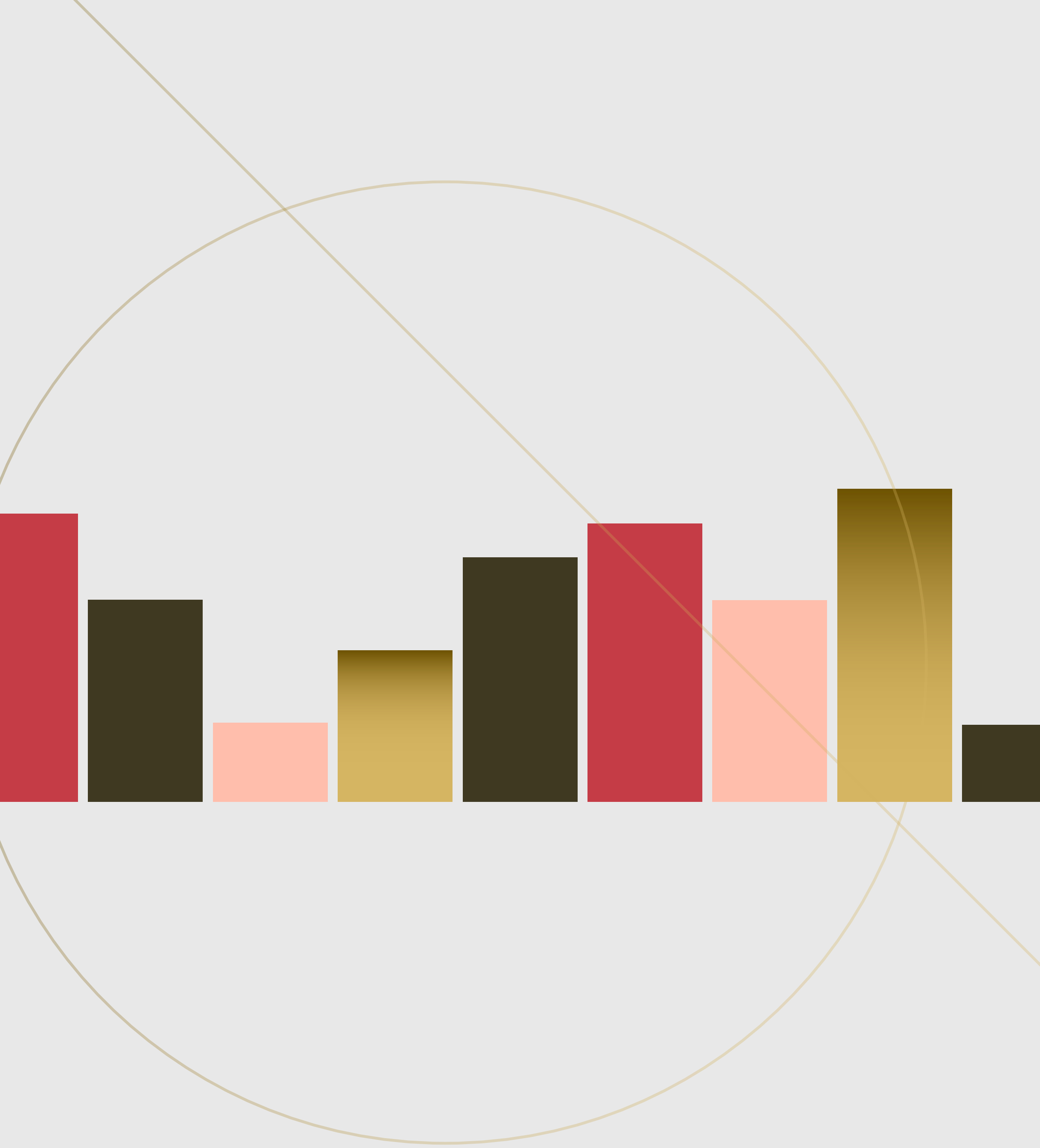


week 1

COMPLEX NUMBERS - STANDARD FORM

- complex number in standard form is $x + yj$ where $x, y \in \mathbb{R}$ & $j^2 = -1$
  - ↳ set of all complex #s: $\mathbb{C} = \{x + yj \mid x, y \in \mathbb{R}\}$
  - ↳ usually use i but engineers use j
- every $x \in \mathbb{R}$ can be expressed as $x = x + 0j \in \mathbb{C}$ so every real # is complex #
  - ↳ denote as $\mathbb{R} \subseteq \mathbb{C}$ ($\mathbb{R}$ is subset of $\mathbb{C}$)
  - ↳ not every complex # is real (e.g. $3 + 4j \notin \mathbb{R}$)
- order of subsets: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

  if $z = x + yj \in \mathbb{C}$ w/ $x, y \in \mathbb{R}$, then $x = \operatorname{Re}(z)$ (i.e. x is real part of z) & $y = \operatorname{Im}(z)$ (i.e. y is imaginary part of z)
  - ↳ if $y = 0$, $z = x$ is purely real
  - ↳ if $x = 0$, $z = yj$ is purely imaginary
  - ↳ e.g. $\operatorname{Im}(3 - 4j) = -4$
- for any $z \in \mathbb{C}$, $\operatorname{Re}(z), \operatorname{Im}(z) \in \mathbb{R}$ meaning both parts of complex # are real #s

- 2 complex #s, $z = x + yj$ & $w = u + vj$, are only equal when $x = u$ & $y = v$
- addition: $(x + yj) + (u + vj) = (x + u) + (y + v)j$
- subtraction: $(x + yj) - (u + vj) = (x - u) + (y - v)j$
- multiplication: $(x + yj)(u + vj) = xu + xvj + yuj + yvj^2$
  - ↳ NOTE: $j^2 = -1$

$$= xu + xvj + yuj - yv$$
$$= (xu - yv) + (xv + yu)j$$

- if $z = x + yj$ is non-zero complex #, then $x, y \neq 0$ & $x - yj \neq 0$, so division: $\frac{1}{z}$

$$= \frac{1}{x + yj}\left(\frac{x - yj}{x - yj}\right)$$
$$= \frac{x - yj}{x^2 - y^2 j^2}$$
$$= \frac{x - yj}{x^2 + y^2}$$
$$= \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} j$$

  - ↳ since $x, y \neq 0$, then $x^2 + y^2 > 0$ which means $\frac{1}{z}$ is always defined
- properties of arithmetic in $\mathbb{C}$ ($u, v, z \in \mathbb{C}$)
  - ↳ $(u + v) + z = u + (v + z)$
    - addition is associative
  - ↳ $u + v = v + u$
    - addition is commutative
  - ↳ $z + 0 = z$
    - zero is additive identity
  - ↳ $z + (-z) = 0$
    - $-z$ is additive inverse of z
  - ↳ $(uv)z = u(vz)$
    - multiplication is associative
  - ↳ $uv = vu$
    - multiplication is commutative
  - ↳ $z(1) = z$
    - 1 is multiplicative identity

↳ for $z \neq 0$, $z^{-1} z = 1$

- z^{-1} is multiplicative inverse of $z \neq 0$

↳ $z(u+v) = zu + zv$

- distributive law

· to solve equations involving complex #s, equate real & imaginary parts of both sides

↳ e.g. find all $z \in \mathbb{C}$ satisfying $z^2 = -7 + 24j$

SOLUTION

Let $z = a + bj$ w/ $a, b \in \mathbb{R}$

$$z^2 = -7 + 24j$$
$$(a+bj)^2 = -7 + 24j$$
$$a^2 + 2abj + b^2 j^2 = -7 + 24j$$
$$a^2 - b^2 + 2abj = -7 + 24j$$

① $a^2 - b^2 = -7$

$$a^2 - \left(\frac{12}{a}\right)^2 = -7$$
$$a^2 - \frac{144}{a^2} = -7$$
$$\frac{a^4 - 144 + 7a^2}{a^2} = 0$$
$$(a^2+16)(a^2-9) = 0$$
$$(a^2+16)(a+3)(a-3) = 0$$
$$a = -3 \text{ or } a = 3$$

② $2ab = 24 \rightarrow a, b \neq 0$

$$b = \frac{12}{a}$$
$$b = \frac{12}{-3} \quad \text{or} \quad b = \frac{12}{3}$$
$$b = -4 \qquad b = 4$$

$\therefore z = -3 - 4j$ or $z = 3 + 4j$

CONJUGATE AND MODULUS

· complex conjugate of $z = x + yj$ w/ $x, y \in \mathbb{R}$ is $\bar{z} = x - yj$

↳ conjugate of purely imaginary # is its negative

↳ conjugate of real # is same #

↳ for complex # $z \neq 0$, $z^{-1} = \frac{1}{z}\left(\frac{\bar{z}}{\bar{z}}\right) = \frac{\bar{z}}{z\bar{z}}$

NOTE: $\Longleftrightarrow$ means "if & only if" (both statements are either T or F)

· properties of conjugates (let $z, w \in \mathbb{C}$ w/ $z = x + yj$ where $x, y \in \mathbb{R}$)

↳ $\bar{\bar{z}} = z$

↳ $z \in \mathbb{R} \Longleftrightarrow \bar{z} = z$

↳ z is purely imaginary $\Longleftrightarrow \bar{z} = -z$

↳ $\overline{z+w} = \bar{z} + \bar{w}$

↳ $\overline{zw} = \bar{z}\,\bar{w}$

↳ $\overline{z^k} = \bar{z}^k$ for $k \in \mathbb{Z}$, $k \geq 0$, ($k \neq 0$ if $z = 0$)

↳ $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ provided $w \neq 0$

↳ $z + \bar{z} = 2x = 2\mathrm{Re}(z)$

↳ $z - \bar{z} = 2yj = 2j\,\mathrm{Im}(z)$

↳ $z\bar{z} = x^2 + y^2$

- $z^{-1} = \frac{\bar{z}}{x^2 + y^2}$

· modulus of $z = x + yj$ w/ $x, y \in \mathbb{R}$ is non-negative real # $|z| = \sqrt{x^2 + y^2}$

↳ modulus of real # x is absolute value

- $|x| = |x + 0j| = \sqrt{x^2 + 0^2} = |x|$

↳ interpreted as size/magnitude of complex #

· properties of modulus (let $z, w \in \mathbb{C}$)

↳ $|z| = 0 \Longleftrightarrow z = 0$

↳ $|\bar{z}| = |z|$

↳ $z\bar{z} = |z|^2$

↳ $|zw| = |z||w|$

↳ $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ provided $w \neq 0$

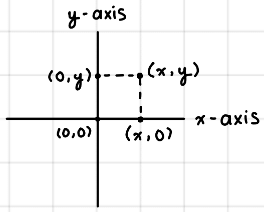
↳ $|z+w| \leq |z| + |w|$

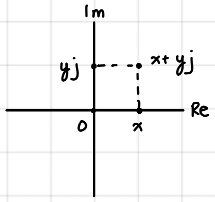
- aka the Triangle Inequality

· for complex # $z \neq 0$, $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

GEOMETRY

- set of real #s is a line but set of complex #s is a plane

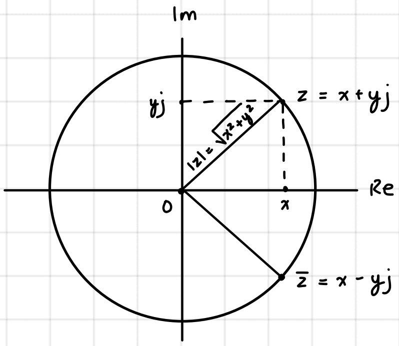




$\mathbb{R}^2$: x-y plane $\qquad$ $\mathbb{C}$: complex plane

↳ $x+yj \in \mathbb{C} = (x,y)\ \mathbb{R}^2$

- geometric interpretation of complex conjugate & modulus

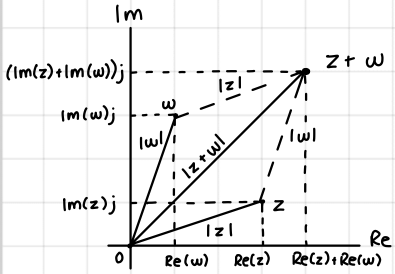


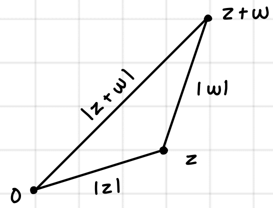
↳ $\bar{z}$ is reflection of z in real axis

↳ $|z|$ is distance btwn 0 & z

↳ for any $w \in \mathbb{C}$ that lies on the circle, $|w| = |z|$
- if w inside, $|w| < |z|$
- if w outside, $|w| > |z|$

- geometric interpretation of addition

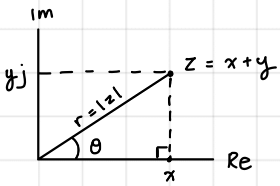




↳ 0, z, w, & z+w form parallelogram w/ $|z+w|$ as diagonal

↳ since length of any triangle side can't exceed sum of other 2 sides, $|z+w| \le |z| + |w|$
- not proof of Triangle Inequality

COMPLEX NUMBERS - POLAR FORM



- for non-zero complex # $z = x+yj$, $r>0$ is radius of z & θ is an argument of z

↳ $\cos\theta = \frac{x}{r}$ $\qquad$ $\sin\theta = \frac{y}{r}$

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

↳ $z = x + yj$

$= (r\cos\theta) + (r\sin\theta)j$

$= r(\cos\theta + j\sin\theta)$

- since $|\cos\theta + j\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, an argument of z gives a point on a circle of radius 1 to move toward while $r>0$ tells how far to move in that direction
- polar form of complex # $z \neq 0$ is $z = r(\cos\theta + j\sin\theta)$

↳ $r = |z| = \sqrt{x^2+y^2}$

↳ θ is argument of z

- unlike standard, z does not have unique polar form
  - ↳ for any $k \in \mathbb{Z}$, $r(\cos\theta + j\sin\theta) = r(\cos(\theta + 2k\pi) + j\sin(\theta + 2k\pi))$
  - ↳ normally choose $0 \leq \theta \leq 2\pi$ or $-\pi \leq \theta \leq \pi$ as domain

- polar form is useful for complex multiplication
  - ↳ angle sum formulas:
    - $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$
    - $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$
  - ↳ if $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ & $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$, then compute $z_1 z_2$

    SOLUTION

$$\begin{aligned} z_1 z_r &= (r_1(\cos\theta_1 + j\sin\theta_1))(r_2(\cos\theta_2 + j\sin\theta_2)) \\ &= r_1 r_2 (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + j(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)) \end{aligned}$$

  - ↳ $z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2))$

- multiplying by $z = r(\cos\theta + j\sin\theta)$ viewed as counterclockwise rotation by θ about 0 in $\mathbb{C}$ plane & then scaling by factor of r

- dividing two complex #s in polar form

$$\hookrightarrow \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)}\left(\frac{\cos\theta_2 - j\sin\theta_2}{\cos\theta_2 - j\sin\theta_2}\right)$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 + j(\sin\theta_1 \cos\theta_2 - \cos\theta_2 \sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right)$$

$$= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + j(\sin(\theta_1 - \theta_2)))$$

  - ↳ $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + j(\sin(\theta_1 - \theta_2)))$

POWERS OF COMPLEX NUMBERS

- for any positive integer n, $z^n = r^n(\cos(n\theta) + j\sin(n\theta))$
  - ↳ e.g.

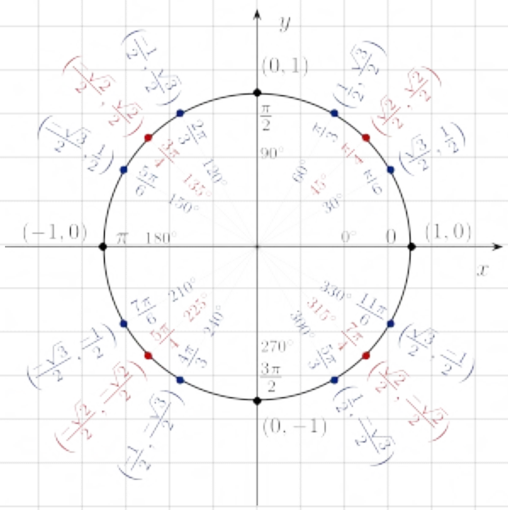
$$\begin{aligned} z^2 &= zz \\ &= r(\cos\theta + j\sin\theta)\, r(\cos\theta + j\sin\theta) \\ &= r^2(\cos(\theta + \theta) + j\sin(\theta + \theta)) \\ &= r^2(\cos(2\theta) + j\sin(2\theta)) \end{aligned}$$

- $z^{-1} = \frac{1}{z}$

  $z^{-1} = \frac{1(\cos 0 + j\sin 0)}{r(\cos\theta + j\sin\theta)}$

  $z^{-1} = \frac{1}{r}(\cos(-\theta) + j\sin(-\theta))$

- de Moivre's Theorem: if $z = r(\cos\theta + j\sin\theta) \neq 0$, then $z^n = r^n(\cos(n\theta) + j\sin(n\theta))$ for any $n \in \mathbb{Z}$
  - ↳ for $n \leq 0$ to be allowed, $z \neq 0$ is added as restriction
  - ↳ theorem holds for $z = 0$ if $n \geq 1$



COMPLEX n^{TH} ROOTS

- to find $w^n = z$, let $z = r(\cos\theta + j\sin\theta)$ & let $w = R(\cos\phi + j\sin\phi)$
  - ↳ $(R(\cos\phi + j\sin\phi))^n = r(\cos\theta + j\sin\theta)$
    $R^n(\cos(n\phi) + j\sin(n\phi)) = r(\cos\theta + j\sin\theta)$
    - $R^n = r$
    - $n\phi = \theta + 2k\pi$ for some $k \in \mathbb{Z}$
  - ↳ $R = r^{\frac{1}{n}}$ and $\phi = \frac{\theta + 2k\pi}{n}$ for some $k \in \mathbb{Z}$
  - ↳ for any $k \in \mathbb{Z}$, $w_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + j\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$
    - $(w_k)^n = \left(r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + j\sin\left(\frac{\theta+2k\pi}{n}\right)\right)\right)^n$
      $= r(\cos(\theta + 2k\pi) + j\sin(\theta + 2k\pi))$
      $= r(\cos\theta + j\sin\theta)$
      $= z$
    - $(w_k)^n = z$ for any integer k but there's exactly n solutions
- let $z = r(\cos\theta + j\sin\theta) \neq 0$ & let n be a positive integer; there's n distinct n^{th} roots of z given by $w_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + j\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$ for $k = 0, 1, \ldots, n-1$
  - ↳ e.g. find 3^{rd} roots of 1 (i.e. find all $w \in \mathbb{C}$ such that $w^3 = 1$)

    SOLUTION

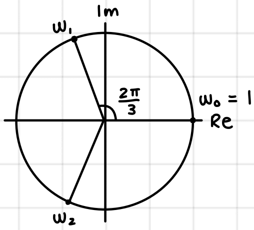
    $z = 1 = 1(1 + 0j) = 1(\cos 0 + j\sin 0)$

    $w_k = 1^{\frac{1}{3}}\left(\cos\left(\frac{0+2k\pi}{3}\right) + j\sin\left(\frac{0+2k\pi}{3}\right)\right)$, $k = 0, 1, 2$

    $= \cos\left(\frac{2k\pi}{3}\right) + j\sin\left(\frac{2k\pi}{3}\right)$

    $w_0 = 1$, $w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$, $w_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$

    - $w_{-1} = w_2$ & $w_3 = w_0$ b/c angle gets rotated $\frac{2\pi}{3}$ for 3 times to come full circle



COMPLEX EXPONENTIAL

- Euler's Formula states that $e^{j\theta} = \cos\theta + j\sin\theta$ for $\theta \in \mathbb{R}$
- complex exponential form of z is $z = re^{j\theta}$
  - ↳ since sine & cosine functions are 2π-periodic, this form is not unique
    - $re^{j\theta} = re^{j(\theta + 2k\pi)}$ for any $k \in \mathbb{Z}$
    - if $r = 1$, then $e^{j\theta}$ is 2π-periodic (i.e. it oscillates like trig functions)
  - ↳ let $z_1 = r_1 e^{j\theta_1}$ & $z_2 = r_2 e^{j\theta_2}$
    - $z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2}$
      $= r_1 r_2 (\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2))$
      $= r_1 r_2 e^{j(\theta_1 + \theta_2)}$
    - if $r_1 = r_2 = 1$, then $e^{j\theta_1} e^{j\theta_2} = e^{j(\theta_1 + \theta_2)}$ (i.e. consistent w/multiplication rules of exponential functions)
  - ↳ let $z = re^{j\theta}$ & $n \in \mathbb{Z}$
    - $(re^{j\theta})^n = (r(\cos\theta + j\sin\theta))^n$
      $= r^n(\cos(n\theta) + j\sin(n\theta))$
      $= r^n e^{j(n\theta)}$
    - if $r = 1$, then $(e^{j\theta})^n = e^{j(n\theta)}$ (i.e. consistent w/rules of powers of exponential functions)
- Euler's Identity states $e^{j\pi} + 1 = 0$
  - ↳ $e^{j\pi} = \cos\pi + j\sin\pi = -1$

week 2

COMPLEX POLYNOMIALS

- polynomial is $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$
  - ↳ if $a_n \neq 0$, $p(x)$ has degree n
  - ↳ c is root of $p(x)$ if $p(c) = 0$
- if $a_0, a_1, \dots, a_n \in \mathbb{C}$, $p(x)$ is complex polynomial
  - ↳ since $\mathbb{R} \subseteq \mathbb{C}$, every real polynomial is also a complex one
  - ↳ often use z instead of x
- let $p(x)$ be a real polynomial; if $z \in \mathbb{C}$ is a root of $p(x)$, then so is $\bar{z}$
  - ↳ e.g. Find roots of $p(x) = x^3 + 16x$

    SOLUTION

    $0 = x^3 + 16x$

    $0 = x(x^2 + 16)$

    $x = 0$ or $x^2 + 16 = 0$

    $$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(16)}}{2(1)} = \frac{\pm\sqrt{-64}}{2} = \frac{\pm\sqrt{64(j)^2}}{2} = \frac{\pm 8j}{2} = \pm 4j$$

    $\therefore$ The roots are $x = 0$, $4j$, and $-4j$.

