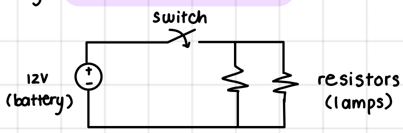
chapter 1

INTRODUCTION

· main purpose of electrical engineering systems are transferring energy from one place to another, communicating info, & processing info for better presentation & analysis

· e.g. electric circuit model of a headlights system



↳ battery converts chemical into electrical energy that forces electrons to flow around circuit & transfer energy

↳ wires made from conducting material (e.g. copper)

↳ switch controls operation of system

- closed means electric current (moving charges) can flow through circuit

↳ resistors convert electric energy into heat & light

ELECTRIC CURRENT

· when electric charges move through element / wire in circuit, create electric current (denoted by i)

↳ dir of current, is dir of movement of +ve charges

- for moving -ve charges, dir of i is opposite dir

current i(t) is defined as time ROC of charge q(t) · $i(t) = \frac{dq(t)}{dt}$

↳ current i(t) is measured in Amperes (A)

↳ charge q(t) is in Coulombs (C)

↳ time t is in seconds (s)

· charge in terms of current : $q(t) = \int_{t_0}^{t} i(\tau)\,d\tau + q(t_0)$

↳ t_0 is usually given

↳ τ is dummy variable

· acc dir of currents for elements in circuit are usually unknown before analysing

value of current for same element can be +ve & -ve at diff times

· reference dirs are assigned arbitrarily to elements before analyzing

↳ if value of current is +ve, then acc dir is same as reference

· in circuit analysis, currents are classified based on how current changes w/time ·

↳ direct current (DC): current flows in one dir

- e.g. constant current i = 3A

↳ alternating current (AC): current reverses dir

- e.g. sinusoidal current $i(t) = 2\sin(5t)$

↳ time varying: any other type of current

- e.g. $i(t) = e^{-t}$

ELECTRIC VOLTAGE

· motion of charge (current) in circuit is created by force called electromotive force (emf), potential diff, or electric voltage

· electric voltage is amount of energy required to move unit charge from one point to another

↳ moving charges transfer energy from one point to another

↳ $v = \frac{dw}{dq}$

- w is energy measured in joules (J)

- q is charge in coulombs (C)
- v is voltage in volts (1 V = 1 J/C)

· when charges move through element, voltage diff is established across it since energy is absorbed/supplied
  - ↳ polarities (+ & -) are used to distinguish higher (+) from lower (-) potential

· actual polarities of voltages for elements are usually unknown so reference polarities are assigned arbitrarily

· voltages are classified similarly to currents
  - ↳ constant over time is DC voltage
    - e.g. $v = 2V$
  - ↳ reverses over time is AC voltage
    - e.g. $v = 3\cos(2t)$

POWER AND ENERGY

· power is time ROC of energy : $p = \frac{dw}{dt}$
  - ↳ w is in joules (J)
  - ↳ p is in watts (W)

· above equation is rarely used since : $p = \frac{dw}{dt}$

$$p = \frac{dw}{dq} \cdot \frac{dq}{dt}$$

$$p = v \cdot i$$

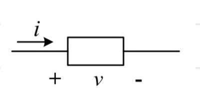
  - ↳ power is product of voltage & current

NOTE

· $E = \int_{t_0}^{t} p(t)dt$ &

1 J = 1 Ws

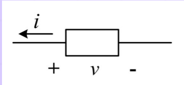
· convention / reference for element to absorb power is when dir of ref current i is from +ve to -ve sign of ref voltage v



  - ↳ ref absorbed power is $p_{abs} = v \cdot i$
    - if $p_{abs} > 0$, power is actually absorbed
    - if $p_{abs} < 0$, power is actually supplied
  - ↳ aka passive sign convention (PSC)

NOTE

· if p_{abs} doesn't comply w/ PSC (i.e. current goes from -ve to +ve sign of voltage), then $p_{abs} = -v \cdot i$



· convention/reference for element to supply / deliver power is when dir of ref current i is from -ve to +ve sign of ref voltage v
  - ↳ ref supplied power is $p_{sup} = v \cdot i$
    - if $p_{abs} > 0$, power is actually supplied
    - if $p_{abs} < 0$, power is actually absorbed
  - ↳ if p_{sup} is computed for element that complies w/ PSC, then $p_{sup} = -v \cdot i$
  - ↳ for every element, $p_{abs} = -p_{sup}$

· for every circuit, Principle of Conservation of Power / Energy states that Σ absorbed powers = Σ supplied powers
  - ↳ i.e. algebraic sum of all powers in circuit is 0 : $\sum_n p_n = 0$

· to compute energy from power, use $p = \frac{dw}{dt}$:

$$dw = p\,dt$$

$$w(t) = \int_{-\infty}^{t} p(\tau)\,d\tau$$

$$= \int_{-\infty}^{t} v(\tau)\, i(\tau)\,d\tau$$

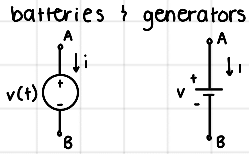
  - ↳ energy from time t_1 to t_2 is $w = w(t_2) - w(t_1)$

$$w = \int_{t_1}^{t_2} p(\tau)\,d\tau$$

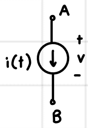
CIRCUIT ELEMENTS

- ideal circuit elements are used to model real physical components in electric circuit
- classified as active / passive elements
  - ↳ active elements generate/supply power/energy to rest of circuit
    - e.g. voltage source & current source
    - sub-classified into independent/dependent sources
  - ↳ passive elements can't supply power/energy
    - e.g. resistors absorb energy
    - e.g. capacitors & inductors store & later release energy
- circuit elements are also classified as linear/ nonlinear elements
  - ↳ linear element has voltage & current in linear relation (e.g. $v = 5i$)

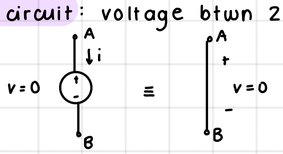
- ideal independent voltage source has specified voltage across its terminals independent of current & other elements
  - ↳ physical sources like batteries & generators can be apx by these
  - ↳ 2 common symbols:



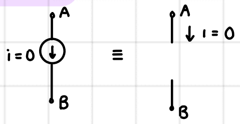
- ideal independent current source has specified current independent of voltage & other elements
  - ↳ good apx is electronic amplifier
  - ↳ common symbol:



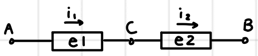
- since models are ideal, they don't exist as separate physical elements
  - ↳ combine w/ other models to more accurately rep real elements
    - e.g. real physical battery modeled by ideal constant voltage source w/ resistor
- short circuit: voltage btwn 2 nodes is 0
  - ↳



- open circuit: current btwn 2 nodes is 0
  - ↳

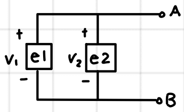


- series connection: 2 elements connected in series if they're only 2 elements connected to node
  - ↳ have same current
  - ↳



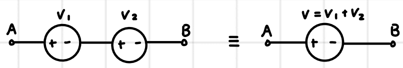
$i_1 = i_2$

- parallel connection: 2+ elements connected in parallel if connected btwn same 2 nodes
  - ↳ have same voltage
  - ↳

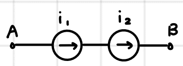


$V_1 = V_2$

- series connection of ideal voltage sources is always valid while of current sources is not valid b/c in general, $i_1 \neq i_2$
  - ↳ valid:

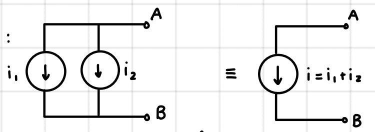


  - ↳ not valid:

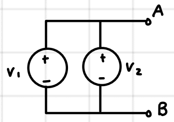


· parallel connection of ideal current sources is always valid while of voltage sources is not valid b/c in general, $V_1 \neq V_2$

↳ valid:



↳ not valid:



· electronic devices (e.g. transistor or amplifier) are modeled by dependent voltage / current sources also called controlled sources

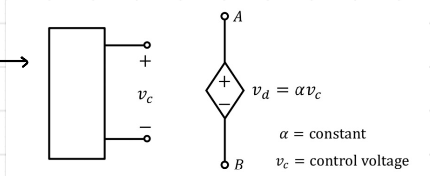
↳ dependent voltage / current sources can be controlled by voltage / current in another circuit element

↳ use diamond shape

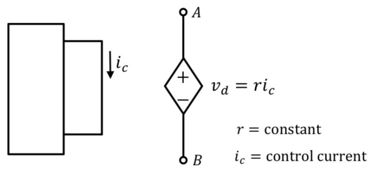
↳ 4 models:

- voltage-controlled voltage source (VCVS)

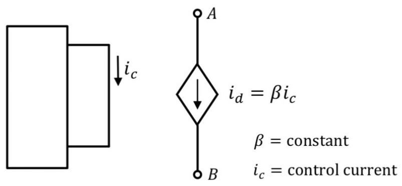
rep rest of circuit b/c V_c is control voltage somewhere in circuit →



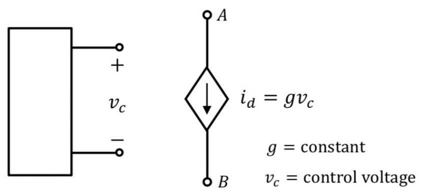
- current-controlled voltage source (CCVS)



- current-controlled current source (CCCS)



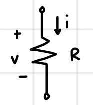
- voltage-controlled current source (VCCS)



chapter 2

RESISTORS AND OHM'S LAW

- flow of current through physical element encounters resistance, denoted by R
  - ↳ results in voltage drop
  - ↳ converts electrical energy to heat
- ideal circuit element is resistor

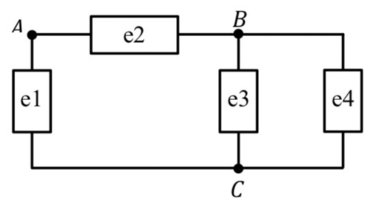


- Ohm's Law: voltage across resistor is proportional to current by a constant of R
  - ↳ $v = Ri$
    - ∘ R is +ve constant & measured in ohms(Ω)
    - ∘ $1\,\Omega = \frac{1V}{1A}$
  - ↳ if resistor doesn't comply w/ PSC, Ohm's law is $v = -Ri$
  - ↳ when $R = 0$, resistor is short circuit
  - ↳ when $R \to \infty$, resistor is open circuit
- power absorbed by resistor is $p = vi = Ri^2 = \frac{v^2}{R}$
  - ↳ since $R \geq 0$, then resistor always absorbs power
- practical physical resistor has 2 values: resistance R & power rating p_{max}
  - ↳ in circuit, ensure each $p < p_{max}$

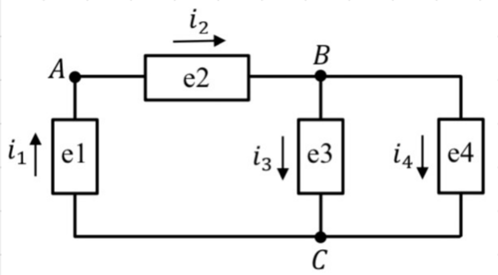
- conductance of resistor, denoted by G, is reciprocal of resistance
  - ↳ $G = \frac{1}{R}$
  - ↳ unit is Siemens(S) & $1\,S = 1\,\Omega^{-1}$
- Ohm's law for conductance:
  - ↳ $v = \frac{i}{G}$
  - ↳ $i = Gv$
- power for conductance is $p = vi = \frac{i^2}{G} = Gv^2$

KIRCHHOFF'S CURRENT LAW (KCL)

- node: point / junction in circuit where 2 or more elements are connected



  - ↳ e.g. circuit above has 3 nodes
- KCL states that algebraic sum of all currents at any node is 0
  - ↳ $\sum_n i_n = 0$
  - ↳ algebraic means if currents entering node are +ve, then currents leaving must be -ve (or vice versa)
  - ↳ based on law of conservation of charge
  - ↳ i.e. Σ entering currents = Σ leaving currents
- e.g. $i_1 = 3A$ & $i_3 = 1A$; find i_2 & i_4



SOLUTION

KCL at node A: $i_1 - i_2 = 0$

$i_2 = i_1$

$i_2 = 3A$

KCL at node B: $i_2 - i_3 - i_4 = 0$

$i_2 - i_3 = i_4$

$i_4 = 3 - 1$

$i_4 = 2\,A$

↳ KCL at node C is $i_1 - i_3 - i_4 = 0$
- not independent b/c it's sum of above 2 KCLs so it's not needed

↳ circuit w/ N nodes has (N-1) independent KCL equations

↳ one current variable is used for all series elements so KCL at node A isn't needed

KIRCHHOFF'S VOLTAGE LAW (KVL)

