\mathcal{R} \cap \mathcal{S} = \emptyset$
    - i.e. $\mathcal{R}$ & are disjoint
  ↳ RS & unambiguous iff there's bijection btwn $\mathcal{RS}$ & $\mathcal{R} \times \mathcal{S}$
    - i.e. for every str $\alpha \in \mathcal{RS}$, there's exactly 1 way to write $\alpha = \rho\sigma$ w/ $\rho \in \mathcal{R}$ & $\sigma \in \mathcal{S}$
  ↳ R^* is unambiguous iff each of concatenation products R^k is unambiguous & all R^k are disjoint
- e.g.
  $0^*(10^*)^*$ is unambiguous expr that prod set of all bin strs
  101011
  $0^*(10^*)(10^*)(10^*)(10^*)$
  $\varepsilon \quad 10 \quad 10 \quad 1\varepsilon \quad 1\varepsilon$
  Any str prod by $0^*(10^*)^*$ has form $\underbrace{0\ldots0}_{m_0}(1\underbrace{0\ldots0}_{m_1})(1\underbrace{0\ldots0}_{m_2})\ldots(1\underbrace{0\ldots0}_{m_k})$ for $k \geq 0$ & $m_0, m_1, \ldots, m_k \geq 0$
  ↳ every bin str $\sigma \in \{0,1\}^*$ can be written in this way. k is #1s & m_i is #0s in block immediately before $(i+1)^{th}$ 1
- e.g. $1^*(01^*)^*$ is also unambiguous expr that prod set of all bin strs
- e.g. $1^*(01111^*)^*$ is unambiguous expr that prod set of all bin strs where every 0 is followed by at least 1s

TRANSLATION INTO GENERATING SERIES

- e.g
  How many strs of length 10 are there where length of every block of 1s is even?
  Let $\mathcal{R}$ be set of such strs. Unambiguous expr prod $\mathcal{R}$ is $R = (0 \cup 11)^*$.
  Since R is unambiguous, there's bijection btwn $\{0, 11\}^*$ & $\mathcal{R}$, so:
  $$\#\text{strs in } \mathcal{R} \text{ of length } 10 = \#\text{strs in } \{0,11\}^* \text{ of length } 10$$
  $$= [x^{10}]\,\Phi_{\{0,11\}^*}(x) \qquad \text{w/ weight fcn = length}$$
  $$= [x^{10}]\,\frac{1}{1 - \Phi_{\{0,11\}}(x)} \qquad \text{string lemma}$$
  $$= [x^{10}]\,\frac{1}{1 - (x + x^2)}$$
  ↳ went from unambiguous expr $R = (0 \cup 11)^*$ to equiv set $\{0, 11\}^*$ so we could count strs prod by R using gen series $\Phi_{\{0,11\}^*}(x) = \frac{1}{1 - x - x^2}$

- reg expr R leads to rational fcn $\frac{1}{1-x-x^2}$
- a reg expr **leads to** a rational fcn, defined recursively as follows:
  - reg exprs ε, 0, & 1 lead to formal power series 1, x, & x
  - if R & S are reg exprs that lead to $f(x)$ & $g(x)$, then:
    - $R \cup S$ leads to $f(x) + g(x)$
    - RS leads to $f(x) \cdot g(x)$
    - R^* leads to $\frac{1}{1-f(x)}$
- e.g.
  - 1100 leads to $x^1 \cdot x^1 \cdot x^1 \cdot x^1 = x^4$
  - $11 \cup 000$ leads to $x^2 + x^3$
  - $0^*(11)(11 \cup 000)^*$ leads to $\frac{1}{1-x} x^2 \frac{1}{1-x^2-x^3}$
- **theorem 9**: let R be reg expr that unambiguously prods lang $\mathcal{R}$ & suppose R leads to $f(x)$, then gen series for $\mathcal{R}$ wrt length is $f(x)$
  - i.e. $\Phi_{\mathcal{R}}(x) = f(x)$

BLOCK DECOMPOSITIONS

- **block** of bin strs is nonempty maximal substr of equal bits
  - e.g. $\underbrace{000}_{1}\underbrace{11}_{2}\underbrace{0}_{3}\underbrace{1}_{4}\underbrace{0000}_{5}$ has 5 blocks
- **proposition 4 (block decomposition)**: reg expr $0^*(11^*00^*)^*1^*$ is unambiguous & prods set of all bin strs
  - same for $1^*(00^*11^*)^*0^*$
  - e.g. 1110011 can be prod from block decomposition: $\underbrace{0^*}_{\varepsilon}\underbrace{(11^*00^*)^*}_{11^2\,00^2}\underbrace{1^*}_{1^2}$ or $\underbrace{1^*}_{1^3}\underbrace{(00^*11^*)^*}_{00^2\,11^1}\underbrace{0^*}_{\varepsilon}$
  - unambiguous b/c the forced 1 & 0 inside parenthesis act as block delimiters
- e.g.
  Give reg expr for set of bin strs where every block of 1s has even length
  $R = (11)^*(00^*11(11)^*)^*0^*$ or $(0 \cup 11)^*$
- e.g.
  Give reg expr for set of all bin strs not containing 0000 as substr
  $R = 1^*((0 \cup 00 \cup 000)11^*)^*(\varepsilon \cup 0 \cup 00 \cup 000)$
  - leads to gen series:
    $\Phi(x) = \frac{1}{1-x} \frac{1}{1-(x+x^2+x^3)(x)(\frac{1}{1-x})}(1+x+x^2+x^3)$
- e.g.
  How many strs of length 15 are there where every block has length 1 or 2?
  $R = (\varepsilon \cup 1 \cup 11)((0 \cup 00)(1 \cup 11))^*(\varepsilon \cup 0 \cup 00)$
  - leads to gen series:

$$\Phi(x) = (1+x+x^2)\left(\frac{1}{1-(x+x^2)(x+x^2)}\right)(1+x+x^2)$$
$$= \frac{(1+x+x^2)^2}{1-(x+x^2)^2}$$
$$= \frac{(1+x+x^2)^2}{(1+x+x^2)(1-(x+x^2))}$$
$$= \frac{1+x+x^2}{1-x-x^2}$$

Thus, # strs of length 15 w/all blocks of length 1 or 2 is $[x^{15}]\frac{1+x+x^2}{1-x-x^2}$ (2x 15th Fibonacci #).

PREFIX DECOMPOSITIONS

- another unambiguous expr for set of all bin is $(0^*1)^*0^*$
  - e.g. 10100101
    $\underbrace{(0^*1)}_{1}\underbrace{(0^*1)}_{01}\underbrace{(0^*1)}_{0^21}\underbrace{(0^*1)}_{01}\underbrace{0^*}_{\varepsilon}$
  - chop str into pieces after each 1
- **prefix decomposition** has form A^*B
  - need to confirm it's unambiguous by checking:
    - A & B are unambiguous
    - there's at most 1 way for str to start w/initial segment prod by A

- if str doesn't start w/segment prod by A, then it's prod by B
- postfix decomposition has form AB^*
  - ↳ e.g. set of all bin strs is also prod by $0^*(10^*)^*$

RECURSIVE DECOMPOSITIONS

- recursive exprs can reference itself
  - ↳ e.g. $S = 0 \cup 1 \cup S(0 \cup 1)$ unambiguously gens all non-empty bin strs
  - ↳ e.g. $S = S \cup (0 \cup 1)^*$ ambiguously gens all bin strs
- e.g. set of all bin strs is $S = \varepsilon \cup S(0 \cup 1)$
  - ↳ generate 1110:
    $S \to S0 \to S10 \to S110 \to S1110 \to \varepsilon 1110$
  - ↳ unambigous expr b/c every non-empty bin str can be uniquely expressed as another bin str w/ 0 or 1 appended
  - ↳ if S leads to gen series $f(x)$, then:
    $$f(x) = x^0 + f(x)(x^1 + x^1)$$
    $$f(x) = 1 + f(x)(2x)$$
    $$1 = f(x)(1-2x)$$
    $$f(x) = \frac{1}{1-2x}$$
- recursive decomposition of set $\mathcal{S}$ describes $\mathcal{S}$ in terms of itself using lang of reg exprs tgt w/symbol S which prod set $\mathcal{S}$
  - ↳ unambiguous if each side of eqn prod each str at most once
    - e.g. $S = 1S1 \cup 0$ prod $\{0, 101, 11011, 1110111, \ldots\}$
      → unambiguous
    - e.g. $S = \varepsilon \cup 0 \cup 1S1$ prod $\{\varepsilon, 0, 11, 101, 1111, 11011, \ldots\}$
      → unambiguous
    - e.g. $S = 0 \cup 00 \cup 0S$ prod $\{0, 00, 000, \ldots\}$
      → ambiguous
- unambiguous recursive decomposition leads to gen series for corresponding set
  - ↳ e.g. $S = \varepsilon \cup 0 \cup 1S1$
    - $\Phi_S(x) = \Phi_{\{\varepsilon\}}(x) + \Phi_{\{0\}}(x) + \Phi_{\{1\}}(x)\,\Phi_S(x)\,\Phi_{\{1\}}(x)$
      $\Phi_S(x) = 1 + x + x^2\,\Phi_S(x)$
      $\Phi_S(x)(1-x^2) = 1+x$
      $\Phi_S(x) = \frac{1+x}{1-x^2}$
      $\Phi_S(x) = \frac{1}{1-x}$
- recursive decompositions allow us to prod sets that aren't rational langs (i.e. can't be prod by reg exprs)
  - ↳ e.g. $\{0^n 1^n, n \geq 0\}$ prod by $S = 0S1 \cup \varepsilon$

EXCLUDED SUBSTRINGS

- e.g. find gen series for set of strs w/o 10100 as substr

  Let A be set of strs that avoid 101000 & let B be set of strs that have exactly 1 occurrence of 10100 at very end.

  Other than ε, every str in $A \cup B$ ends w/ 0 or 1. If it ends w/ 1, then prefix (excluding last 1) must alr avoid 10100 so prefix is in A. If it ends w/ 0, either last 5 bits were 10100 or they weren't. If they were, then str was in B but prefix is still in A. If they weren't, str is in A & prefix also in A.