A VERY BRIEF INTRODUCTION TO PROOFS

- statements of "if... then..." form are implications
  - ↳ to prove true, assume hypothesis is true & show conclusion must be true
- in a proof:
  - ↳ state your assumptions
  - ↳ state where hypothesis is used
  - ↳ don't assume conclusion is true
  - ↳ e.g. Let $z \in \mathbb{C}$. Show that $|Re(z)| + |Im(z)| \leq \sqrt{2}|z|$

    SOLUTION

    Let $z \in \mathbb{C}$ so $z = x + yj$ w/ $x, y \in \mathbb{R}$. Then, $x = Re(z)$ & $y = Im(z)$

    $(\sqrt{2}|z|)^2 = 2|z|^2 = 2(x^2 + y^2) = 2x^2 + 2y^2 = 2|x|^2 + 2|y|^2$

    $(|Re(z)| + |Im(z)|)^2$

    $= (|x| + |y|)^2$

    $= |x|^2 + 2|x||y| + |y|^2$

    $(\sqrt{2}|z|)^2 - (|Re(z)| + |Im(z)|)^2$

    $= 2|x|^2 + 2|y|^2 - |x|^2 - 2|x||y| - |y|^2$

    $= |x|^2 - 2|x||y|^2 + |y|^2$

    $= (|x| - |y|)^2$

    $= (|Re(z)| - |Im(z)|)^2$

    ≥ 0

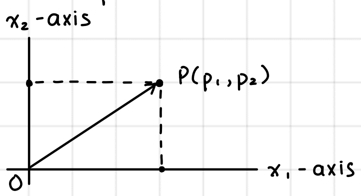
    $\therefore$ Since $(\sqrt{2}|z|)^2 - (|Re(z)| + |Im(z)|)^2 \geq 0$ & both $\sqrt{2}|z|$ & $|Re(z)| + |Im(z)|$ are non-negative real numbers, we can conclude $|Re(z)| + |Im(z)| \leq \sqrt{2}|z|$

ROUNDOFF ERROR

- phenomenon where small change to input value leads to relatively large change in output value

VECTOR ALGEBRA

- Cartesian plane



- assign each point a vector
  - ↳ $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$
  - ↳ $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \, x_1, \ldots, x_n \in \mathbb{R} \right\}$ is set of all vectors w/ n components (each is real #)
    - ◦ $x_1, \ldots, x_n$ are entries / components
- 2 vectors $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ & $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$ are equal if their corresponding entries are equal
  - ↳ $\vec{x} = \vec{y}$
  - ↳ if $\vec{x} \in \mathbb{R}^n$ & $\vec{y} \in \mathbb{R}^m$ w/ $n \neq m$, then $\vec{x}$ can never equal $\vec{y}$
- zero vector is denoted by $\vec{0}_{\mathbb{R}^n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$
- vector addition is $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \in \mathbb{R}^n$
  - ↳ add corresponding entries
- scalar multiplication is $c\vec{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_2 \end{bmatrix} \in \mathbb{R}^n$
  - ↳ let $c \in \mathbb{R}$
  - ↳ multiply each entry of $\vec{x}$ by c
- 2 non-zero vectors in $\mathbb{R}^n$ are parallel if they're scalar multiples of each other

- properties when $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ & $c, d \in \mathbb{R}$:
  - ↳ $\vec{x} + \vec{y} \in \mathbb{R}^n$
    - • $\mathbb{R}^n$ is closed under addition
  - ↳ $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
    - • addition is commutative
  - ↳ $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
    - • addition is associative
  - ↳ there exists a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{v} + \vec{0} = \vec{v}$ for every $\vec{v} \in \mathbb{R}^n$
    - • zero vector
  - ↳ for each $\vec{x} \in \mathbb{R}^n$ there exists a $(-\vec{x}) \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
    - • additive inverse
  - ↳ $c\vec{x} \in \mathbb{R}^n$
    - • $\mathbb{R}^n$ is closed under scalar multiplication
  - ↳ $c(d\vec{x}) = (cd)\vec{x}$
    - • scalar multiplication is associative
  - ↳ $(c+d)\vec{x} = c\vec{x} + d\vec{x}$
    - • distributive law
  - ↳ $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
    - • distributive law
  - ↳ $1\vec{x} = \vec{x}$
    - • scalar multiplicative identity

- let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$ & $c_1, c_2, \ldots, c_k \in \mathbb{R}$ for some +ve integer k ; vector $c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_k\vec{x}_k$ is a linear combination of the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$
  - ↳ $\mathbb{R}^n$ is closed under linear combinations, meaning that every linear combination of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ will

again be vector of $\mathbb{R}^n$

↳ e.g. in $\mathbb{R}^3$, let $\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, & $\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$; then for any $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3$, $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$

- i.e. every $\vec{x} \in \mathbb{R}^3$ can be expressed as linear combo of $\vec{e}_1, \vec{e}_2,$ & $\vec{e}_3$

- given a point P, $\vec{p} = \vec{OP}$ is position vector of P & $\vec{p}$ is in standard position
- in $\mathbb{R}^n$ where $A(a_1, \ldots, a_n)$ & $B(b_1, \ldots, b_n)$, $\vec{AB} = \begin{bmatrix}b_1 - a_1\\ \vdots \\ b_n - a_n\end{bmatrix} = \begin{bmatrix}b_1\\ \vdots \\ b_n\end{bmatrix} - \begin{bmatrix}a_1\\ \vdots \\ a_n\end{bmatrix} = \vec{OB} - \vec{OA}$

NORMS AND DOT PRODUCTS

- norm of $\vec{x} = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} \in \mathbb{R}^n$ is non-negative real # $\|\vec{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$
  - length/magnitude of vector
- properties of norms where $\vec{x}, \vec{y} \in \mathbb{R}^n$ & $c \in \mathbb{R}$:
  - ↳ $\|\vec{x}\| \geq 0$
    - has equality iff $\vec{x} = \vec{0}$
  - ↳ $\|c\vec{x}\| = |c|\,\|\vec{x}\|$
  - ↳ $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
    - aka the Triangle Inequality
- vector $\vec{x} \in \mathbb{R}^n$ is unit vector if $\|\vec{x}\| = 1$
  - ↳ for a nonzero vector $\vec{x} \in \mathbb{R}^n$, $\vec{y} = \frac{1}{\|\vec{x}\|}(\vec{x})$ is unit vector in direction of $\vec{x}$

- dot product of $\vec{x} = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}$ & $\vec{y} = \begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix}$ is $\vec{x} \cdot \vec{y} = x_1y_1 + \ldots + x_ny_n$
  - ↳ aka scalar/standard inner product
  - ↳ always real #
- properties of dot products where $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ & $c \in \mathbb{R}$:
  - ↳ $\vec{x} \cdot \vec{y} \in \mathbb{R}$
  - ↳ $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
  - ↳ $\vec{x} \cdot \vec{0} = 0$
  - ↳ $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2$
  - ↳ $(c\vec{x}) \cdot \vec{y} = c(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (c\vec{y})$
  - ↳ $\vec{w} \cdot (\vec{x} \pm \vec{y}) = \vec{w} \cdot \vec{x} \pm \vec{w} \cdot \vec{y}$

- for 2 nonzero vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, determine an angle θ by $\vec{x} \cdot \vec{y} = \|\vec{x}\|\|\vec{y}\| \cos\theta$
  - ↳ θ is angle btwn vectors
    - $\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|\|\vec{y}\|}$
  - ↳ Corollary: for any 2 vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$
    - Cauchy-Schwarz Inequality states that size of 2 vectors' dot product can't exceed product of their norms (holds for any vectors in $\mathbb{R}^n$)

NOTE: Corollary is a result following a preceeding theorem

- often interested if θ is acute, obtuse, or orthogonal
  - ↳ $\vec{x} \cdot \vec{y} > 0 \Leftrightarrow 0 \leq \theta < \frac{\pi}{2} \Leftrightarrow \vec{x}$ & $\vec{y}$ determine acute angle
  - ↳ $\vec{x} \cdot \vec{y} = 0 \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \vec{x}$ & $\vec{y}$ are orthogonal
  - ↳ $\vec{x} \cdot \vec{y} < 0 \Leftrightarrow \frac{\pi}{2} < \theta \leq \pi \Leftrightarrow \vec{x}$ & $\vec{y}$ determine obtuse angle

COMPLEX VECTORS

- for $z_1, \ldots, z_n \in \mathbb{C}$, $\vec{z} = \begin{bmatrix}z_1\\ \vdots \\ z_n\end{bmatrix}$ is a complex vector
  - ↳ $\mathbb{C}^n = \left\{ \begin{bmatrix}z_1\\ \vdots \\ z_n\end{bmatrix} \middle| z_1, \ldots, z_n \in \mathbb{C} \right\}$ is set of all complex vectors w/n entries
  - ↳ addition & scalar multiplication for vectors in $\mathbb{C}^n$ are defined in the same way as for vectors in $\mathbb{R}^n$
- if $\vec{z} = \begin{bmatrix}z_1\\ \vdots \\ z_n\end{bmatrix}$ & $\vec{w} = \begin{bmatrix}w_1\\ \vdots \\ w_n\end{bmatrix}$ are vectors in $\mathbb{C}^n$
  - ↳ the complex inner product of $\vec{z}$ & $\vec{w}$ is $\langle \vec{z}, \vec{w} \rangle = \bar{z}_1 w_1 + \ldots + \bar{z}_n w_n$

- ↳ norm of $\vec{z}$ is $\|\vec{z}\| = \sqrt{\bar{z}_1 z_1 + \ldots + \bar{z}_n z_n}$
- for $\vec{z} \in \mathbb{C}^n$, $\langle \vec{z}, \vec{z} \rangle = \bar{z}_1 z_1 + \ldots + \bar{z}_n z_n = |z_1|^2 + \ldots + |z_n|^2$
  - ↳ $\langle \vec{z}, \vec{z} \rangle \in \mathbb{R}$ w/ $\langle \vec{z}, \vec{z} \rangle \geq 0$
  - ↳ $\|\vec{z}\|^2 = \langle \vec{z}, \vec{z} \rangle$ so $\|\vec{z}\| = \sqrt{\langle \vec{z}, \vec{z} \rangle}$ is a non-negative real #
- if $\vec{z}, \vec{w} \in \mathbb{R}^n$, then $\langle \vec{z}, \vec{w} \rangle = \vec{z} \cdot \vec{w}$
- complex inner product of $\vec{z}, \vec{w} \in \mathbb{C}^n$ can be viewed as dot product of $\bar{\vec{z}}$ & $\vec{w}$
  - ↳ $\langle \vec{z}, \vec{w} \rangle = \bar{\vec{z}} \cdot \vec{w}$
- properties of complex inner products where $\vec{v}, \vec{w}, \vec{z} \in \mathbb{C}^n$ & $\alpha \in \mathbb{C}$:
  - ↳ $\langle \vec{z}, \vec{z} \rangle \geq 0$
    - equality iff $\vec{z} = 0$
  - ↳ $\langle \vec{z}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{z} \rangle}$
  - ↳ $\langle \vec{v} + \vec{w}, \vec{z} \rangle = \langle \vec{v}, \vec{z} \rangle + \langle \vec{w}, \vec{z} \rangle$ & $\langle \vec{z}, \vec{v} + \vec{w} \rangle = \langle \vec{z}, \vec{v} \rangle + \langle \vec{z}, \vec{w} \rangle$
  - ↳ $\langle \alpha\vec{z}, \vec{w} \rangle = \bar{\alpha} \langle \vec{z}, \vec{w} \rangle$ & $\langle \vec{z}, \alpha\vec{w} \rangle = \alpha \langle \vec{z}, \vec{w} \rangle$
  - ↳ $|\langle \vec{z}, \vec{w} \rangle| \leq \|\vec{z}\| \|\vec{w}\|$
    - Cauchy-Schwarz Inequality
  - ↳ $\|\vec{z} + \vec{w}\| \leq \|\vec{z}\| + \|\vec{w}\|$
    - Triangle Inequality

CROSS PRODUCT IN $\mathbb{R}^3$

- cross product of 2 vectors in $\mathbb{R}^3$ is vector in $\mathbb{R}^3$
  - ↳ only valid $\mathbb{R}^3$ (also valid in $\mathbb{R}^7$)
- if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are 2 vectors in $\mathbb{R}^3$, then cross product (aka vector product) is

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_2 y_3 - y_2 x_3 \\ -(x_1 y_3 - y_1 x_3) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

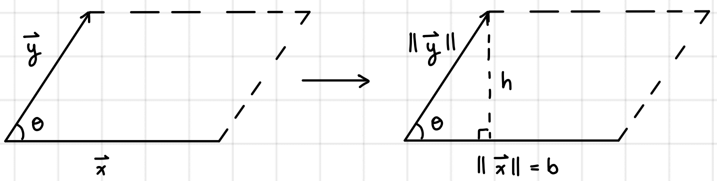
  - ↳ $\vec{x} \times \vec{y}$ is orthogonal to both $\vec{x}$ & $\vec{y}$

week 3

PROPERTIES OF CROSS PRODUCTS

- let $x, y, \vec{w} \in \mathbb{R}^3$ & $c \in \mathbb{R}$; properties are:
  - ↳ $\vec{x} \times \vec{y} \in \mathbb{R}^3$
  - ↳ $\vec{x} \times \vec{y}$ is orthogonal to both $\vec{x}$ & $\vec{y}$
  - ↳ $\vec{x} \times \vec{0} = \vec{0} = \vec{0} \times \vec{x}$
  - ↳ $\vec{x} \times \vec{x} = \vec{0}$
  - ↳ $\vec{x} \times \vec{y} = -(\vec{y} \times \vec{x})$
    - anti - commutative
  - ↳ $(c\vec{x}) \times \vec{y} = c(\vec{x} \times \vec{y}) = \vec{x} \times (c\vec{y})$
  - ↳ $\vec{w} \times (\vec{x} \pm \vec{y}) = (\vec{w} \times \vec{x}) \pm (\vec{w} \times \vec{y})$
  - ↳ $(\vec{x} \pm \vec{y}) \times \vec{w} = (\vec{x} \times \vec{w}) \pm (\vec{y} \times \vec{w})$
- cross product isn't associative
  - ↳ $\vec{x} \times \vec{y} \times \vec{w}$ is undefined & must always include brackets

- Lagrange Identity: let $\vec{x}, \vec{y} \in \mathbb{R}^3$; then $\|\vec{x} \times \vec{y}\|^2 = \|\vec{x}\|^2 \|\vec{y}\|^2 - (\vec{x} \cdot \vec{y})^2$
  - ↳ let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be non-zero vectors so $\vec{x} \cdot \vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$ where $0 \le \theta \le \pi$
    - $\|\vec{x} \times \vec{y}\| = \|\vec{x}\|^2 \|\vec{y}\|^2 - (\|\vec{x}\|\|\vec{y}\|\cos\theta)^2$
      $= \|\vec{x}\|^2 \|\vec{y}\|^2 - \|\vec{x}\|^2 \|\vec{y}\|^2 \cos^2\theta$
      $= \|\vec{x}\|^2 \|\vec{y}\|^2 (1 - \cos^2\theta)$
      $= \|\vec{x}\|^2 \|\vec{y}\|^2 \sin^2\theta$
    - since $\sin\theta \ge 0$ for $0 \le \theta \le \pi$, then take +ve square root to get $\|\vec{x} \times \vec{y}\| = \|\vec{x}\|\|\vec{y}\|\sin\theta$
- parallelogram determined by non-zero vectors $\vec{x}$ & $\vec{y}$



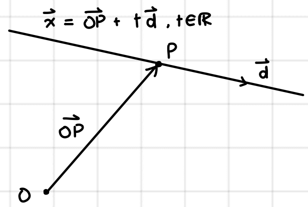
  - ↳ $A = bh = \|\vec{x}\|\|\vec{y}\|\sin\theta = \|\vec{x} \times \vec{y}\|$

  norm of cross product of 2 non-zero vectors gives area of parallelogram
  - ↳ if $\vec{x}$ & $\vec{y}$ were parallel, parallelogram they determine is line segment & area is 0
  - ↳ if $\vec{x}$ or $\vec{y}$ were $\vec{0}$, area is also 0

VECTOR EQUATION OF A LINE

$x_2 = x_1$ is line in $\mathbb{R}^2$ but in $\mathbb{R}^3$, there's no restrictions on x_3 (i.e. $x_3 \in \mathbb{R}$) so $x_2 = x_1$ rep a plane

- require 2 things to describe a line.
  - ↳ point P on line
  - ↳ vector $\vec{d}$ in direction of line
- a line in $\mathbb{R}^n$ through a point P w/ direction $\vec{d}$ (where $\vec{d} \in \mathbb{R}^n$ is non-zero) is given by vector equation: $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{OP} + t\vec{d},\ t \in \mathbb{R}$
  - ↳ first moves from origin to point P
  - ↳ then moves as far as needed from P in direction $\vec{d}$



NOTE: vector equation for a line isn't unique since $\vec{BA}$ could've been used as $\vec{d}$ & B as point on line

↳ e.g. find a vector equation of line through points $A(1,1,-1)$ & $B(4,0,-3)$

SOLUTION

$\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix}4\\0\\-3\end{bmatrix} - \begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}3\\-1\\-2\end{bmatrix}$

$\vec{x} = \vec{OA} + t\vec{AB} = \begin{bmatrix}1\\1\\-1\end{bmatrix} + t\begin{bmatrix}3\\-1\\-2\end{bmatrix}, \; t\in\mathbb{R}$

$= \begin{bmatrix}1\\1\\-1\end{bmatrix} + \begin{bmatrix}3t\\-t\\-2t\end{bmatrix}$

$= \begin{bmatrix}1+3t\\1-t\\-1-2t\end{bmatrix}$

$x_1 = 1+3t$

$x_2 = 1-t$ for $t\in\mathbb{R}$

$x_3 = -1-2t$

↳ parametric equations of line give x_1-, x_2-, & x_3-coordinates of point on line for each choice of $t\in\mathbb{R}$

VECTOR EQUATION OF A PLANE

- vector equation of plane in $\mathbb{R}^n$ through point P is $\vec{x} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \vec{OP} + s\vec{u} + t\vec{v}$, $s,t\in\mathbb{R}$
  - ↳ $\vec{u},\vec{v}\in\mathbb{R}^n$ are non-zero non-parallel vectors
  - ↳ parameters s & t are chosen independently of one another
  - ↳ e.g. find vector equation for plane containing $A(1,1,1)$, $B(1,2,3)$, & $C(-1,1,2)$

SOLUTION

$\vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$

$\vec{AC} = \vec{OC} - \vec{OA} = \begin{bmatrix}-1\\1\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\0\\1\end{bmatrix}$

* $\vec{AB}$ & $\vec{AC}$ are non-zero & non-parallel *

$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \vec{OA} + s\vec{AB} + t\vec{AC}$

$= \begin{bmatrix}1\\1\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\2\end{bmatrix} + t\begin{bmatrix}-2\\0\\1\end{bmatrix}$, $s,t\in\mathbb{R}$

- parametric equations: $x_1 = 1-2t$

$x_2 = 1+s$ for $s,t\in\mathbb{R}$

$x_3 = 1+2s+t$

SCALAR EQUATION OF PLANE IN $\mathbb{R}^3$

- a non-zero vector $\vec{n}\in\mathbb{R}^3$ is normal vector for a plane if for any 2 points P & Q on plane, $\vec{n}$ is orthogonal to $\vec{PQ}$
  - ↳ not unique b/c any non-zero scalar multiple can work

if normal vector $\vec{n} = \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}$ & $P(a,b,c)$ is given point on plane, $Q(x_1,x_2,x_3)$ lies on plane iff $0 = \vec{n}\cdot\vec{PQ} = n\cdot(\vec{OQ}-\vec{OP})$

$= \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}\cdot\begin{bmatrix}x_1-a\\x_2-b\\x_3-c\end{bmatrix}$

$= n_1(x_1-a) + n_2(x_2-b) + n_3(x_3-c)$

- scalar equation of plane in $\mathbb{R}^3$ w/ normal vector $\vec{n} = \begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}$ containing point $P(a,b,c)$ is: $n_1a + n_2b + n_3c = d$, $d\in\mathbb{R}$

given vector equation $\vec{x} = \vec{OP} + s\vec{u} + t\vec{v}$ for plane in $\mathbb{R}^3$ w/ point P, can compute normal vector $\vec{n} = \vec{u}\times\vec{v}$

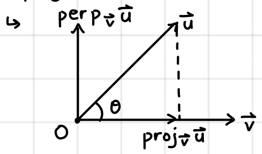
- scalar equation is preferable when verifying if given point lies on plane
- vector equation is preferable when asked to generate points that lie on a plane
- 2 lines in $\mathbb{R}^n$ are parallel if their direction vectors are parallel
- 2 planes in $\mathbb{R}^3$ are parallel if their normal vectors are parallel