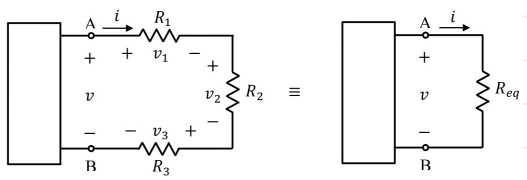
- loop: closed path in circuit in which no element is encountered more than once
- mesh: loop that doesn't enclose another loop
- KVL states that algebraic sum of all voltages around any loop is 0
  - ↳ $\sum_n V_n = 0$
  - ↳ based on law of conservation of energy
- voltage drop: move from +ve to -ve side of voltage
- voltage rise: move from -ve to +ve side of voltage

  \# of independent KVL equations = # of meshes
- one voltage variable is used for all parallel elements so # of KVL equations can be reduced
- another statement of KVL is sum of voltage drops btwn 2 nodes is same, regardless of path taken

NOTE
- in every circuit, there must be at least 1 source supplying power

EQUIVALENT SERIES AND PARALLEL RESISTORS

- 2+ resistors connected in series have same current



↳ applying KVL & Ohm's Law: $v = v_1 + v_2 + v_3$

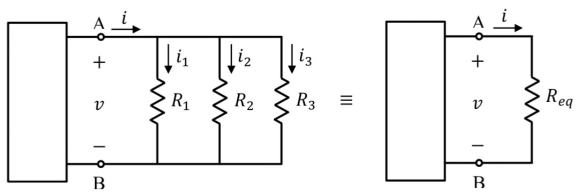
$= R_1 i + R_2 i + R_3 i$

$= (R_1 + R_2 + R_3) i$

$= R_{eq} i$

↳ $R_{eq} = R_1 + R_2 + R_3$

- 2+ resistors connected in parallel have same voltage



↳ applying KCL & Ohm's Law: $i = i_1 + i_2 + i_3$

$$= \frac{v}{R_1} + \frac{v}{R_2} + \frac{v}{R_3}$$
$$= \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)v$$
$$= \frac{1}{R_{eq}}(v)$$

↳ $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

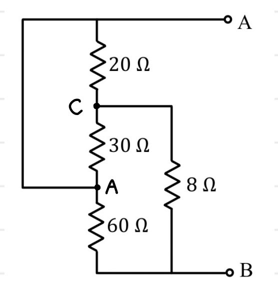
- in terms of conductances: $G_{eq} = G_1 + G_2 + G_3$

· for special case of 2 parallel resistors: $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$

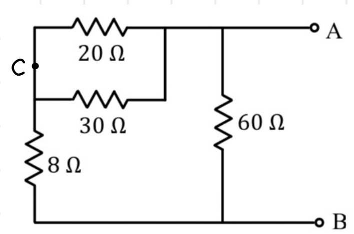
$$= \frac{R_1 + R_2}{R_1 R_2}$$
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

· for special case of short circuit in parallel w/ resistor, $R_{eq} = 0$

· e.g. find R_{eq} btwn A & B for below circuit:



redraw circuit →



SOLUTION

$R_{eq} = ((20 \| 30) + 8) \| 60$

$= \left(\frac{20 \cdot 30}{20+30} + 8\right) \| 60$

$= (12 + 8) \| 60$

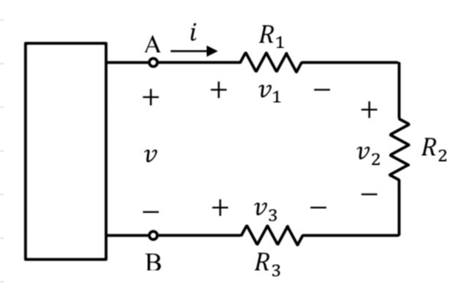
$= \frac{20 \cdot 60}{20 + 60}$

$= 15\Omega$

VOLTAGE DIVISION

sometimes, we know total voltage of 2+ resistors in series & want to know individual voltage across each resistor

↳ voltage division relations provide shortcut w/o computation of current & Ohm's Law



↳ current is $i = \frac{v}{R_1 + R_2 + R_3}$

↳ individual voltages are $v_1 = R_1 i$, $v_2 = R_2 i$, & $v_3 = -R_3 i$

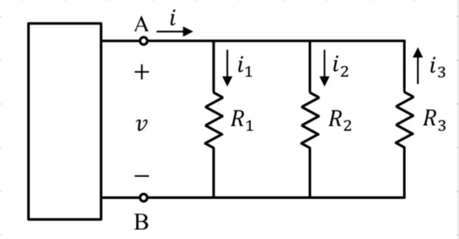
↳ sub current to get voltage divider relations:

- $v_1 = \frac{R_1}{R_1 + R_2 + R_3} v$
- $v_2 = \frac{R_2}{R_1 + R_2 + R_3} v$
- $v_3 = \frac{-R_3}{R_1 + R_2 + R_3} v$

CURRENT DIVISION

sometimes, we know total current of 2+ resistors in parallel & want to know individual current through each resistor

↳ current division relations provide shortcut w/o computation of voltage & Ohm's Law



NOTE
· more convenient to use conductance instead of resistance

↳ voltage is $v = \frac{i}{G_1+G_2+G_3}$

↳ individual currents are $i_1 = G_1 v$, $i_2 = G_2 v$, & $i_3 = -G_3 v$

↳ sub voltage to get current divider relations

- $i_1 = \frac{G_1}{G_1+G_2+G_3} i$
- $i_2 = \frac{G_2}{G_1+G_2+G_3} i$
- $i_3 = \frac{-G_3}{G_1+G_2+G_3} i$

· for special case of 2 resistors in parallel, current divider relations can be written in terms of resistances

↳ $i_1 = \frac{G_1}{G_1+G_2} i$

$= \frac{1/R_1}{1/R_1 + 1/R_2} i$

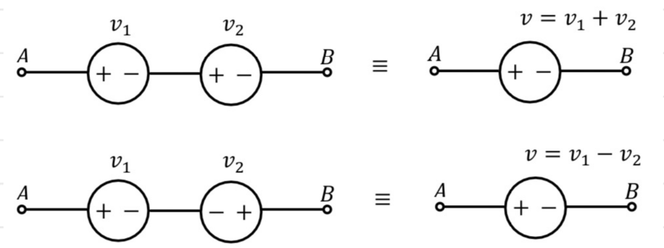
$= \frac{1}{R_1}\left(\frac{R_1 R_2}{R_2+R_1}\right) i$

$i_1 = \frac{R_2}{R_1+R_2} i$

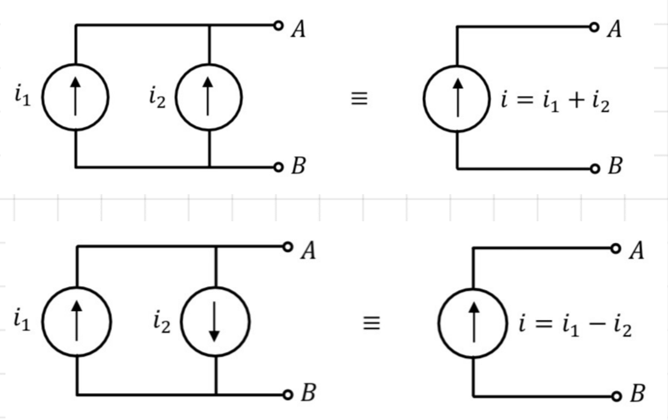
↳ $i_2 = \frac{R_1}{R_1+R_2} i$

SERIES AND PARALLEL CONNECTIONS OF SOURCES

· voltage sources can be connected in series but parallel connection is invalid



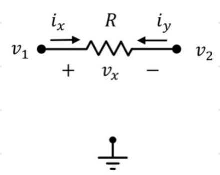
· current sources can be connected in parallel but series connection is invalid



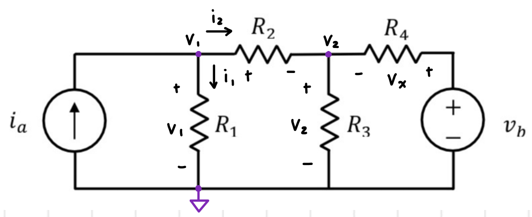
chapter 3

NODE - VOLTAGE (NODAL) ANALYSIS

- essential node: node where 3+ circuit elements are connected
- reference node: chosen arbitrarily & assumed to have zero-voltage
  - ↳ one of essential nodes
  - ↳ symbol is $\equiv$ or $\triangledown$
  - ↳ usually chosen to have most branches
- node-voltage: voltage drop from non-ref node to ref node
- essential node-voltage: voltage drop from essential non-ref node to ref node
- procedure for nodal analysis on circuit w/ N essential nodes:
  1) choose ref node & label (N-1) node-voltages
  2) optional: assign currents to branches
  3) apply KCL at each non-ref essential node to get (N-1) equations
  4) use Ohm's Law & KVL to express currents in terms of node-voltages
     - ↳ use 3-points rule:
       - $i_x = \frac{v_x}{R} = \frac{v_1 - v_2}{R}$
       - $i_y = -\frac{v_x}{R} = \frac{v_2 - v_1}{R}$



  5) solve resulting (N-1) linear independent equations for (N-1) unknown node-voltages
- e.g. find all currents & voltages using nodal analysis



$i_a = 3.75\,A$

$v_b = 6\,V$

$R_1 = R_3 = 2\,\Omega$

$R_2 = R_4 = 4\,\Omega$

NOTE
- branches are connections btwn nodes
- # of branches in circuit = # of circuit elements

NOTE
- when writing KCL equation at node, assume all currents leave node (unless indicated otherwise)
- leaving currents are +ve
- entering currents are -ve

SOLUTION

KCL at node 1: $-i_a + i_1 + i_2 = 0$

$$\frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2} = i_a$$

$$v_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - v_2\left(\frac{1}{R_2}\right) = i_a$$

$$v_1\left(\frac{1}{2} + \frac{1}{4}\right) - v_2\left(\frac{1}{4}\right) = 3.75$$

$$0.75v_1 - 0.25v_2 = 3.75 \quad (1)$$

KCL at node 2: $\frac{v_2 - v_1}{R_2} + \frac{v_2}{R_3} + \frac{v_2 - v_b}{R_4} = 0$

$$v_1\left(-\frac{1}{R_2}\right) + v_2\left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) - v_b\left(\frac{1}{R_4}\right) = 0$$

$$v_1\left(-\frac{1}{4}\right) + v_2\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right) = \frac{6}{4}$$

$$v_2 - 0.25v_1 = 1.5$$

$$v_2 = 1.5 + 0.25v_1 \quad (2)$$

Sub (2) into (1):

$$0.75v_1 - 0.25v_2 = 3.75$$

$$0.75v_1 - 0.25(1.5 + 0.25v_1) = 3.75$$

$$0.75v_1 - 0.375 - 0.0625v_1 = 3.75$$

$$0.6875v_1 = 4.125$$

$$v_1 = 6\,V$$

Use (2):

$$v_2 = 1.5 + 0.25v_1$$

$$= 1.5 + 0.25(6)$$

$$= 3V$$

Compute all other currents & voltages:

$i_1 = \frac{v_1}{R_1} = \frac{6}{2} = 3A$

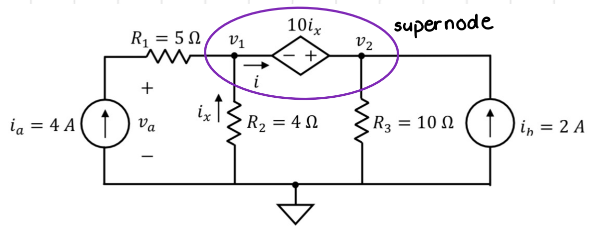
$i_2 = \frac{v_2}{R_3} = \frac{3}{4} = 0.75A$

$V_x = V_b - V_2 = 6 - 3 = 3V$

- algebraic sum of all currents entering/leaving any closed surface (i.e. supernode) is 0
  - ↳ general KCL statement

↳ supernode: when voltage source is located btwn 2 essential non-ref nodes, treat them as one node tgt

· e.g. use nodal analysis to find i_x



NOTE

· R_1 has no effect on rest of circuit b/c its current is fixed by current source i_a

· R_1 has effect on voltage & power for current source i_a

SOLUTION

KCL at supernode: $-i_a + \frac{v_1}{R_2} + \frac{v_2}{R_3} - i_b = 0$

$$\frac{v_1}{4} + \frac{v_2}{10} = 4 + 2$$

$$0.25v_1 + 0.1v_2 = 6 \qquad (1)$$

From dependent voltage source: $10i_x = v_2 - v_1$

$$10\left(-\frac{v_1}{4}\right) = v_2 - v_1$$

$$v_2 = -\frac{5v_1}{2} + v_1$$

$$v_2 = -1.5v_1 \qquad (2)$$

Sub (2) into (1):

$$0.25v_1 + 0.1(-1.5v_1) = 6$$

$$0.1v_1 = 6$$

$$v_1 = 60\,V$$

Use (2):

$$v_2 = -1.5(60)$$

$$= -90\,V$$

$$i_x = -\frac{v_1}{4}$$

$$= -\frac{60}{4}$$

$$= -15\,A$$

MESH ANALYSIS

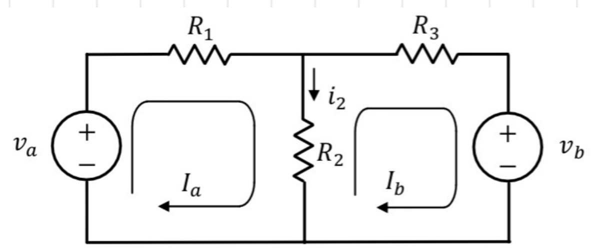
· mesh: loop that doesn't enclose another loop inside it