  $A \cup B = \varepsilon \cup A(0 \cup 1)$

  Gen series:

  $A(x) + B(x) = 1 + 2xA(x)$ (1)

  Every str in B can be uniquely obtained by appending 10100 to str in A & appending 10100 to any str in A always gives str in B.
  - ↳ for $b \in B$, then it has form $b = \sigma 10100$ where $\sigma \in A$
  - ↳ for $\sigma \in A$, $\sigma 10100 \in B$ b/c σ doesn't contain 10100 and for the str 10100, there's no possibility of overlap when we append it to the end of any str

  Write $B = A10100$ & gen series:

  $B(x) = x^5 A(x)$ (2)

Sub (2) into (1).

$$A(x) + x^5 A(x) = 1 + 2x A(x)$$

$$1 = A(x) + x^5 A(x) - 2x A(x)$$

$$1 = A(x)(1 + x^5 - 2x)$$

$$A(x) = \frac{1}{1 - 2x + x^5}$$

- e.g. find gen series for $\mathcal{S}$ = {strings w/o 101 as substr}

  Define 2 cases b/c of potential overlap 10101:

  ↳ $\mathcal{A}$ = {strs w/exactly 1 occurrence of 101 appearing in last 3 bits}

  ↳ $\mathcal{B}$ = {strs w/ exactly 2 occurrences of 101 appearing in last 5 bits}

  We derive:

  ↳ $\varepsilon \cup S(0 \cup 1) = S \cup A$

  - leads to $2x\,\Phi_S(x) = \Phi_S(x) + \Phi_A(x)$

  ↳ $S101 = A \cup B$

  - leads to $x^3 \Phi_S(x) = \Phi_A(x) + \Phi_B(x)$

  ↳ $A01 = B$

  - leads to $x^2 \Phi_A(x) = \Phi_B(x)$

  Solving for $\Phi_S(x)$ gets us $\Phi_S(x) = \dfrac{1 + x^2}{1 - 2x + x^2 - x^3}$

PARTIAL FRACTIONS AND RECURRENCE RELATIONS

- e.g. extracting coeffs when gen series has multi-term denom

$$[x^7]\frac{1+7x}{1-x-6x^2} = [x^7]\frac{1+7x}{1+2x-3x-6x^2}$$

$$= [x^7]\frac{1+7x}{1(1+2x)-3x(1+2x)}$$

$$= [x^7]\frac{1+7x}{(1+2x)(1-3x)}$$ factoring

$$= [x^7]\left(\frac{-1}{1+2x} + \frac{2}{1-3x}\right)$$ partial fractions

$$= [x^7]\left(-1\sum_{n\geq 0}(-2x)^n + 2\sum_{n\geq 0}(3x)^n\right)$$ geometric series

$$= -1(-2)^7 + 2(3)^7$$

- theorem 10 (partial fracs simple ver): let $G(x) = \frac{P(x)}{(1-\lambda_1 x)(1-\lambda_2 x)\ldots(1-\lambda_s x)}$ where P is polynomial of degree $< s$ & $\lambda_i \in \mathbb{C}$ are distinct; then there exist $C_1, C_2, \ldots, C_s \in \mathbb{C}$ st $G(x) = \frac{C_1}{1-\lambda_1 x} + \frac{C_2}{1-\lambda_2 x} + \ldots + \frac{C_s}{1-\lambda_s x}$
  - ↳ to find these C_i, cross-multiply & equate coeffs
  - ↳ e.g.

$$\frac{1+7x}{(1+2x)(1-3x)} = \frac{C_1}{1+2x} + \frac{C_2}{1-3x}$$

$$1+7x = C_1(1-3x) + C_2(1+2x)$$
$$1+7x = C_1 - C_1 3x + C_2 + C_2 2x$$
$$1+7x = (C_1+C_2) + x(-3C_1+2C_2)$$

1) $1 = C_1 + C_2$
2) $7 = -3C_1 + 2C_2$

Solve to get $C_1 = -1$, $C_2 = 2$

- theorem 11 (partial fractions full ver): let $G(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{(1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2}\ldots(1-\lambda_s x)^{d_s}}$ where $\deg(P) < \deg(Q)$, $\lambda_i \in \mathbb{C}$ are distinct, & $d_i \geq 1$; then there exist $C_1^{(1)}, C_1^{(2)}, \ldots, C_1^{(d_1)}, C_2^{(1)}, \ldots, C_2^{(d_2)}, \ldots, C_s^{(d_s)} \in \mathbb{C}$ st $G(x) = \sum_{i=1}^{s}\sum_{j=1}^{d_i}\frac{C_i^{(j)}}{(1-\lambda_i x)^j}$
  - ↳ e.g.

$$G(x) = \frac{2+5x}{1-3x^2-2x^3}$$

$$= \frac{2+5x}{(1+x)^2(1-2x)}$$ factoring

$$= \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-2x}$$ partial fractions

$$2+5x = A(1+x)(1-2x) + B(1-2x) + C(1+x)^2$$
$$2+5x = A(1+x-2x-2x^2) + B(1-2x) + C(1+2x+x^2)$$
$$2+5x+0x^2 = (A+B+C) + (-A-2B+2C)x + (-2A+0B+C)x^2$$

System of eqns by equating coeffs:

$2 = A+B+C$
$5 = -A-2B+2C$
$0 = -2A+0B+C$

Solving gives $A=1$, $B=-1$, $C=2$ so $\frac{2+5x}{1-3x^2-2x^3} = \frac{1}{1+x} - \frac{1}{(1+x)^2} + \frac{2}{1-2x}$

Use geometric series & -ve binomial theorem to get single sum:

$$\frac{2+5x}{1-3x^2-2x^3} = \sum_{n\geq 0}(-x)^n - \sum_{n\geq 0}\binom{n+2-1}{2-1}(-x)^n + 2\sum_{n\geq 0}(2x)^n$$

$$= \sum_{n\geq 0}((-1)^n - (n+1)(-1)^n + 2\cdot 2^n)x^n$$

$$= \sum_{n\geq 0}(2^{n+1} - n(-1)^n)x^n$$

- in summary, to analyze coeffs of gen series like $G(x) = \frac{2+5x}{1-3x^2-2x^3}$:
  1) factor denom: $G(x) = \frac{2+5x}{(1+x)^2(1-2x)}$
  2) write as sum of -ve powers of polynomials using partial fracs: $G(x) = \frac{A}{(1+x)} + \frac{B}{(1+x)^2} + \frac{C}{1-2x}$
  3) solve for A, B, C by clearing denom: $(A,B,C) = (1,-1,2)$
  4) use geometric series & -ve binomial theorem to write G(x) as single sum: $G(x) = \sum_{n\geq 0} (2^{n+1} - n(-1)^n)x^n$

RECURRENCE RELATIONS

- recall Fibonacci seq defined by $f_0 = 1$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$
  - ↳ to get closed form expr of recurrence rltn:

    Let $F(x) = \sum_{n\geq 0} f_n x^n$ be gen series for Fibonacci seq.

$$f_n = f_{n-1} + f_{n-2}$$
$$0 = f_n - f_{n-1} - f_{n-2}$$
$$= [x^n]F(x) - [x^{n-1}]F(x) - [x^{n-2}]F(x) \quad \text{since } f_i = [x^i]F(x)$$
$$= [x^n]F(x) - [x^n]xF(x) - [x^n]x^2F(x) \quad \text{rules of coeff extraction}$$
$$= [x^n](1-x-x^2)F(x)$$

So, for $n \geq 2$, $[x^n](1-x-x^2)F(x) = 0$ (i.e. all coeffs ≥ 2 are 0)

$$(1-x-x^2)F(x) = C_1 + C_2x + C_3x^2 + C_4x^3 + \dots \quad \text{formal power series}$$
$$(1-x-x^2)F(x) = C_1 + C_2x$$
$$F(x) = \frac{C_1 + C_2x}{1-x-x^2}$$
$$= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \qquad \alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$$
$$= \sum_{n\geq 0} (A\cdot\alpha^n + B\cdot\beta^n)x^n \quad \text{apply geometric series}$$

So, $f_n = [x^n]F(x) = A\cdot\alpha^n + B\beta^n$

Sub in for init conditions $f_0 = 1$, $f_1 = 1$ to get:

$$f_0 = 1 = A\alpha^0 + B\beta^0$$
$$1 = A + B$$
$$f_1 = 1 = A\alpha^1 + B\beta^1$$
$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

Solve 2 linear eqns to get $A = \frac{5+\sqrt{5}}{10}$ & $B = \frac{5-\sqrt{5}}{10}$

Finally, we can conclude $f_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n$ for all $n \geq 0$.

  - ↳ in summary:
    1) start from recurrence: $f_n - f_{n-1} - f_{n-2} = 0$ for $n \geq 2$
    2) write rational expr & factor denom: $F(x) = \frac{C_1 + C_2x}{1-x-x^2} = \frac{C_1 + C_2x}{(1-\alpha x)(1-\beta x)}$
    3) apply partial fracs to write expr as sum: $F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$
    4) apply geometric & -ve binomial theorem to obtain coeff formula: $f_n = [x^n]F(x) = A\alpha^n + B\beta^n$
    5) sub init conditions $f_0 = 1$, $f_1 = 1$ to solve for A & B

- theorem 12: let $c_1, \dots, c_k, \lambda_1, \dots, \lambda_s \in \mathbb{C}$ w/ λ_i distinct, st

$$\underbrace{1 + c_1x + c_2x^2 + \dots + c_kx^k}_{\text{characteristic polynomial}} = (1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2} \dots (1-\lambda_s x)^{d_s}$$

  - ↳ if $a_0, a_1, \dots$ is seq satisfying recurrence rltn, $\underbrace{a_n + c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} = 0}_{\text{homogeneous linear recurrence}}$ for all $n \geq k$, then

    $a_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \dots + p_s(n)\lambda_s^n$
    - each p_i is polynomial of deg $< d_i$
  - ↳ note that λ_i is simply coeff in front of each x

- e.g.

  Suppose a_n is seq given by $(a_0, a_1, a_2) = (0, -5, -1)$ & $a_n = 3a_{n-2} - 2a_{n-3}$ for $n \geq 3$. Give formula for a_n as fcn of n.

$$0 = a_n - 3a_{n-2} + 2a_{n-3}$$

  Characteristic polynomial of recurrence rltn:

$$1 - 3x^2 + 2x^3 = (1-x)^2(1+2x)$$

$\lambda_1 = 1$, $d_1 = 2$ $(\deg(p_1) < 2)$; $\lambda_2 = -2$, $d_2 = 1$ $(\deg(p_2) < 1)$

  By theorem, we can write: $a_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n$

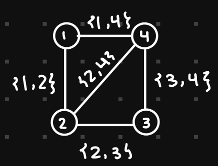
$$= (A + Bn)1^n + C(-2)^n$$

Subbing in init conditions a_0, a_1, a_2 & solving for A, B, C gives $(A, B, C) = (-2, -1, 1)$