PROJECTIONS

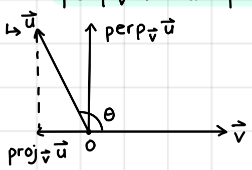
- given 2 vectors $\vec{u},\vec{v}\in\mathbb{R}^n$ w/ $\vec{v}\neq\vec{0}$, $\vec{u} = \vec{u}_1 + \vec{u}_2$
  - ↳ $\vec{u}_1$ is scalar multiple of $\vec{v}$

↳ $\vec{u}_2$ is orthogonal to $\vec{v}$

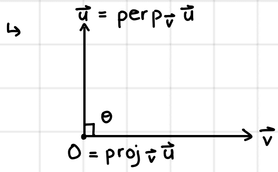
- let $\vec{u}, \vec{v} \in \mathbb{R}^n$ w/ $\vec{v} \neq \vec{0}$ so the projection of $\vec{u}$ onto $\vec{v}$ is $\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$
  - ↳ projection of $\vec{u}$ $\perp$ to $\vec{v}$ is $\text{perp}_{\vec{v}}\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\vec{u}$

↳ 

- $0 \leq \theta < \frac{\pi}{2}$
- $\vec{u}\cdot\vec{v} > 0$
- proj$_{\vec{v}}\vec{u}$ is same dir as $\vec{v}$

↳ 

- $\frac{\pi}{2} < \theta \leq \pi$
- $\vec{u}\cdot\vec{v} < 0$
- proj$_{\vec{v}}\vec{u}$ is opp dir as $\vec{v}$

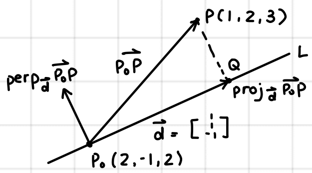
↳ 

- $\theta = \frac{\pi}{2}$
- $\vec{u}\cdot\vec{v} = 0$
- proj$_{\vec{v}}\vec{u} = \vec{0}$
- perp$_{\vec{v}}\vec{u} = \vec{u}$

DISTANCES FROM POINTS TO LINES AND PLANES

- e.g. find shortest distance from point $P(1,2,3)$ to line L which passes through point $P_0(2,-1,2)$ w/ $\vec{d} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$; also find point Q on L that's closest to P

SOLUTION



$$\vec{P_0P} = \vec{OP} - \vec{OP_0} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\-1\\2\end{bmatrix} = \begin{bmatrix}-1\\3\\1\end{bmatrix}$$

$$\text{proj}_{\vec{d}}\vec{P_0P} = \frac{\vec{P_0P}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{-1+3-1}{1^2+1^2+(-1)^2}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/3\\1/3\\-1/3\end{bmatrix}$$

$$\text{perp}_{\vec{d}}\vec{P_0P} = \vec{P_0P} - \text{proj}_{\vec{d}}\vec{P_0P} = \begin{bmatrix}-1\\3\\1\end{bmatrix} - \begin{bmatrix}1/3\\1/3\\-1/3\end{bmatrix} = \begin{bmatrix}-4/3\\8/3\\4/3\end{bmatrix}$$

Shortest distance from P to L is: $\|\text{perp}_{\vec{d}}\vec{P_0P}\| = \frac{1}{3}\sqrt{(-4)^2+8^2+4^2} = \frac{1}{3}(4\sqrt{6}) = \frac{4}{3}\sqrt{6}$

NOTE: another way to get $\vec{OQ}$ is $\vec{OQ} = \vec{OP} - \text{perp}_{\vec{d}}\vec{P_0P}$

$$\vec{OQ} = \vec{OP_0} + \text{proj}_{\vec{d}}\vec{P_0P} = \begin{bmatrix}2\\-1\\2\end{bmatrix} + \begin{bmatrix}1/3\\1/3\\-1/3\end{bmatrix} = \begin{bmatrix}7/3\\-2/3\\5/3\end{bmatrix}$$

- e.g. find shortest distance from point $P(1,2,3)$ to plane T w/ equation $x_1 + x_2 - 3x_3 = -2$; also find point Q on T that's closest to P

SOLUTION

$P_0(-2,0,0)$ lies on T b/c $-2+0-3(0) = -2$

$\vec{n} = \begin{bmatrix}1\\1\\-3\end{bmatrix}$ is normal vector for T

$$\vec{P_0P} = \vec{OP} - \vec{OP_0} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}-2\\0\\0\end{bmatrix} = \begin{bmatrix}3\\2\\3\end{bmatrix}$$

$$\text{proj}_{\vec{n}}\vec{P_0P} = \frac{\vec{P_0P}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \frac{3+2-9}{1^2+1^2+(-3)^2}\begin{bmatrix}1\\1\\-3\end{bmatrix} = -\frac{4}{11}\begin{bmatrix}1\\1\\-3\end{bmatrix} = \begin{bmatrix}-4/11\\-4/11\\12/11\end{bmatrix}$$

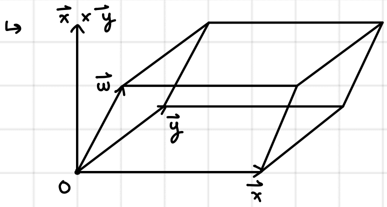
Shortest distance from P to T: $\|\text{proj}_{\vec{n}}\vec{P_0P}\| = \left|-\frac{4}{11}\right|\sqrt{1^2+1^2+(-3)^2} = \frac{4\sqrt{11}}{11}$

∗ $\|\vec{QP}\| = \|\text{proj}_{\vec{n}}\vec{P_0P}\|$ ∗

$$\vec{OQ} = \vec{OP} - \text{proj}_{\vec{n}}\vec{P_0P} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}-4/11\\-4/11\\12/11\end{bmatrix} = \begin{bmatrix}15/11\\26/11\\21/11\end{bmatrix}$$

VOLUMES OF PARALLELEPIPEDS IN $\mathbb{R}^3$

- 3 non-zero vectors $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^3$ are not linear combos of each other determine a parallelepiped
  - ↳ $\vec{w}, \vec{x}, \vec{y}$ are non-parallel & no one of them lies on plane determined by other 2
    - set $\{\vec{w}, \vec{x}, \vec{y}\}$ is linearly independent



- volume of parallelepiped is height × base area
  - ↳ base area = $\|\vec{x} \times \vec{y}\|$
  - ↳ height is length of projection of $\vec{w}$ onto $\vec{x} \times \vec{y}$
  - ↳ $V = \|\vec{x} \times \vec{y}\| \, \|\text{proj}_{\vec{x} \times \vec{y}} \vec{w}\|$

$$= \|\vec{x} \times \vec{y}\| \, \left\| \frac{\vec{w} \cdot (\vec{x} \times \vec{y})}{\|\vec{x} \times \vec{y}\|^2} (\vec{x} \times \vec{y}) \right\|$$

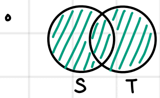
$$= \|\vec{x} \times \vec{y}\| \, \frac{|\vec{w} \cdot (\vec{x} \times \vec{y})|}{\|\vec{x} \times \vec{y}\|^2} \, \|\vec{x} \times \vec{y}\|$$

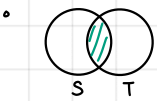
$$V = |\vec{w} \cdot (\vec{x} \times \vec{y})|$$

INTRODUCTION TO SET THEORY

set is collection of objects & objects are elements of set
- ↳ when set is empty, write $\emptyset$ instead of $\{\}$
- ↳ $x \in S$ when x is element of S
- ↳ $x \notin S$ when x isn't element of S

- set builder notation is when arbitrary element is described
  - ↳ e.g. $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$

if S, T are 2 sets:
- ↳ union of S & T is $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$
  - 

- ↳ intersection of S & T is $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$
  - 

- let S, T be sets; S is subset of T ($S \subseteq T$) if for every $x \in S$, there's an $x \in T$
  - ↳ if not, write $S \not\subseteq T$
- distinguish btwn element of set & subset of set
  - ↳ e.g. $1 \in \{1, 2, 3\}$ but $1 \not\subseteq \{1, 2, 3\}$
  - ↳ e.g. $\{1\} \notin \{1, 2, 3\}$ but $\{1\} \subseteq \{1, 2, 3\}$
  - ↳ e.g. $\{1, 2\} \in \{1, 2, \{1, 2\}\}$ & $\{1, 2\} \subseteq \{1, 2, \{1, 2\}\}$
- for any set S, $\emptyset \subseteq S$
  - ↳ vacuously true b/c can't show it's false
- if S, T are sets, $S = T$ when $S \subseteq T$ & $T \subseteq S$

week 4

SPANNING SETS

let $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$; span of B is

$\text{Span}\, B\{c_1\vec{v}_1 + \ldots c_k\vec{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$

↳ set SpanB is spanned by B & B is spanning set for SpanB

↳ SpanB is set of all linear combinations of vectors $\vec{v}_1, \ldots, \vec{v}_k$

↳ for $i = 1, \ldots, k$, $\vec{v}_i = 0\vec{v}_1 + \ldots + 0\vec{v}_{i-1} + 1\vec{v}_i + 0\vec{v}_{i+1} + \ldots + 0\vec{v}_k$ so it shows that $B \subseteq \text{Span}B$

e.g. describe subset $S = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ of $\mathbb{R}^3$ geometrically

SOLUTION

By definition, $S = \left\{s\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}1\\1\\0\end{bmatrix} \mid s, t \in \mathbb{R}\right\}$

↳ vector equation for S is $\vec{x} = s\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}1\\1\\0\end{bmatrix}$, $s, t \in \mathbb{R}$

↳ since 2 vectors aren't parallel, S is plane in $\mathbb{R}^3$ through origin

- if one vector in spanning set is linear combo of other vectors in spanning set, can be removed & resulting smaller set would still span S

- let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$; one of these vectors $\vec{v}_i$ can be expressed as linear combo of others $(\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots \vec{v}_k)$ iff $\text{Span}\{\vec{v}_1, \ldots \vec{v}_k\} = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \vec{v}_k\}$

PROOF

Assume $i = k$ (i.e. $\vec{v}_i$ = last vector in set $\vec{v}_k$)

Let $A = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$

Let $B = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$

Implication 1: if $\vec{v}_k$ can be expressed as linear combo of $\vec{v}_1, \ldots, \vec{v}_{k-1}$, then $A = B$

↳ assume $\vec{v}_k$ can be expressed as linear combo of $\vec{v}_1, \ldots, \vec{v}_{k-1}$ so there exists $c_1, \ldots, c_{k-1} \in \mathbb{R}$ such that $\vec{v}_k = c_1\vec{v}_1 + \ldots + c_{k-1}\vec{v}_{k-1}$

↳ to prove $A = B$

1) let $\vec{x} \in A$

- there exist $d_1, \ldots, d_{k-1}, d_k \in \mathbb{R}$ such that

$\vec{x} = d_1\vec{v}_1 + \ldots + d_{k-1}\vec{v}_{k-1} + d_k\vec{v}_k$

$= d_1\vec{v}_1 + \ldots + d_{k-1}\vec{v}_{k-1} + d_k(c_1\vec{v}_1 + \ldots + c_{k-1}\vec{v}_{k-1})$

$= (d_1 + d_kc_1)\vec{v}_1 + \ldots + (d_{k-1} + d_kc_{k-1})\vec{v}_{k-1}$

- $\vec{x}$ can be expressed as linear combo of $\vec{v}_1, \ldots, \vec{v}_{k-1}$ so $\vec{x} \in B$ & $A \subseteq B$

2) let $\vec{y} \in B$

- there exist $a_1, \ldots, a_{k-1} \in \mathbb{R}$ such that

$\vec{y} = a_1\vec{v}_1 + \ldots + a_{k-1}\vec{v}_{k-1}$

$= a_1\vec{v}_1 + \ldots + a_{k-1}\vec{v}_{k-1} + 0\vec{v}_k$

- $\vec{y}$ can be expressed as linear combo of $\vec{v}_1, \ldots, \vec{v}_k$ so $\vec{y} \in A$ & $B \subseteq A$

↳ since $A \subseteq B$ & $B \subseteq A$, we conclude that $A = B$

Implication 2: if $A = B$, then $\vec{v}_k$ can be expressed as linear combo of $\vec{v}_1, \ldots, \vec{v}_{k-1}$

↳ assume $A = B$

↳ since $\vec{v}_k \in A$ & $A = B$, then $\vec{v}_k \in B$

- this means there exist $b_1, \ldots, b_{k-1} \in \mathbb{R}$ such that $\vec{v}_k = b_1\vec{v}_1 + \ldots + b_{k-1}\vec{v}_{k-1}$ as required

Both implications have been proven so entire theorem has been proven true

LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

- let $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ be set of vectors in $\mathbb{R}^n$; B is linearly dependent if there exist $c_1, \dots, c_k \in \mathbb{R}$ (at least one is non-zero) so that $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$
  - ↳ B is linearly independent if only solution to $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$ is $c_1 = \dots = c_k = 0$
    - aka trivial solution
- a set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ is linearly dependent iff $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \vec{v}_k\}$ for some $i = 1, \dots, k$
  - ↳ i.e. the set is linearly dependent iff one of its vectors can be expressed as linear combo of others

PROOF

Implication 1:

↳ assume set $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ is linearly dependent so there exist $c_1, \dots, c_k \in \mathbb{R}$ (not all zero) such that $c_1\vec{v}_1 \dots + c_{i-1}\vec{v}_{i-1} + c_i\vec{v}_i + c_{i+1}\vec{v}_{i+1} + \dots + c_k\vec{v}_k = 0$

↳ assume $c_i \neq 0$ so isolate for $\vec{v}_i$

$$\vec{v}_i = -\frac{c_1}{c_i}\vec{v}_1 - \dots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1} - \frac{c_{i+1}}{c_i}\vec{v}_{i+1} - \dots \frac{c_k}{c_i}\vec{v}_k$$

↳ proves that $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \vec{v}_k\}$

Implication 2:

↳ assume that $\vec{v}_i \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \vec{v}_k\}$ for some $i = 1, \dots, k$ so there exist $d_1, \dots, d_{i-1}, d_{i+1}, d_k \in \mathbb{R}$ such that $\vec{v}_i = d_1\vec{v}_1 + \dots + d_{i-1}\vec{v}_{i-1} + d_{i+1}\vec{v}_{i+1} + \dots + d_k\vec{v}_k$

↳ rearranging gives $d_1\vec{v}_1 + \dots + d_{i-1}\vec{v}_{i-1} - 1\vec{v}_i + d_{i+1}\vec{v}_{i+1} + \dots + d_k\vec{v}_k = \vec{0}$

↳ proves that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent

Both implications have been proven so entire theorem has been proven true

- any subset of $\mathbb{R}^n$ containing $\vec{0} \in \mathbb{R}^n$ will be linearly dependent

- e.g. let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a linearly independent set of vectors in $\mathbb{R}^n$; prove that $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ is linearly independent

SOLUTION

Using proof by contradiction, suppose $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ is linearly dependent

↳ there must exist $c_1, \dots, c_{k-1}$ not all zero such that

$$c_1\vec{v}_1 + \dots + c_{k-1}\vec{v}_{k-1} = \vec{0}$$
$$c_1\vec{v}_1 + \dots + c_{k-1}\vec{v}_{k-1} + 0\vec{v}_k = \vec{0}$$

↳ adding $0\vec{v}_k$ to both sides shows that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent, which is a contradiction

↳ hence, supposition was wrong so $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$ is linearly independent

- given any linearly independent set B, every subset of B is linearly independent as well
  - ↳ empty subset contain no vectors so it's vacuously linearly independent

BASES

- let S be subset of $\mathbb{R}^n$; if $B = \{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent set of vectors in S such that $S = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then B is basis for S
  - ↳ basis for $\{\vec{0}\}$ is empty set $\emptyset$
  - ↳ e.g. prove $B = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is basis for $\mathbb{R}^2$

SOLUTION

First, prove Span $B = \mathbb{R}^2$:

- for $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \in \mathbb{R}^2$, $\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}$
  - → shows $\vec{x} \in \text{Span}B$ so $\mathbb{R}^2 \subseteq \text{Span}B$
- since $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \in \mathbb{R}^2$ & $\mathbb{R}^2$ is closed under linear combos, Span$B \subseteq \mathbb{R}^2$
- hence, Span$B = \mathbb{R}^2$

Next, show B is linearly independent:

- let $c_1, c_2 \in \mathbb{R}$
- $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
- $c_1 = c_2 = 0$ so B is linearly independent

$\therefore$ B is basis for $\mathbb{R}^2$

- for $i = 1, \ldots, n$, let $\vec{e}_i \in \mathbb{R}^n$ be a vector whose i^{th} entry is 1 & other $n-1$ entries be 0
  - ↳ set $\{e_1, \ldots, e_n\}$ is standard basis for $\mathbb{R}^n$
- if $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for a set $S \subseteq \mathbb{R}^n$, then every $\vec{x} \in S$ can be expressed as a linear combo of $\vec{v}_1, \ldots, \vec{v}_k$ in a unique way
  - ↳ basis B for subset S of $\mathbb{R}^n$ is minimal spanning set
    - i.e. B spans S but since it's linearly independent, can't remove vector from B

SUBSPACES OF $\mathbb{R}^n$

- since $\mathbb{R}^n$ is closed under linear combos, subsets of $\mathbb{R}^n$ w/a basis act like $\mathbb{R}^n$ itself under addition & scalar multiplication
- subset S of $\mathbb{R}^n$ is subspace of $\mathbb{R}^n$ if for every $\vec{w}, \vec{x}, \vec{y} \in S$ & $c, d \in \mathbb{R}$:
  - S1) $\vec{x} + \vec{y} \in S$
    - S is closed under addition
  - S2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
    - addition is commutative
  - S3) $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$
    - addition is associative
  - S4) zero vector: there exists $\vec{0} \in S$ such that $\vec{v} + \vec{0} = \vec{v}$ for every $\vec{v} \in S$
  - S5) for each $\vec{x} \in S$, there exists a $(-\vec{x}) \in S$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
    - additive inverse
  - S6) $c\vec{x} \in S$
    - S is closed under scalar multiplication
  - S7) $c(d\vec{x}) = (cd)\vec{x}$
    - scalar multiplication is associative
  - S8) $(c+d)\vec{x} = c\vec{x} + d\vec{x}$
    - distributive law
  - S9) $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
    - distributive law
  - S10) $1\vec{x} = \vec{x}$
    - scalar multiplicative identity
- $\mathbb{R}^n$ is subspace of $\mathbb{R}^n$
- $S = \{\vec{0}\}$ is subspace of $\mathbb{R}^n$ called trivial subspace

  $\emptyset$ is not subspace of $\mathbb{R}^n$ since $\vec{0} \notin \emptyset$ so S4 fails to hold

- to show non-empty subset S of $\mathbb{R}^n$ is subspace of $\mathbb{R}^n$, verify S1, S4, S5, & S6 b/c they depend on S & not on $\mathbb{R}^n$
  - ↳ once S1 & S6 hold, can conclude that for any $\vec{x} \in S$, $0\vec{x} = \vec{0} \in S$ & $(-\vec{x}) = (-1)x \in S$ which prove S4 & S5 respectively
- Subspace Test: let S be non-empty subset of $\mathbb{R}^n$; if for every $\vec{x}, \vec{y} \in S$ & for every $c \in \mathbb{R}$ $\vec{x} + \vec{y} \in S$ & $c\vec{x} \in S$, then S is subspace of $\mathbb{R}^n$
  - ↳ e.g. prove $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_1 + x_2 = 0 \text{ \& } x_2 - x_3 = 0 \right\}$ is subspace of $\mathbb{R}^3$

    SOLUTION

    By definition, $S \subseteq \mathbb{R}^3$ & since $0 + 0 = 0$ & $0 - 0 = 0$, $\vec{0} \in S$ so S is non-empty

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ be 2 vectors in $\mathbb{S}$

- $x_1 + x_2 = 0 = y_1 + y_2$ & $x_2 - x_3 = 0 = y_2 - y_3$

Prove $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ belongs to $\mathbb{S}$ by showing $(x_1 + y_1) + (x_2 + y_2) = 0$ & $(x_2 + y_2) - (x_3 + y_3) = 0$

- $(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = 0 + 0 = 0$
- $(x_2 + y_2) - (x_3 + y_3) = (x_2 - x_3) + (y_2 - y_3) = 0 - 0 = 0$
- thus, $\vec{x} + \vec{y} \in \mathbb{S}$

Prove $c\vec{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$ belongs to $\mathbb{S}$ by showing $cx_1 + cx_2 = 0$ & $cx_2 - cx_3 = 0$

- $cx_1 + cx_2 = c(x_1 + x_2) = c(0) = 0$
- $cx_2 - cx_3 = c(x_2 - x_3) = c(0) = 0$
- thus, $c\vec{x} \in \mathbb{S}$

$\therefore$ $\mathbb{S}$ is subspace of $\mathbb{R}^3$

let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$; $\mathbb{S} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is subspace of $\mathbb{R}^n$

↳ every subspace $\mathbb{S}$ of $\mathbb{R}^n$ can be expressed as $\mathbb{S} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ for some $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{S}$

BASES OF SUBSPACES

- $\{\vec{v}_1, \dots, \vec{v}_k\}$ is spanning set for subspace $\mathbb{S}$
  - ↳ remove any dependencies & we're left w/a basis for $\mathbb{S}$
- when finding spanning set for subspace $\mathbb{S}$ of $\mathbb{R}^n$, choose arbitrary $\vec{x} \in \mathbb{S}$ & try to decompose $\vec{x}$ as linear combo of some $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{S}$
  - ↳ this shows $\mathbb{S} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$
  - ↳ $\mathbb{S} \subseteq \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ implies $\mathbb{S} = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ b/c $\mathbb{S}$ is subspace so it contains all linear combos of $\vec{v}_1, \dots, \vec{v}_k$
    - don't normally show $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \subseteq \mathbb{S}$
- e.g. find a basis for subspace $\mathbb{S} = \left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix}, a, b, c \in \mathbb{R} \right\}$ of $\mathbb{R}^3$

<u>SOLUTION</u>

Let $\vec{x} \in \mathbb{S}$ so that for some $a, b, c \in \mathbb{R}$.