· mesh current: imaginary current that flows around mesh w/ same value & dir regardless of physical branch currents

· procedure for mesh analysis on circuit w/ N meshes:

1) draw & label mesh currents
2) write KVL around each mesh; use Ohm's Law to express voltages in terms of mesh currents, using relations btwn mesh currents & branch currents
   - ↳ in KVL equations, voltage-drops are +ve & voltage-rises are -ve
   - ↳ when encounter resistor, assume voltage-drop in dir of movement over resistor
   - ↳ assume branch current through resistor to be in dir of movement (i.e. branch current expressed directly in terms of mesh currents)
3) solve resulting set of N linear equations for N unknown mesh currents

· e.g. use mesh analysis to find current i_2



$v_a = 12\,V$

$v_b = 3\,V$

$R_1 = 6\,k\Omega$

$R_2 = 6\,k\Omega$

$R_3 = 3\,k\Omega$

SOLUTION

KVL around I_a: $-v_a + R_1 I_a + R_2(I_a - I_b) = 0$

$$6I_a + 6(I_a - I_b) = 12$$

$$12I_a - 6I_b = 12$$

$$2I_a - I_b = 2$$

$$I_b = 2(I_a - 1) \qquad (1)$$

KVL around I_b: $R_2(I_b - I_a) + R_3 I_b + V_b = 0$

$$6(I_b - I_a) + 3I_b = -3$$
$$9I_b - 6I_a = -3$$
$$3I_b - 2I_a = -1 \qquad (2)$$

Sub (1) into (2):

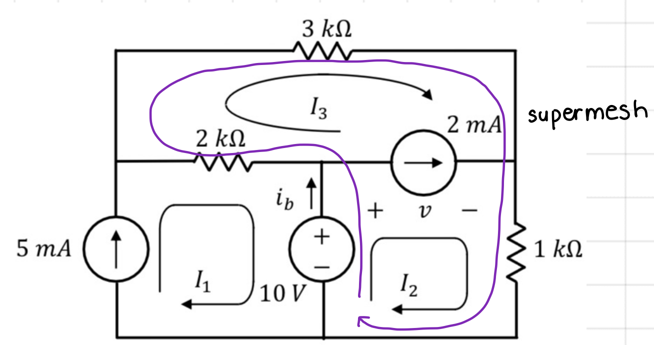
$$3(2(I_a - 1)) - 2I_a = -1$$
$$6I_a - 6 - 2I_a = -1$$
$$4I_a = 5$$
$$I_a = \frac{5}{4} \text{ mA}$$

$$I_b = 2\left(\frac{5}{4} - 1\right)$$
$$= 2\left(\frac{1}{4}\right)$$
$$= \frac{1}{2} \text{ mA}$$

$$i_2 = I_a - I_b$$
$$= \frac{5}{4} - \frac{1}{2}$$
$$= \frac{3}{4} \text{ mA}$$

Thus, $i_2 = \frac{3}{4}$ mA.

- supermesh: formed when 2+ meshes have common current source so they're grouped tgt & loop doesn't pass through current source
- e.g. find current i_b



SOLUTION

$I_1 = 5\text{mA}$

KVL around supermesh: $-10 + 2(I_3 - 5) + 3I_3 + I_2 = 0$

$$2I_3 - 10 + 3I_3 + I_2 = 10$$
$$5I_3 + I_2 = 20 \qquad (1)$$

From current source: $2 = I_2 - I_3$

$$I_2 = 2 + I_3 \qquad (2)$$

Sub (2) into (1):

$$5I_3 + 2 + I_3 = 20$$
$$6I_3 = 18$$
$$I_3 = 3 \text{ mA}$$

$$I_2 = 2 + 3$$
$$= 5\text{mA}$$

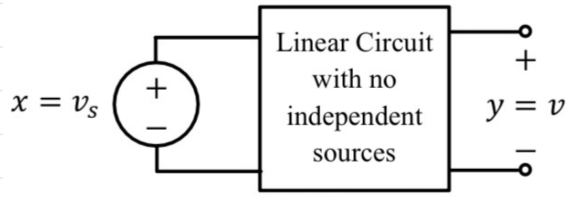
$$i_b = I_2 - I_1$$
$$= 5 - 5$$
$$= 0$$

Thus, $i_b = 0$mA.

chapter 4

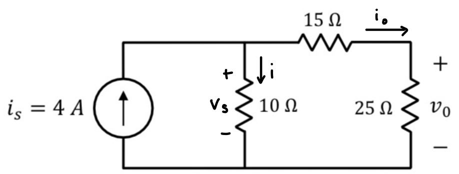
LINEARITY AND SUPERPOSITION

- system is entity (hardware/software) that's characterized by input(s) (i.e. excitations/causes), output(s) (effects/responses), & description of relation(s) btwn inputs & outputs
  - ↳ SISO is single-input single-output system
- linear system has to satisfy 2 properties of homogeneity/scaling & additivity
- homogeneity property requires that if input is multiplied by constant, output must be multiplied by same constant
  - ↳ if input $x_1 = cx$, then output must be $y_1 = cy$ (for some constant c)
- additivity requires if y_1 is output due to input x_1 & y_2 is output due to input x_2, then output due to input $x = x_1 + x_2$ is $y = y_1 + y_2$
- in circuits, input is independent voltage / current source & output is voltage/current somewhere in circuit



↳ $v = c\,v_s$

- e.g. In the circuit shown below, find the actual voltage v_0 by assuming $v_0 = 1\,V$ and using the linearity property.



SOLUTION

$i_o = \frac{v_o}{25} = \frac{1}{25}$

$v_s = (15+25)i_o = \frac{40}{25}$

$i = \frac{v_s}{10} = \frac{4}{25}$

$i_s = i + i_o = \frac{4}{25} + \frac{1}{25} = \frac{1}{5}\,A$

↳ linearity constant of multiplication is $c = \frac{\text{actual } i_s}{\text{assumed } i_s} = \frac{4}{\frac{1}{5}} = 20$

$V_o = 20(1) = 20V$

linearity doesn't apply to computation of power since it's non-linear function of current/voltage

↳ $p = vi = Ri^2 = \frac{v^2}{R}$

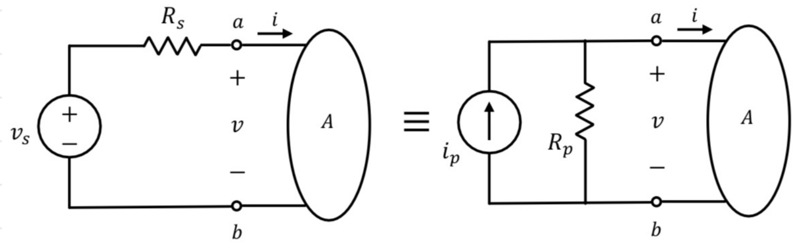
- for linear circuits w/ 2+ independent current/voltage sources, linearity is applied through principle of superposition: current/voltage in linear circuit can be computed as algebraic sum of individual contributions of each independent source acting alone
- to deactivate independent sources:
  - ↳ voltage source $v_s = 0$: short circuit
  - ↳ current source $i_s = 0$: open circuit

↳ dependent sources aren't deactivated b/c they're not inputs to circuit system

· for DC circuits, superposition isn't always easier but for AC circuits w/ independent sources that have diff frequencies, superposition must be used

SOURCE TRANSFORMATION

· combo of voltage source in series w/ resistance is equivalent to combo of current source in parallel w/ resistance



↳ given i_p & R_p, can find v_s & R_s easily:

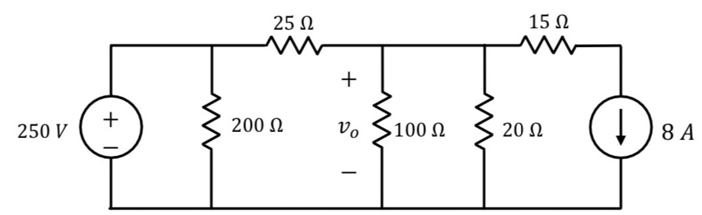
- $R_s = R_p$
- $v_s = R_p i_p$

↳ tip of arrow must point to terminal at which voltage source has +ve polarity

↳ source transformation can be applied to independent sources but control variable must be kept visible in circuit

· resistor in parallel w/ voltage source & resistor in series w/ current source have no effect on rest of circuit

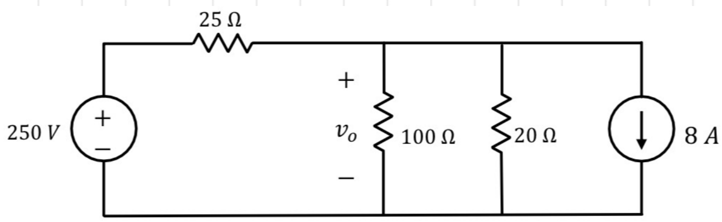
· e.g. Find the voltage v_o using source transformation.



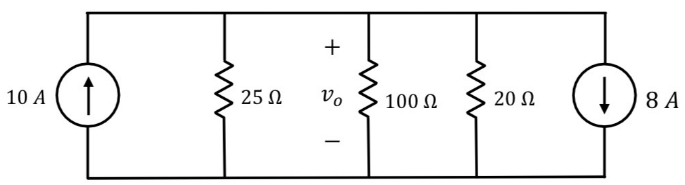
SOLUTION

↳ no source transformation on 250V in parallel w/ 200Ω & on 8 A in series w/ 15Ω

↳ replace 200Ω w/ open circuit & 15Ω w/ short circuit



↳ use source transformation:



- $10A = \frac{250V}{25\Omega}$

↳ combine 2 current sources & all resistors

- $I_T = 10 - 8$

  $= 2A$

- $\frac{1}{R_{eq}} = \frac{1}{25} + \frac{1}{100} + \frac{1}{20}$

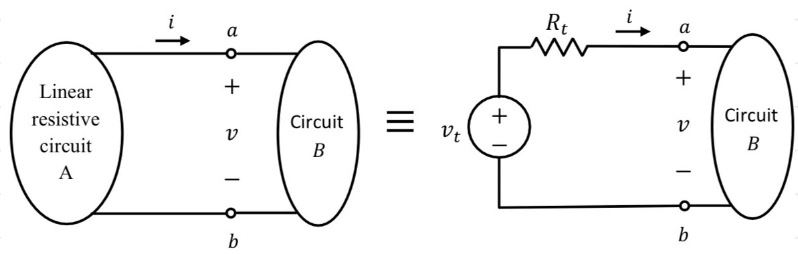
  $R_{eq} = 10\,\Omega$
- $V_0 = I_T R_{eq}$

  $= 2(10)$

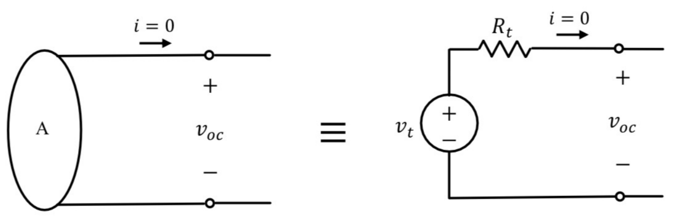
  $= 20\text{ V}$

THEVENIN AND NORTON EQUIVALENT CIRCUITS

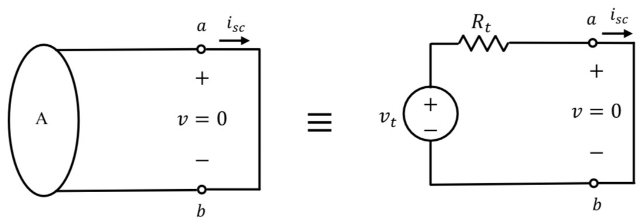
- Thevenin's Theorem: any resistive linear circuit btwn 2 terminals is equivalent to an independent voltage source in series w/ resistance



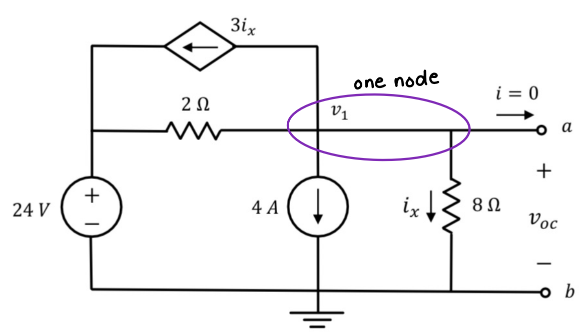
- from open-circuit:



  ↳ since i = 0, then $V_t = V_{oc}$
- from short-circuit:



  ↳ since v = 0, $i_{sc} = \frac{V_t}{R_t} = \frac{V_{oc}}{R_t}$
  - $R_t = \frac{V_{oc}}{i_{sc}}$
- to ensure relations are +ve, polarity of V_t should correspond to polarity of V_{oc} & dir of i_{sc} must be from +ve to −ve of V_{oc} through short
- e.g. Find Thevenin equivalent circuit between the terminals *a* and *b*.