Thus, $a_n = -2n - 1 + (-2)^n$

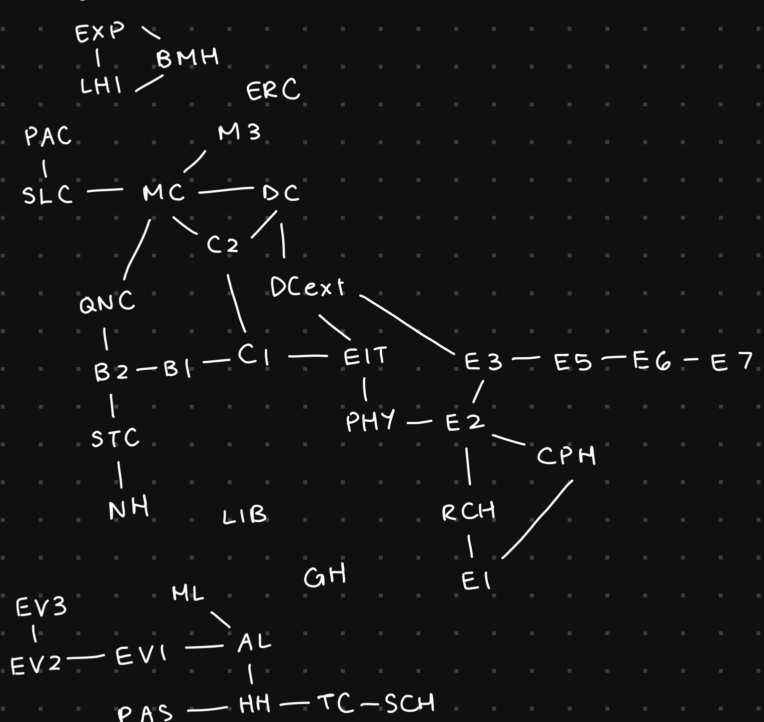
INTRO TO GRAPH THEORY

- graph G consists of finite non-empty set $V(G)$ of objs, called vertices, & set $E(G)$ of edges, which are unordered pairs of distinct vertices
  - ↳ e.g. $V(G) = \{1,2,3,4\}$, $E(G) = \{\{1,2\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

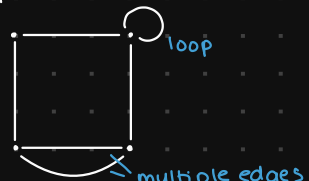


← not graph, only a rep

  - ↳ graph is rltnships, which is set of vertices & edges
- 2 vertices u & v are adjacent if $\{u,v\}$ is an edge
  - ↳ e.g. 1 & 2 are adjacent, but 1 & 3 are not
- if $e = \{u,v\}$ is edge, then e is incident w/ vertices u & v
  - ↳ e.g. $e = \{1,2\}$ is incident w/ vertices 1 & 2
- vertices adjacent to vertex v are neighbours of v
  - ↳ set of neighbours of v is denoted by $N_G(v)$
  - ↳ e.g. $N(2) = \{1,3,4\}$
- use $e = uv$ to rep edge $e = \{u,v\}$
  - ↳ edges in course are unordered / undirected
    - i.e. $e = uv = vu$
- e.g. graph of UW



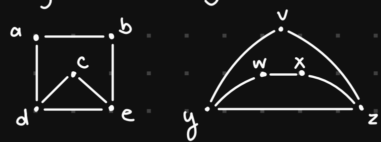
- we'll focus on simple graphs that have no loops / multiple edges b/c edges are a set (not multiset) of pairs of distinct vertices
  - ↳



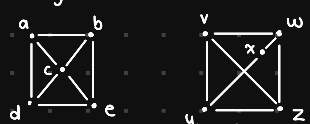
- graph is planar if it can be drawn w/o edges crossing

ISOMORPHISM

- e.g. are 2 graphs same?



  - ↳ not the same b/c there's diff labels on vertices & edges
  - ↳ if relabel $a \to w$, $b \to x$, $c \to v$, $d \to y$, $e \to z$ & rearrange, then we get same graph
- 2 graphs G_1 & G_2 are isomorphic if there exists a bijection $f: V(G_1) \to V(G_2)$ st u, v are adjacent in G_1 iff $f(u)$, $f(v)$ are adjacent in G_2
  - ↳ this bijection is called isomorphism
- to show 2 graphs are isomorphic, write down isomorphism & argue it preserves edge rltnships
  to show 2 graphs aren't isomorphic, find some property that should be preserved by isomorphism but isn't same in 2 graphs:
  1) deg seq
     - unordered
  2) 2^{nd} lvl deg seq
     - i.e. connections to other vertices & their # neighbours
  3) # vertices
  4) # edges
  - ↳ e.g.



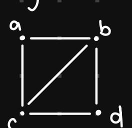
G_1 G_2

$G_1 = 3, 3, 4, 3, 3$
$G_2 = 3, 3, 2, 3, 3$

  - 2 graphs aren't isomorphic b/c G_1 has vertex c that's adjacent to 4 vertices, but in G_2, every vertex is adjacent to at most 3 vertices

DEGREE

- deg of vertex v in graph G is # edges incident w/ v & is denoted $\deg(v)$ / $\deg_G(v)$
  - ↳ deg is size of neighbourhood (i.e. $\deg(v) = |N(v)|$)
- e.g.

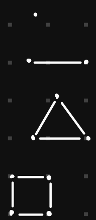


  - ↳ $\deg(a) = 2$
  - ↳ $\deg(b) = 3$
  - ↳ $\deg(c) = 3$
  - ↳ $\deg(d) = 2$
  - ↳ note $2+3+3+2 = 10$ & # edges = 5
- theorem 4.3.1 (handshaking lemma): for every graph G, $\sum_{v \in G} \deg(v) = 2|E(G)|$
  - ↳ proof: every edge = uv has 2 ends so when we sum deg of vertices, we're counting each edge twice, once at each end (one in $\deg(u)$ & one in $\deg(v)$)
  - ↳ corollary 1: # vertices of odd deg in any graph is even
    - proof:
      Let O be set of vertices w/ odd degree. Let E be set of vertices w/ even deg.
      $\sum_{v \in G} \deg(v) = \sum_{v \in O} \deg(v) + \sum_{v \in E} \deg(v)$

      By handshaking lemma, LHS is even. By defn of E, $\deg(v)$ is even for all $v \in E$ so $\sum_{v \in E} \deg(v)$ is even. Thus, $\sum_{v \in O} \deg(v)$ is even. By defn of O, $\deg(v)$ is odd for all $v \in O$. Since they sum to even #, there must be even # of them so $|O|$ is even.
  - ↳ corollary 2: avg deg of vertex in G is $\frac{2|E(G)|}{|V(G)|}$
    - measure of density of graph

- k-regular graph has all vertices w/deg k
  - ↳ complete graph has all vertices adjacent to every other vertex
    - i.e $|V-1|$-regular graph
    - complete graph on n vertices is denoted K_n
    - e.g.

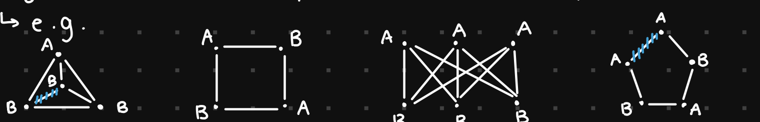
K_1:
K_2:
K_3:
K_4:



- in k-regular graph, there's $\frac{nk}{2}$ edges
  - ↳ in complete graph K_n, there's $\frac{n(n-1)}{2} = \binom{n}{2}$ edges

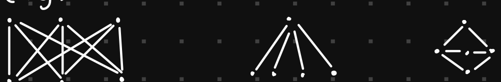
BIPARTITE GRAPHS

- graph is bipartite if its vertex set can be partitioned into 2 disjoint sets A & B st $V = A \cup B$ & every edge in G has 1 endpoint in A & 1 endpoint in B
  - ↳ e.g.



not bipartite　　bipartite ($K_{2,2}$)　　bipartite ($K_{3,3}$)　　not bipartite

  - ↳ graph is bipartite iff it contains no odd cycle (odd # edges)
    - cycle is seq of edges w/no repetition
- for +ve ints m & n, complete bipartite graph $K_{m,n}$ is graph w/ bipartition A, B where $|A| = m$ & $|B| = n$, containing all possible edges joining vertices in A w/vertices in B
  - ↳ e.g.



$K_{3,3}$　　$K_{1,4}$　　$K_{2,3}$

  - ↳ mn edges in $K_{m,n}$
- for $n \geq 0$, n-cube (aka hypercube) is graph whose vertex set consists of all bin strs of length n & 2 vertices are adjacent iff the 2 bin strs differ in exactly 1 pos
  - ↳ e.g.

1-cube　　2-cube　　3-cube

  - ↳ characteristics:
    - #vertices is 2^n (#bin strs of length n)
    - n-regular b/c for each str s, can get neighbour of s by changing one of n pos of s
    - #edges is $n2^{n-1}$
      - → handshaking lemma:

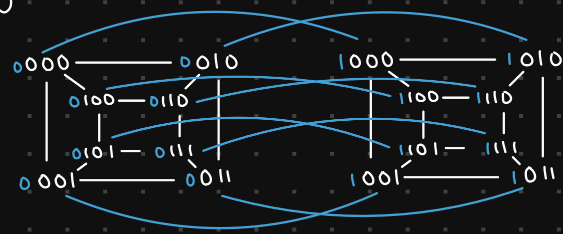
$$|E(G)| = \frac{1}{2} \sum_{v \in G} \deg(v)$$
$$= \frac{1}{2} 2^n \cdot n$$
$$= n 2^{n-1}$$

    - bipartite
      - → proof:

Let A be set of all bin strs of length n w/even # 1s. Let B be set of all bin strs of length n w/odd # 1s. (A, B) partitions vertex set of n-cube. Let st be edge in graph. By defn, t can be obtained from s by changing 1 pos. In either case, parity of # 1s changes so parity of #1s in s & t are diff. Hence, one end of st is in A & other end is in B so n-cube is bipartite.

- To construct n-cube recursively:
  1) make 2 copies of (n-1)-cube

2) prepend 0 in front of all strs in one copy & 1 in front of all strs in other copy
3) join corresponding copies of vertices w/ new edges
↳ e.g. 4-cube (aka tesseract)

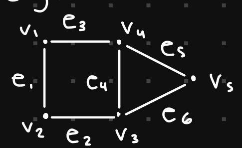


HOW TO SPECIFY GRAPH

- adjacency matrix of graph w/ vertices $\{v_1, \dots, v_n\}$ is $n \times n$ matrix A where

  $A_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ \& } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$

  ↳ e.g.



$$A = \begin{array}{cc} \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} & \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} \\ \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} & \end{array}$$

- incidence matrix of graph w/ vertices $\{v_1, \dots, v_n\}$ & edges $\{e_1, \dots, e_m\}$ is $n \times m$ B where

  $B_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is incident w/ } e_j \\ 0, & \text{otherwise} \end{cases}$

  ↳ e.g.

$$B = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} & \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} \\ \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} & \end{array}$$

  - each col contains exactly two 1s
  - each row sums to deg of that vertex

- adjacency list is table listing each vertex & vertices adjacent to it

  ↳ e.g.

vertex	adjacent vertices
1	2, 4
2	1, 3
3	2, 4, 5
4	1, 3, 5
5	3, 4

- diff specs are more useful / efficient in diff contexts

  ↳ e.g. in sparse graph (many vertices, but very few edges per vertex), adjacency list might be more efficient than incidence / adjacency matrix

SUBGRAPHS, PATHS, AND CYCLES

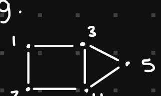
- graph H is subgraph of graph G if vertex set of H is non-empty subset of vertex set of G (i.e. $V(H) \subseteq V(G)$) & edge set of H is subset of edge set of G (i.e. $E(H) \subseteq E(G)$) where both endpoints of any edge in $E(H)$ are in $V(H)$

  ↳ if $V(H) = V(G)$, then H is spanning subgraph

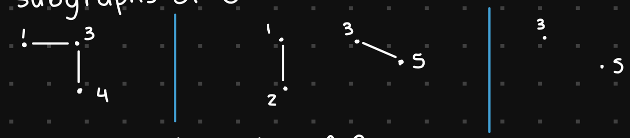
  ↳ if $H \neq G$, then H is proper subgraph

  ↳ e.g.