$\vec{x} = \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

↳ $\mathbb{S} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Since $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, then $\mathbb{S} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

↳ $\mathcal{B}$ is basis for $\mathbb{S}$ b/c it's linearly independent (neither vector in $\mathcal{B}$ is scalar multiple of the other)

↳ $\mathbb{S}$ is plane through origin & vector equation is $\vec{x} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $s, t \in \mathbb{R}$

- $x_1 + x_2 + x_3 = 0$ is scalar equation

week 5

K-FLATS AND HYPERPLANES

- if P is point in $\mathbb{R}^n$ & $\vec{p} = \vec{OP}$:
  - ↳ let $\vec{v}_1 \in \mathbb{R}^n$ be such that $\{\vec{v}_1\}$ is linearly independent (i.e. $\vec{v}_1 \neq \vec{0}$),
    - set w/ vector equation $\vec{x} = \vec{p} + c_1\vec{v}_1$, $c_1 \in \mathbb{R}$ is line in $\mathbb{R}^n$ through P
  - ↳ let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ be such that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent
    - set w/ vector equation $\vec{x} = \vec{p} + c_1\vec{v}_1 + c_2\vec{v}_2$, $c_1, c_2 \in \mathbb{R}$ is plane in $\mathbb{R}^n$ through P
- for some +ve integer $k \leq n-1$, let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ be such that $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent
  - ↳ set w/ vector equation $\vec{x} = \vec{p} + c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$, $c_1, \ldots, c_k \in \mathbb{R}$ is k-flat in $\mathbb{R}^n$ through P
    - e.g. 0-flat is point, 1-flat is line, & 2-flat is plane
- let $\vec{v}_1, \ldots, \vec{v}_{n-1} \in \mathbb{R}^n$ be such that $\{\vec{v}_1, \ldots, \vec{v}_{n-1}\}$ is linearly independent
  - ↳ set w/ vector equation $\vec{x} = \vec{p} + c_1\vec{v}_1 + \ldots + c_{n-1}\vec{v}_{n-1}$, $c_1, \ldots, c_{n-1} \in \mathbb{R}$ is hyperplane in $\mathbb{R}^n$ through P
    - hyperplanes are only k-flats in $\mathbb{R}^n$ w/ scalar equations
- subspaces in $\mathbb{R}^n$ are all k-flats through origin for $k = 0, 1, \ldots, n-1$ along w/ $\mathbb{R}^n$ itself
  - ↳ k-flat w/ vector equation $\vec{x} = \vec{p} + c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$ is subspace of $\mathbb{R}^n$ iff $\vec{p} \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$

ORTHOGONAL SETS AND BASES

NOTE: in an orthogonal set, all vectors in set must be orthogonal to every other vector in set

- orthogonal set $\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is orthonormal set (OS) if $\|\vec{v}_i\| = 1$ for $i = 1, \ldots, k$
  - ↳ if orthonormal set B is basis for subspace S of $\mathbb{R}^n$, then B is orthonormal basis (OB) for S
  - ↳ OS must be linearly independent
- given orthogonal basis $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ for subspace S of $\mathbb{R}^n$, get OB: $C = \{\vec{w}_1, \ldots, \vec{w}_k\}$ for S by letting $\vec{w}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i$ for $i = 1, \ldots, k$
- if B is OB for subspace S of $\mathbb{R}^n$, then B is orthogonal basis for S
  - ↳ for any $\vec{x} \in S$, $\vec{x} = \frac{\vec{x}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \ldots + \frac{\vec{x}\cdot\vec{v}_k}{\|\vec{v}_k\|^2}\vec{v}_k$
    - since $\|\vec{v}_i\| = 1$ for any $i = 1, \ldots, k$, then $\vec{x} = (\vec{x}\cdot\vec{v}_1)\vec{v}_1 + \ldots + (\vec{x}\cdot\vec{v}_k)\vec{v}_k$

NOTE: if $\vec{z} \in \mathbb{R}^n$ & $D = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is basis for $\mathbb{R}^n$, then there exist unique $c_1, \ldots, c_n \in \mathbb{R}$ such that $\vec{z} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$; $[\vec{z}]_D = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is called D-vector of $\vec{z}$

- e.g. $B = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}6\\-2\end{bmatrix}\right\}$ is orthogonal basis for $\mathbb{R}^2$; obtain orthonormal basis C for $\mathbb{R}^2$ from B & express $\vec{x} = \begin{bmatrix}-1\\1\end{bmatrix}$ as linear combo of vectors in C

SOLUTION

$$\left\|\begin{bmatrix}1\\3\end{bmatrix}\right\| = \sqrt{1^2+3^2} = \sqrt{10} \qquad \left\|\begin{bmatrix}6\\-2\end{bmatrix}\right\| = \sqrt{6^2+(-2)^2} = \sqrt{40} = 2\sqrt{10}$$

$C = \left\{\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}, \begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{10}\end{bmatrix}\right\}$ is OB for $\mathbb{R}^2$

$$\begin{bmatrix}-1\\1\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix} = -\frac{1}{\sqrt{10}} + \frac{3}{\sqrt{10}} = \frac{2}{\sqrt{10}}$$

$$\begin{bmatrix}-1\\1\end{bmatrix}\cdot\begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{10}\end{bmatrix} = -\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}} = -\frac{4}{\sqrt{10}}$$

$$\therefore \vec{x} = \frac{2}{\sqrt{10}}\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix} - \frac{4}{\sqrt{10}}\begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{10}\end{bmatrix}.$$

SYSTEMS OF LINEAR EQUATIONS

- linear equation in n variables is equation of form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$
  - ↳ $x_1, \ldots, x_n \in \mathbb{R}$ are variables/unknowns
  - ↳ $a_1, \ldots, a_n \in \mathbb{R}$ are coeffs
  - ↳ $b \in \mathbb{R}$ is constant term

- system of linear equations is collection of finitely many linear equations
  - ↳ system of m linear equations in n variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad & \;\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n &= b_m
\end{aligned}
$$

  - a_{ij} is coeff of x_j in i^{th} equation
  - b_i is constant term in i^{th} equation
  - each of m equations is scalar equation of hyperplane in $\mathbb{R}^n$
- vector $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \in \mathbb{R}^n$ is solution to system of m equations in n variables if all m equations are satisfied when $x_j = s_j$ for $j = 1, \dots, n$
  - ↳ solution set is set of all solutions to system of equations
  - ↳ solution set of a system of m equations in n variables is intersection of m hyperplanes determined by system
- for any linear system of m equations in n variables, can either have no solutions, exactly 1 solution, or infinitely many solutions
- linear system of equations is consistent if it has at least 1 solution
  - ↳ otherwise, called inconsistent

- augmented matrix of linear system, denoted by $[A \mid \vec{b}]$, has 2 parts:
  - ↳ coefficient matrix denoted by A
  - ↳ constant matrix/vector denoted by $\vec{b}$
- allowed to perform Elementary Row Operations (EROs) to augmented matrix:
  - ↳ swap 2 rows
  - ↳ add scalar multiple of 1 row to another
  - ↳ multiply any row by non-zero scalar
- 2 systems are equivalent if they have same solution set
  - ↳ system derived from doing EROs on augmented matrix is equivalent to given system
  - ↳ then, can read off solutions from simplified system
- e.g. solve linear system of equations:

$$
\begin{aligned}
2x_1 + x_2 + 9x_3 &= 31 \\
x_2 + 2x_3 &= 8 \\
x_1 \qquad + 3x_3 &= 10
\end{aligned}
$$

SOLUTION

$$
\left[\begin{array}{ccc|c} 2 & 1 & 9 & 31 \\ 0 & 1 & 2 & 8 \\ 1 & 0 & 3 & 10 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 2 & 1 & 9 & 31 \end{array}\right] \xrightarrow[R_3 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & 3 & 11 \end{array}\right] \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_1 - 3R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2 - 2R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

$x_1 = 1$, $x_2 = 2$, $x_3 = 3$ or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ or $(x_1, x_2, x_3) = (1, 2, 3)$ is solution

Another technique is back substitution:

- ↳ from $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$, we get

$$
\begin{aligned}
x_1 \qquad + 3x_3 &= 10 \\
x_2 + 2x_3 &= 8 \\
x_3 &= 3
\end{aligned}
$$

- ↳ since $x_3 = 3$:
  - $x_2 = 8 - 2x_3 = 8 - 2(3) = 2$
  - $x_1 = 10 - 3x_3 = 10 - 3(3) = 1$
- ↳ arrive at same solution of $x_1 = 1$, $x_2 = 2$, & $x_3 = 3$

SOLVING SYSTEMS OF LINEAR EQUATIONS

- 1^{st} non-zero entry in each row of matrix is leading entry/pivot

- matrix is in **Row Echelon Form (REF)** if:
  - ↳ all rows whose entries are all zero appear below all rows that contain non-zero entries
  - ↳ each leading entry is to the right of leading entries above it
- matrix is in **Reduced Row Echelon Form (RREF)** if:
  - ↳ it's in REF
  - ↳ each leading entry is 1 (aka **leading one**)
  - ↳ eaching leading one is only non-zero entry in column
- any matrix has many REFs but RREF is unique for any matrix
- e.g. solve

$$\begin{aligned} 3x_1 + x_2 &= 10 \\ 2x_1 + x_2 + x_3 &= 6 \\ -3x_1 + 4x_2 + 15x_3 &= -20 \end{aligned}$$

SOLUTION

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 10 \\ 2 & 1 & 1 & 6 \\ -3 & 4 & 15 & -20 \end{array}\right] \xrightarrow[R_1 - R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 2 & 1 & 1 & 6 \\ -3 & 4 & 15 & -20 \end{array}\right] \xrightarrow[R_3 + 3R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 2 & 1 & 1 & 6 \\ 0 & 4 & 12 & 8 \end{array}\right] \xrightarrow[R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 4 & 12 & 8 \end{array}\right] \xrightarrow[R_3 - 4R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 \quad - x_3 = 4 \longrightarrow x_1 = 4 + x_3$$

$$x_2 + 3x_3 = -2 \longrightarrow x_2 = -2 - 3x_3$$

$$0 = 0$$

↳ no restriction on x_3 so let $x_3 = t$, $t \in \mathbb{R}$

↳ solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$

- ◦ line in $\mathbb{R}^3$

- for consistent system of equations w/ augmented matrix $[A \mid \vec{b}]$ & let $[R \mid \vec{c}]$ be any REF of $[A \mid \vec{b}]$:
  - ↳ if j^{th} column of R has leading entry, x_j is **leading variable**
  - ↳ if j^{th} column of R doesn't have leading entry, x_j is **free variable**
- existence of free variable guarantees infinitely many solutions
- if row reducing augmented matrix reveals row of form $[0 \, .. \, 0 \mid c]$ w/ $c \neq 0$, then system is inconsistent
- e.g. solve

$$\begin{aligned} jz_1 - z_2 - z_3 + (-1 + j) z_4 &= -1 \\ -(1+j) z_3 - 2jz_4 &= -1 - 3j \\ 2jz_1 - 2z_2 - z_3 - (1-3j) z_4 &= j \end{aligned}$$

SOLUTION

$$\left[\begin{array}{cccc|c} j & -1 & -1 & -1+j & -1 \\ 0 & 0 & -1-j & -2j & -1-3j \\ 2j & -2 & -1 & -1+3j & j \end{array}\right] \xrightarrow[R_3 - 2R_1]{} \left[\begin{array}{cccc|c} j & -1 & -1 & -1+j & -1 \\ 0 & 0 & -1-j & -2j & -1-3j \\ 0 & 0 & 1 & 1+j & 2+j \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{-jR_1} \left[\begin{array}{cccc|c} 1 & j & j & 1+j & j \\ 0 & 0 & 1 & 1+j & 2+j \\ 0 & 0 & -1-j & -2j & -1-3j \end{array}\right] \xrightarrow[R_3 + (1+j)R_2]{}$$

$$\left[\begin{array}{cccc|c} 1 & j & j & 1+j & j \\ 0 & 0 & 1 & 1+j & 2+j \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - jR_2]{} \left[\begin{array}{cccc|c} 1 & j & 0 & 2 & 1-j \\ 0 & 0 & 1 & 1+j & 2+j \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

↳ system is consistent

↳ z_1 & z_3 are leading variables

↳ z_2 & z_4 are free variables

$$z_1 + jz_2 \quad + 2z_4 = 1 - j$$

$$z_3 + (1+j) z_4 = 2 + j$$

Let $z_2 = s$ & $z_4 = t$ where $s, t \in \mathbb{R}$.

$$\begin{aligned} z_1 &= (1-j) - js - 2t \\ z_2 &= s \\ z_3 &= (2+j) - (1+j)t \\ z_4 &= t \end{aligned} \quad , \; s, t \in \mathbb{R} \quad \text{or} \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1-j \\ 0 \\ 2+j \\ 0 \end{bmatrix} + s \begin{bmatrix} -j \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1-j \\ 1 \end{bmatrix}, \; s, t \in \mathbb{R}$$

- let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$; $\vec{b} \in \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ iff system w/ augmented matrix $[\vec{v}_1 \ldots \vec{v}_k \mid \vec{b}]$ is consistent

RANK

rank of matrix A, denoted by $\operatorname{rank}(A)$, is # of leading entries in any REF of A

↳ if matrix has m rows & n columns, then **$\operatorname{rank}(A) \leq \min\{m, n\}$** b/c there can be at most 1 leading entry in each row & column

- **System-Rank Theorem**: let $[A|\vec{b}]$ be augmented matrix of system of m linear equations in n variables
  1) system is consistent iff $\text{rank}(A) = \text{rank}([A|\vec{b}])$
  2) if system is consistent, then # of parameters $= n - \text{rank}(A)$
  3) system is consistent for all $\vec{b} \in \mathbb{R}^m$ iff $\text{rank}(A) = m$

↳ e.g. analyze system of $m=3$ linear equations in $n=3$ variables

$$\begin{aligned} 2x_1 + 12x_2 - 8x_3 &= -4 \\ 2x_1 + 13x_2 - 6x_3 &= -5 \\ -2x_1 - 14x_2 + 4x_3 &= 7 \end{aligned}$$

SOLUTION

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 2 & 12 & -8 & -4 \\ 2 & 13 & -6 & -5 \\ -2 & -14 & 4 & 7 \end{array}\right] \xrightarrow[R_2 - R_1]{} \left[\begin{array}{ccc|c} 2 & 12 & -8 & -4 \\ 0 & 1 & 2 & -1 \\ -2 & -14 & 4 & 7 \end{array}\right] \xrightarrow[R_3 + R_1]{} \left[\begin{array}{ccc|c} 2 & 12 & -8 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & 3 \end{array}\right] \xrightarrow[R_3 + 2R_2]{} \left[\begin{array}{ccc|c} 2 & 12 & -8 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

↳ inconsistent

From System-Rank Theorem:

1) $\text{rank}(A) = 2 < 3 = \text{rank}([A|\vec{b}])$ so system is inconsistent
2) system is inconsistent so N/A
3) $\text{rank}(A) = 2 < 3 = m$ so system won't be consistent for every $\vec{b} \in \mathbb{R}^3$

↳ e.g. find equation that $b_1, b_2, b_3 \in \mathbb{R}$ must satisfy so

$$\begin{aligned} 2x_1 + 12x_2 - 8x_3 &= b_1 \\ 2x_1 + 13x_2 - 6x_3 &= b_2 \\ -2x_1 - 14x_2 + 4x_3 &= b_3 \end{aligned}$$

is consistent

SOLUTION

$$\left[\begin{array}{ccc|c} 2 & 12 & -8 & b_1 \\ 2 & 13 & -6 & b_2 \\ -2 & -14 & 4 & b_3 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 12 & -8 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & -2 & -4 & b_3 + b_1 \end{array}\right] \xrightarrow[R_3 + 2R_2]{} \left[\begin{array}{ccc|c} 2 & 12 & -8 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & -b_1 + 2b_2 + b_3 \end{array}\right]$$

- since $\text{rank}(A) = 2$, then $\text{rank}([A|\vec{b}]) = 2$ for consistency
- so $-b_1 + 2b_2 + b_3 = 0$ for system to be consistent

week 6

- e.g. for which values of $k, \ell \in \mathbb{R}$ does $\begin{aligned} 2x_1 + 6x_2 &= 5 \\ 4x_1 + (k+15)x_2 &= \ell + 8 \end{aligned}$ have no, unique, & infinite solutions?

SOLUTION

Let $A = \begin{bmatrix} 2 & 6 \\ 4 & k+15 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ \ell+8 \end{bmatrix}$

Carry $[A|\vec{b}]$ to REF:

$$\left[\begin{array}{cc|c} 2 & 6 & 5 \\ 4 & k+15 & \ell+8 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & 6 & 5 \\ 0 & k+3 & \ell-2 \end{array}\right]$$

↳ if $k+3 \neq 0$, then $\operatorname{rank}(A) = \operatorname{rank}([A|\vec{b}])$ so system is consistent
  - $k \neq -3$
  - # of parameters $= n - \operatorname{rank}(A) = 2 - 2 = 0$ so unique solution

↳ if $k+3=0$, $\left[\begin{array}{cc|c} 2 & 6 & 5 \\ 0 & k+3 & \ell-2 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 6 & 5 \\ 0 & 0 & \ell-2 \end{array}\right]$
  - if $\ell - 2 \neq 0 \rightarrow \ell \neq 2$, then $\operatorname{rank}(A) = 1 < 2 = \operatorname{rank}([A|\vec{b}])$ so the system is inconsistent & has no solutions
  - if $\ell - 2 = 0 \rightarrow \ell = 2$, then system is consistent w/ # of parameters $= 2 - \operatorname{rank}(A) = 2 - 1 = 1$; this means there's infinitely many solutions

Summary:

↳ unique: $k \neq -3$

↳ none: $k = -3$ and $\ell \neq 2$

↳ infinite: $k = -3$ and $\ell = 2$

- linear system of m equations in n variables is underdetermined if $n > m$
  - ↳ i.e. more variables than equations
  - ↳ a consistent underdetermined linear system of equations has infinitely many solutions
- linear system of m equations in n variables is overdetermined if $n < m$
  - ↳ i.e. more equations than variables
  - ↳ often inconsistent

HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

- homogeneous linear equation is when constant term is 0
  - ↳ system of homogeneous linear equations is collection of finitely many homogeneous equations
  - ↳ written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

  - ↳ every homogeneous system has trivial solution ($\vec{0} \in \mathbb{R}^n$) so they're always consistent
- given non-homogeneous system of equations w/ augmented matrix $[A|\vec{b}]$ ($\vec{b} \neq \vec{0}$), associated homogeneous system has augmented matrix $[A|\vec{0}]$
  - ↳ solution to a consistent non-homogeneous system of equations is a particular solution, which is a point in $\mathbb{R}^n$ the solution goes through, plus the solution to the associated homogeneous system of equations

- let S be solution set to homogeneous system of m equations in n variables; then S is a subspace of $\mathbb{R}^n$
  - ↳ solution set of homogeneous system is solution space
- let $[A|\vec{b}]$ be augmented matrix for consistent system of m linear equations in n variables; if $\operatorname{rank}(A) = k < n$, then general solution of system is: $\vec{x} = \vec{d} + t_1\vec{v}_1 + \dots t_{n-k}\vec{v}_{n-k}$

- ↳ $\vec{d} \in \mathbb{R}^n$
- ↳ $t_1, \ldots, t_{n-k} \in \mathbb{R}$
- ↳ $\{\vec{v}_1, \ldots, \vec{v}_{n-k}\} \subseteq \mathbb{R}^n$ is linearly independent
- ↳ solution set is $(n-k)$-flat in $\mathbb{R}^n$

- spanning set for solution space of homogeneous system is always linearly independent
- given a spanning set $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ for subspace $\mathbb{S}$ of $\mathbb{R}^n$, to find basis B' for $\mathbb{S}$ w/ $B' \subseteq B$, construct matrix $[\vec{v}_1 \ \ldots \ \vec{v}_k]$ & carry to REF / RREF
  - ↳ for $i = 1, \ldots, k$, take $\vec{v}_i \in B'$ iff i^{th} column of any REF of matrix has leading entry
  - ↳ when $\vec{v}_j \notin B'$, $\vec{v}_j$ is linear combo of vectors in $\{\vec{v}_1, \ldots, \vec{v}_{j-1}\} \cap B'$
  - ↳ e.g. let $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} \right\}$; find basis B' for Span$B$ w/ $B' \subseteq B$

SOLUTION

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ -1 & 2 & 5 & 6 \\ 1 & -3 & -7 & -9 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 + R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 6 & 9 \\ 0 & -4 & -8 & -12 \end{bmatrix} \xrightarrow[R_3/(-4)]{R_2/3} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 - R_2]{} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 - R_2]{} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since first 2 columns contain leading entries, first 2 vectors in B make up B'.
$B' = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$ is basis for Span$B$.