SOLUTION

Find V_{oc} w/open-circuit.

$V_{oc} = V_1$, $i_x = \frac{V_1}{8}$

KCL at v_1: $0 = 3i_x + \frac{v_1 - 24}{2} + 4 + i_x$

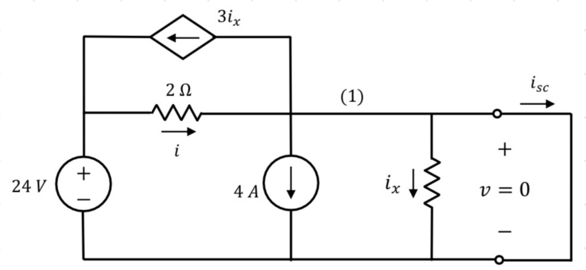
$0 = \frac{3v_1}{8} + \frac{v_1}{2} - 12 + 4 + \frac{v_1}{8}$

$0 = v_1 - 8$

$v_1 = 8V$

$v_1 = v_t = v_{oc} = 8V$

Find i_{sc} w/short-circuit.



Since $v=0$, then $i_x = 0$ & $3i_x = 0$ b/c i_x is in parallel w/short-circuit.

KCL at node 1: $0 = -i + 3i_x + 4 + i_x + i_{sc}$

$0 = -i + 4 + i_{sc}$

$i_{sc} = i - 4$

$= \frac{24}{2} - 4$

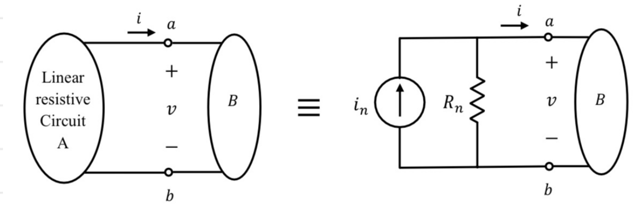
$= 8A$

$R_t = \frac{V_{oc}}{i_{sc}}$

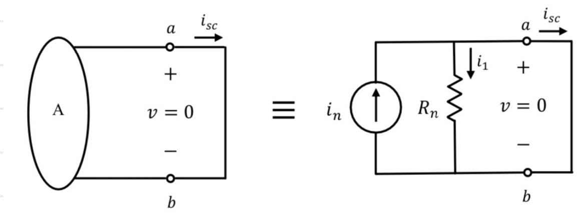
$= \frac{8}{8}$

$= 1\Omega$

· Norton's Theorem: any resistive linear circuit btwn 2 terminals is equivalent to current source in parallel w/ resistance



· from short-circuit.

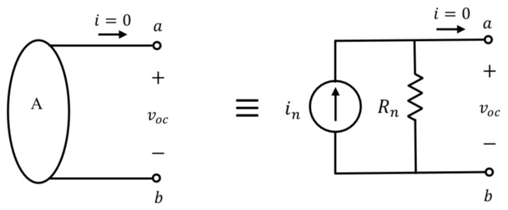


↳ since $v=0$:

- $i_1 = \frac{v}{R_n} = \frac{0}{R_n} = 0$
- $i_{sc} = i_n - i_1$

$i_n = i_{sc}$

from open-circuit·

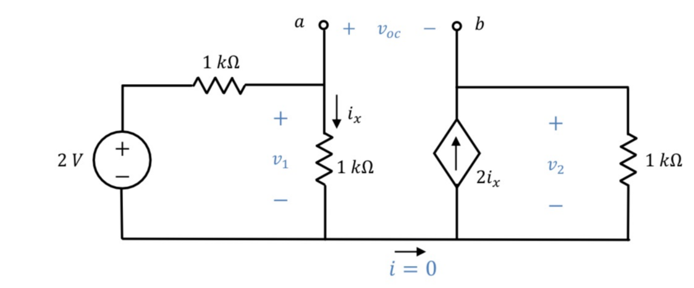


↳ since $i = 0$:

- $V_{oc} = R_n i_n$

$$R_n = \frac{V_{oc}}{i_{sc}}$$

- Thevenin & Norton equivalents are source transformation of each other
- when circuit has no independent sources, all currents & voltages are 0 so relation $R_T = R_n = \frac{V_{oc}}{i_{sc}}$ can't be applied
- e.g. Find Norton equivalent circuit between the terminals a and b.



SOLUTION

From open-circuit:

$V_{oc} = V_1 - V_2$

$V_1 = 2\left(\frac{1}{1+1}\right)$ ⟵ voltage division

$= 1$

$V_2 = (2i_x)(1000)$

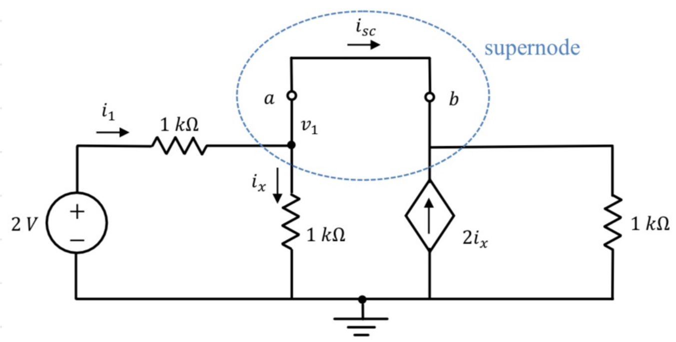
$= 2\left(\frac{2}{1000+1000}\right)(1000)$

$= 2$

$V_{oc} = 1 - 2$

$= -1\,V$

From short-circuit:



$i_x = \frac{V_1}{1000}$

KCL at supernode: $0 = \frac{V_1 - 2}{1000} + i_x - 2i_x + \frac{V_1}{1000}$

$0 = \frac{V_1}{1000} - \frac{2}{1000} - \frac{V_1}{1000} + \frac{V_1}{1000}$

$0 = -\frac{2}{1000} + \frac{V_1}{1000}$

$V_1 = 2\,V$

$0 = -i_1 + i_x + i_{sc}$

$i_{sc} = i_1 - i_x$

$= \frac{2-2}{1000} - \frac{2}{1000}$

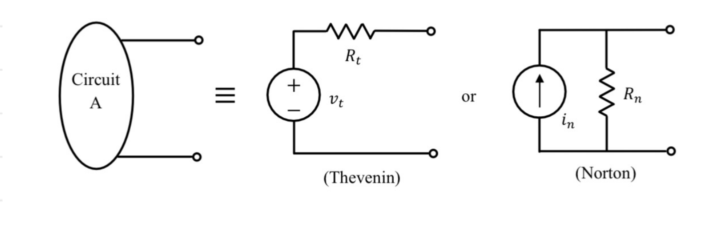
$= -2\,mA$

$R_n = \frac{V_{oc}}{i_{sc}}$

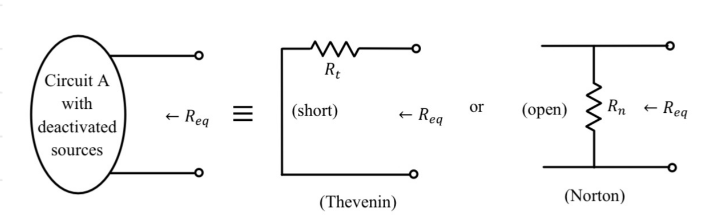
$= \frac{-1\,V}{-2\,mA}$

$= 0.5\,k\Omega$

· can find Thevenin / Norton resistance w/ equivalent resistance iff circuits have only independent sources



By *deactivating the sources*:



↳ for Thevenin :
- find V_{oc} as usual so $V_t = V_{oc}$
- R_{eq} from circuit A so $R_t = R_{eq}$

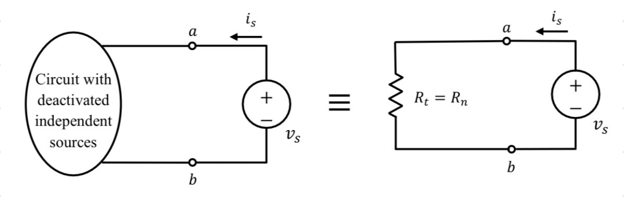
↳ for Norton:
- find i_{sc} as usual so $i_n = i_{sc}$
- R_{eq} from circuit A so $R_n = R_{eq}$

· finding R_t or R_n using test source can be applied to any circuit w/ independent & dependent sources

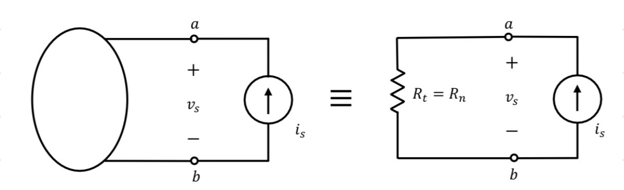
↳ deactivate all independent sources

↳ apply external test voltage source V_s or current source i_s

↳



or



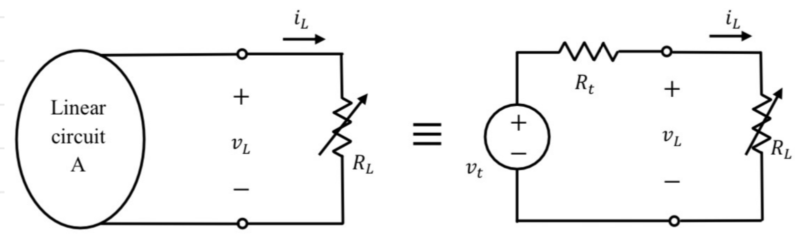
NOTE
- test source method must be used for circuits w/ dependent sources only
- in this case, $V_{oc} = 0$ & $i_{sc} = 0$

↳ assume V_s / i_s & find the other value wrt assumed value , then $R_t = R_n = \frac{V_s}{i_s}$

MAXIMUM POWER TRANSFER (LOAD MATCHING)

· in many practical applications , given linear circuit is required to transfer max power to load resistance R_L

· replace linear circuit w/ Thevenin equivalent :



· power delivered by circuit A / absorbed by load R_L is $P_L = i_L^2 R_L$

$$= \left(\frac{V_t}{R_t + R_L}\right)^2 R_L$$

$$= \frac{V_t^2 R_L}{(R_t + R_L)^2}$$

↳ to find R_L st P_L is max, differentiate wrt R_L & equate to 0.

$$\frac{dP_L}{dR_L} = \frac{V_t^2 (R_t + R_L)^2 - 2(R_t + R_L) V_t^2 R_L}{(R_t + R_L)^4} = 0$$

$$0 = V_t^2 (R_t + R_L)^2 - 2(R_t + R_L) V_t^2 R_L$$

$$0 = V_t^2 (R_t + R_L)(R_t + R_L - 2R_L)$$

$$0 = R_t - R_L$$

$$R_{Lmax} = R_t$$

↳ max power is $P_{Lmax} = \frac{V_t^2 R_L}{(R_t + R_L)^2}$

$$= \frac{V_t^2 R_t}{(2R_t)^2}$$

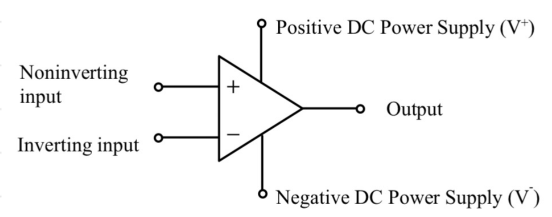
$$P_{Lmax} = \frac{V_t^2}{4R_t}$$

· using Norton equivalent, $R_{Lmax} = R_n$ & $P_{Lmax} = \frac{R_n i_n^2}{4}$

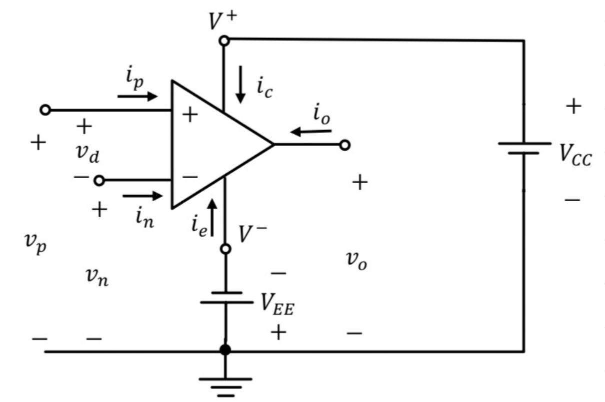
chapter 5

OPERATIONAL AMPLIFIER

- operational amplifier's main operations include amplification, addition, subtraction, differentiation, integration, & filtering
  - ↳ aka op-amp
  - ↳ symbol:



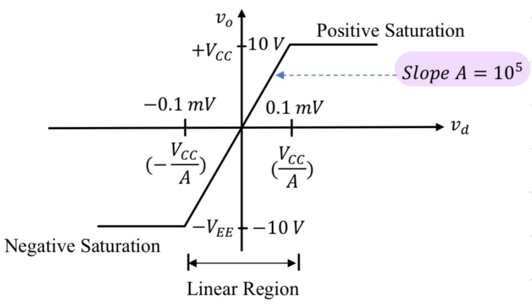
- terminal voltages & currents conventions:



- ↳ v_p & i_p rep noninverting input (+ve)
- ↳ v_n & i_n rep inverting input (-ve)
- ↳ v_o & i_o rep output
- ↳ V_{cc} & V_{EE} are batteries
- ↳ v_d is differential input voltage: $v_d = v_p - v_n$
  - not a node voltage
- v_o is function of v_d:

$$v_o = \begin{cases} -V_{EE} & \text{for } Av_d < -V_{EE} & \text{(non-linear)} \\ Av_d & \text{for } -V_{EE} \leq Av_d \leq +V_{cc} & \text{(linear)} \\ +V_{cc} & \text{for } Av_d > +V_{cc} & \text{(non-linear)} \end{cases}$$

- ↳ A is open-circuit voltage gain (large constant)
  - typical values of A: 10^5 to 10^8
- ↳ e.g. plot of v_o vs v_d w/ $V_{EE} = V_{cc} = 10V$:

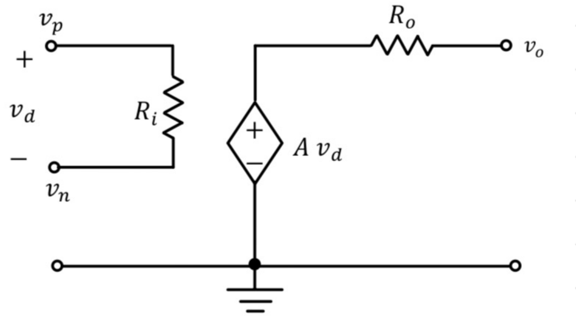


  - linear region is actually very small
  - for $A = 10^5$ & $V_{cc} = 10\,V$: $\frac{V_{cc}}{A} = \frac{10\,V}{10^5} = 10^{-4}\,V = 0.1\,mV$

- conclude that to operate op-amp in linear region, $|v_d| \leq 0.1 mV$
- condition for linear region in op-amp: $v_p \simeq v_n$

NON-IDEAL LINEAR MODEL OF OP-AMP

- when $v_d = v_p - v_n$ is very small (i.e. $v_p \simeq v_n$), op-amp can be modeled by linear circuit:



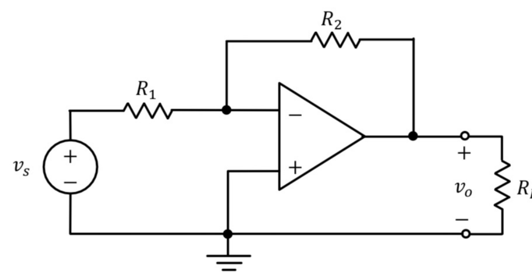
↳ 2 Thevenin equivalent circuits
↳ R_i is input resistance (very large)
  - typical values: 10^5 to 10^{12} Ω
↳ R_o is output resistance (very small)
  - typical values: 50 to 150 Ω
↳ A is open-circuit voltage gain (very large)
  - typical values: 10^5 to 10^8

- R_i, R_o, & A change widely w/ diff temps & aging
- feedback voltage from v_o to v_n (i.e. -ve feedback) is used b/c.
  ↳ makes op-amp circuit invariant to parameter variations in R_i, R_o, & A
  ↳ restricts operation in linear region
  ↳ makes circuit more stable
- above linear model can be used to analyze any op-amp circuit, given that it's operating in linear region
- e.g

Using the *linear model*:

(a) Find the *voltage gain* $G_v = \frac{v_o}{v_s}$ ← inverting amplifier

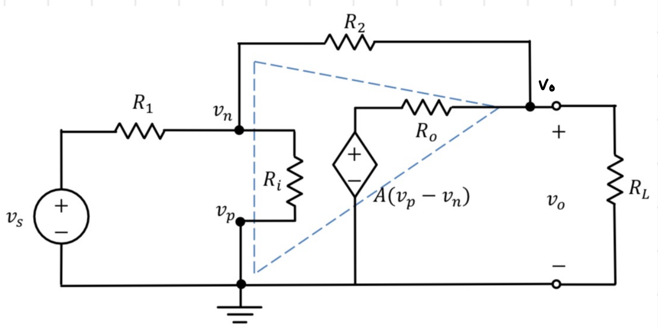
(b) Compute v_o for $v_s = 1\,V$ and $0.5\,V$



$R_1 = 1\,k\Omega$
$R_2 = 5\,k\Omega$
$R_L = 1\,k\Omega$
$R_i = 10^8\,\Omega$
$R_o = 50\,\Omega$
$A = 10^5$

SOLUTION

Replace op-amp w/ its linear model:



$v_p = 0$

KCL at v_n: $0 = \frac{v_n - v_s}{R_1} + \frac{v_n - v_p}{R_i} + \frac{v_n - v_o}{R_2}$

KCL at v_o: $0 = \frac{v_o - A(v_p - v_n)}{R_o} + \frac{v_o - v_n}{R_2} + \frac{v_o}{R_L}$

Rearranging gives: $G_v = \frac{v_o}{v_s}$

$$= \frac{-\frac{R_2}{R_1}}{1 + \frac{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_i}\right)\left(\frac{1}{R_2} + \frac{1}{R_o} + \frac{1}{R_L}\right)}{\left(\frac{A}{R_2 R_o} - \frac{1}{R_2^2}\right)}}$$

$\simeq -4.99968$

$\simeq -5$

NOTE

- since A is very large & ideally $A \to \infty$, then $\lim_{A\to\infty}\left(\frac{v_o}{v_s}\right) = -\frac{R_2}{R_1}$

$= -\frac{5k\Omega}{1k\Omega}$

$= -5$

- expression is independent of R_i, R_o, & A & can be derived more easily w/ ideal op-amp

For $v_s = 1V$:

$v_o = G_v \times v_s$

$= 4.99968 (1\ V)$

$= -4.99968\ V$

$\simeq -5\ V$

For $v_s = -0.5V$:

$v_o = G_v \times v_s$

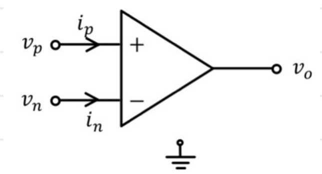
$= 4.99968 (-0.5\ V)$

$= 2.49984\ V$

$\simeq 2.5\ V$

IDEAL OP-AMP MODEL

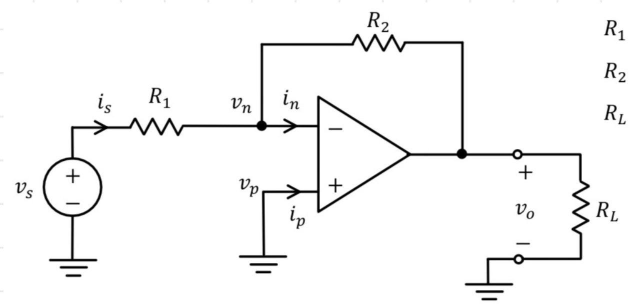
- simple model that makes analysis of op-amp circuits easier



- derivation of model:
  1) R_i is very large (ideally, $R_i \to \infty$)
     - $i_p = i_n = 0$
     - input looks like open circuit
  2) voltage gain A is very large (ideally, $A \to \infty$)
     - since $v_o = A(v_p - v_n)$ must be finite, then $v_p - v_n = 0$ (i.e. $v_p = v_n$)
     - input looks like short circuit

  ↳ conditions rep assumptions for ideal op-amp

e.g same inverting amplifier but assume ideal op-amp



$R_1 = 1\ k\Omega$

$R_2 = 5\ k\Omega$

$R_L = 2\ k\Omega$

VOLTAGE GAIN ($G_v = \frac{v_o}{v_s}$)

↳ by circuit connection, $v_p = 0$

↳ by ideal op-amp, $v_n = v_p = 0$

↳ KCL at v_n: $0 = \frac{v_n - v_s}{R_1} + \frac{v_n - v_o}{R_2} + i_n$

$0 = -\frac{v_s}{R_1} - \frac{v_o}{R_2}$ ⟵ since $v_n = 0$ & $i_n = 0$

$v_o = -\frac{R_2}{R_1} v_s$

↳ $G_v = \frac{v_o}{v_s}$

$= -\frac{R_2}{R_1} v_s \left(\frac{1}{v_s}\right)$

$G_v = -\frac{R_2}{R_1}$

$= -5$

INPUT RESISTANCE (IMPEDANCE) ($R_{in} = \frac{v_s}{i_s}$)

↳ Thevenin equivalent resistance as seen by source

↳ KVL at input: $-v_s + R_1 i_s + v_n = 0$

$$v_s = R_1 i_s \quad \longleftarrow \text{since } v_n = 0$$

$$R_1 = \frac{v_s}{i_s}$$

$$R_{in} = R_1$$

$$= 1\,k\Omega$$

OUTPUT RESISTANCE

↳ Thevenin equivalent resistance as seen by load

↳ since $v_o = -\frac{R_2}{R_1} v_s$ regardless of R_L, then $R_{out} = 0$

- i.e. R_L sees perfect controlled voltage source
- even if R_L changes, v_o doesn't

↳ ideally, R_L has no effect on G_v, R_{in}, & R_{out}

- i.e. R_L is isolated from v_s

· e.g. non-inverting amplifier

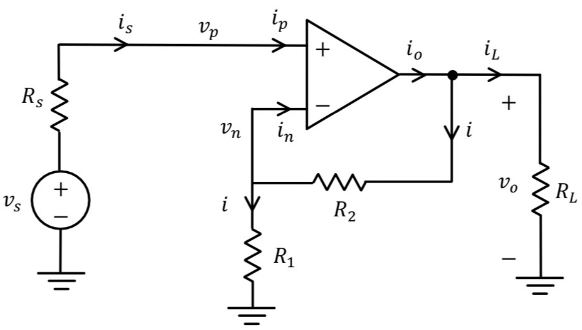
Assuming *ideal* op-amp, find:

(a) The *voltage gain* $G_v = \frac{v_o}{v_s}$.

(b) The voltage v_o and the current i_o for $v_s = 0.5\,V$.

(c) The *power supplied* by the source v_s and the *power absorbed* by the load R_L for $v_s = 0.5\,V$.

(d) The *input resistance* $R_{in} = \frac{v_s}{i_s}$ seen by the source v_s and the *output resistance* R_{out} seen by the load R_L.