G



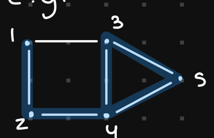
- subgraphs of G



- spanning subgraphs of G

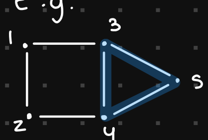
- $H-e$ is subgraph of H w/ vertex set $V(H)$ & edge set $E(H) \setminus \{e\}$
- $H+e$ is graph w/ vertex set $V(H)$ & edge set $E(H) \cup \{e\}$

- a **u,v-walk** is seq of alternating vertices & edges $v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots, v_{k-1}, v_{k-1}v_k, v_k$ where $u = v_0$ & $v = v_k$
  - ↳ walk has **length k**
  - ↳ walk is **closed** if $v_0 = v_k$
  - ↳ e.g.

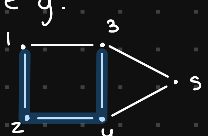


    1,4 walk: 1, 12, 2, 24, 4, 34, 3, 35, 5, 45, 4
    - length 5

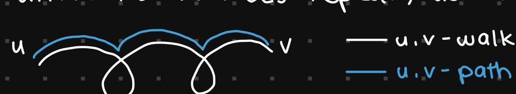
  - ↳ e.g.



    closed 3,3-walk: 3, 34, 4, 45, 5, 35, 3
    - length 3
  - ↳ allowed to repeat vertices & edges in walk
  - ↳ for simple graphs, suffices to list seq of vertices
    - edges are implied
- **u,v-path** is u,v-walk w/ no repeated vertices & edges
  - ↳ e.g.



    1,3-path: 1, 2, 4, 3
    - length 3
- can have **trivial/empty** path/walk of length 0
- **Theorem 4.6.2**: if there's u,v-walk in G, then there's u,v-path in G
  - ↳ idea: find vertices that are repeated in walk & remove parts btwn repetition, repeat until no vertices repeat, use induction on # repeated vertices

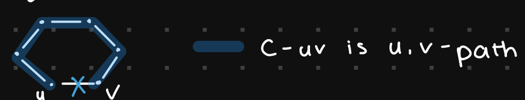


  - ↳ **proof by contradiction**:
    Suppose $v_0, \ldots, v_t$ is u,v-walk in G w/ minimal length among all u,v-walks. If walk is not a path, then there exist some repeated vertex $v_i = v_j$ for $i < j$. Then $v_0, \ldots v_i, v_{j+1}, \ldots, v_t$ is shorter u,v-walk, which is contradiction. So, all vertices in og minimal length walk must be distinct, so it's u,v-path.
  - ↳ **proof by strong induction on k** on stmt "if there's u,v-walk w/ k repeated vertices, then there's u,v-path":

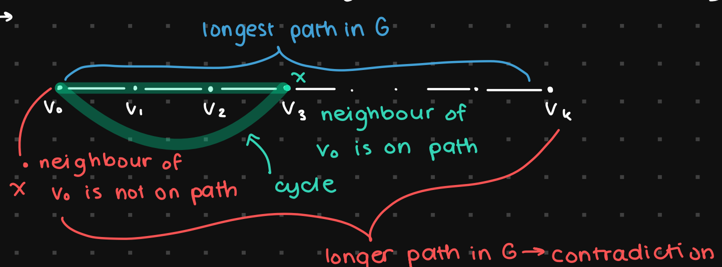
Base case: when $k=0$, u,v-walk w/ $k=0$ repeated vertices is u,v-path
Inductive hypothesis: if there's u,v-walk w/ $<k$ repeated vertices, then there's u,v-path
Inductive step: Suppose there's u,v-walk w/ $v_0, v_1, \ldots, v_t$ w/ k repeated vertices. Let w be repeated vertex. Let i & j be smallest & largest indices where $w = v_i = v_j$. Then $v_0, v_1, \ldots, v_i, v_{j+1}, \ldots, v_t$ is also u,v-walk. Since w is no longer repeated, this u,v-walk has $<k$ repeated vertices & by induction, there's u,v-path.

- corollary 4.6.3: if there's u,v-path & v,w-path in G, then there's u,w-path in G
  - ↳ proof: join u,v-path & v,w-path to make u,w-walk, then apply thm 4.6.2 to get u,w-path
    - path doesn't necessarily go thru v
- cycle of length n is subgraph w/ n distinct vertices $v_0, \ldots, v_{n-1}$ & n distinct edges $v_0v_1, v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0$
  - ↳ e.g.

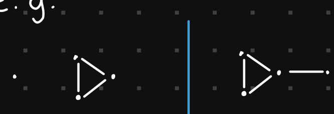
  - ↳ in simple graph, min length of cycle is 3
  - ↳ can rep cycle as closed walk, starting at any vertex
  - ↳ if uv is edge of cycle C, then $C-uv$ is u,v-path
    - e.g.



  - ↳ if P is u,v-path, then $P+uv$ is cycle
  - ↳ every cycle is 2-regular graph (i.e. every vertex has deg 2)
- theorem 4.6.4: if every vertex in G has deg at least 2, then G contains cycle
  - ↳



    - v_0 must have deg ≥ 2
  - ↳ proof:
    Let $v_0, v_1, \ldots, v_k$ be longest path in G (it exists b/c G is finite & paths can't repeat vertices). Since v_0 has deg ≥ 2, it has 2 neighbours v_1 & x.
    - case 1: x is not on path, then $x, v_0, \ldots, v_k$ is longer path in G which is contradiction
    - case 2: $x = v_i$ for $i \geq 2$ (i.e. x is on path), then $v_0, v_1, \ldots, v_i, v_0$ is cycle
  - ↳ converse isn't true: if graph has cycle, it's not necessarily the case that every vertex has deg ≥ 2
    - e.g.

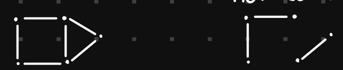


- girth of graph is length of its shortest cycle
  - ↳ if G has no cycle, then girth is ∞
- Hamiltonian cycle is a cycle that is a spanning subgraph (i.e. visits all vertices)
  - ↳ e.g.

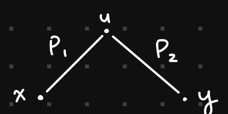
CONNECTEDNESS

- graph is connected if there exists a u,v-path for every pair of vertices u,v
  - ↳ e.g.

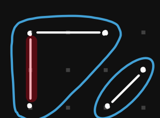
connected / not connected



  - ↳ to check if graph is connected, could check for existence of all $\binom{|V(G)|}{2}$ paths, but can simplify to check $|V(G)|-1$ paths
- theorem 4.8.2: let G be graph & let $u \in V(G)$; a u,v-path exists for each $v \in V(G)$ iff G is connected
  - ↳ proof:
    - $\Leftarrow$: By defn
    - $\Rightarrow$: To show G is connected, need to show there's an x,y-path for any $x,y \in V(G)$. By assumption, there exists a u,x-path P_1 & u,y-path P_2. Reverse P_1 to get x,u-path P_1'. Then P_1', P_2 is x,y-walk so by thm 4.6.2, there's an x,y-path.

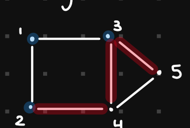


  - ↳ e.g. Show n-cube is connected for every $n \geq 0$
    Use thm 4.8.2. Let $u = 00\ldots0$, & let $x \in V(G)$. Suppose x has k 1s in positions $i_1, \ldots, i_k$. Let v_j be $\{0,1\}$ str w/ 1s in positions $i_1, \ldots, i_j$ & 0s everywhere else. Then, $uv_1v_2\ldots v_k$ is u,x-path. By thm 4.8.2, n-cube is connected
- component C of graph G is a connected subgraph of G st C is not proper subgraph of any other connected subgraph of G
  - ↳ i.e. C is maximally connected subgraph of G

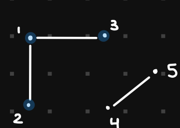


— components
— not a component

  - ↳ connected graph has exactly 1 component
  - ↳ graph w/ exactly 1 component is connected
  - ↳ disconnected graph has at least 2 components
  - ↳ there are no edges joining vertex of component w/ vertex outside that component
    - otherwise, it's not maximally connected
- let $X \subseteq V(G)$; cut induced by X in G is set of all edges in G w/ exactly 1 end in X
  - ↳ e.g.



$X = \{1,2,3\}$
cut induced by $X = \{24, 34, 35\}$

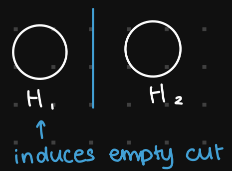


$Y = \{1,2,3\}$
cut induced by $Y = \emptyset$

  - ↳ even though cut is generated by set of vertices, cut itself is set of edges
  - ↳ looking at 2nd example w/ Y:
    - if graph is disconnected, cut induced by vertices of a component is empty
    - converse is not necessarily true: if graph has empty cut induced by some set of vertices X, then it doesn't have to be disconnected
      → $X = \emptyset$ induces empty cut
    - if graph has empty cut induced by non-empty set of vertices X, then graph is still not necessarily disconnected
      → $X = V(G)$ induces empty cut
- thm 4.8.5: graph G is disconnected iff there exists a non-empty proper subset X of $V(G)$ st cut induced by X is empty

↳ proof:

$\Rightarrow$: Let G be disconnected & let H_1 be component of G. Since H_1 is connected & G is not, there exists some vertex not in H_1. So, $V(H_1)$ is non-empty proper subset of $V(G)$. Since H_1 is component, if y is adjacent to any vertex in H_1, y must be in H_1. Hence, cut induced by H_1 is empty.



$\Leftarrow$: Use contrapositive that we suppose G is connected & let X be proper non-empty subset of $V(G)$. Choose $u \in X$ & $v \in V(G) \notin X$. Since G is connected, there exists a u,v-path $u = x_0, x_1, \ldots, x_n = v$. Let k be largest index st $x_k \in X$. Since $x_n = v \notin X$, it must be that $k < n$ & $x_{k+1} \notin X$. Thus $x_k x_{k+1}$ is in cut induced by X, so cut is non-empty.