DIMENSION

- let $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be basis for subspace $\mathbb{S}$ of $\mathbb{R}^n$; if $C = \{\vec{w}_1, \ldots, \vec{w}_\ell\}$ is set in $\mathbb{S}$ w/ $\ell > k$, then C is linearly dependent
  - ↳ if C is linearly independent, then $\ell \leq k$
- if $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ & $C = \{\vec{w}_1, \ldots, \vec{w}_\ell\}$ are both bases for subspace $\mathbb{S}$ of $\mathbb{R}^n$, then $k = \ell$
  - ↳ given non-trivial subspace $\mathbb{S}$ of $\mathbb{R}^n$, all bases have same # of vectors
- if $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is basis for subspace $\mathbb{S}$ of $\mathbb{R}^n$, dimension of $\mathbb{S}$ is k
  - ↳ $\dim(\mathbb{S}) = k$
  - ↳ if $\mathbb{S} = \{\vec{0}\}$, $\dim(\mathbb{S}) = 0$ since $\emptyset$ is basis for $\mathbb{S}$
  - ↳ e.g. standard basis for $\mathbb{R}^n$ is $\{\vec{e}_1, \ldots, \vec{e}_n\}$ so $\dim(\mathbb{R}^n) = n$
- if $\mathbb{S}$ is k-dimensional subspace of $\mathbb{R}^n$ w/ $k > 0$, then:
  - ↳ set of more than k vectors in $\mathbb{S}$ is linearly dependent
  - ↳ set of fewer than k vectors in $\mathbb{S}$ can't span $\mathbb{S}$
  - ↳ set of k vectors in $\mathbb{S}$ span $\mathbb{S}$ iff it's linearly independent

APPLICATION: CHEMICAL REACTIONS

- chemical reaction is process by which molecules combine to form new molecules
- e.g. photosynthesis is rep by $CO_2 + H_2O \to C_6H_{12}O_6 + O_2$; balance this equation

SOLUTION

Let x_1 rep. # of CO_2 molecules. Let x_2 rep. # of H_2O molecules. Let x_3 rep. # of $C_6H_{12}O_6$ molecules. Let x_4 rep. # of O_2 molecules.

$$x_1 CO_2 + x_2 H_2O \to x_3 C_6H_{12}O_6 + x_4 O_2$$

Equate # of atoms of each type:

C : $x_1 = 6x_3$

O : $2x_1 + x_2 = 6x_3 + 2x_4$

H : $2x_2 = 12x_3$

Homogeneous system:

$$\begin{aligned} x_1 - 6x_3 &= 0 \\ 2x_1 + x_2 - 6x_3 - 2x_4 &= 0 \\ 2x_2 - 12x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -6 & 0 \\ 2 & 1 & -6 & -2 \\ 0 & 2 & -12 & 0 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 6 & -2 \\ 0 & 2 & -12 & 0 \end{bmatrix} \xrightarrow[R_3 - 2R_2]{} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 6 & -2 \\ 0 & 0 & -24 & 4 \end{bmatrix} \xrightarrow[R_3/4]{} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 6 & -2 \\ 0 & 0 & -6 & 1 \end{bmatrix} \xrightarrow[R_1 - R_3]{} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 6 & -2 \\ 0 & 0 & -6 & 1 \end{bmatrix} \xrightarrow[R_2 + R_3]{}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -6 & 1 \end{bmatrix} \xrightarrow[R_3/-6]{} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/6 \end{bmatrix}$$

↳ x_4 is free variable

Let $x_4 = t$, $t \in \mathbb{R}$.

$x_1 - x_4 = 0 \qquad x_2 - x_4 = 0 \qquad x_3 - \frac{1}{6}x_4 = 0$

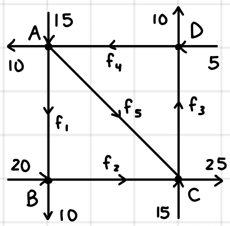
$x_1 = x_4 \qquad x_2 = x_4 \qquad x_3 = \frac{1}{6}x_4$

The solution set is $x_1 = t$, $x_2 = t$, $x_3 = \frac{1}{6}t$, $x_4 = t$ for $t \in \mathbb{R}$. Although there's infinite solutions, x_1, x_2, x_3, & x_4 must be not -ve integers & we want smallest solution. Let $t = 6$ $x_1 = x_2 = x_4 = 6$ and $x_3 = 1$.

$6CO_2 + 6H_2O \longrightarrow C_6H_{12}O_6 + 6O_2$ balances the chemical equation.

APPLICATION: NETWORK FLOW

- network consists of system of junctions/nodes connected by directed line segments
- Junction Rule: at each junction, flow into it must equal flow out of it
  - ↳ system where every junction obeys this rule is in steady state/equilibrium
- e.g. in figure, directed line segments rep train tracks & #s rep # of trains travelling on that track per day (assume tracks are one-way); find all values of $f_1, \ldots, f_5$ so system is in equilibrium; if f_4 & f_5 are closed due to maintenance, can system still be in equilibrium?



SOLUTION

	Flow in		Flow out
A:	$15 + f_4$	=	$10 + f_1 + f_5$
B:	$20 + f_1$	=	$10 + f_2$
C:	$15 + f_2 + f_5$	=	$25 + f_3$
D:	$5 + f_3$	=	$10 + f_4$

$$\begin{aligned} f_1 \qquad\qquad - f_4 + f_5 &= 5 \\ f_1 - f_2 \qquad\qquad\qquad &= -10 \\ f_2 - f_3 \qquad + f_5 &= 10 \\ f_3 - f_4 \qquad &= 5 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 5 \\ 1 & -1 & 0 & 0 & 0 & -10 \\ 0 & 1 & -1 & 0 & 1 & 10 \\ 0 & 0 & 1 & -1 & 0 & 5 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 5 \\ 0 & -1 & 0 & 1 & -1 & -15 \\ 0 & 1 & -1 & 0 & 1 & 10 \\ 0 & 0 & 1 & -1 & 0 & 5 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 5 \\ 0 & -1 & 0 & 1 & -1 & -15 \\ 0 & 0 & -1 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 0 & 5 \end{array}\right] \xrightarrow{R_4 + R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 5 \\ 0 & -1 & 0 & 1 & -1 & -15 \\ 0 & 0 & -1 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3(-1)]{R_2(-1)} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 5 \\ 0 & 1 & 0 & -1 & 1 & 15 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

↳ f_4 & f_5 are free variables

$f_1 - f_4 + f_5 = 5$

$f_2 - f_4 + f_5 = 15$

$f_3 - f_4 = 5$

Let $f_4 = s$ & $f_5 = t$ for $s, t \in \mathbb{Z}$ (no fractional trains).

$f_1 = 5 + s - t$, $f_2 = 15 + s - t$, $f_3 = 5 + s$, $f_4 = s$, and $f_5 = t$

$f_1 \geq 0 \longrightarrow 5 + s - t \geq 0 \longrightarrow s - t \geq -5$

$f_2 \geq 0 \longrightarrow 15 + s - t \geq 0 \longrightarrow s - t \geq -15$

$f_3 \geq 0 \longrightarrow 5 + s \geq 0 \longrightarrow s \geq -5$

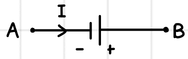
$f_4 \geq 0 \longrightarrow s \geq 0$

$f_5 \geq 0 \longrightarrow t \geq 0$

Thus, we have the solution set: $f_1 = 5 + s - t$, $f_2 = 15 + s - t$, $f_3 = 5 + s$, $f_4 = s$, and $f_5 = t$ for all $s, t \in \mathbb{Z}$ provided that $s \geq 0$, $t \geq 0$, and $s - t \geq -5$.

If f_4 & f_5 were shut down, $f_4 = f_5 = 0$. So, $s = t = 0$ and $s - t = 0 \geq -5$ so it's a valid solution. We have the unique solution: $f_1 = 5$, $f_2 = 15$, $f_3 = 5$, $f_4 = 0$, and $f_5 = 0$.

APPLICATION: ELECTRICAL NETWORKS

- electrical networks consist of voltage sources, resistors, & wires
- voltage source provides electromotive force V measured in volts
  - ↳ force moves e^- along wire at a rate referred to as the current I
    - measured in amperes/amps
  - ↳ e.g. 

  - ↳ if current passes from − to +, voltage increases
  - ↳ if current passes from + to −, voltage decrease
- resistors retard current by slowing flow of e^-
  - ↳ e.g. 

  - ↳ measured in ohms Ω
- nodes are intersection points btwn 3+ wires
- Ohm's Law: potential difference V across resistor is given by $V = IR$
  - ↳ current travelling in reference dir will result in voltage drop
  - ↳ current travelling against reference dir will result in voltage gain
- Kirchoff's Laws:
  1) Conservation of Energy: around any closed voltage loop in network, algebraic sum of voltage drops & increases is zero
  2) Conservation of Charge: at each node, total inflow of I = total outflow of I

week 7

MATRIX ALGEBRA

- an $m \times n$ matrix A is rectangular array w/ m rows & n columns
  - ↳ a_{ij} denotes entry in i^{th} row & j^{th} column
  - ↳ $A = [a_{ij}]$ when size of matrix is known
  - ↳ 2 $m \times n$ matrices A & B are equal if $a_{ij} = b_{ij}$ for all $i = 1, 2, \dots, m$ & $j = 1, 2, \dots, n$
    - write $A = B$
- set of all $m \times n$ matrices w/ real #s is denoted by $M_{m \times n}(\mathbb{R})$
  - ↳ for matrix $A \in M_{m \times n}(\mathbb{R})$, A has size $m \times n$ & a_{ij} is called (i,j)-entry of A
    - can write $(A)_{ij}$ instead of a_{ij}
  - ↳ if $m = n$, A is square matrix
- $m \times n$ matrix w/ all 0 entries is zero matrix
  - ↳ denoted by $O_{m \times n}$ or O if size is clear

- for $A, B \in M_{m \times n}(\mathbb{R})$ & $c \in \mathbb{R}$ ·
  - ↳ matrix addition is $(A + B)_{ij} = (A)_{ij} + (B)_{ij}$
  - ↳ scalar multiplication is $(cA)_{ij} = c(A)_{ij}$
- for any $A = M_{m \times n}(\mathbb{R})$ & any $c \in \mathbb{R}$:
  - ↳ $0A = O_{m \times n}$
  - ↳ $cO_{m \times n} = O_{m \times n}$
- properties of matrices where $A, B, C \in M_{m \times n}(\mathbb{R})$ & $c, d \in \mathbb{R}$:
  1) $A + B \in M_{m \times n}(\mathbb{R})$
     - $M_{m \times n}(\mathbb{R})$ is closed under addition
  2) $A + B = B + A$
     - addition is commutative
  3) $(A + B) + C = A + (B + C)$
     - addition is associative
  4) there exists $O_{m \times n} \in M_{m \times n}(\mathbb{R})$ such that $A + O_{m \times n} = A$ for every $A \in M_{m \times n}(\mathbb{R})$
     - zero matrix
  5) for each $A \in M_{m \times n}(\mathbb{R})$, there exists a $(-A) \in M_{m \times n}(\mathbb{R})$ such that $A + (-A) = O_{m \times n}$
     - additive inverse
  6) $cA \in M_{m \times n}(\mathbb{R})$
     - $M_{m \times n}(\mathbb{R})$ is closed under scalar multiplication
  7) $c(dA) = (cd)A$
     - scalar multiplication is associative
  8) $(c + d)A = cA + dA$
     - distributive law
  9) $c(A + B) = cA + cB$
     - distributive law
  10) $1A = A$
     - scalar multiplicative identity

- transpose of $A \in M_{m \times n}(\mathbb{R})$, denoted by A^T, is $n \times m$ matrix satisfying $(A^T)_{ij} = (A)_{ji}$
  - ↳ e.g $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ & $A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
  - ↳ e.g. $C = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$ & $C^T \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$

- properties of transpose where $A, B \in M_{m\times n}(\mathbb{R})$ & $c \in \mathbb{R}$:
  - ↳ $A^T \in M_{m\times n}(\mathbb{R})$
  - ↳ $(A^T)^T = A$
  - ↳ $(A+B)^T = A^T + B^T$
  - ↳ $(cA)^T = cA^T$
- e.g. solve for A if $(2A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix})^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

  SOLUTION

$$(2A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix})^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$
$$(2A^T)^T - (3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix})^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$
$$2A - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$
$$2A - 3\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$
$$2A - \begin{bmatrix} 3 & -3 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$
$$2A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 6 & 3 \end{bmatrix}$$
$$2A = \begin{bmatrix} 5 & 0 \\ 5 & 5 \end{bmatrix}$$
$$A = \begin{bmatrix} 5/2 & 0 \\ 5/2 & 5/2 \end{bmatrix}$$

matrix A is symmetric if $A^T = A$
- ↳ must be square matrix b/c $n = m$

MATRIX-VECTOR PRODUCT

- let $A = [\vec{a}_1 \dots \vec{a}_n] \in M_{m\times n}(\mathbb{R})$ ($\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^m$) & $\vec{x} = [x_1 \dots x_n]^T \in \mathbb{R}^n$ so that vector $A\vec{x} = x_1\vec{a} + \dots + x_n\vec{a}_n \in \mathbb{R}^m$
  - ↳ known as matrix-vector product
- every linear system of equations can be expressed as $A\vec{x} = \vec{b}$ for some matrix A & some vector $\vec{b}$
- system $A\vec{x} = \vec{b}$ is consistent iff $\vec{b}$ can be expressed as linear combo of columns of A
- if $\vec{a}_1, \dots, \vec{a}_n$ are columns of $A \in M_{m\times n}(\mathbb{R})$ & $\vec{x} = [x_1 \dots x_n]^T$, then $\vec{x}$ satisfies $A\vec{x} = \vec{b}$ iff $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{b}$
- sizes of matrices & vectors are important: $\underbrace{A}_{m\times n}\underbrace{\vec{x}}_{\mathbb{R}^n} = \underbrace{\vec{b}}_{\mathbb{R}^m}$
- for $A \in M_{m\times n}(\mathbb{R})$ & $\vec{x} \in \mathbb{R}^n$ w/ $A \neq 0_{m\times n}$ & $\vec{x} \neq \vec{0}_{\mathbb{R}^n}$, it's not guaranteed that $A\vec{x} \neq 0$
  - ↳ e.g. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- properties of matrix-vector product where $A, B \in M_{m\times n}(\mathbb{R})$, $\vec{x}, \vec{y} \in \mathbb{R}^n$, & $c \in \mathbb{R}$:
  - ↳ $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
  - ↳ $A(c\vec{x}) = c(A\vec{x}) = (cA)\vec{x}$
  - ↳ $(A+B)\vec{x} = A\vec{x} + B\vec{x}$
- e.g. let $A \in M_{m\times n}(\mathbb{R})$ & $\vec{x} \in \mathbb{R}^n$; let S be solution set to homogeneous system of equations $A\vec{x} = \vec{0}$; show S is subspace of $\mathbb{R}^n$

  SOLUTION

  Since $\vec{x} \in \mathbb{R}^n$, then $S \subseteq \mathbb{R}^n$. $A\vec{0}_{\mathbb{R}^n} = \vec{0}$ so $\vec{0}_{\mathbb{R}^n} \in S$ & S is non-empty.

  Let $\vec{y}, \vec{z} \in \mathbb{R}$.

$$\begin{aligned} A(\vec{y} + \vec{z}) &= A\vec{y} + A\vec{z} \\ &= \vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

  $\vec{y} + \vec{z} \in S$ so S is closed under vector addition.

  Let $c \in \mathbb{R}$.

$$\begin{aligned} A(c\vec{y}) &= c(A\vec{y}) \\ &= c\vec{0} \\ &= \vec{0} \end{aligned}$$

$c\vec{y} \in S$ so S is closed under scalar multiplication.

Hence, S is subspace of $\mathbb{R}^n$.

- given $A \in M_{m\times n}(\mathbb{R})$, there's vectors $\vec{r}_1, \ldots, \vec{r}_m \in \mathbb{R}^n$ so $A = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix}$ & for any $\vec{x} \in \mathbb{R}^n$:
  $A\vec{x} = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{r}_1^{\,T}\vec{x} \\ \vdots \\ \vec{r}_m^{\,T}\vec{x} \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$
  - ↳ i^{th} entry of $A\vec{x}$ is dot product $\vec{r}_i \cdot \vec{x}$ where $\vec{r}_i^{\,T}$ is i^{th} row of A
- $n \times n$ identity matrix, denoted by I_n (i.e. $I_{n\times n}$ or I if size is clear), is square matrix w/ $a_{ii} = 1$ for $i = 1, 2, \ldots, n$ (entries are main diagonal of matrix) & zeros elsewhere
  - ↳ e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\vec{e}_1 \;\; \vec{e}_2 \;\; \vec{e}_3]$
  - ↳ $I_n \vec{x} = \vec{x}$ for every $\vec{x} \in \mathbb{R}^n$
- e.g. let $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}$, & $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
  - ↳ $A\vec{x} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = B\vec{x}$
    - $A\vec{x} = B\vec{x}$ but $A \neq B$
  - ↳ as can be seen, $A\vec{x} = B\vec{x}$ for a given $\vec{x} \in \mathbb{R}^n$ isn't sufficient to guarantee $A = B$
- Matrices Equal Theorem: let $A, B \in M_{m\times n}(\mathbb{R})$; if $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$, then $A = B$

FUNDAMENTAL SUBSPACES OF A MATRIX

- fundamental subspaces of matrix are 3 sets: nullspace, column space, & row space
  - ↳ let $A \in M_{m\times n}(\mathbb{R})$; nullspace (i.e. kernel) of A is subset of $\mathbb{R}^n$ defined by $\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$
    - solution space of homogeneous system of equations $A\vec{x} = \vec{0}$
  - ↳ let $A \in [\vec{a}_1 \cdots \vec{a}_n] M_{m\times n}(\mathbb{R})$; column space of A is subset of $\mathbb{R}^m$ defined by $\text{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$
    $= \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$
  - ↳ let $A = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} \in M_{m\times n}(\mathbb{R})$; row space of A is subset of $\mathbb{R}^n$ defined by $\text{Row}(A) = \{A^T\vec{x} \mid \vec{x} \in \mathbb{R}^m\}$
    $= \text{Span}\{\vec{r}_1, \ldots, \vec{r}_m\}$
- $A\vec{x} = \vec{b}$ is consistent iff $\vec{b} \in \text{Col}(A)$
- to find basis for Null(A), solve homogeneous system of equations $A\vec{x} = \vec{0}$
- to find basis for Col(A), remove dependencies among columns of A

let $A \in M_{m\times n}(\mathbb{R})$; if R is obtained from A by a series of elementary row operations, then $\text{Row}(R) = \text{Row}(A)$
  - ↳ to find basis for Row(A), find basis for Row(R)
- e.g. let $A = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 3 & 12 & 3 \end{bmatrix}$; find basis for Null(A), Col(A), & Row(A), & state dimensions of each subspace

SOLUTION

$$\begin{bmatrix} 1 & 1 & 5 & 1 \\ 1 & 2 & 7 & 2 \\ 2 & 3 & 12 & 3 \end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↳ solution to homogeneous system $A\vec{x} = \vec{0}$.

$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} \cdot \vec{x} = \vec{0}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \cdot \vec{x} = \vec{0}$

$x_1 \quad + 3x_3 \quad = 0$

$x_2 + 2x_3 + x_4 = 0$

Let $x_3 = s$ & $x_4 = t$ for $s, t \in \mathbb{R}$.

$x_1 = -3s$

$x_2 = -2s - t$

$x_3 = s$

$x_4 = t$

The solution is $\vec{x} = s\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $s, t \in \mathbb{R}$

Hence $B_1 = \left\{\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is basis for Null(A) & $\dim(\text{Null}(A)) = 2$.

↳ first 2 columns of RREF have leading entries so $B_2 = \left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$ is basis for Col(A) & $\dim(\text{Col}(A)) = 2$

- 2 vectors in B_2 are first 2 columns of original A

↳ rows of RREF of A span Row(A):

$Row(A) = Span\left\{\begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right\}$

Since each non-zero vector in spanning set has 1 where others have 0, those vectors are linearly independent & still span $Row(A)$.