$R_s = 1.8\,k\Omega$

$R_1 = 2\,k\Omega$

$R_2 = 18\,k\Omega$

$R_L = 5\,k\Omega$

SOLUTION

a) By ideal op-amp: $v_n = v_p$ & $i_n = i_p = 0$

$$v_n = v_p = v_s - R_s i_p$$
$$= v_s - R_s(0)$$
$$= v_s$$

KCL at v_n: $0 = \frac{v_n}{R_1} + \frac{v_n - v_o}{R_2}$

$$0 = v_n\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{v_o}{R_2}$$

$$\frac{v_o}{R_2} = v_s\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$v_o = R_2 v_s\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\frac{v_o}{v_s} = R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$G_v = \frac{R_2}{R_1} + 1 \quad \longleftarrow \text{independent of } R_L$$

$$= \frac{18}{2} + 1$$

$$= 10 \quad \longleftarrow 10\times \text{ amplification}$$

b) $v_o = G_v v_s = 10(0.5\,V)$

$$= 5\,V$$

$0 = -i_o + i + i_L$

$i_o = i + i_L$

$= \frac{V_o}{R_1 + R_2} + \frac{V_o}{R_L}$ ⟵ since i_n is open circuit, others ways to rep i are $\frac{V_o - V_n}{R_2}$ or $\frac{V_n}{R_1}$

$= \frac{5}{2+18} + \frac{5}{5}$

$= 1.25\text{mA}$

c) $p_s = V_s i_s = V_s i_p = 0\text{W}$

$p_L = \frac{V_o^2}{R_L}$

$= \frac{5^2}{5k}$

$= 5\text{mW}$ ⟵ very large power gain (comes from DC)

d) Since $i_s = i_p = 0$, then:

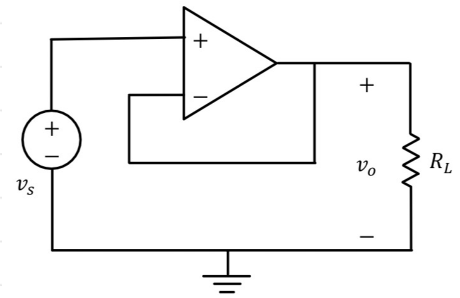
↳ $R_{in} = \frac{V_s}{i_s} = \frac{V_s}{0} \rightarrow \infty$

- very large

↳ $R_{out} = R_t = 0$

- b/c V_o is independent of R_L

· for $R_1 = \infty$ & $R_2 = 0$ of above non-inverting amplifier, we get unity-gain buffer



NOTE

· for summing amplifier,

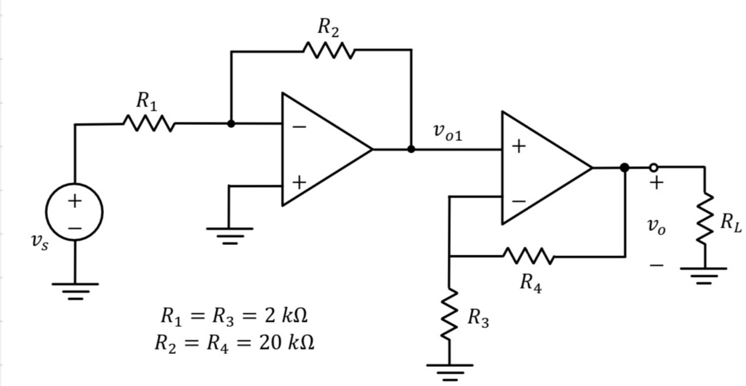
$V_o = -\left(\frac{R_f}{R_1} V_1 + \frac{R_f}{R_2} V_2\right)$ &

can be applied to any # of input sources

↳ used when source needs to be isolated from effect & change of load

· e.g. cascading op-amp circuit

Find the voltage gain $= \frac{v_o}{v_s}$, assuming *ideal* op-amp.



SOLUTION

↳ circuit is cascade of inverting amplifier & non-inverting amplifier

First op-amp: $V_{o1} = -\frac{R_2}{R_1} V_s$

Second op-amp: $V_o = \left(1 + \frac{R_4}{R_3}\right) V_{o1}$

$V_o = \left(1 + \frac{R_4}{R_3}\right)\left(-\frac{R_2}{R_1}\right) V_s$

$\frac{V_o}{V_s} = -\frac{R_2}{R_1}\left(1 + \frac{R_4}{R_3}\right)$

$G = -\frac{R_2}{R_1}\left(1 + \frac{R_4}{R_3}\right)$

$= -10(1+10)$

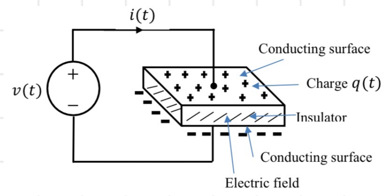
$= -110$

↳ voltage gain is product of 2 op-amps' voltage gains

chapter 6

CAPACITOR

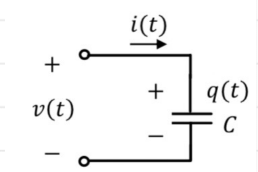
· capacitor consists of 2 conducting surfaces separated by insulator

↳ 

↳ e^- move from voltage source to be collected on lower plate as charge $q(t)$
  - creates E-field which forces e^- to leave upper plate & make it +ve w/ charge $q(t)$

↳ capacitor energy stored in its E-field can be later supplied to outside circuit

· ideal linear capacitor has following linear relation for charge: $q(t) = C\,v(t)$

↳ 

↳ $q(t)$ = charge in Coulombs (C)

↳ $v(t)$ = voltage in Volts (V)

↳ C = capacitance in Farad ($1F = \frac{1C}{1V}$)
  - constant that depends on physical dimensions of capacitor & material of insulator
  - practical capacitors have range of $1pF\,(10^{-12}F)$ to $10\,000\,\mu F\,(10^{-6}F)$

· taking derivative of above relation gives $i(t) = \frac{dq(t)}{dt} = C\frac{dv(t)}{dt}$

↳ if capacitor doesn't comply w/ PSC (i.e. current goes from +ve to -ve), $i(t) = -C\frac{dv(t)}{dt}$

↳ if voltage $v(t)$ is constant (i.e. DC voltage), then $i(t) = 0$
  - i.e. capacitor looks like open circuit

↳ capacitor voltage must be continuous function of time
  - can't have jump since $\frac{dv(t)}{dt}$ isn't defined there & infinite current is required

↳ capacitor current can have jump

· to find $v(t)$ from $i(t)$:

↳ from $i(t) = C\frac{dv(t)}{dt}$, $dv(t) = \frac{1}{C}i(t)\,dt$

↳ change of variables $(t \to \tau)$: $dv(\tau) = \frac{1}{C}i(\tau)\,d\tau$

↳ integrate from $\tau = t_0$ to $\tau = t$: $\int_{v(t_0)}^{v(t)} dv(\tau) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau$

$$v(t) - v(t_0) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$$

· power can be defined as $p(t) = v(t)\,i(t) = C\,v(t)\frac{dv(t)}{dt}$

· to find energy:

↳ $p(t) = \frac{dw(t)}{dt}$

$dw(t) = p(t)\,dt$

↳ change of variables $(t \to \tau)$: $dw(\tau) = p(\tau)\,d\tau$

↳ energy delivered to capacitor from $\tau = t_1$ to $\tau = t_2$:

$$\int_{w(t_1)}^{w(t_2)} dw(\tau) = \int_{t_1}^{t_2} p(\tau)\,d\tau$$

$$\int_{w(t_1)}^{w(t_2)} dw(\tau) = \int_{t_1}^{t_2} C\,v(\tau)\frac{dv(\tau)}{d\tau}\,d\tau$$

$$w(t_2) - w(t_1) = C\int_{v(t_1)}^{v(t_2)} v(\tau)\,dv(\tau)$$

$$w(t_2) - w(t_1) = C\left[\tfrac{1}{2}v^2(\tau)\right]_{v(t_1)}^{v(t_2)}$$

$$w = \tfrac{1}{2}C\left(v^2(t_2) - v^2(t_1)\right)$$

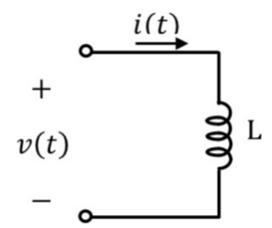
- rep energy stored in capacitor from t_1 to t_2
- energy as function of time is $w(t) = \frac{1}{2} C v^2(t) = \frac{q^2(t)}{2C}$

INDUCTOR

inductor consists of conducting wire in form of coil
↳ usually coil wound around iron core
↳ stores energy in magnetic field

· voltage of ideal linear conductor is $v(t) = L \frac{di(t)}{dt}$

↳



↳ L = inductance in Henry ($H = \frac{Vs}{A}$)
- constant that depends on physical dimensions of inductor, # of coil turns, & material of core
- practical inductors have range of a few μH to tens of H

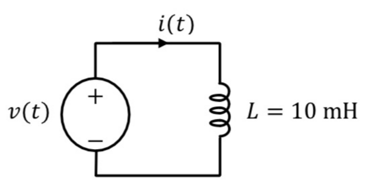
↳ if $i(t)$ is constant (i.e. DC current), then $v(t) = 0$
- i.e. inductor looks like short circuit

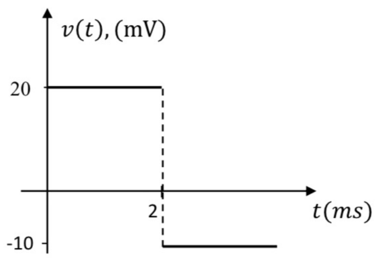
↳ inductor current must be continuous function of time
- can't have jump since $\frac{di(t)}{dt}$ isn't defined there & infinite voltage is required

· inductor relations are similar to capacitor relations but w/ replacements of $v(t) \to i(t)$, $i(t) \to v(t)$, & $C \to L$

↳ $i(t) = \frac{1}{L} \int_{t_0}^{t} v(\tau) d\tau + i(t_0)$
↳ power: $p(t) = v(t) i(t) = L i(t) \frac{di(t)}{dt}$
↳ stored energy from t_1 to t_2: $w = w(t_2) - w(t_1) = \frac{1}{2} L (i^2(t_2) - i^2(t_1))$
↳ stored energy as function of t: $w(t) = \frac{1}{2} L i^2(t)$

e.g. Find $i(t)$ and $w(t)$ for $t \geq 0$, assuming $i(0^-) = 0$.





SOLUTION

For $0 \leq t < 2$ ms:
↳ $t_0 = 0$
↳ $i(0) = i(0^-) = 0$ b/c current is continuous function

$i(t) = \frac{1}{L} \int_{t_0}^{t} v(\tau) d\tau + i(t_0)$
$= \frac{1}{10 \cdot 10^{-3}} \int_0^t 20 \cdot 10^{-3} \, d\tau + 0$

$i(t) = 2t$ A

At $t = 2^-$ ms, $i(2^- \text{ms}) = 2(2 \cdot 10^{-3})$
$= 4 \cdot 10^{-3}$ A

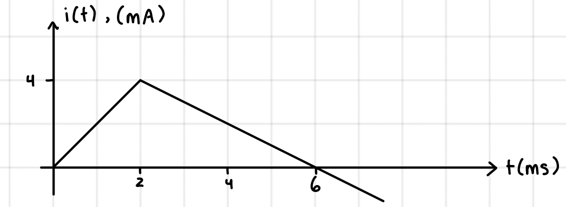
For $2 \text{ ms} \leq t < \infty$:
↳ $t_0 = 2$ ms

↳ $i(2ms) = i(2^- ms) = 4 \cdot 10^{-3}$ A

$$i(t) = \frac{1}{10 \cdot 10^{-3}} \int_{2 \cdot 10^{-3}}^{t} (-10 \cdot 10^{-3})\, d\tau + i(2ms)$$

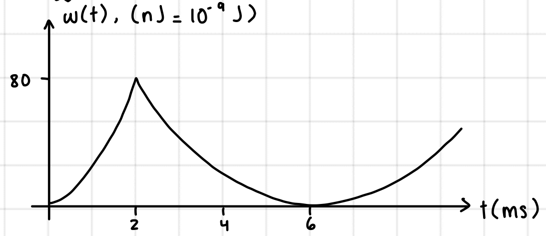
$$= -(t - 2 \cdot 10^{-3}) + 4 \cdot 10^{-3}$$

$i(t) = -t + 6 \cdot 10^{-3}$ A

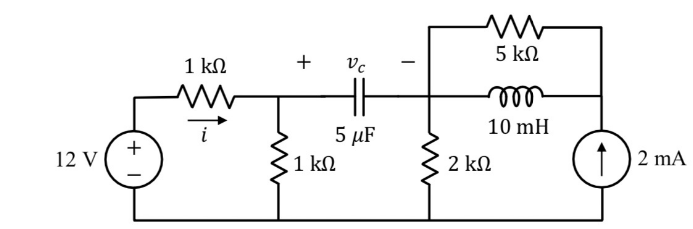


Energy: $w(t) = \frac{1}{2} L i^2(t)$

$$w(t) = \begin{cases} (20 \cdot 10^{-3}) t^2 & , \; 0 \le t < 2\text{ ms} \\ \frac{1}{2}(10 \cdot 10^{-3})(-t + 6 \cdot 10^{-3})^2 & , \; 2\text{ ms} \le t \end{cases}$$



· e.g.

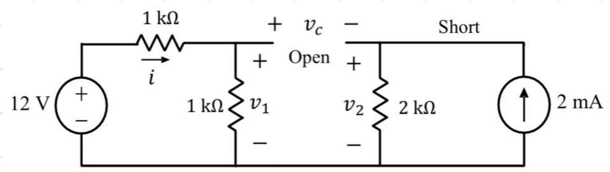


(a) Find the current i and the voltage v_C.

(b) Find the energy stored in each of L and C.