EULERIAN CIRCUITS

- **Eulerian circuit** (aka Euler tour) of graph G is closed walk that contains every edge of G exactly once
- if G has Eulerian circuit, doesn't necessarily have to be connected b/c may have **isolated vertex**
  - ↳ for convenience, we'll focus on **connected graphs**
- **thm 4.9.2**: let G be connected graph; G has Eulerian circuit iff every vertex has even deg
  - ↳ proof:

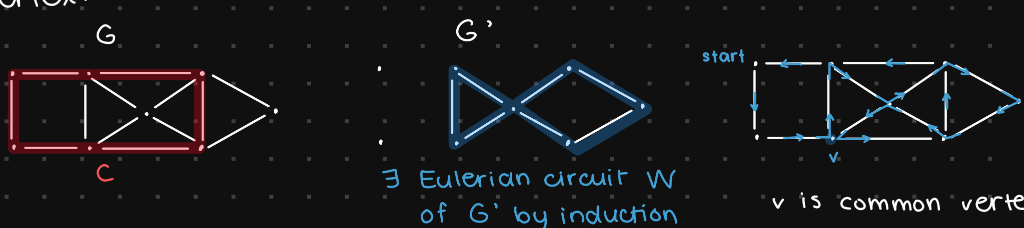
$\Rightarrow$: Closed walk contributes 2 to degree of a vertex for each visit

$\Leftarrow$: We use proof by strong induction on # edges m in G.

Base case: G has 0 edges so G is isolated vertex which is trivial Eulerian circuit.

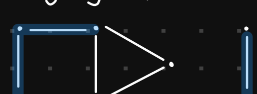
Inductive hypothesis: Assume any connected graph w/ all even degs & $< m$ edges has Eulerian circuit.

Inductive step: Let G be connected graph w/ $m \geq 1$ edges & all even degs. Each vertex has deg ≥ 2 so by thm 4.6.4, there exists a cycle C. If C contains all edges of G, then we're done. Otherwise, remove edges of C from G to get G'. Since G & C both have even deg vertices, so does G'. Each component of G' has fewer edges than G. By induction, each component of G' has Eulerian circuit. Since G is connected, each component of G' has at least 1 vertex in component w/ C. Obtain Eulerian circuit of G by attaching Eulerian circuit of each component to C at their common vertex.



BRIDGES

- an edge e of G is **bridge** (aka cut-edge) if G − e has more components than G
  - ↳ e.g. graph G w/ 2 components & 3 bridges



- **lemma 4.10.2**: if $e = xy$ is bridge in connected graph G, then G − e has exactly 2 components & x & y are in diff components of G − e
  - ↳ proof:

If e is bridge of G, then G − e has at least 2 components. Let H be component of G − e containing x. Let z be any vertex of G − e not in H. Since G is connected, there's path P in G from x to z. However, there's no path from x to z in G − e, so P must have included $e = xy$.

So, P is of form $P = xyv_1v_2 \dots z$. So there's a path from y to any $z \notin H$, so there can be only one component besides H in $G - e$. Thus, $G - e$ has exactly 2 components & x & y are in diff components of $G - e$.

thm 4.10.3 : an edge e is bridge of G iff it's not contained in any cycle of G

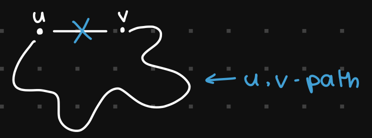
↳ edge in cycle $\Leftrightarrow$ edge is not a bridge

↳ edge not in cycle $\Leftrightarrow$ edge is a bridge

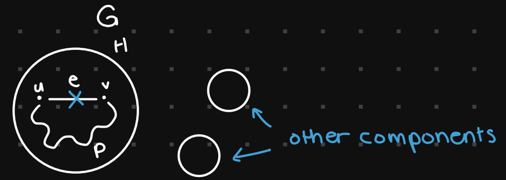
↳ proof:

Contrapositive is e is in cycle iff e is not a bridge.

$\Rightarrow$: Suppose $e = uv$ is in a cycle $C = uvv_1v_2 \dots v_k u$. Then $C - uv$ contains a v,u-path $v, v_1, \dots, v_k, u$ which is also in $G - uv$. So, u & v are in same component of $G - e$. By lemma 4.10.2, e is not a bridge.

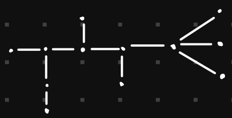


$\Leftarrow$: Suppose $e = uv$ is not a bridge. Suppose H is component of G containing e. Then, $H - e$ is connected so there exists a u,v-path in P in $H - e$. Then $P + uv$ is a cycle in H & also in G.

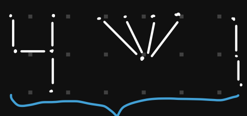


TREES

- **tree** is connected graph w/ no cycles
  - ↳ e.g.

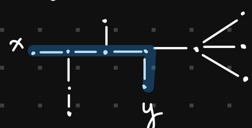


- **forest** is graph w/ no cycles
  - ↳ e.g.

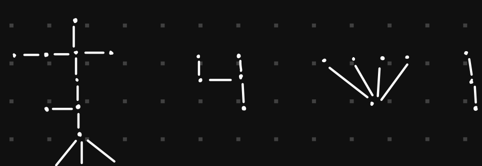


1 forest w/ 3 components

- **lemma 5.1.4**: every edge in tree / forest is bridge
  - ↳ proof: Since tree / forest has no cycles, every edge is not in a cycle. By thm 4.10.3, every edge must be a bridge.
- **lemma 5.1.3**: let x, y be vertices in a tree T, then there exists a unique x,y-path in T
  - ↳ e.g.



  - ↳ proof:
    Suppose there are 2 x,y-paths in T. Then, there exists some edge uv in one x,y-path but not the other. Remove edge uv. We can still find a walk u to v, so u & v are in same component of $T - uv$. Thus, uv is not a bridge, then it's in a cycle. Trees don't have cycles, so we have a contradiction.
- tree is **minimally-connected graph**
  - ↳ it's connected, but removing any edge disconnects it
  - ↳ e.g. How many edges in a tree?



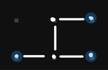
$\lvert V(T)\rvert$:	12	5	5	3
$\lvert E(T)\rvert$:	11	4	4	2

- **thm 5.1.5**: if T is a tree, then $\lvert E(T)\rvert = \lvert V(T)\rvert - 1$
  - ↳ prove by strong induction on # edges; remove 1 edge, apply induction on 2 trees, then add it up
  - ↳ proof:
    Use strong induction on # edges m.
    Base case: When $m = 0$, there are no edges & only 1 vertex so result holds.
    Inductive hypothesis: Assume result holds for any tree w/ $< m$ edges.
    Inductive step: Suppose T is tree w/ $m \geq 1$ edges. Let $e \in E(T)$. By lemma 5.1.4, e is bridge. By lemma 4.10.2, $T - e$ has exactly 2 components, T_1 & T_2, each of which are trees w/ $< m$ edges.

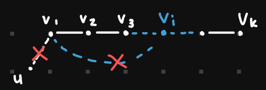
$$
\begin{aligned}
\lvert E(T)\rvert &= \lvert E(T_1)\rvert + \lvert E(T_2)\rvert + 1 \\
&= \lvert V(T_1)\rvert - 1 + \lvert V(T_2)\rvert - 1 + 1 \quad \text{by induction} \\
&= \lvert V(T_1)\rvert + \lvert V(T_2)\rvert - 1 \\
&= \lvert V(T)\rvert - 1
\end{aligned}
$$

2 smaller trees

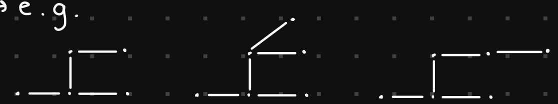
- in a forest w/ k components, $|E(F)| = |V(F)| - k$
- leaf in a tree is vertex of deg 1
  - ↳ e.g.



- thm 5.1.8: if T is tree w/ at least 2 vertices, then it has at least 2 leaves



  - ↳ proof:
    Let $P = v_1 v_2 \dots v_k$ be longest path in T. Since T contains at least 1 edge, P has at least 2 vertices. Consider v_1. It has neighbour v_2 on P. It can't have any other neighbour b/c if it had neighbour u not on P, then we'd contradict P being longest path. If v_1 has another neighbour $v_i \neq v_2$ on P, then we'd have cycle. So v_1 has deg 1 & is a leaf. Same for v_k. Thus, T has at least 2 leaves.
- how many leaves in a tree?
  - ↳ e.g.



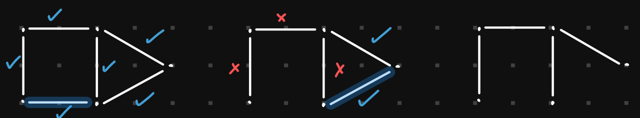
3 leaves   4 leaves   3 leaves

  - ↳ proof:
    Let T be tree & n_i be # vertices in T of deg i for $i \geq 1$.
    $|V(T)| = n_1 + n_2 + n_3 + n_4 + \dots$
    By thm 5.1.5:
    $|E(T)| = |V(T)| - 1$
    $= -1 + n_1 + n_2 + n_3 + \dots$
    By handshaking lemma:
    $2|E(T)| = n_1 + 2n_2 + 3n_3 + \dots$
    Solve for n_1:
    $2|E(T)| = -2 + 2n_1 + 2n_2 + \dots$
    $2|E(T)| = n_1 + 2n_2 + 3n_3 + \dots$
    $0 = -2 + n_1 - n_3 - 2n_4 - 3n_5 - \dots$
    $n_1 = 2 + n_3 + 2n_4 + 3n_5 + \dots$
    $n_1 = 2 + \sum_{i \geq 3} (i+2) n_i$
  - ↳ every tree has at least 2 leaves
    - $n_1 = 2 +$ (some more stuff)
  - ↳ vertices of deg 2 have no effect on # leaves
  - ↳ higher deg vertices lead to more leaves
- lemma: every tree is bipartite

SPANNING TREES

- spanning tree of G is spanning subgraph of G that's a tree
  - ↳ i.e. $V(T) = V(G)$, $E(T) \subseteq E(G)$, & T is tree
  - ↳ e.g.

- thm 5.2.1: graph G is connected iff G has spanning tree
  - ↳ proof:
    ⇐: Let T be spanning tree of G. Since T is connected, for every $x, y \in V(T)$, there exists an x,y-path in T. Since T is spanning subgraph of G, $V(T) = V(G)$ & $E(T) \subseteq E(G)$. This x,y-path in T is also x,y-path in G. Thus, G is connected.
    ⇒: Idea is to remove edges until we get spanning tree. Can remove edges that aren't bridges (i.e. edges in cycles). Do induction on # cycles



✓ ok to remove edge
✗ not ok to remove edge

Prove by strong induction on #cycles k: if G is connected w/ k cycles, then G has spanning tree.