Hence, $B_3 = \left\{\begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\\1\end{bmatrix}\right\}$ is basis for $Row(A)$ & $\dim(Row(A)) = 2$

- non-zero rows of any REF / RREF of A form basis for $Row(A)$
- $\dim(Col(A)) = rank(A) = \dim(Row(A))$
- by System-Rank Theorem, $\dim(Null(A)) = n - rank(A)$
  - n is # of variables

MATRIX MULTIPLICATION

- if $A \in M_{m\times n}(\mathbb{R})$ & $B = [\vec{b}_1 \cdots \vec{b}_k] \in M_{n\times k}(\mathbb{R})$, then matrix product AB is $m\times k$ matrix

  $AB = [A\vec{b}_1 \cdots A\vec{b}_k]$

  ↳ for product AB to be defined, # of columns in A must equal # of rows in B
  - i.e. $A \in M_{m\times n}(\mathbb{R})$ & $B \in M_{n\times k}(\mathbb{R}) \Rightarrow AB \in M_{m\times k}(\mathbb{R})$
- can simplify w/ dot products; let $A = \begin{bmatrix}\vec{r}_1^{\,T}\\ \vdots \\ \vec{r}_m^{\,T}\end{bmatrix} \in M_{m\times n}(\mathbb{R})$ & $B = [\vec{b}_1 \cdots \vec{b}_k] \in M_{n\times k}(\mathbb{R})$

  ↳ $AB = \begin{bmatrix}\vec{r}_1^{\,T}\\ \vdots \\ \vec{r}_m^{\,T}\end{bmatrix}[\vec{b}_1 \cdots \vec{b}_k] = \begin{bmatrix}\vec{r}_1\cdot\vec{b}_1 & \cdots & \vec{r}_1\cdot\vec{b}_k \\ \vdots & & \vdots \\ \vec{r}_m\cdot\vec{b}_1 & \cdots & \vec{r}_m\cdot\vec{b}_k\end{bmatrix}$

  ↳ (i,j)-entry of AB is $\vec{r}_i \cdot \vec{b}_j$ for $i = 1, 2, \ldots, m$ & $j = 1, 2, \ldots, k$
- e.g. let $A = \begin{bmatrix}1&2\\3&4\end{bmatrix}$ & $B = \begin{bmatrix}1&1&3\\4&-2&1\end{bmatrix}$; find AB

  SOLUTION

  $AB = \begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&1&3\\4&-2&1\end{bmatrix} = \begin{bmatrix}1(1)+2(4) & 1(1)+2(-2) & 1(3)+2(1)\\ 3(1)+4(4) & 3(1)+4(-2) & 3(3)+4(1)\end{bmatrix} = \begin{bmatrix}9&-3&5\\19&-5&13\end{bmatrix}$

  ↳ $A \in M_{2\times 2}(\mathbb{R})$ & $B \in M_{2\times 3}(\mathbb{R})$ so $AB \in M_{2\times 3}(\mathbb{R})$

  ↳ # of columns in B isn't equal to # of rows in A so BA is undefined
- matrix multiplication isn't commutative ($AB \neq BA$)
- properties of matrix products where $c \in \mathbb{R}$ & A, B, C are matrices in which the following are defined:
  1) $IA = A$
     - I is identity matrix
  2) $AI = A$
     - I is identity matrix
  3) $A(BC) = (AB)C$
     - matrix multiplication is associative
  4) $A(B+C) = AB + AC$
     - left distributive law
  5) $(B+C)A = BA + CA$
     - right distributive law
  6) $(cA)B = c(AB) = A(cB)$
  7) $(AB)^T = B^T A^T$

  since we've defined matrix products in terms of matrix-vector product, 3) also holds:

  $A(B\vec{x}) = (AB)\vec{x}$

  ↳ $\vec{x}$ has same # of entries as B has columns
- 7) can be generalized as $(A_1 A_2 \cdots A_k)^T = A_k^T \cdots A_2^T A_1^T$

  ↳ $A_1, \ldots, A_k$ are matrices of right sizes

  ↳ if $A_1 = \ldots = A_k = A$ for some square matrix A, then $(A^k)^T = (A^T)^k$ for any $k \in \mathbb{Z}^+$
  - $A^k = \underbrace{A \cdots A}_{k \text{ times}}$
- e.g. simplify $A(3B - C) + (A - 2B)C + 2B(C + 2A)$

  SOLUTION

  $A(3B - C) + (A - 2B)C + 2B(C + 2A)$

$= 3AB - AC + AC - 2BC + 2BC + 4BA$

$= 3AB + 4BA$

COMPLEX MATRICES

- denote set of $m \times n$ matrices w/ complex entries by $M_{m\times n}(\mathbb{C})$
  - ↳ same rules for addition, scalar multiplication, matrix-vector product, matrix multiplication, & transpose as real matrices
- when computing matrix-vector products & matrix products, use dot product & not complex inner product

let $A = [a_{ij}] \in M_{m\times n}(\mathbb{C})$; conjugate of A is $\bar{A} = [\bar{a}_{ij}]$

↳ conjugate transpose of A is $A^* = \bar{A}^T$

- let $A \in M_{m\times n}(\mathbb{C})$; A is Hermitian if $A^* = A$

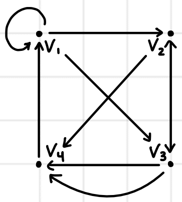
if $A \in M_{m\times n}(\mathbb{C})$ & $\vec{z} \in \mathbb{C}^n$, then $\overline{A\vec{z}} = \bar{A}\bar{\vec{z}}$

- properties of conjugate transpose where $A, B \in M_{m\times n}(\mathbb{C})$ of right sizes, $\vec{z} \in \mathbb{C}^n$, & $\alpha \in \mathbb{C}$:
  1) $(A^*)^* = A$
  2) $(A+B)^* = A^* + B^*$
  3) $(\alpha A)^* = \bar{\alpha} A^*$
  4) $(AB)^* = B^* A^*$
  5) $(A\vec{z})^* = \vec{z}^* A^*$

APPLICATION: ADJACENCY MATRICES FOR DIRECTED GRAPHS

- directed graph (i.e. digraph) is set of vertices & set of directed edges btwn some of pairs of vertices

↳ e.g.



and matrix A whose (i,j)-entry is # of directed edges from V_i to V_j : $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

- adjacency matrix of digraph w/ n vertices $V_1, V_2, \ldots, V_n$ is $n \times n$ matrix A whose (i,j)-entry is # of directed edges from V_i to V_j
- if there's digraph w/ n vertices $V_1, V_2, \ldots, V_n$, then for any $k \in \mathbb{Z}^+$, # of distinct k-edged paths from V_i to V_j is given by (i,j)-entry of A^k

week 8

APPLICATION: MARKOV CHAINS

- vector $\vec{s} \in \mathbb{R}^n$ is called probability vector if entries in vector are not -ve & sum to 1
  - ↳ square matrix is stochastic if its columns are probability vectors
  - ↳ Markov Chain is sequence of probability vectors $\vec{s}_0, \vec{s}_1, \vec{s}_2, \ldots$ where $\vec{s}_{k+1} = P\vec{s}_k$ for every not -ve $k \in \mathbb{Z}$
    - P is stochastic matrix
    - $\vec{s}_k$ are state vectors
- if P is stochastic matrix, then state vector $\vec{s}$ is steady-state vector for P if $P\vec{s} = \vec{s}$
  - ↳ every stochastic matrix has steady-state vector
- an $n \times n$ stochastic matrix P is regular if for some $k \in \mathbb{Z}$, matrix P^k has all +ve entries
  - ↳ since matrix has all entries btwn 0 & 1 inclusive, P is not regular when P^k contains a zero entry for every $k \in \mathbb{Z}^+$
- if P is a regular $n \times n$ stochastic matrix, then P has unique steady-state vector $\vec{s}$ & for any initial state vector $\vec{s}_0 \in \mathbb{R}^n$, resulting Markov Chain converges to steady-state vector $\vec{s}$
- to solve Markov Chain problem:
  1) read & understand problem
  2) determine stochastic matrix P & verify it's regular
  3) determine initial state vector $\vec{s}_0$ if required
  4) solve homogeneous system $(P - I)\vec{s} = \vec{0}$
     - I is identity matrix
  5) choose values for any parameters resulting from solving above system so $\vec{s}$ is probability vector
     - i.e. all entries in $\vec{s}$ sum up to 1
  6) conclude w/above theorem that $\vec{s}$ is steady-state vector
  7) interpret entries of $\vec{s}$ in terms of original problem
- notation for state vectors: $\vec{s}_k = \begin{bmatrix} s_1^{(k)} \\ \vdots \\ s_n^{(k)} \end{bmatrix}$ for $k = 0, 1, 2, \ldots$
  - ↳ steady-state vector: $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$

MATRIX INVERSES

NOTE: inverse means multiplicative inverse b/c additive inverse of $A \in M_{m \times n}(\mathbb{R})$ is $-A$

- let $A \in M_{n \times n}(\mathbb{R})$; if there exists a $B \in M_{n \times n}(\mathbb{R})$ such that $AB = I = BA$, then A is invertible & B is inverse of A (& vice versa)
  - ↳ $AB = BA$ means A must be square matrix
- let $A, B \in M_{n \times n}(\mathbb{R})$ be such that $AB = I$; then $BA = I$ & $\text{rank}(A) = \text{rank}(B) = n$
  - ↳ RREF of A is I
- let $A, B \in M_{n \times n}(\mathbb{R})$ be invertible; if $B, C \in M_{n \times n}(\mathbb{R})$ are both inverses of, then $B = C$
  - ↳ i.e if A is invertible, inverse of A is unique
  - ↳ inverse is denoted as A^{-1}
- properties of inverse where $A, B \in M_{n \times n}(\mathbb{R})$ are invertible & $c \in \mathbb{R}$ w/ $c \neq 0$:
  1) $(cA)^{-1} = \frac{1}{c} A^{-1}$
  2) $(AB)^{-1} = B^{-1} A^{-1}$
     - ↳ generalizes for 2+ matrices so $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$
  3) $(A^k)^{-1} = (A^{-1})^k$
     - k is +ve integer
  4) $(A^T)^{-1} = (A^{-1})^T$
  5) $(A^{-1})^{-1} = A$

MATRIX INVERSION ALGORITHM

- carry $A \in M_{n \times n}(\mathbb{R})$ to RREF
  - ↳ if $[A \mid I]$ is $[I \mid B]$ for some $B \in M_{n \times n}(\mathbb{R})$, then $B = A^{-1}$
  - ↳ if RREF of A is not I, then A is not invertible
    - rank $(A) < n$; goes against theorem that rank $(A) = n$
- e.g. let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$; find A^{-1} if it exists

SOLUTION

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 + 3R_2} \left[\begin{array}{cc|cc} 2 & 0 & -5 & 3 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow[R_2 = -R_2]{R_1 = (\frac{1}{2})R_1} \left[\begin{array}{cc|cc} 1 & 0 & -5/2 & 3/2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Since RREF of A is I, A is invertible ; $A^{-1} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix}$.

PROPERTIES OF MATRIX INVERSES

- let $A \in M_{n \times n}(\mathbb{R})$ be invertible
  - ↳ for all $B, C \in M_{n \times k}(\mathbb{R})$, if $AB = AC$, then $B = C$
    - left cancellation
  - ↳ for all $B, C \in M_{k \times n}(\mathbb{R})$, if $BA = CA$, then $B = C$
    - right cancellation
  - ↳ mixed cancellation (e.g. $AB = CA$ means $B = C$) isn't possible b/c matrix multiplication isn't commutative
- Invertible Matrix Theorem where $A \in M_{n \times n}(\mathbb{R})$; following are equivalent:
  - ↳ A is invertible
  - ↳ rank $(A) = n$
  - ↳ RREF of A is I
  - ↳ for all $\vec{b} \in \mathbb{R}^n$, system $A\vec{x} = \vec{b}$ is consistent ; has unique solution
    - $A\vec{x} = \vec{b}$
      $A^{-1}A\vec{x} = A^{-1}\vec{b}$
      $I\vec{x} = A^{-1}\vec{b}$
      $\vec{x} = A^{-1}\vec{b}$
  - ↳ $\mathrm{Null}(A) = \{\vec{0}\}$
  - ↳ columns of A form linearly independent set
  - ↳ columns of A span $\mathbb{R}^n$
  - ↳ A^T is invertible
  - ↳ $\mathrm{Null}(A^T) = \{\vec{0}\}$
  - ↳ rows of A form linearly independent set
  - ↳ rows of A span $\mathbb{R}^n$

LINEAR TRANSFORMATIONS

- function is a rule that assigns to every element in one set (domain) a unique element in another set (codomain)
  - ↳ given sets U ; V, write $f: U \to V$ to indicate f is function w/domain U ; codomain V
    - f assigns each element $u \in U$ a unique element $v \in V$
    - f maps u to v
    - v is image of u under f
    - write $v = f(u)$
- for $A \in M_{m \times n}(\mathbb{R})$, function $f_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $f_A(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$ is called matrix transformation corresponding to A
  - ↳ $\mathbb{R}^n$ is domain of f_A
  - ↳ $\mathbb{R}^m$ is codomain of f_A
  - ↳ f_A maps to $A\vec{x}$; $A\vec{x}$ is image of $\vec{x}$ under f_A

↳ aka matrix mapping

- for $A \in M_{m\times n}(\mathbb{R})$, function f_A sends vectors in $\mathbb{R}^n$ to $\mathbb{R}^m$ so we write $f_A\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$
  - ↳ functions often viewed as sending points to points, prefer notation $f(x_1, \dots, x_n) = (y_1, \dots, y_m)$ or $f(x_1, \dots, x_n) = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$
  - ↳ e.g. let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ so $A \in M_{2\times 3}(\mathbb{R})$ & $f_A : \mathbb{R}^3 \to \mathbb{R}^2$; compute $f_A(1,1,4)$

SOLUTION

$$\begin{aligned} f_A(1,1,4) &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1(1)+2(1)+3(4) \\ 1(1)+(-1)(1)+1(4) \end{bmatrix} \\ &= \begin{bmatrix} 15 \\ 4 \end{bmatrix} \\ &= (15, 4) \end{aligned}$$

- let $A \in M_{m\times n}(\mathbb{R})$ & let f_A be matrix transformation corresponding to A; for every $\vec{x}, \vec{y} \in \mathbb{R}^n$ & $c \in \mathbb{R}$:
  - ↳ $f_A(\vec{x} + \vec{y}) = f_A(\vec{x}) + f_A(\vec{y})$
  - ↳ $f_A(c\vec{x}) = c f_A(\vec{x})$

- a function $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation/mapping if for every $\vec{x}, \vec{y} \in \mathbb{R}$ & $s, t \in \mathbb{R}$, we have $L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$
  - ↳ i.e. vector sums & scalar multiplication are preserved
  - ↳ for $m = n$, linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ is called linear operator on $\mathbb{R}^n$
  - ↳ every matrix transformation is linear transformation
- linear transformation always maps zero vector of domain to zero vector of codomain
  - ↳ i.e. $L(\vec{0}_{\mathbb{R}^n}) = \vec{0}_{\mathbb{R}^m}$
  - ↳ doesn't guarantee function is linear
- for linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, if given $L(\vec{x}_1), \dots, L(\vec{x}_k)$ for $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{R}^n$, then can compute $L(\vec{x})$ for any $\vec{x} \in \operatorname{Span}\{\vec{x}_1, \dots, \vec{x}_k\}$ since L preserves linear combos
  - ↳ if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is basis for $\mathbb{R}^n$ & we know $L(\vec{v}_1), \dots, L(\vec{v}_n)$, then can compute $L(\vec{v})$ for any $\vec{v} \in \mathbb{R}^n$
- if $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation, then L is matrix transformation w/ corresponding matrix $[L] = [L(\vec{e}_1) \cdots L(\vec{e}_n)] \in M_{m\times n}(\mathbb{R})$
  - ↳ i.e. $L(\vec{x}) = [L]\vec{x}$ for every $\vec{x} \in \mathbb{R}^n$
  - ↳ $[L] \in M_{m\times n}(\mathbb{R})$ is standard matrix of L
- transformation is linear iff it's a matrix transformation
- e.g. let $\vec{d} \in \mathbb{R}^2$ be non-zero & define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{x}) = \operatorname{proj}_{\vec{d}}\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$, show L is linear & find standard matrix of L w/ $\vec{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

SOLUTION

Let $\vec{x}, \vec{y} \in \mathbb{R}^2$ & $s, t \in \mathbb{R}$

$$\begin{aligned} L(s\vec{x} + t\vec{y}) &= \operatorname{proj}_{\vec{d}}(s\vec{x} + t\vec{y}) \\ &= \frac{(s\vec{x} + t\vec{y}) \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} \\ &= s\frac{\vec{x} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} + t\frac{\vec{y} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} \\ &= s\operatorname{proj}_{\vec{d}}\vec{x} + t\operatorname{proj}_{\vec{d}}\vec{y} \\ &= sL(\vec{x}) + tL(\vec{y}) \end{aligned}$$

We've proven L is linear. Let $\vec{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

$$\begin{aligned} L(\vec{e}_1) &= \operatorname{proj}_{\vec{d}}\vec{e}_1 \\ &= \frac{\vec{e}_1 \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} \\ &= \frac{1}{(-1)^2+3^2}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \frac{1}{10}(1(-1) + 0(3))\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= -\frac{1}{10}\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1/10 \\ -3/10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} L(\vec{e}_2) &= \operatorname{proj}_{\vec{d}}\vec{e}_2 \\ &= \frac{\vec{e}_2 \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} \\ &= \frac{1}{10}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \frac{0(-1) + 1(3)}{10}\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \frac{3}{10}\begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -3/10 \\ 9/10 \end{bmatrix} \end{aligned}$$

Thus, $[L] = [L(\vec{e}_1) \ \ L(\vec{e}_2)] = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$

- $\text{proj}_{\vec{d}} : \mathbb{R}^n \to \mathbb{R}^n$ is projection onto $\vec{d} \neq \vec{0}$ in $\mathbb{R}^n$
- $\text{perp}_{\vec{d}} : \mathbb{R}^n \to \mathbb{R}^n$ is perpendicular onto $\vec{d} \neq \vec{0}$ in $\mathbb{R}^n$
- $\text{refl}_{\vec{n}} : \mathbb{R}^n \to \mathbb{R}^n$ is reflection in a hyperplane through origin w/normal vector $\vec{n} \neq \vec{0}$ in $\mathbb{R}^n$

week 9

- let $\mathbb{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise (ccw) rotation abt origin by angle of θ ; to show $\mathbb{R}_\theta$ is linear
  - ↳ write $\vec{x} \in \mathbb{R}^2$ as $\vec{x} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$
    - $r \in \mathbb{R} = \|\vec{x}\| \geq 0$
    - $\phi \in \mathbb{R}$ is angle $\vec{x}$ makes w/ +ve x_1-axis measured ccw
  - ↳ $\|\mathbb{R}_\theta(\vec{x})\| = r$ & $\mathbb{R}_\theta(\vec{x})$ makes angle of $\theta + \phi$ w/ +ve x_1-axis measured ccw
  - ↳ $R_\theta(\vec{x}) = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix}$

$$= \begin{bmatrix} r(\cos\phi\cos\theta - \sin\phi\sin\theta) \\ r(\sin\phi\cos\theta + \cos\phi\sin\theta) \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta(r\cos\phi) - \sin\theta(r\sin\phi) \\ \sin\theta(r\cos\phi) + \cos\theta(r\sin\phi) \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\vec{x}$$

  - R_θ is matrix transformation & thus, linear
  - ↳ $[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
    - $[R_{-\theta}] = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

- to apply $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$, consider these matrices:
  - ↳ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
  - ↳ $B = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
  - ↳ $C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - ↳ linear transformations are:
    - $L_1 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L_1(\vec{x}) = A\vec{x}$ is ccw rotation abt x_1-axis
    - $L_2 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L_2(\vec{x}) = B\vec{x}$ is ccw rotation abt x_2-axis
    - $L_3 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L_3(\vec{x}) = C\vec{x}$ is ccw rotation abt x_3-axis

- for $t \in \mathbb{R}$ & $t > 0$, let $A = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$ & define $L : \mathbb{R}^2 \to \mathbb{R}^2$ as $L(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$
  - ↳ for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $L(\vec{x}) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ x_2 \end{bmatrix}$
    - if $t > 1$, L is stretch in x_1-dir by factor of t
    - if $0 < t < 1$, L is compression in x_1-dir by factor of t
  - ↳ if $t = 0$, L is projection onto x_2-axis
  - ↳ if $t < 0$, L is reflection in x_2-axis followed by stretch/compression by factor of $-t > 0$
  - ↳ stretch/compression in x_2-dir would be $L(\vec{x}) = \begin{bmatrix} x_1 \\ tx_2 \end{bmatrix}$
- for $t \in \mathbb{R}$ & $t > 0$, let $B = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$ & define $L : \mathbb{R}^2 \to \mathbb{R}^2$ as $L(\vec{x}) = B\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$
  - ↳ for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $L(\vec{x}) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix} = t\vec{x}$
    - if $t > 1$, L is dilation
    - if $0 < t < 1$, L is contraction
    - if $t = 1$, $B = I$ & $L(\vec{x}) = \vec{x}$
- for $s \in \mathbb{R}$, let $C = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ & define $L : \mathbb{R}^2 \to \mathbb{R}^2$ as $L(\vec{x}) = C\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$
  - ↳ for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $L(\vec{x}) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + sx_2 \\ x_2 \end{bmatrix}$
    - L is shear in x_1-dir (horizontal shear) by factor of s
  - ↳ shear in x_2-dir (vertical shear) by factor of $s \in \mathbb{R}$ has standard matrix $\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$

OPERATIONS ON LINEAR TRANSFORMATIONS

- let $L, M : \mathbb{R}^n \to \mathbb{R}^m$ be (linear) transformations ; if $L(\vec{x}) = M(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$, then L & M are equal so we write $L = M$
  - ↳ if for some $\vec{x} \in \mathbb{R}^n$ we have $L(\vec{x}) \neq M(\vec{x})$, then L & M aren't equal so we write $L \neq M$

↳ if L & M are linear transformations: $L = M \iff L(\vec{x}) = M(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$
$\iff [L]\vec{x} = [M]\vec{x}$
$\iff [L] = [M]$ by Matrices Equal Theorem

- let $L, M : \mathbb{R}^n \to \mathbb{R}^m$ be (linear) transformations & let $c \in \mathbb{R}$
  - ↳ $(L+M) : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $(L+M)(\vec{x}) = L(\vec{x}) + M(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$
  - ↳ $(cL) : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $(cL)(\vec{x}) = cL(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$
- let $L, M : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations & $c \in \mathbb{R}$; then, $L+M : \mathbb{R}^n \to \mathbb{R}^m$ & $cL : \mathbb{R}^n \to \mathbb{R}^m$ are also linear transformations
  - ↳ $[L+M] = [L] + [M]$
  - ↳ $[cL] = c[L]$
- let $L : \mathbb{R}^n \to \mathbb{R}^m$ & $M : \mathbb{R}^m \to \mathbb{R}^p$ be (linear) transformations; composition of $M \circ R : \mathbb{R}^n \to \mathbb{R}^p$ is defined by $(M \circ L)\vec{x} = M(L(\vec{x}))$ for every $\vec{x} \in \mathbb{R}^n$
  - ↳ domain of M must contain codomain of L
- let $L : \mathbb{R}^n \to \mathbb{R}^m$ & $M : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations; then, $M \circ L : \mathbb{R}^n \to \mathbb{R}^p$ is a linear transformation
  - ↳ $[M \circ L] = [M][L]$
  - ↳ $[L] \in M_{m \times n}(\mathbb{R})$ & $[M] \in M_{p \times m}(\mathbb{R})$ so $[M \circ L] = [M][L] \in M_{p \times n}(\mathbb{R})$
  - ↳ $[L][M]$ is not defined unless $n = p$
    - even if they're both defined, not guaranteed $[L][M] = [M][L]$ (i.e. L & M don't commute)

e.g. let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be ccw rotation abt origin by angle of $\frac{\pi}{4}$ & let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto x_1-axis; find standard matrices for $M \circ L$ & $L \circ M$