SOLUTION

DC equivalent circuit :



↳ L is short & C is open

$i = \frac{12\text{ V}}{1k + 1k}$ $= 6$ mA

$V_1 = 1k(6mA)$ $= 6V$

$V_2 = 2k(2mA)$ $= 4V$

$V_C = V_1 - V_2$ $= 6 - 4$

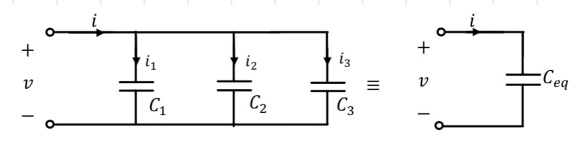
$V_C = 2$ V

$W_L = \frac{1}{2} L i_L^2$
$= \frac{1}{2}(10mH)(2mA)^2$
$= 20 \; 10^{-9}$ J
$= 20$ nJ

$W_C = \frac{1}{2} C v_C^2$
$= \frac{1}{2}(5\mu F)(2)^2$
$= 10 \cdot 10^{-6}$ J
$= 10 \mu$J

PARALLEL AND SERIES CAPACITORS

· parallel capacitors:



↳ use KCL : $i = i_1 + i_2 + i_3$

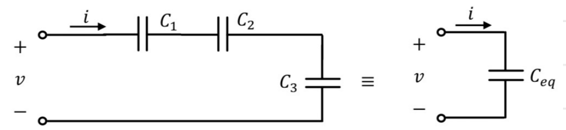
$$= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt}$$

$$= (C_1 + C_2 + C_3)\frac{dv}{dt}$$

↳ $C_{eq} = C_1 + C_2 + C_3$

- i.e. capacitors in parallel are added

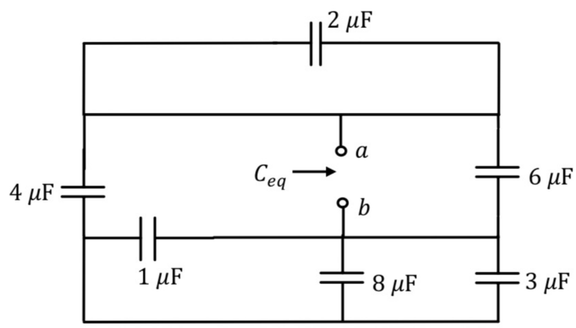
· series capacitors:



↳ $C_{eq} = \dfrac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$

- similar to resistors in parallel

· e.g. Find equivalent capacitance C_{eq} between a and b.



SOLUTION

↳ 2 μF is in parallel w/ short

↳ 1 μF, 8 μF, & 3 μF are in parallel w/ one another

$C_x = 1\,\mu F \parallel 8\,\mu F \parallel 3\,\mu F$

$= 1 + 8 + 3$

$= 12\,\mu F$

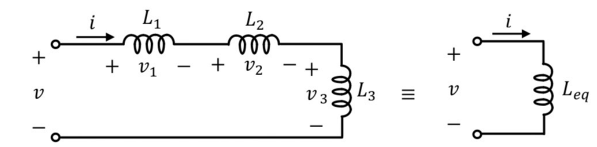
↳ 4 μF is in series w/ C_x which in turn is in parallel w/ 6 μF

$C_{eq} = \dfrac{4 \cdot 12}{4 + 12} + 6$

$C_{eq} = 9\,\mu F$

SERIES AND PARALLEL INDUCTORS

· series inductors:



↳ use KVL: $v = v_1 + v_2 + v_3$

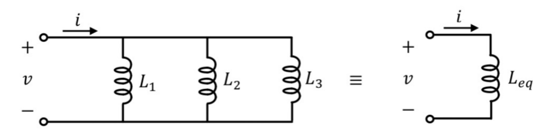
$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt}$$

$$= (L_1 + L_2 + L_3)\frac{di}{dt}$$

↳ $L_{eq} = (L_1 + L_2 + L_3)$

- inductors in series are added

· parallel inductors:



↳ $L_{eq} = \dfrac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}}$

- similar to resistors in parallel

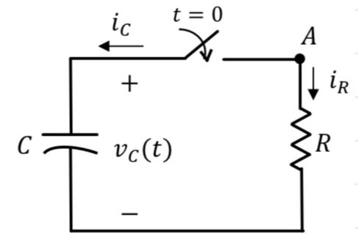
chapter 7

DISCHARGING OF RC CIRCUIT

· time-varying currents & voltages resulting from abrupt change of circuit / due to switching are called transients

↳ generally, transient analysis requires solving differential equations

·



↳ before $t = 0$, capacitor is assumed to be charged to initial voltage V_i

• i.e $v_C(0^-) = V_i$

↳ at $t = 0$, switch closes & $v_C(0) = v_C(0^-) = V_i$

↳ for $t \geq 0$, current flows thru resistor discharging capacitor

• eventually, all energy will be dissipated from resistor

SOLUTION

KCL at node A:

$$i_C + i_R = 0$$

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0 \quad , \quad t \geq 0$$

$$RC\frac{dv_C(t)}{dt} + v_C(t) = 0 \quad \longleftarrow \quad 1^{st} \text{ order differential equation (DE)}$$

↳ solution of DE $v_C(t)$ must be function that has form that's similar to its derivative so exponential function ($y = e^x$) satisfies requirement

↳ assume $v_C(t) = Ke^{st}$, $t \geq 0$

• K & s are constants

$$\frac{dv_C(t)}{dt} = Kse^{st}$$

$$RC\frac{dv_C(t)}{dt} + v_C(t) = 0$$

$$RCKse^{st} + Ke^{st} = 0$$

$$Ke^{st}(RCs + 1) = 0$$

↳ for non-trivial solution $Ke^{st} \neq 0$, then $RCs + 1 = 0$

$$s = -\frac{1}{RC}$$

• $v_C(t) = Ke^{st}$

$v_C(t) = Ke^{-\frac{t}{RC}}$

↳ find K from initial condition: $v_C(0) = V_i$

$$V_i = Ke^{-\frac{0}{RC}}$$

$$K = V_i$$

Solution is $v_C(t) = V_i e^{-\frac{t}{RC}}$, $t \geq 0$

· rapidity of decay is constant $\tau = RC$

↳ τ is time-constant of circuit

↳ response decays faster for smaller τ

↳ $v_C(t) = V_i e^{-\frac{t}{\tau}}$, $t \geq 0$

· theoretically, capacitor will never be completely discharged but practically, we assume it is when $v_c(t) \simeq 0$

↳ e.g. for $t > 5\tau$, capacitor assumed to be completely discharged

- discharging of RC circuit aka source-free RC circuit
- current thru capacitor is: $i_c(t) = C\frac{dv_c(t)}{dt}$

$$= -C\frac{V_i}{\tau}e^{-\frac{t}{\tau}}$$

$$= -C\frac{V_i}{RC}e^{-\frac{t}{\tau}}$$

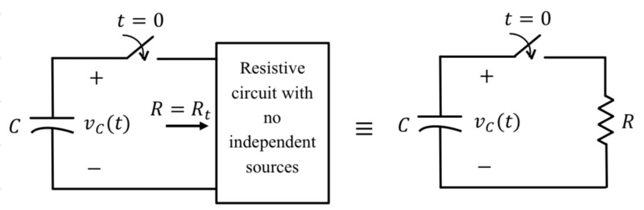
$$i_c(t) = -\frac{V_i}{R}e^{-\frac{t}{\tau}}, \quad t \geq 0$$

↳ jump at $t=0$, where $i_c(0^-) = 0$ & $i_c(0) = -\frac{V_i}{R}$

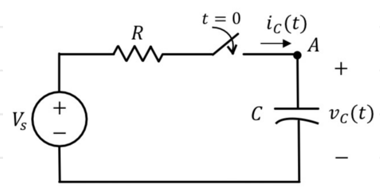
- energy w_R that's dissipated by R from $t: 0 \to \infty$ is equal to initial energy w_c stored in capacitor

↳ i.e. $w_R = w_c = \frac{1}{2}CV_i^2$

R in RC circuit can be Thevenin equivalent resistance of any circuit w/ no independent sources



CHARGING RC CIRCUIT



↳ source is DC source so V_s = constant

↳ capacitor is assumed to be initially discharged so $v_c(0^-) = 0$

↳ at $t=0$, switch closes & $v_c(0) = v_c(0^-) = 0$

↳ for $t \geq 0$, current flows charging capacitor

SOLUTION

KCL at node A:

$$i_c + \frac{v_c(t) - V_s}{R} = 0$$

$$C\frac{dv_c(t)}{dt} + \frac{v_c(t) - V_s}{R} = 0$$

$$RC\frac{dv_c(t)}{dt} + v_c(t) = V_s \quad , t \geq 0 \longleftarrow \text{1st order DE}$$

↳ since RHS is constant, then assume solution: $v_c(t) = K_1 + K_2e^{st}$

- K_1, K_2, & s are constants

$$RC\frac{dv_c(t)}{dt} + v_c(t) = V_s$$

$$RCK_2se^{st} + K_1 + K_2e^{st} = V_s$$

$$K_2e^{st}(RCs + 1) + K_1 = V_s$$

↳ match functions on both sides:

- $K_1 = V_s$
- $(RCs + 1)K_2e^{st} = 0$

$$RCs + 1 = 0$$

$$s = -\frac{1}{RC}$$

↳ find K_2 from initial condition: $v_c(0) = 0$

$$0 = K_1 + K_2e^{st}$$

$$0 = V_s + K_2e^{-\frac{0}{RC}}$$

$$K_2 = -V_s$$

Solution is $v_c(t) = V_s - V_s e^{-\frac{t}{RC}} = V_s(1 - e^{-\frac{t}{\tau}})$, $t \geq 0$

↳ $\tau = RC$ is time-constant

- for $t > 5\tau$, capacitor is assumed practically to be completely charged
  - ↳ i.e. $v_c(t) \simeq V_s$
  - ↳ DC value V_s is called DC steady-state value of $v_c(t)$

charging of RC circuit aka step response of RC circuit

- current thru capacitor is given by: $i_c(t) = C \frac{dv_c(t)}{dt}$

$$= C \frac{V_s}{\tau} e^{-\frac{t}{\tau}}$$

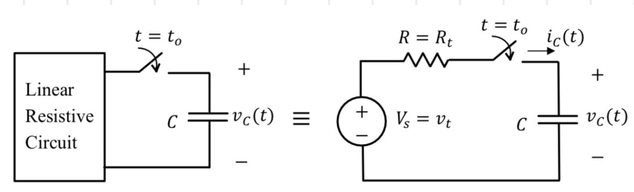
$$= C \frac{V_s}{RC} e^{-\frac{t}{\tau}}$$

$$i_c(t) = \frac{V_s}{R} e^{-\frac{t}{\tau}} \quad , t \geq 0$$

  - ↳ jump at $t = 0$, where $i_c(0^-) = 0$ & $i_c(0) = \frac{V_s}{R}$
- if initial value is non-zero (i.e. $v_c(0^-) = V_i$), then complete solution is $v_c(t) = V_s + (V_i - V_s)e^{-\frac{t}{\tau}}$, $t \geq 0$
- complete solution for any circuit can be decomposed into 2 components: $v_c(t) = v_{ss}(t) + v_{tr}(t)$
  - ↳ $v_{ss}(t)$ is steady-state response, which persists forever
    - e.g. V_s in above RC circuit
  - ↳ $v_{tr}(t)$ is transient response, which decays to 0 after a long time
    - e.g. $(V_i - V_s)e^{-\frac{t}{\tau}}$ in above RC circuit
- instead of specifying equation is valid for $t \geq 0$, step function $u(t)$ is used
  - ↳ $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$
  - ↳ rewrite function as $v_c(t) = [V_s + (V_i - V_s)e^{-\frac{t}{\tau}}]u(t)$
- if switch closes at time $t = t_o$ instead of $t = 0$, complete solution becomes $v_c(t) = V_s + (V_i - V_s)e^{-\frac{t - t_o}{\tau}}$, $t \geq t_o$
- instead of solving differential equation for every 1st order RC circuit, use general solution:

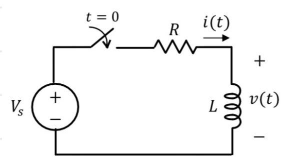
$v_c(t) = v_c(\infty) + [v_c(t_o) - v_c(\infty)]e^{-\frac{t - t_o}{\tau}}$, $t \geq t_o$

  - ↳ $v_c(\infty)$ is final value / DC steady-state value when $t \to \infty$
  - ↳ $v_c(t_o)$ is initial value
  - ↳ $\tau = RC$ is time-constant
  - ↳ can be applied for both charging & discharging RC circuits
- if capacitor's connected to more complicated linear circuit, use Thevenin equivalent
  - ↳



  - ↳ in general solution:
    - $v_c(\infty) = v_t$
    - $\tau = R_t \cdot C$

RL CIRCUIT



TIP

for RL circuit, always find eq for i(t) first then derive v(t) from it

- ↳ assume V_s = constant & $i(0^-) = 0$
- ↳ at $t = 0$, switch closes & $i(0) = i(0^-) = 0$ b/c no jump

↳ for $t \geq 0$, current flows thru R & L

SOLUTION

Using KVL: $Ri(t) + L\frac{di(t)}{dt} = V_s$

$\frac{L}{R}\frac{di(t)}{dt} + i(t) = \frac{V_s}{R}$, $t \geq 0$

↳ DE has same form as prev DE from RC circuit w/replacements:

- $v_c(t) \rightarrow i(t)$
- $RC \rightarrow \frac{L}{R}$
- $V_s \rightarrow \frac{V_s}{R}$

Solution is $i(t) = \frac{V_s}{R} - \frac{V_s}{R}e^{-\frac{t}{\tau}}$, $t \geq 0$

↳ $\tau = \frac{L}{R}$ is time-constant of RL circuit

- to find voltage: $v(t) = L\frac{di(t)}{dt}$

$= L\left(\frac{V_s}{R\tau}e^{-\frac{t}{\tau}}\right)$

$v(t) = V_s e^{-\frac{t}{\tau}}$, $t \geq 0$

↳ at $t = 0$, inductor behaves as open circuit since $v(0) = V_s$

↳ after long time, inductor behaves as short circuit (i.e. $v(t) = 0$)

- general solution for RL circuit: $i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-\frac{t-t_0}{\tau}}$, $t \geq t_0$

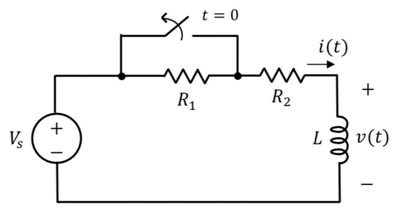
↳ $i(\infty)$ is final value / DC steady-state of inductor current

↳ $i(t_0)$ is initial current value

↳ $\tau = \frac{L}{R}$ is time-constant

- R is Thevenin R_{eq} as seen by inductor

- e.g. Assume switch is closed for a long time before $t = 0$. Find $i(t)$ and $v(t)$ for $t \geq 0$.