- base case: $k = 0$ so G is connected & has no cycles, so G is alr spanning tree.
- inductive hypothesis: Assume any connected graph w/ $< k$ cycles has spanning tree.
- inductive step: Let G be connected graph w/ $k \geq 1$ cycles. Let e be edge in cycle C of G. By thm 4.10.3, e is not a bridge, so $G - e$ is connected. C is no longer cycle in $G - e$, so $G - e$ has $< k$ cycles. By induction, $G - e$ has spanning tree T. Since $V(T) = V(G - e)$ & $E(T) \subseteq E(G-e) \subseteq E(G)$, T is spanning tree of G.

corollary 5.2.2: if G is connected graph w/ n vertices & $n-1$ edges, then G is tree

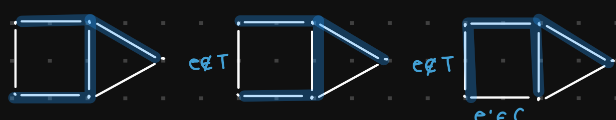
↳ proof: Since G is connected, it has spanning tree T that visits all n vertices. A tree on n vertices has $n-1$ edges. So, $E(T) = E(G)$ & thus, G is tree.

corollary 2: let G be graph w/ n vertices; if any 2/3 conditions hold, then G is tree:

1) G is connected
2) G has no cycles
3) G has $n-1$ edges

thm 5.2.3: if T is spanning tree of G & $e \notin E(T)$, then $T + e$ contains exactly 1 cycle C; if $e' \in E(C)$, then $T + e - e'$ is spanning tree of G

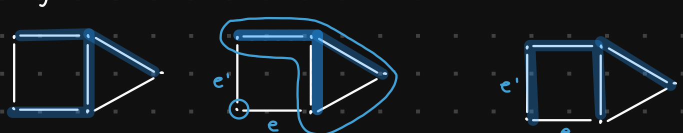
↳ e.g.



spanning tree T — T + e contains a cycle C — T + e − e' is spanning tree

thm 5.2.4: if T is spanning tree of G & $e \in E(T)$, then $T - e$ has 2 components; if e' is an edge in cut induced by vertices of one component, then $T - e + e'$ is spanning tree of G

↳ e.g.



spanning tree T — — components of T − e; e': in cut induced by one of components of T − e — T − e + e' is spanning tree

CHARACTERIZING BIPARTITE GRAPHS

thm 5.3.2: graph G is bipartite iff it has no odd cycles

↳ proof:

$\Rightarrow$: if G is bipartite, G has no odd cycle

Suppose G is bipartite w/ bipartition (A, B). Let $C = v_1 v_2 \dots v_k v_1$ be cycle of length k. WLOG, assume $v_1 \in A$. Since $v_1 v_2$ is an edge, $v_2 \in B$, $v_3 \in A$, etc. So, if $v_i \in A$ iff i is odd. Since $v_k v_1$ is edge & v_1 is in A, $v_k \in B$ so k is even. Thus, every cycle has even length.

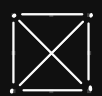
$\Leftarrow$: if G has no odd cycles, then G is bipartite

Proof by contrapositive: suppose G is not bipartite, show that G has an odd cycle. Since G is not bipartite, it has a non-bipartite component H. Since H is connected, there's a spanning T of H (thm 5.2.1). Trees are bipartite, so let (A, B) be bipartition of T. Since H is not bipartite, so (A, B) is not bipartition of H. Thus, there's an edge uv of H st both $u, v \in A$ or both $u, v \in B$. WLOG, assume $u, v \in A$. Since T is connected, there's a u,v-path $P = u x_1 x_2 \dots x_k v$ in T. Since T is bipartite, we have $u \in A$, $x_1 \in B$, ..., $x_{2i} \in A$, $x_{2i+1} \in B$, ..., $v \in A$. So, x_k must be in B & thus, k is odd. So, P has length $k+1$ (even) & $P + uv = u x_1 x_2 \dots x_k v u$ is of length $k+2$, which is odd.

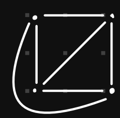
PLANAR GRAPHS

PLANARITY

- **planar embedding** of graph G is drawing of graph in plane so that its edges intersect only at their ends (i.e. edges don't cross) & no 2 vertices occupy same point
  - ↳ graph that has planar embedding is called **planar**
  - ↳ e.g.



not planar embedding of K_4

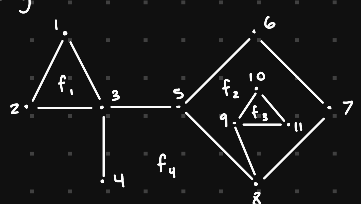


planar embedding of K_4

  - ↳ e.g.

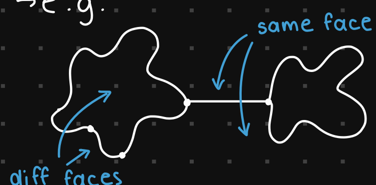
not planar embeddings of K_5

    - no planar embedding of K_5 exists
- graph is planar iff each of its components is planar
- **face** of planar embedding is connected region of plane
  - ↳ e.g.



Face	Boundary walk	Deg
f_1	1, 2, 3, 1	3
f_2	5, 6, 7, 8, 9, 11, 10, 9, 8, 5	9
f_3	9, 10, 11, 9	3
f_4	1, 2, 3, 4, 3, 5, 8, 7, 6, 5, 3, 1	11

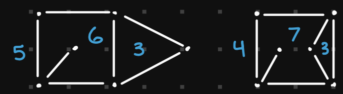
- **boundary** of face is subgraph formed from all vertices & edges that touch the face
  - ↳ 2 faces are **adjacent** if their boundaries have at least 1 edge in common
- for planar embedding of connected graph, **boundary walk** of a face is closed walk once around perimeter of face boundary
  - ↳ **deg** of face f is length of boundary walk of face
    - denoted **deg(f)**
- if graph is not connected, then boundary may be disconnected too
  - ↳ sum lengths of boundary walks on each component of boundary to get deg of face
- **thm 7.12 (faceshaking lemma)**: let G be connected planar graph w/ planar embedding, where F is set of all faces, then $\sum_{f \in F} \deg(f) = 2|E(G)|$
  - ↳ proof:
    Each edge has 2 sides & each side contributes to 1 to a boundary walk so edge contributes 2 to sum of degree.
- **lemma 1**: in planar embedding, an edge e is bridge iff 2 sides of e are in same face
  - ↳ e.g.



- **Jordan curve thm**: every planar embedding of a cycle separates plane into 2 parts, one on inside & one on outside
  - ↳ true for plane & surface of sphere, but not necessarily true for other surfaces (e.g. torus/donut)

EULER'S FORMULA

- e.g. 2 planar embeddings of same graph



  ↳ same # faces, but diff face degs
- thm 7.2.1 (Euler's formula): let G be connected graph w/ n vertices & m edges; on planar embedding of G w/ f faces, then $n - m + f = 2$
  ↳ as such, # faces f in planar embedding (if it exists) of a graph w/ n vertices & m edges is $f = 2 - n + m$
  ↳ idea:

Fix # vertices & induct on # edges. Remove non-bridge edge to keep it connected. Removing non-bridge edge causes 2 faces to merge, so faces & edges both dec by 1. Base case is tree w/ 1 face, which satisfies Euler's formula.

↳ proof: fix # vertices n & use induction on # edges m

Base case: The graph is connected so it has a spanning tree (thm 5.2.1) which has $n-1$ edges (thm 5.1.15). Base case is $m = n-1$. A connected graph w/ $n-1$ edges is a tree. Any planar embedding of a tree has 1 face so $f = 1$.

$$n - m + f = n - (n-1) + 1$$
$$= 2$$

Inductive hypothesis: Assume Euler's formula holds for any connected planar graph w/ n vertices & $m-1$ edges.

Inductive step: Consider a connected planar graph G w/ n vertices, m edges, & f faces. Let e be non-bridge edge. Then $G - e$ is connected, planar, & has $m-1$ edges. By induction, $G-e$ has $2 - n + (m-1)$ faces. Since e is not bridge, by prev lemma, 2 sides of e are diff faces. When we remove e from embedding of G to get embedding of $G - e$, these 2 faces merge into 1. So, $G - e$ has $f - 1$ faces. So, # faces in G is $f = 2 - n + (m-1) + 1 = 2 - n + m$ as required.

↳ if G has c components, Euler's formula becomes $n - m + f = 1 + c$

NONPLANAR GRAPHS

- verifying that graph is planar is easy given planar embedding
- one way to prove nonplanarity is to show there's too many edges
- lemma 7.5.2: let G be planar embedding w/ n vertices & m edges; if there's planar embedding of G where every face has deg at least $d \geq 3$, then $m \leq \frac{d(n-2)}{d-2}$
  ↳ proof:

Case 1: G is connected.

Let F be set of all faces in planar embedding w/ $|F| = s$. By faceshaking lemma:

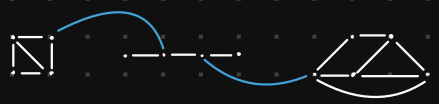
$$2m = \sum_{f \in F} \deg(f)$$
$$\geq \sum_{f \in F} d$$
$$\geq ds$$

By Euler's formula, $s = 2 - n + m$. We combine to get:

$$2m \geq ds$$
$$2m \geq d(2 - n + m)$$
$$m \leq \frac{d(n-2)}{d-2}$$

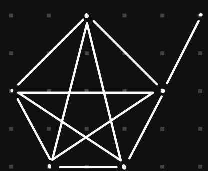
Case 2: G is not connected.

Make new connected planar graph by adding edges in outer face to join components. Face degs haven't dec so inequality holds for new graph. Since og graph had fewer edges, inequality holds in it too.

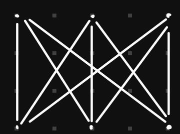


— edges added to components

- lemma 7.5.1: if G contains cycle, then in any planar embedding of G, every face boundary contains cycle
- thm 7.5.3: let G be planar graph w/ $n \geq 3$ vertices & m edges; $m \leq 3n-6$
  - ↳ proof:
    Suppose G doesn't have cycle. Then G is a forest so $m \leq n-1 \leq 3n-6$ when $n \geq 3$.
    Suppose G has cycle. Every face boundary has cycle by lemma 7.5.1. Each cycle has length ≥ 3, so every face has deg ≥ 3. By lemma 7.5.2., $m \leq \frac{3(n-2)}{3-2} = 3n-6$.
  - ↳ can prove graph is non-planar, but not that graph is planar b/c there's graphs that are nonplanar where $m \leq 3n-6$
    - e.g.



$n=6$
$m=11$
$m=11 \leq 3(6)-6=12$



$K_{3,3}$

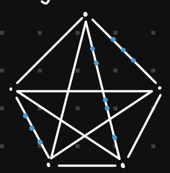
$n=6$
$m=9$
$m=9 \leq 3(6)-6=12$

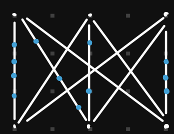
- corollary 7.5.4: K_5 is not planar
  - ↳ proof:
    $m = |E(K_5)| = \binom{5}{2} = 10$
    $n = |V(K_5)| = 5$
    $3n-6 = 3(5)-6$
    $= 9$
    $m = 10 > 9$ so by thm 7.5.3., K_5 is not planar.
- thm 7.5.6: let G be bipartite planar graph w/ $n \geq 3$ vertices & m edges; $m \leq 2n-4$
  - ↳ proof:
    Suppose G doesn't have cycle. Then $m \leq n-1 \leq 2n-4$ whenever $n \geq 3$.
    Suppose G has cycle. Then every face boundary has cycle by lemma 7.5.1. Since each cycle in bipartite graph has length ≥ 4, every face has deg ≥ 4. By lemma 7.5.2., $m \leq \frac{4(n-2)}{4-2} = 2n-4$.
- corollary 7.5.7: $K_{3,3}$ is not planar
  - ↳ proof:
    $m = |E(K_{3,3})| = 9$
    $n = |V(K_{3,3})| = 6$
    $2n-4 = 2(6)-4$
    $= 8$
    $m = 9 > 8$ so by thm 7.5.6, bipartite graph $K_{3,3}$ can't be planar.