SOLUTION

Since L & M are linear:

$$[L] = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$[M] = [\operatorname{proj}_{\vec{e}_1}\vec{e}_1 \quad \operatorname{proj}_{\vec{e}_1}\vec{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[M \circ L] = [M][L] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 \end{bmatrix}$$

$$[L \circ M] = [L][M] = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 \end{bmatrix}$$

INVERSE LINEAR TRANSFORMATIONS

- linear transformation $Id : \mathbb{R}^n \to \mathbb{R}^n$ defined by $Id(\vec{x}) = \vec{x}$ for every $\vec{x} \in \mathbb{R}^n$ is called identity transformation
  - ↳ $[Id] = [Id(\vec{e}_1) \cdots Id(\vec{e}_n)] = [\vec{e}_1 \cdots \vec{e}_n] = I$
- if $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear transformation & there exists another linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$ such that $M \circ L = Id = L \circ M$, then L is invertible & M is inverse of L
  - ↳ $[L]^{-1} = [M]$ & $[M]^{-1} = [L]$
- if $L, M : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations, then M is inverse of L iff $[M]$ is inverse of $[L]$
  - ↳ $[L^{-1}] = [L]^{-1}$
  - ↳ geometrically, L^{-1} "undoes" what L does

COMPLEX LINEAR TRANSFORMATIONS

- theory of linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^m$ mirrors those from $\mathbb{R}^n$ to $\mathbb{R}^m$

e.g. let $L : \mathbb{C}^3 \to \mathbb{C}^2$ be linear transformation such that $L(\vec{e}_1) = \begin{bmatrix} 1-j \\ 2+j \end{bmatrix}$, $L(\vec{e}_2) = \begin{bmatrix} j \\ 3 \end{bmatrix}$, & $L(\vec{e}_3) = \begin{bmatrix} 2 \\ 1+j \end{bmatrix}$; compute $L(1, j, 1+j)$

SOLUTION

Compute standard matrix for L:

$$[L] = [L(\vec{e}_1) \quad L(\vec{e}_2) \quad L(\vec{e}_3)] = \begin{bmatrix} 1-j & j & 2 \\ 2+j & 3 & 1+j \end{bmatrix}$$

$$L(1, j, 1+j) = \begin{bmatrix} 1-j & j & 2 \\ 2+j & 3 & 1+j \end{bmatrix}\begin{bmatrix} 1 \\ j \\ 1+j \end{bmatrix} = \begin{bmatrix} (1-j)(1) + j(j) + 2(1+j) \\ (2+j)(1) + 3(j) + (1+j)(1+j) \end{bmatrix}$$

$$= \begin{bmatrix} 1-j+j^2+2+2j \\ 2+j+3j+1+2j+j^2 \end{bmatrix}$$
$$= \begin{bmatrix} 2+j \\ 2+6j \end{bmatrix}$$

APPLICATION: LINEAR TRANSFORMATIONS

- translations (i.e shift in dir of non-zero vector) are nonlinear transformations
  - ↳ for $\ell_1 \neq 0$, define $L(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \ell_1 \\ 0 \end{bmatrix}$
    - $L(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \ell_1 \\ 0 \end{bmatrix}$
      $= \begin{bmatrix} \ell_1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
    - since L doesn't map $\vec{0}$ to $\vec{0}$, it's nonlinear
  - ↳ translations can't be accomplished using matrix multiplication in $\mathbb{R}^n$
- to each point (x_1, x_2) in x_1x_2-plane, there's corresponding point $(x_1, x_2, 1)$ lying in plane $x_3 = 1$ in $\mathbb{R}^3$
  - ↳ homogeneous coordinates of point (x_1, x_2) are $(x_1, x_2, 1)$
  - ↳ for any linear transformation $L: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(\vec{x}) = A\vec{x}$, where $A \in M_{2\times 2}(\mathbb{R})$, construct 3×3 matrix $\begin{bmatrix} A & \vec{0}_{\mathbb{R}^2} \\ \vec{0}_{\mathbb{R}^2} & 1 \end{bmatrix}$ to rep transformation in homogeneous coordinates
- if polynomials are written as $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$ where $n \in \mathbb{Z}^+$, then $p(x)$ can be rep as vector in $\mathbb{R}^{n+1}$: $p(x) \to \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}$
  - ↳ since $\frac{d}{dx}p(x) = a_1 + 2a_2x + \ldots + (n-1)a_{n-1}x^{n-2} + na_nx^{n-1}$, then $\frac{d}{dx}p(x) \longrightarrow \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix}$
- given arbitrary vector $[a_0\ a_1\ a_2 \cdots a_{n-1}\ a_n]^T \in \mathbb{R}^{n+1}$, we have:

$$\begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}$$

THE KERNEL AND THE RANGE OF A LINEAR TRANSFORMATION

- let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a (linear) transformation; kernel (i.e. nullspace) of L is $\text{Ker}(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = \vec{0}\}$
  - ↳ $\text{Ker}(L) \subseteq \mathbb{R}^n$
- let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a (linear) transformation; range of L is $\text{Range}(L) = \{L(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$
  - ↳ $\text{Range}(L) \subseteq \mathbb{R}^m$
  - ↳ e.g. let $L: \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformation defined by $L(x_1, x_2) = (x_1 + x_2, 2x_1 + x_2, 3x_2)$, determine which of $\vec{y}_1 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ & $\vec{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ belong to Range(L)

    <u>SOLUTION</u>

    To see if $\vec{y}_1 \in \text{Range}(L)$, find $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ st $L(\vec{x}) = \vec{y}_1$.

    $L(x_1, x_2) = (2, 3, 3)$

    $x_1 + x_2 = 2$

    $2x_1 + x_2 = 3$

    $3x_2 = 3$

    Augmented matrix to RREF: $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 0 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$

    Since $x_1 = x_2 = 1$, then $L(1,1) = (2,3,3)$ so $\vec{y}_1 \in \text{Range}(L)$.

    To see if $\vec{y}_2 \in \text{Range}(L)$, find $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ st $L(\vec{x}) = \vec{y}_2$.

    $L(x_1, x_2) = (1, 1, 2)$

    $x_1 + x_2 = 1$

    $2x_1 + x_2 = 1$

    $3x_2 = 2$

    Augmented matrix to REF: $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{array}\right]$

    System is inconsistent so there's no $\vec{x} \in \mathbb{R}^2$ st $L(\vec{x}) = \vec{y}_2$ & so $\vec{y}_2 \notin \text{Range}(L)$.
- let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation; then, Ker(L) is subspace of $\mathbb{R}^n$ & Range(L) is subspace of $\mathbb{R}^m$
- let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation w/ standard matrix $[L]$
  - ↳ $\text{Ker}(L) = \text{Null}([L])$

↳ $\text{Range}(L) = \text{Col}([L])$

ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

- let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a (linear) transformation; L is one-to-one (i.e. injective) if $L(\vec{x}_1) = L(\vec{x}_2)$ implies that $\vec{x}_1 = \vec{x}_2$ for any $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$
  - ↳ can't send distinct/different elements of domain to same element in range
- if $L: \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation, then L is one-to-one iff $\text{Ker}(L) = \{\vec{0}\}$
- let $L: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation, then L is one-to-one iff $\text{rank}([L]) = n$

- let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a (linear) transformation; L is onto (i.e. surjective) if for every $\vec{y} \in \mathbb{R}^m$ there exists an $\vec{x} \in \mathbb{R}^n$ st $L(\vec{x}) = \vec{y}$
  - ↳ since $\text{Range}(L) \subseteq \mathbb{R}^m$, if $L: \mathbb{R}^n \to \mathbb{R}^m$ is onto, then $\text{Range}(L) = \mathbb{R}^m$
- let $L: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation, then L is onto iff $\text{rank}([L]) = m$

- a (linear) transformation $L: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one correspondence (i.e. bijective) if it's both one-to-one & onto
  - ↳ only linear operators can be bijective
- let $L: \mathbb{R}^n \to \mathbb{R}^n$ be linear transformation, then L is one-to-one correspondence iff L is invertible

DETERMINANTS, ADJUGATES, AND MATRIX INVERSES

- let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$
  - ↳ determinant of A is $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
  - ↳ adjugate of A is $\text{adj}\, A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- let $A \in M_{2\times 2}(\mathbb{R})$, then $A(\text{adj}\, A) = (\det A) I = (\text{adj}\, A) A$
  - ↳ A is invertible iff $\det A \neq 0$ so $A^{-1} = \frac{1}{\det A} \text{adj}\, A$

week 10

- let $A \in M_{n \times n}(\mathbb{R})$ & let $A(i,j)$ be $(n-1) \times (n-1)$ matrix obtained from A by deleting i^{th} row & j^{th} column of A
  - ↳ (i,j)-cofactor of A is $C_{ij} = (-1)^{i+j} \det A(i,j)$
  - ↳ e.g. let $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 4 \\ 4 & 1 & 1 \end{bmatrix}$ & determine $(3,2)$-cofactor of A

    SOLUTION

    $$\begin{aligned} C_{32} &= (-1)^{3+2} \det A(3,2) \\ &= (-1)^5 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \\ &= -1(4-3) \\ &= -1 \end{aligned}$$

- let $A \in M_{n \times n}(\mathbb{R})$
  - ↳ for any $i = 1, \ldots, n$, determinant of A is $\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in}$
    - cofactor expansion of A along i^{th} row of A
  - ↳ for any $j = 1, \ldots, n$, determinant of A is $\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \ldots + a_{nj} C_{nj}$
    - cofactor expansion of A along j^{th} column of A
  - ↳ can do cofactor expansion along any row/column we choose
  - ↳ e.g. compute $\det A$ where $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & 9 \end{bmatrix}$

    SOLUTION

    We will do cofactor expansion along 1^{st} row:

    $$\begin{aligned} \det A = \begin{vmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ -7 & 8 & 9 \end{vmatrix} &= 1 \begin{vmatrix} -5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ -7 & 9 \end{vmatrix} - 3 \begin{vmatrix} 4 & -5 \\ -7 & 8 \end{vmatrix} \\ &= 1(-45-48) - 2(36-(-42)) - 3(32-35) \\ &= -93 - 2(78) - 3(-3) \\ &= -240 \end{aligned}$$

- let $A = [a_{ij}] \in M_{n \times n}(\mathbb{R})$
  - ↳ $C_{ij} = (-1)^{i+j} \det A(i,j)$ is (i,j)-cofactor of A
  - ↳ cofactor matrix of A is $\operatorname{cof} A = [C_{ij}] \in M_{n \times n}(\mathbb{R})$
  - ↳ adjugate of A is $\operatorname{adj} A = [C_{ij}]^T \in M_{n \times n}(\mathbb{R})$

let $A \in M_{n \times n}(\mathbb{R})$, then $A(\operatorname{adj} A) = (\det A) I = (\operatorname{adj} A) A$

- ↳ A is invertible iff $\det A \neq 0$ so $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$

e.g. find $\det A$, $\operatorname{adj} A$, & A^{-1} if $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 2 & 4 \end{bmatrix}$

SOLUTION

Using cofactor expansion along 1^{st} row:

$$\begin{aligned} \det A &= 1 \begin{vmatrix} 1 & 4 \\ 2 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 1(4-8) - 1(4-4) + 2(2-1) \\ &= -4 - 0 + 2(1) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \operatorname{adj} A &= \begin{bmatrix} \begin{vmatrix} 1 & 4 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^T \\ &= \begin{bmatrix} 4-8 & -(4-4) & 2-1 \\ -(4-4) & 4-2 & -(2-1) \\ 4-2 & -(4-2) & 1-1 \end{bmatrix}^T \\ &= \begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & -1 \\ 2 & -2 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} -4 & 0 & 2 \\ 0 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^{-1} &= -\frac{1}{2} \begin{bmatrix} -4 & 0 & 2 \\ 0 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 1 \\ -1/2 & 1/2 & 0 \end{bmatrix} \end{aligned}$$

- determinants are also defined for complex square matrices & computations are identical

ELEMENTORY ROW/COLUMN OPERATIONS

NOTE: don't perform EROs & ECOs at the same time

properties of discriminant under EROs/ECOs where $A \in M_{n\times n}(\mathbb{R})$

1) if A has a row/column of 0s, then $\det A = 0$
2) if B is obtained from A by swapping 2 distinct rows/columns, then $\det B = -\det A$
3) if B is obtained from A by adding a multiple of a row/column to another, then $\det B = \det A$
4) if 2 distinct rows/columns of A are equal, then $\det A = 0$
5) if B is obtained from A by multiplying a row/column by $c \in \mathbb{R}$, then $\det B = c \det A$

- e.g. find $\det A$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$

SOLUTION

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} \overset{\substack{R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 7R_1}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix} \overset{-\frac{1}{3}C_1 \to C_1}{=} (-3) \begin{vmatrix} 1 & -6 \\ 2 & -11 \end{vmatrix} = -3(-11-(-6)(2)) = -3$$

↳ $-\frac{1}{3}C_1 \to C_1$ is viewed as "factoring out" & usually writing out operation explicitly

NOTE: for $A = [a_{ij}] \in M_{m\times n}(\mathbb{R})$, main diagonal of A consists of entries $a_{11}, a_{22}, \dots, a_{kk}$ w/ k being min of m & n

- let $A \in M_{m\times n}(\mathbb{R})$
  - ↳ A is upper triangular if every entry below main diagonal is 0
    - e.g. $\begin{bmatrix} 4 & -7 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & -2 \end{bmatrix}$, & $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
  - ↳ A is lower triangular if every entry above main diagonal is 0
    - e.g. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}$, & $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- if $A = [a_{ij}] \in M_{n\times n}(\mathbb{R})$ is triangular matrix (upper/lower), then $\det(A) = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$

PROPERTIES OF DETERMINANTS

- if $A \in M_{n\times n}(\mathbb{R})$ & $k \in \mathbb{R}$, then $\det(kA) = k^n \det A$
- if $A, B \in M_{n\times n}(\mathbb{R})$, then $\det(AB) = (\det A)(\det B)$
  - ↳ i.e. determinant distributes over matrix multiplication
  - ↳ $\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)$
    - although A & B don't commute in general, $\det(AB) = \det(BA)$ for any $A, B \in M_{n\times n}(\mathbb{R})$
  - ↳ extends to more than 2 matrices: for $A_1, A_2, \dots, A_k \in M_{n\times n}(\mathbb{R})$, $\det(A_1A_2 \dots A_k) = (\det A_1)(\det A_2)\cdots(\det A_k)$
    - if $A_1 = A_2 = \dots = A_k$ for any $k \in \mathbb{Z}^+$, then $\det(A^k) = (\det A)^k$
- let $A \in M_{n\times n}(\mathbb{R})$ be invertible, then $\det(A^{-1}) = \frac{1}{\det A}$
  - ↳ $\det(A^k) = (\det A)^k$ for any $k \in \mathbb{Z}$ st $k \le 0$ requires A to be invertible
- let $A \in M_{n\times n}(\mathbb{R})$, then $\det(A^T) = \det(A)$
- e.g. if $\det(A) = 3$, $\det(B) = -2$, & $\det(C) = 4$ for $A, B, C \in M_{n\times n}(\mathbb{R})$, find $\det(A^2B^TC^{-1}B^2(A^{-1})^2)$

SOLUTION

$$\begin{aligned} \det(A^2B^TC^{-1}B^2(A^{-1})^2) &= \det(A^2)\det(B^T)\det(C^{-1})\det(B^2)\det((A^{-1})^2) \\ &= (\det A)^2(\det B)\frac{1}{\det C}(\det B)^2\frac{1}{(\det A)^2} \\ &= \frac{(\det B)^3}{\det C} \\ &= \frac{(-2)^3}{4} \\ &= -\frac{8}{4} \\ &= -2 \end{aligned}$$

- for $A, B \in M_{n\times n}(\mathbb{R})$, $\det(A+B) \neq \det A + \det B$ in general
  - ↳ i.e. determinant doesn't distribute over matrix addition

APPLICATION: POLYNOMIAL INTERPOLATION

- given n data points, seek polynomial of degree $n-1$
- given n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, construct polynomial $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$ st $p(x_i) = y_i$ for each $i = 1, 2, \dots, n$
  - ↳ system of equations:
$$\begin{matrix} a_0 & + & a_1x_1 & + & a_2x_1^2 & + & \cdots & + & a_{n-1}x_1^{n-1} & = & y_1 \\ a_0 & + & a_1x_2 & + & a_2x_2^2 & + & \cdots & + & a_{n-1}x_2^{n-1} & = & y_2 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ a_0 & + & a_1x_n & + & a_2x_n^2 & + & \cdots & + & a_{n-1}x_n^{n-1} & = & y_n \end{matrix}$$
  - ↳ matrix equation:
$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- for $x_1, x_2, \dots, x_n \in \mathbb{R}$ & $n \geq 2$, $n \times n$ matrix $A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$ is Vandermonde matrix
  - ↳ for $n \times n$ Vandermonde matrix, $\det A = \prod_{1 \leq i < j \leq n} (x_j - x_i)$
    - i.e. $\det A$ is product of terms $(x_j - x_i)$ where $j > i$ & i, j both lie btwn 1 & n inclusively
    - e.g for $n = 3$, $A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$ & $\det A = (x_3 - x_1)(x_3 - x_2)(x_2 - x_1)$
  - ↳ $n \times n$ Vandermonde matrix is invertible iff $x_1, x_2, \dots, x_n$ are all distinct so matrix equation has unique solution

for n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ where $x_1, x_2, \dots, x_n$ are all distinct, there exists a unique polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$ satisfying $p(x_i) = y_i$ for $i = 1, 2, \dots, n$

DETERMINANTS AND AREA

- $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$ so parallelogram determined by $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ & $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is subset of $\mathbb{R}^2$ while parallelogram determined by $\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$ is subset of $\mathbb{R}^3$
  - 2 parallelograms have same area
- area of parallelogram $\vec{u}, \vec{v} \in \mathbb{R}^2$ can be computed as

$$\begin{aligned} A &= \left\| \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 0 \\ 0 \\ u_1 v_2 - v_1 u_2 \end{bmatrix} \right\| \\ &= \sqrt{(u_1 v_2 - v_1 u_2)^2} \\ &= |u_1 v_2 - v_1 u_2| \\ &= \left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right| \end{aligned}$$

$$A = |\det [\vec{u} \ \ \vec{v}]|$$

- consider $\vec{u}, \vec{v} \in \mathbb{R}^2$ & linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ w/ standard matrix $[L]$; area of parallelogram determined by $L(\vec{u})$ & $L(\vec{v})$ is

$$\begin{aligned} A &= |\det [L(\vec{u}) \ \ L(\vec{v})]| \\ &= |\det [[L]\vec{u} \ \ [L]\vec{v}]| \\ &= |\det ([L][\vec{u} \ \ \vec{v}])| \end{aligned}$$

$$A = |\det [L]| \, |\det [\vec{u} \ \ \vec{v}])|$$

  - ↳ can be generalized to any shape in $\mathbb{R}^2$
    - e.g. consider circle of radius $r = 1$ centred at origin in $\mathbb{R}^2$ so $A_{circle} = \pi r^2 = \pi(1)^2 = \pi$; linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ is stretch in x_1- dir by factor of 2; compute area of image of circle

SOLUTION

$[L] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned} A_{ellipse} &= |\det [L]| \, A_{circle} \\ &= |2| \pi \\ &= 2\pi \end{aligned}$$

DETERMINANTS AND VOLUME

- volume of parallelepiped determined by $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ is $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$
  - ↳ $V = |\det [\vec{u} \ \ \vec{v} \ \ \vec{w}]|$
- for 3 vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ & any linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$, volume of parallelepiped determined by $L(\vec{u})$, $L(\vec{v})$, & $L(\vec{w})$ is $V = |\det [L(\vec{u}) \ \ L(\vec{v}) \ \ L(\vec{w})]|$

$$V = |\det [L]| \, |\det [\vec{u} \ \ \vec{v} \ \ \vec{w}]|$$

  - generalizes to any shape in $\mathbb{R}^3$

EIGENVALUES AND EIGENVECTORS

- for $A \in M_{n \times n}(\mathbb{R})$, a scalar λ is eigenvalue of A if $A\vec{x} = \lambda \vec{x}$ for some non-zero vector $\vec{x}$
  - ↳ $\vec{x}$ is eigenvector of A corresponding to λ
  - ↳ if $A = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$ & $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find eigenvalue & eigenvector

SOLUTION

$A\vec{x} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/5(1) + 4/5(2) \\ 4/5(1) + 3/5(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\vec{x}$

So, $\lambda = 1$ is an eigenvalue of A & $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is corresponding eigenvector

- let $A \in M_{n \times n}(\mathbb{R})$; a # λ is eigenvalue of A iff λ satisfies equation $\det(A - \lambda I) = 0$
  - ↳ if λ is eigenvalue of A, then all non-zero solutions of homogeneous system of equations $(A - \lambda I)\vec{x} = \vec{0}$ are all of eigenvectors corresponding to λ
- let $A \in M_{n \times n}(\mathbb{R})$; characteristic polynomial of A is $C_A(\lambda) = \det(A - \lambda I)$
  - ↳ λ is eigenvalue of A iff $C_A(\lambda) = 0$
  - ↳ $C_A(\lambda)$ will have real coeffs but may have non-real roots

week 11

- let λ be eigenvalue of $A \in M_{n\times n}(\mathbb{R})$; set containing all of eigenvectors of A corresponding to λ together w/zero vector of $\mathbb{R}^n$ is eigenspace of A corresponding to λ & denoted by $E_\lambda(A)$
  - ↳ $E_\lambda(A) = \text{Null}(A - \lambda I)$
  - ↳ $E_\lambda(A)$ is subspace of $\mathbb{R}^n$
- once we have basis for $E_\lambda(A)$, can construct all eigenvectors of A corresponding to λ by taking all non-zero linear combos of basis vectors
- e.g. find eigenvalues & a basis for each eigenspace of A where $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

NOTE · verify work by checking basis vectors satisfy $A\vec{x} = \lambda\vec{x}$ for corresponding λ

SOLUTION

$$C_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right)$$

$$= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \quad \begin{matrix} R_1 = R_1 + \lambda R_2 \\ R_3 = R_3 - R_2 \end{matrix}$$

$$= \begin{vmatrix} 0 & 1-\lambda^2 & 1+\lambda \\ 1 & -\lambda & 1 \\ 0 & 1+\lambda & -\lambda-1 \end{vmatrix} \longleftarrow \text{cofactor expansion along } C_1$$

$$= -\begin{vmatrix} 1-\lambda^2 & 1+\lambda \\ 1+\lambda & -\lambda-1 \end{vmatrix}$$

$$= -\begin{vmatrix} (1-\lambda)(1+\lambda) & 1+\lambda \\ 1+\lambda & -(1+\lambda) \end{vmatrix}$$

$$= -(1+\lambda)^2 \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -(1+\lambda)^2(-(1-\lambda) - 1)$$

$$= -(1+\lambda)^2(-1+\lambda-1)$$

$$= -(\lambda+1)^2(\lambda-2)$$

Eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$.