PLANAR GRAPHS

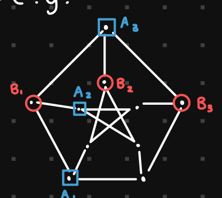
KURATOWSKI'S THEOREM

- any graph that looks like / contains $K_{3,3}$ / K_5 is nonplanar
- edge subdivision of G is obtained by replacing each edge of G w/new path of length at least 1
  - ↳ e.g.

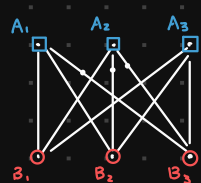




  - ↳ i.e. drawing edge on plane is equiv to drawing path on plane
- fact 1: graph is planar iff any edge subdiv of graph is planar
- corollary 1: if G contains edge subdiv of $K_{3,3}$ / K_5 as subgraph, then G is not planar
- thm 7.6.1 (Kuratowski's Thm): graph is planar iff it doesn't contain edge subdiv of $K_{3,3}$ or K_5 as subgraph
- show graph is planar by drawing planar embedding
- show graph is nonplanar by finding edge subdiv of $K_{3,3}$ / K_5 as subgraph
  - ↳ e.g.



Peterson graph

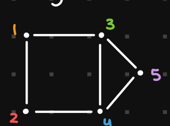


$K_{3,3}$

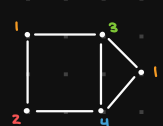
  - ↳ when finding edge subdiv to demonstrate nonplanarity, only 5/6 vertices of K_5 / $K_{3,3}$ can be repeated & all other vertices/edges can't be repeated

COLOURING

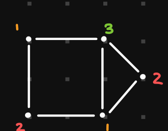
- k-colouring of graph G is asgnt of colour to each vertex using 1 of k colours st adjacent vertices have diff colours
  - ↳ i.e. if C is set of size k, then k-colouring is fcn $f: V(G) \to C$ st $f(u) \neq f(v)$ for all $uv \in E(G)$
  - ↳ graph that has this is called k-colourable
  - ↳ e.g.



5-colouring of G



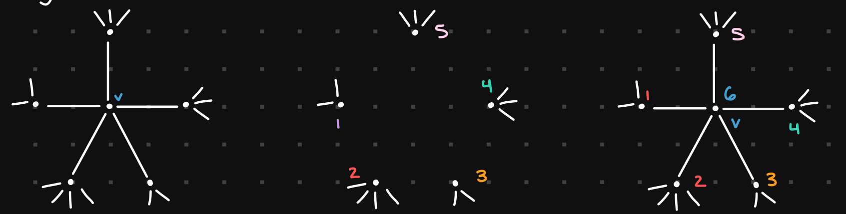
4-colouring of G



3-colouring of G

- k-colouring doesn't have to use all k colours so k-colourable graph is also (k+1)-colourable
- extreme cases:
  - ↳ graph w/ no edges can be 1-coloured
  - ↳ graph w/ at least 1 edge requires ≥ 2 colours
  - ↳ bipartite graphs need only 2 colours
- thm 7.7.2: graph is 2-colourable iff it's bipartite
- thm 7.7.3: complete graph K_n is n-colourable & not k-colourable for any $k < n$
- thm 7.7.4 (6-colour thm): every planar graph is 6-colourable
  - ↳ use corollary 7.5.5: every planar graph has at least 1 vertex of deg at most 5
    - ◦ proof:
      Let G be planar graph w/n vertices. If every vertex has deg ≥ 6, then by handshaking lemma, #edges $m \geq \frac{6n}{2} = 3n$. By thm 7.5.3, every planar graph has at most $3n-6$ edges, so this is contradiction. Thus, there exists a vertex of at most 5.
  - ↳ idea for proof is that there's vertex of deg at most 5 so we can colour it even if all of its ≤ 5

neighbours have diff colours ; remove that vertex , inductively 6-colour rest , & add it back on



G   G - v   at least 1 available colour for v

↳ proof:

Proof by induction on stmt: every planar graph w/n vertices is 6-colourable.

Base case : n = 1

Graph w/1 vertex can be coloured w/no more than 6 colours.

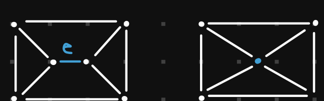
Inductive hypothesis: Assume any planar graph w/n-1 vertices is 6 colourable.

Inductive step:

Let G be planar graph w/n vertices. Let v be vertex of deg at most 5 (which exists by corollary 7.5.5). Let $G' = G - v$. G' is planar w/n-1 vertices so it's 6-colourable. Let $f' : V(G') \to \{1, \dots, 6\}$ be 6-colouring of G'. We obtain 6-colour $f : V(G) \to \{1, \dots, 6\}$ as follows. Keep same colours for all vertices except v (i.e. $f(x) = f'(x)$ for all $x \in V(G) \setminus \{v\}$). Let $U \subseteq \{1, \dots, 6\}$ be set of colours used by neighbours of v. $U = \{f(w) : vw \in E(G)\}$. Since v has deg ≤ 5, then $|U| \leq 5$. So, there's at least 1 colour c that's not used (i.e. $\{1, \dots, 6\} \setminus U$ is not empty). Set $f(v) = c$.

· let e be edge in graph G ; graph G/e is formed by contracting $e = uv$ removes e & squeezes 2 ends of e into 1 vertex, preserving all edges incident w/either end

↳ e.g.



G   G / e

↳ G/e may not be simple graph (i.e. may have multiple edges)

↳ if G is planar, then G/e is planar

· thm 7.7.6 (5-colour thm): every planar graph is 5-colourable

↳ proof:

Proof by strong induction on stmt: every planar graph w/n vertices is 5-colourable.

Base case : n = 1 / n = 2

Trivial

Inductive hypothesis: Assume any planar graph w/ < n vertices can be coloured w/ no more than 5 colours.

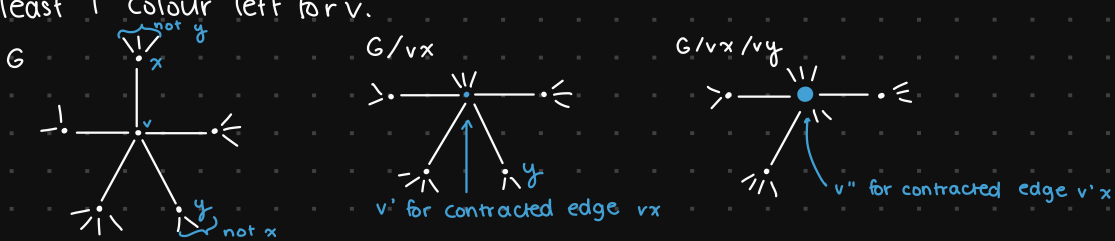
Inductive step:

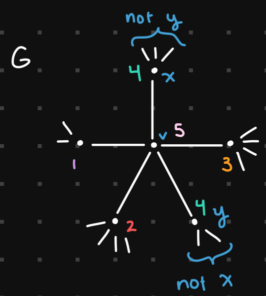
Let G be planar graph w/n vertices. Let v be vertex of deg at most 5 (which exists by corollary 7.5.5). If $\deg(v) \leq 4$, then use similar argument as in 6-colour thm to prove graph is 5-colourable.

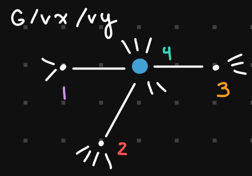
Suppose $\deg(v) = 5$. There are 2 neighbours x & y of v that aren't adjacent otherwise 5 neighbours of v form K_5 complete subgraph, which can't exist inside planar graph by Kuratowski's thm. Let G' be graph obtained from G by contracting vx & vy. Let v^* be contracted vertex. Then, G' is planar w/n-2 vertices. By induction, G' is 5-colourable w/colouring f'.

Create 5-colouring f of G as follows. For any $w \in V(G) \setminus \{v, x, y\}$, set $f(w) = f'(w)$. Set $f(x) = f(y) = f'(v^*)$. This is valid since x & y aren't adjacent in G. Since 2 neighbours of v have same colour, at most 4 colours have been used in 5 neighbours of v. Thus, we have at least 1 colour left for v.

G   G/vx   G/vx/vy



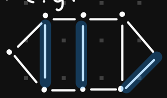


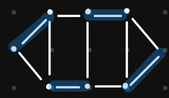


- 4-colour thm (thm 7.7.7): every planar graph is 4-colourable

MATCHINGS

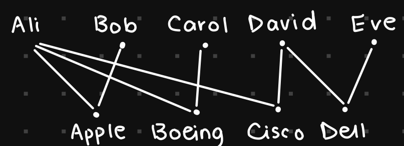
- matching of graph is set of edges in which no 2 edges share common vertex
  - ↳ e.g.





matching of size 3   matching of size 4

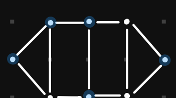
  - ↳ e.g. coop matching where there's set A of students & set B of jobs; there's edge btwn $a \in A$ & $b \in B$ if student a & job b both rank each other



A matching is ≤ 1 job per student & ≤ 1 student per job.
  - ↳ small matchings are easy to find (e.g. empty edge set)
  - ↳ harder q is that given a graph, what's largest size of matching?
- in matching M, vertex v is saturated by M if v is incident w/edge in M
- perfect matching is if every vertex is saturated
  - ↳ not every graph has perfect matching
  - ↳ if graph has odd # vertices, then it doesn't have perfect matching
  - ↳ it's a max matching, but it's not always the case that a max matching is a perfect matching

COVERS

- cover of graph G is set of vertices C st every edge of G has at least 1 endpoint in C
  - ↳ e.g.



cover of size 6

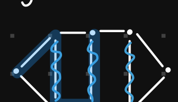
  - ↳ easier to find larger cover so main q is what's smallest size of cover?
- lemma 8.2.1: if M is a matching of G & C is cover of G, then $|M| \leq |C|$
  - ↳ proof:

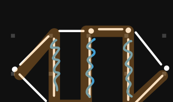
    For each edge $uv \in M$, at least 1 of u/v is in C since C is cover. No 2 edges in M saturate same vertex in C, since M is matching. Every edge in M leads to distinct vertex in C so $|M| \leq |C|$.
- lemma 8.2.2: if M is a matching, C is a cover, & $|M| = |C|$, then M is max matching & C is min cover
  - ↳ proof:

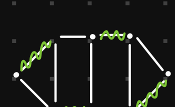
    Let M' be any matching. By lemma 8.2.1, $|M'| \leq |C| = |M|$. So M is max matching. Let C' be any cover. By lemma 8.2.1, $|C'| \geq |M| = |C|$. So C is min cover.