For $\lambda_1 = -1$, solve $(A+I)\vec{x} = \vec{0}$.

$$A + I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As such, $\vec{x} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $s, t \in \mathbb{R}$. So, a basis for $E_{\lambda_1}(A)$ is $B_1 = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = 2$, solve $(A - 2I)\vec{x} = \vec{0}$:

$$A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_2]{R_1 = R_1 + 2R_2} \begin{bmatrix} 0 & -3 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_1} \begin{bmatrix} 0 & -3 & 3 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = -\frac{1}{3}R_1} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

As such, $\vec{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. So, a basis for $E_{\lambda_2}(A)$ is $B_2 = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

- in general, for $A \in M_{n\times n}(\mathbb{R})$, $C_A(\lambda)$ will have degree n
- let $A \in M_{n\times n}(\mathbb{R})$ w/eigenvalue λ
  - ↳ algebraic multiplicity of λ is # of times λ appears as root of $C_A(\lambda)$
    - denoted by a_λ
  - ↳ geometric multiplicity of λ is dimension of eigenspace $E_\lambda(A)$
    - denoted by g_λ
- for any $A \in M_{n\times n}(\mathbb{R})$ & any eigenvalue λ of A, $1 \leq g_\lambda \leq a_\lambda \leq n$
- for square upper/lower triangular matrix, eigenvalues of A are entries on main diagonal of A
- given $A \in M_{n\times n}(\mathbb{R})$, $C_A(\lambda)$ is real polynomial of degree n
  - ↳ since real polynomial can have non-real roots, real matrix can have non-real eigenvalues
- if real nxn matrix A has complex eigenvalue λ, then $\bar{\lambda}$ is also eigenvalue of A
  - ↳ if $\vec{x}$ is eigenvector of A corresponding to complex eigenvalue λ, then $\bar{\vec{x}}$ is eigenvector of A corresponding to complex eigenvalue $\bar{\lambda}$

DIAGONALIZATION

- diagonal matrix is an nxn matrix D st $d_{ij} = 0$ for all $i \neq j$
  - ↳ denoted by $D = \text{diag}(d_{11}, \ldots, d_{nn})$
  - ↳ are both upper & lower triangular matrices
- if $D = \text{diag}(d_{11}, \ldots, d_{nn})$ & $E = \text{diag}(e_{11}, \ldots, e_{nn})$, then:
  - ↳ $D + E = \text{diag}(d_{11} + e_{11}, \ldots, d_{nn} + e_{nn})$

- $DE = \mathrm{diag}(d_{11}e_{11}, \ldots, d_{nn}e_{nn}) = \mathrm{diag}(e_{11}d_{11}, \ldots, e_{nn}d_{nn}) = ED$
- for any $k \in \mathbb{Z}^+$, $D^k = \mathrm{diag}(d_{11}^k, \ldots, d_{nn}^k)$
  - holds for any $k \in \mathbb{Z}$ iff none of $d_{11}, \ldots, d_{nn}$ are 0 (i.e. D is invertible)

- $n \times n$ matrix A is **diagonalizable** if there exists an $n \times n$ invertible matrix P & $n \times n$ diagonal matrix D st $P^{-1}AP = D$
  - i.e. P diagonalizes A to D
  - in general, $P^{-1}AP = D$ doesn't imply $A = D$ b/c matrix multiplication doesn't commute
- if A & B are $n \times n$ matrices st $P^{-1}AP = B$ for some invertible $n \times n$ matrix P, then:
  - $\det A = \det B$
  - A & B have same eigenvalues
  - $\mathrm{rank}(A) = \mathrm{rank}(B)$
  - $\mathrm{tr}(A) = \mathrm{tr}(B)$
    - $\mathrm{tr}(A) = a_{11} + \ldots + a_{nn}$ is **trace** of A
- if A & B are $n \times n$ matrices st $P^{-1}AP = B$ for some $n \times n$ invertible matrix P, then A & B are **similar**
  - $n \times n$ matrix is diagonalizable if it's similar to diagonal matrix
- let A be $n \times n$ matrix & let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of A; if B_i is basis for eigenspace $E_{\lambda_i}(A)$ for $i = 1, \ldots, k$, then $B = B_1 \cup B_2 \cup \ldots \cup B_k$ is linearly independent
  - since # of vectors in each $E_{\lambda_i}(A)$ is g_{λ_i}, there are $k \le g_{\lambda_1} + \ldots + g_{\lambda_k} \le n$ vectors in B
    - if there's n vectors in B, then B is basis for $\mathbb{R}^n$ consisting of eigenvectors of A

> NOTE: eigenvalue λ of diagonalizable matrix A appears in diagonal matrix D a_λ times

- **Diagonalization Theorem**: $n \times n$ matrix A is diagonalizable iff there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A
- given that A is diagonalizable, to construct invertible matrix P & diagonal matrix D st $P^{-1}AP = D$:
  - j^{th} column of P contains j^{th} vector from basis of eigenvectors
  - j^{th} column of D contains corresponding eigenvalue in (j,j)-entry
- $n \times n$ matrix A is diagonalizable iff $a_\lambda = g_\lambda$ for every eigenvalue λ of A
- if $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable
- e.g. diagonalize matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

  SOLUTION

  From above example, eigenvalues of A are $\lambda_1 = -1$ w/ $a_{\lambda_1} = 2$ & $\lambda_2 = 2$ w/ $a_{\lambda_2} = 1$. Also, we know $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is basis for $E_{\lambda_1}(A)$ so $\lambda_1 = -1$ has $g_{\lambda_1} = 2$ & $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ is basis for $E_{\lambda_2}(A)$ so λ_2 has $g_{\lambda_2} = 1$.

  Since $a_{\lambda_1} = g_{\lambda_1}$ & $a_{\lambda_2} = g_{\lambda_2}$, A is diagonalizable.

  $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ so $P^{-1}AP = \mathrm{diag}(-1,-1,2) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$

POWERS OF MATRICES

- $A^k = PD^kP^{-1}$ for any $k \in \mathbb{Z}^+$
- e.g. find A^k for $A = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$

  SOLUTION

  $$\begin{aligned} C_A(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} 3-\lambda & -4 \\ -2 & 1-\lambda \end{vmatrix} \\ &= (3-\lambda)(1-\lambda) - (-4)(-2) \\ &= 3 - 3\lambda - \lambda + \lambda^2 - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1) \end{aligned}$$

  $\lambda_1 = -1$ & $\lambda_2 = 5$ are eigenvalues of A. Since A is 2×2 matrix & has 2 distinct eigenvalues, A is diagonalizable.

  For $\lambda_1 = -1$, solve $(A + I)\vec{x} = \vec{0}$:

  $A + I = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \xrightarrow[R_2 = \frac{1}{2}R_2]{R_1 = \frac{1}{4}R_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

As such, $\vec{x} = t\begin{bmatrix}1\\1\end{bmatrix}$, $t \in \mathbb{R}$ so $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ is basis for $E_{\lambda_1}(A)$.

For $\lambda_2 = 5$, solve $(A-5I)\vec{x} = \vec{0}$:

$$A-5I = \begin{bmatrix}-2 & -4\\-2 & -4\end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix}-2 & -4\\0 & 0\end{bmatrix} \xrightarrow{R_1 = -\frac{1}{2}R_1} \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

As such, $\vec{x} = t\begin{bmatrix}-2\\1\end{bmatrix}$, $t \in \mathbb{R}$ so $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$ is basis for $E_{\lambda_2}(A)$.

Now, $P = \begin{bmatrix}1 & -2\\1 & 1\end{bmatrix}$ so it follows $D = \begin{bmatrix}-1 & 0\\0 & 5\end{bmatrix}$

$$\det P = \begin{vmatrix}1 & -2\\1 & 1\end{vmatrix} = 1(1) - (-2)(1) = 1+2 = 3$$

$$\operatorname{adj} P = \begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}$$

$$P^{-1} = \frac{1}{\det P}\operatorname{adj} P = \frac{1}{3}\begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}$$

$$\begin{aligned}A^k &= PD^kP^{-1}\\
&= \begin{bmatrix}1 & -2\\1 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 5\end{bmatrix}^k \frac{1}{3}\begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}\\
&= \frac{1}{3}\begin{bmatrix}1 & -2\\1 & 1\end{bmatrix}\begin{bmatrix}(-1)^k & 0\\0 & 5^k\end{bmatrix}\begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}\\
&= \frac{1}{3}\begin{bmatrix}(-1)^k & (-2)5^k\\(-1)^k & 5^k\end{bmatrix}\begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}\\
&= \frac{1}{3}\begin{bmatrix}(-1)^k - 2(5^k) & 2(-1)^k - 2(5^k)\\(-1)^k - 5^k & 2(-1)^k + 5^k\end{bmatrix}\end{aligned}$$

- can use eigenvalues of $n \times n$ matrix A to compute determinant & trace of A; let A have k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ w/ algebraic multiplicities $a_{\lambda_1}, \dots, a_{\lambda_k}$
  - ↳ $\det A = \lambda_1^{a_{\lambda_1}} \cdots \lambda_k^{a_{\lambda_k}} = \prod_{i=1}^{k} \lambda_i^{a_{\lambda_i}}$
  - ↳ $\operatorname{tr} A = \lambda_1 a_{\lambda_1} + \dots + \lambda_k a_{\lambda_k} = \sum_{i=1}^{k} \lambda_i a_{\lambda_i}$
- A is invertible iff 0 isn't eigenvalue of A

VECTOR SPACES

- a set $\mathbb{V}$ w/ operation of addition, denoted $\vec{x} + \vec{y}$, & operation of scalar multiplication, denoted $c\vec{x}$ ($c \in \mathbb{R}$), is vector space over $\mathbb{R}$ if for every $\vec{v}, \vec{x}, \vec{y} \in \mathbb{V}$ & every $c, d \in \mathbb{R}$:
  - V1) $\vec{x} + \vec{y} \in \mathbb{V}$
  - V2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
  - V3) $(\vec{x} + \vec{y}) + \vec{v} = \vec{x} + (\vec{y} + \vec{v})$
  - V4) there exists a $\vec{0} \in \mathbb{V}$ st $\vec{x} + \vec{0} = \vec{x}$ for every $\vec{x} \in \mathbb{V}$
    - zero vector
  - V5) for every $\vec{x} \in \mathbb{V}$, there exists a $(-\vec{x}) \in \mathbb{V}$ st $\vec{x} + (-\vec{x}) = \vec{0}$
  - V6) $c\vec{x} \in \mathbb{V}$
  - V7) $c(d\vec{x}) = (cd)\vec{x}$
  - V8) $(c+d)\vec{x} = c\vec{x} + d\vec{x}$
  - V9) $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
  - V10) $1\vec{x} = \vec{x}$
- elements of $\mathbb{V}$ are called vectors

- let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be set of vectors in vector space $\mathbb{V}$; span of $\mathcal{B}$ is $\operatorname{Span}\mathcal{B} = \{c_1\vec{v}_1 + \dots + c_k\vec{v}_k \mid c_1, \dots, c_k \in \mathbb{R}\}$
  - ↳ set $\operatorname{Span}\mathcal{B}$ is spanned by $\mathcal{B}$
  - ↳ $\mathcal{B}$ is spanning set for $\operatorname{Span}\mathcal{B}$
- let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be set of vectors in vector space $\mathbb{V}$
  - ↳ $\mathcal{B}$ is linearly dependent if there exist $c_1, \dots, c_k \in \mathbb{R}$ not all zero st $\vec{0} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$
  - ↳ $\mathcal{B}$ is linearly independent if only solution to $\vec{0} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$ is $c_1 = \dots = c_k = 0$
    - trivial solution
- subset $\mathbb{S}$ of $\mathbb{V}$ is subspace of $\mathbb{V}$ if properties V1 - V10 hold for every $\vec{x}, \vec{y}, \vec{v} \in \mathbb{S}$ & for all $c, d \in \mathbb{R}$
  - ↳ e.g. $\{\vec{0}\}$ is trivial subspace of $\mathbb{V}$
  - ↳ e.g $\mathbb{V}$ is subspace of $\mathbb{V}$
  - ↳ Subspace Test: let $\mathbb{S}$ be non-empty subset of $\mathbb{V}$; if for every $\vec{x}, \vec{y} \in \mathbb{S}$ & every $c \in \mathbb{R}$,

$\vec{x}+\vec{y} \in S$ & $c\vec{x} \in S$, then S is subspace of $\mathbb{V}$

- let S be subspace of $\mathbb{V}$ & let $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be set of vectors in S; B is basis for S if B is linearly independent & $S = \text{Span}B$
  - ↳ if $S = \{\vec{0}\}$, then $B = \emptyset$ is basis for S
- dimension of subspace S of $\mathbb{V}$ is # of vectors in any basis for S
  - ↳ denoted by $\dim(S)$

NOTE: dimension for $M_{m\times n}(\mathbb{R})$ matrix is $\dim(M) = mn$

- e.g. consider set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$; show B is basis for vector space $M_{2\times 2}(\mathbb{R})$

SOLUTION

For $c_1, c_2, c_3, c_4 \in \mathbb{R}$, consider:

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = c_4 = 0$ is the only solution & B is linearly independent. Since B has 4 vectors & $\dim(S) = 4$, B is basis for S.

Note that for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $\text{Span}B = M_{2\times 2}(\mathbb{R})$. We call B standard basis for $M_{2\times 2}(\mathbb{R})$.

- set $P(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + \ldots \mid a_0, a_1, a_2, \ldots \in \mathbb{R}$ where only finitely many of a_is are non-zero$\}$ denotes set of all real polynomials
  - ↳ $P_n(\mathbb{R}) = \{a_0 + a_1 x + \ldots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\}$ denotes set of all real polynomials of degree at most n
    - $P_n(\mathbb{R}) \subseteq P(\mathbb{R})$
  - ↳ zero polynomial is $0 = 0 + 0x + \ldots + 0x^n \in P_n(\mathbb{R})$
- if $p(x), q(x) \in P_n(\mathbb{R})$, then $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ & $q(x) = b_0 + b_1 x + \ldots + b_n x^n$ for some $a_1, \ldots, a_n, b_0, \ldots, b_n \in \mathbb{R}$
  - ↳ $p(x) = q(x)$ iff $a_i = b_i$ for $i = 0, 1, \ldots, n$
  - ↳ $(p+q)(x) = p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$
  - ↳ $(kp)(x) = kp(x) = ka_0 + ka_1 x + \ldots + ka_n x^n$ for any $k \in \mathbb{R}$
- $P_n(\mathbb{R})$ is vector space w/above operations
- consider set $B = \{1, x, \ldots, x^n\} \subseteq P_n(\mathbb{R})$
  - ↳ for $c_0, c_1, \ldots, c_n \in \mathbb{R}$: $c_0(1) + c_1 x + \ldots + c_n x^n = 0$

    $0 + 0x + \ldots + 0x^n = 0$
    - $c_0 = c_1 = \ldots = c_n = 0$ so B is linearly independent
  - ↳ for any polynomial $p(x) = a_0 + a_1 x + \ldots + a_n x^n \in P_n(\mathbb{R})$, $p(x)$ is trivially linear combo of elements in B
    - $p(x) = a_0(1) + a_1(x) + \ldots + a_n(x^n)$
    - $\text{Span}B = P_n(\mathbb{R})$
  - ↳ B is standard basis of $P_n(\mathbb{R})$
  - ↳ $\dim(P_n(\mathbb{R})) = n+1$
  - ↳ $B = \{1, x, x^2, \ldots\}$ is standard basis for $P(\mathbb{R})$ & it's infinite dimensional
- e.g. consider subspace $S = \{p(x) \in P_2(\mathbb{R}) \mid p(2) = 0\}$ of $P_2(\mathbb{R})$; find basis for S

SOLUTION

Let $p(x) \in S$. We know $p(2) = 0$ so $x-2$ is factor of $p(x)$. Since $p(x) \in P_2(\mathbb{R})$, there are $a, b \in \mathbb{R}$ st:

$$\begin{aligned} p(x) &= (x-2)(ax+b) \\ &= ax^2 + bx - 2ax - 2b \\ &= a(x^2 - 2x) + b(x-2) \end{aligned}$$

$S = \text{Span}\{x^2 - 2x, x-2\}$. Since $x^2 - 2x$ & $x-2$ aren't scalar multiples of each other, $B = \{x^2 - 2x, x-2\}$ is linearly independent & B is basis for S.

let $\mathbb{V} = \{x \in \mathbb{R} \mid x > 0\}$; under standard operations of addition & scalar multiplication of real #s, $\mathbb{V}$ is not vector space over $\mathbb{R}$

↳ e.g. $2 \in \mathbb{V}$ & $-1 \in \mathbb{R}$ but $(-1)(2) = -2 \notin \mathbb{V}$ so V6 fails

- define new addition $x \oplus y = xy$ & scalar multiplication $c \odot x = x^c$ for all $x, y \in \mathbb{V}$ & $c \in \mathbb{R}$; we show $\mathbb{V}$ is vector space over $\mathbb{R}$ under new operations w/ $x, y, z \in \mathbb{V}$ & $c, d \in \mathbb{R}$:
  - V1) $x \oplus y = xy > 0$ since $x, y > 0$
  - V2) $x \oplus y = xy = yx = y \oplus x$
  - V3) $(x \oplus y) \oplus z = (xy) \oplus z = (xy)z = x(yz) = x \oplus (yz) = x \oplus (y \oplus z)$
  - V4) since $1 \in \mathbb{V}$ & $x \oplus 1 = x(1) = x$ for all $x \in \mathbb{V}$, 1 is zero vector
  - V5) since $x \in \mathbb{V}$, $x > 0$ so $\frac{1}{x} > 0$ & $\frac{1}{x} \in \mathbb{V}$
    - $x \oplus \frac{1}{x} = x(\frac{1}{x}) = 1$ so $\frac{1}{x}$ is additive inverse of x
  - V6) $c \odot x = x^c > 0$ since $x > 0$
  - V7) $c \odot (d \odot x) = c \odot (x^d) = (x^d)^c = x^{cd} = (cd) \odot x$
  - V8) $(c+d) \odot x = x^{c+d} = x^c x^d = (c \odot x) \oplus (d \odot x)$
  - V9) $c \odot (x \oplus y) = c \odot (xy) = (xy)^c = x^c y^c = x^c \oplus y^c = (c \odot x) \oplus (c \odot y)$
  - V10) $1 \odot x = x^1 = x$
- if $\mathbb{V}$ is vector space , then for every $\vec{x} \in \mathbb{V}$:
  - ↳ $0\vec{x} = \vec{0}$
  - ↳ $-\vec{x} = (-1)\vec{x}$

VECTOR SPACES OVER $\mathbb{C}$

- $\mathbb{C}^n$ is vector space over $\mathbb{C}$, for $\vec{z} \in \mathbb{C}^n$, $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = z_1\vec{e}_1 + \dots + z_n\vec{e}_n$
  - ↳ $\{\vec{e}_1, \dots, \vec{e}_n\}$ is standard basis for $\mathbb{C}^n$
  - ↳ $\dim(\mathbb{C}^n) = n$
- $M_{m \times n}(\mathbb{C})$ is vector space over $\mathbb{C}$
  - ↳ standard basis for $M_{m \times n}(\mathbb{C})$ is same as for $M_{m \times n}(\mathbb{R})$
  - ↳ $\dim(M_{m \times n}(\mathbb{C})) = mn$
- $P_n(\mathbb{C})$ is set of polynomials of degree at most n & is vector space over $\mathbb{C}$
  - ↳ standard basis is $\{1, x, \dots, x^n\}$ where $x \in \mathbb{C}$
  - ↳ $\dim(P_n(\mathbb{C})) = n+1$