KÖNIG'S THEOREM

- alternating path wrt matching M is a match where we consecutively alternate btwn being in M & not in M
- augmenting path is alting path that starts & ends w/ distinct unsaturated vertices
  - ↳ single edge joining 2 unsaturated vertices is augmenting path
  - ↳ every augmenting path has odd length
- e.g.







matching
alting path

matching
augmenting path

larger matching by swapping edges of augmenting path

↳ augmenting path can be used to make larger matching by removing matching edges from path & add non-matching edges on path
- saturate 2 more vertices so matching size +1
- given matching M & augmenting path P, new matching is $M' = (M \cup (E(P) \setminus M)) \setminus (E(P) \cap M)$

- **lemma 8.1.1**: if matching M has augmenting path, then M is not max
- **thm 8.1.1 (König's thm)**: in a bipartite graph, size of max matching is equal to size of min cover

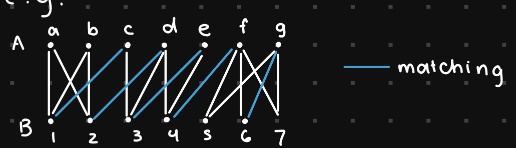
↳ proof: algo that looks for augmenting paths

Suppose G is bipartite graph w/ bipartition (A, B). WLOG, suppose path starts in A. It will end in B. To find augmenting path, start w/ unsaturated vertex in A, then fan out to find unsaturated vertex B via alternating path.

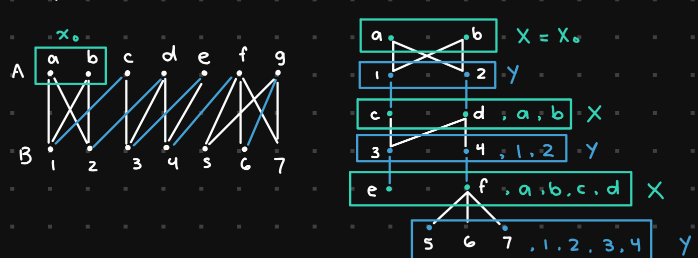
Main idea: Find all possible altng paths starting from unsaturated vertices in A. If augmenting path is found, use it to make larger matching & start over. If no augmenting path is found, then we've found a cover that has same size is matching. Thus, current matching is max.

- **Bipartite Matching Algo (aka XY-Construction)**: given bipartite graph G w/ bipartition (A, B) & matching M of G
  1) Let X_0 be set of all unsaturated vertices in A.
  2) Set $X \leftarrow X_0$ & $Y \leftarrow \emptyset$.
  3) Find all neighbours of X in B & not currently in Y
     a) If 1 such vertex is unsaturated, then we've found augmenting path. Make larger matching by swapping edges in augmenting path. Start over at step 1.
     b) If all such vertices are saturated, put all of them in Y. Add matching neighbours to X. Go to step 3.
     c) If no such vertices exist, stop. Matching is max & min cover is $Y \cup (A \setminus X)$.

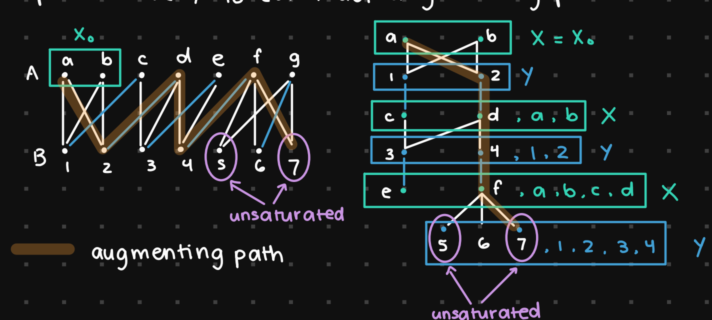
↳ e.g.



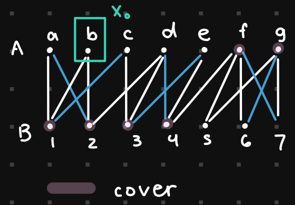
Steps 1, 2, 3b, 3b:

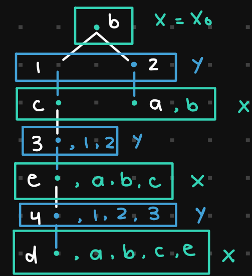


Step 3a (use 7 to construct augmenting path):



Update matching, then do steps 1, 2, 3b, 3b, 3c:





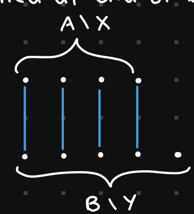
Now that we've reached step 3c w/ $X = \{a, b, c, d, e\}$ & $Y = \{1, 2, 3, 4\}$, algo outputs cover $C = Y \cup (A \setminus X) = \{1, 2, 3, 4, f, g\}$. Since $|C| = |M|$, cover must be min & matching must be max.

↳ proof of König's thm:

At end of algo:

- X_0 is set of all unsaturated vertices in A
- X is set of all vertices in A reachable via alting path starting w/ vertex in X_0
- Y is set of all vertices in B reachable via alting path starting w/ vertex in X_0

Let M be matching obtained at end of bipartite matching algo. Let X_0, X, & Y be as below.



Claim 1: There's no edge btwn X & B \ Y. Otherwise, this edge can be used to extend alting path. Every edge has at least 1 end in A \ X or Y so $Y \cup (A \setminus X)$ is cover.

Claim 2: Every vertex in Y is saturated. Otherwise, we can find augmenting path & algo wouldn't have ended.

Claim 3: Every vertex in A \ X is saturated. All unsaturated vertices of A are in X_0, which is in X. For a matching that saturates vertex in A / X, other end is in B \ Y. Edges of matching that saturate Y & A \ X are distinct. So, $|M| = |Y| + |A \setminus X| = |C|$. By lemma 8.2.2, we have max matching & min cover.

- corollary 1: let G be bipartite graph on m edges w/ bipartition (A, B) st $|A| = |B| = n$; then, G has matching of size at least $\frac{m}{n}$

↳ proof:

Let C be cover of G. There can be at most n edges incident to any elmt of C, meaning there's at most $n|C|$ edges incident w/ all of C. Since C is cover, every edge must be incident w/ some elmt of C. Then, $n|C| \geq m$ so $|C| \geq \frac{m}{n}$. Since every cover has size at least $\frac{m}{n}$, min cover has cover size at least $\frac{m}{n}$. By König's thm, max matching has size at least $\frac{m}{n}$.

HALL'S THEOREM

- what prevents us from finding matching for everyone in A?
  - ↳ isolated vertex
  - ↳ group of vertices in A that match w/ fewer vertices in B
  - ↳ e.g.

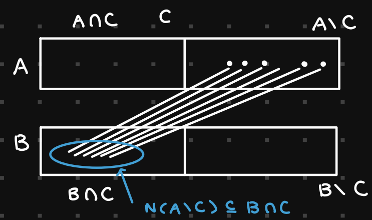
- let $D \subseteq V(G)$; neighbour set N(D) of D is set of all vertices adjacent to at least 1 vertex of D
  - ↳ $N(D) = \{v : uv \in E(G) \wedge u \in D\}$
- thm 8.4.1 (Hall's thm): a bipartite graph G w/ bipartion (A, B) has matching saturating every vertex in A iff every subset $D \subseteq A$ satisfies $|N(D)| \geq |D|$

↳ proof:

$\Rightarrow$: Suppose M is matching that saturates A. For any $D \subseteq A$, each matching edge that saturates a vertex in D must have other end in N(D). These neighbours are distinct so $|N(D)| \geq |D|$.

$\Leftarrow$: Prove contrapositive, which is if G doesn't have matching saturating A, then there exists

$D \subseteq A$ st $|N(D)| < |D|$. Let M be max matching in G. Then $|M| < |A|$ b/c M doesn't saturate A.
By König's thm, there exists cover C st $|M| = |C| < |A|$. Consider $A \cap C$, $A \setminus C$, $B \cap C$, $B \setminus C$.



Since C is cover, no edge joins $A \setminus C$ w/ $B \setminus C$. All neighbours of $A \setminus C$ are in $B \cap C$ so $N(A \setminus C) \subseteq B \cap C$
$C = (C \cap A) \cup (C \cap B)$ & $A = (A \cap C) \cup (A \setminus C)$ We also know $|C| < |A|$.

$$|C| < |A|$$
$$|C \cap A| + |C \cap B| < |A \cap C| + |A \setminus C|$$
$$|C \cap B| < |A \setminus C|$$

However, $|N(A \setminus C)| \le |B \cap C|$ so $|N(A \setminus C)| < |A \setminus C|$. We let $D = A \setminus C$ since $A \setminus C \subseteq A$ & we've proven the contrapositive.

PERFECT MATCHINGS IN BIPARTITE GRAPHS

- **corollary 8.6.1**: a bipartite graph G w/ bipartition (A, B) has perfect matching iff $|A| = |B|$ & every subset $D \subseteq A$ satisfies $|N(D)| \ge |D|$
- **thm 8.6.2**: if G is k-regular bipartite graph w/ $k \ge 1$, then G has perfect matching
  - ↳ proof:

    By handshaking lemma:

    $$\sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$$

    $$k|A| = k|B|$$
    $$|A| = |B| \qquad \text{since } k \ge 1$$

    Let $D \subseteq A$. Every edge incident w/ vertex in D has other end in $N(D) \subseteq B$ so:

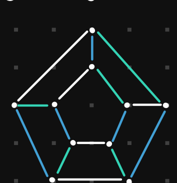
    $$\sum_{v \in D} \deg(v) \le \sum_{v \in N(D)} \deg(v)$$

    $$k|D| \le k|N(D)|$$
    $$|D| \le |N(D)|$$

    By corollary 8.6.1, G has perfect matching.

EDGE COLOURINGS

- **edge k-colouring** of graph G assigns 1 of k colours to each edge of G st 2 edges incident w/ same vertex are assigned diff colours
  - ↳ e.g. edge 3-colouring



  - ↳ each set of coloured edges is a matching
    - i.e. an edge k-colouring partitions sets of edges into k matchings
- **lemma 8.7.2**: let G be bipartite graph having at least 1 edge; G has matching saturating each vertex of max deg
- **thm 8.7.1**: every bipartite graph w/ max deg Δ has edge Δ-colouring
  - ↳ proof: Use induction on Δ.

    Base case: $\Delta = 0$

    G has no edges so it's edge 0-colourable.

    Inductive hypothesis: Assume every bipartite graph w/ $\Delta < m$ has edge Δ-colouring.

    Inductive step:

Let G be bipartite w/ $\Delta = m$. By lemma 8.7.2, G has matching M saturating all vertices of deg m. Delete M from G to obtain H, which is still bipartite & has $\Delta = m-1$, so it has edge $(m-1)$-colouring. We use this colouring & add 1 more for edges in M. Thus, we get an edge m-colouring for G.