$V_s = constant$

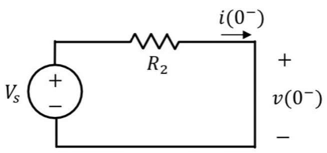
SOLUTION

For $t = 0^-$, switch is closed & circuit's in DC steady-state

↳ $i(0^-) = \frac{V_s}{R_2}$

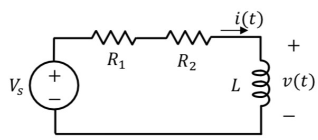
↳ $v(0^-) = 0$

↳



For $t \geq 0$, switch is open:

↳



↳ $t_0 = 0$

↳ $i(0) = i(0^-) = \frac{V_s}{R_2}$ b/c no jump

↳ $i(\infty) = \frac{V_s}{R_1 + R_2}$

- inductor becomes short circuit again & circuit's at DC steady-state

↳ $\tau = \frac{L}{R}$

$= \frac{L}{R_1 + R_2}$

Subbing values into general formula:

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-\frac{t}{\tau}} \quad , \; t \geq 0$$

$$i(t) = \frac{V_s}{R_1+R_2} + \left[\frac{V_s}{R_2} - \frac{V_s}{R_1+R_2}\right]e^{-\frac{t}{\tau}} \quad , \; t \geq 0$$

$$\hookrightarrow \tau = \frac{L}{R_1+R_2}$$

To find voltage:

$$v(t) = L\frac{di(t)}{dt}$$

$$= -\frac{L}{\tau}\left[\frac{V_s}{R_2} - \frac{V_s}{R_1+R_2}\right]e^{-\frac{t}{\tau}}$$

$$= -\frac{L}{\frac{L}{R_1+R_2}}\left[\frac{V_s}{R_2} - \frac{V_s}{R_1+R_2}\right]e^{-\frac{t}{\tau}}$$

$$= (R_1+R_2)\left(\frac{V_s}{R_1+R_2} - \frac{V_s}{R_2}\right)e^{-\frac{t}{\tau}}$$

$$v(t) = -V_s\frac{R_1}{R_2}e^{-\frac{t}{\tau}} \quad , \; t \geq 0$$

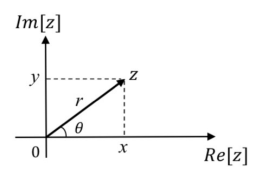
chapter 8

SINUSOIDS

- sinusoidal current is usually called Alternating Current (AC)
  - ↳ electric circuits w/ sinusoidal current &/or voltage sources are called AC circuits
- for AC circuits, transient component ($v_{tr}(t)$) isn't important b/c it decays to 0 quickly
  - ↳ only consider steady-state analysis of AC circuits
- everlasting sinusoidal voltage is $v(t) = V_m \cos(\omega t + \theta)$, $-\infty < t < +\infty$
  - ↳ V_m is amp, peak, or max value of sinusoid
  - ↳ ω is angular freq
    - radians/s
  - ↳ θ is phase angle
    - rads
- period of sinusoid is $T = \frac{2\pi}{\omega}$ s
- frequency is $f = \frac{1}{T}$ cycles/s (Hertz)
  - ↳ $\omega = \frac{2\pi}{T} = 2\pi f$

useful math formulas:
- ↳ $\sin\alpha = \cos(\alpha - \frac{\pi}{2})$
- ↳ $-\sin\alpha = \cos(\alpha + \frac{\pi}{2})$
- ↳ $-\cos\alpha = \cos(\alpha \pm \pi)$
- ↳ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- ↳ $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

- complex # z defined in rectangular form is $z = x + jy$
  - ↳ $x = Re[z]$ is real part
  - ↳ $y = Im[z]$ is imaginary part
  - ↳ $j = \sqrt{-1}$ is imaginary operator
- complex # rep graphically:



- complex # in polar form is $z = r\angle\theta$
  - ↳ $r = |z| = \sqrt{x^2+y^2}$ is magnitude
  - ↳ $\theta = \angle z = \tan^{-1}(\frac{y}{x})$ is angle
- complex # in exponential form is $z = re^{j\theta}$
  - ↳ using Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$, can convert exp form to rectangular form:
    - $z = re^{j\theta}$
    - $z = r\cos\theta + jr\sin\theta$
      - $x = r\cos\theta$
      - $y = r\sin\theta$
- polar & exp form are essentially same b/c both are specified by r & θ
- since $j = \sqrt{-j}$:
  - ↳ $j^2 = -1$

↳ $j^3 = -j$
↳ $j^4 = 1$
↳ $\frac{1}{j} = -j$

· using Euler's identity:
↳ $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
↳ $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

· conjugate of z is $z^* = x - jy = r(-\theta) = re^{-j\theta}$
↳ $zz^* = r^2 = x^2 + y^2$

· addition & subtraction are easier in rectangular form
↳ $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
↳ $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

· multiplication & division are easier in polar form
↳ $z_1 \cdot z_2 = r_1\angle\theta_1 \cdot r_2\angle\theta_2 = (r_1 r_2)\angle(\theta_1 + \theta_2)$
↳ $\frac{z_1}{z_2} = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \left(\frac{r_1}{r_2}\right)\angle(\theta_1 - \theta_2)$

PHASORS

· given sinusoid $v(t) = V_m\cos(\omega t + \theta)$, its phasor is complex # $\tilde{V} = V_m e^{j\theta} = V_m\angle\theta$
↳ transformation from sinusoid in time-domain to phasor in complex-domain (aka phasor-domain or frequency-domain)

· phasor rep amp V_m & phase θ but not angular freq ω
↳ every phasor is associated w/ specific freq (but not explicitly rep)
↳ when several phasors are manipulated tgt, must be associated w/ same freq

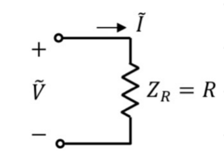
· phasors are useful for adding sinusoids w/ same freq & solving AC circuits

· relation of $v(t)$ & $\tilde{V}$ is $v(t) = Re\{\tilde{V}e^{j\omega t}\}$
$= Re\{V_m e^{j\theta} e^{j\omega t}\}$
$= Re\{V_m e^{j(\omega t + \theta)}\}$
$= V_m\cos(\omega t + \theta)$

BASIC PHASOR RELATIONS

· for resistor, we have Ohm's law: $v(t) = Ri(t)$
↳ $i(t) = I_m\cos(\omega t + \theta_i)$
↳ $v(t) = RI_m\cos(\omega t + \theta_i)$

· in phasor-domain:



↳ $\tilde{I} = I_m\angle\theta_i$
↳ $\tilde{V} = V_m\theta_v$
$= RI_m\angle\theta_i$

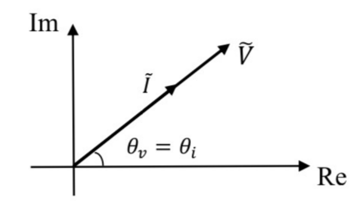
$\tilde{V} = R\tilde{I}$
- Ohm's Law
- R is a real #

↳ $V_m = RI_m$
↳ $\theta_v = \theta_i$

· impedance is $Z = \frac{\tilde{V}}{\tilde{I}}$
↳ for resistance, $Z_R = R$ (i.e. real)

↳ $\tilde{V} = Z_R \tilde{I}$

· phasor diagram:



↳ $\tilde{V}$ & $\tilde{I}$ are in-phase since $\theta_v = \theta_i$

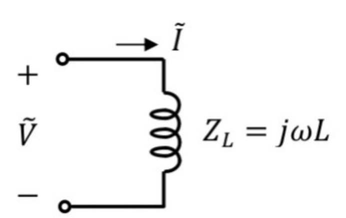
· for inductors, in time-domain, $v(t) = L\frac{di(t)}{dt}$

↳ $i(t) = I_m \cos(\omega t + \theta_i)$

↳ $v(t) = L I_m [-\omega \sin(\omega t + \theta_i)]$

- since $-\sin\alpha = \cos(\alpha + \frac{\pi}{2})$, $v(t) = \omega L I_m \cos(\omega t + \theta_i + \frac{\pi}{2})$
- $\omega L I_m = V_m$
- $\theta_i + \frac{\pi}{2} = \theta_v$

in phasor-domain:



↳ $\tilde{I} = I_m \angle \theta_i$

$= I_m e^{j\theta_i}$

↳ $\tilde{V} = V_m \angle \theta_v$

$= V_m e^{j\theta_v}$

$= \omega L I_m e^{j(\theta_i + \frac{\pi}{2})}$

$= \omega L (I_m e^{j\theta_i}) \underbrace{e^{j\frac{\pi}{2}}}_{j}$

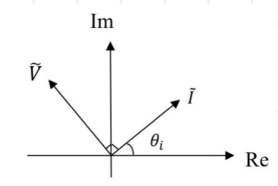
$\tilde{V} = j\omega L \tilde{I}$

· impedance of inductance is $Z_L = \frac{\tilde{V}}{\tilde{I}} = j\omega L$

↳ purely imaginary & +ve

↳ $\tilde{V} = Z_L \tilde{I}$

· phasor diagram:



NOTE

- $j = 1\angle 90°$
- $-j = 1\angle -90°$

↳ $V_m = \omega L I_m$

↳ $\theta_v = \theta_i + \frac{\pi}{2}$

↳ $\tilde{V}$ leads $\tilde{I}$ by $\frac{\pi}{2}$

↳ $\tilde{I}$ lags $\tilde{V}$ by $\frac{\pi}{2}$

· for capacitors, in time-domain, $i(t) = C\frac{dv(t)}{dt}$

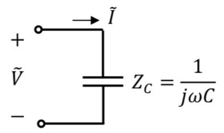
↳ $v(t) = V_m \cos(\omega t + \theta_v)$

↳ $i(t) = C V_m [-\omega \sin(\omega t + \theta_v)]$

$= \omega C V_m \cos(\omega t + \theta_v + \frac{\pi}{2})$

- $\omega C V_m = I_m$
- $\theta_v + \frac{\pi}{2} = \theta_i$

· in phasor - domain:



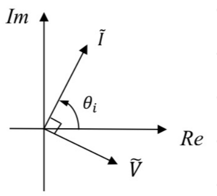
↳ $\tilde{I} = I_m e^{j\theta_i}$
$= \omega C V_m e^{j(\theta_v + \frac{\pi}{2})}$
$= j\omega C \tilde{V}$

↳ $\tilde{V} = \frac{1}{j\omega C} \tilde{I}$

· impedance of capacitance is $Z_c = \frac{\tilde{V}}{\tilde{I}}$
$= \frac{1}{j\omega C}$
$Z_c = -j \frac{1}{\omega C}$

- purely imaginary & -ve
- $\tilde{V} = Z_c \tilde{I}$

· phasor diagram:



↳ $\tilde{V} = V_m \angle \theta_v$
$= \frac{1}{j\omega C} \tilde{I}$
$= -j \frac{1}{\omega C} (I_m \angle \theta_i)$
$= (\frac{1}{\omega C} I_m) \angle (\theta_i - \frac{\pi}{2})$

- $V_m = \frac{1}{\omega C} I_m$
- $\theta_v = \theta_i - \frac{\pi}{2}$

↳ $\tilde{V}$ lags $\tilde{I}$ by $\frac{\pi}{2}$

· Kirchhoff's Laws:

↳ KCL in phasor domain is $\sum_n \tilde{I}_n = 0$
↳ KVL in phasor domain is $\sum_n \tilde{V}_n = 0$

IMPEDANCE AND ADMITTANCE

· for R, L, C, or combo, impedance Z is defined in phasor domain to be $Z = \frac{\tilde{V}}{\tilde{I}}$

↳ Z is complex but it's not phasor b/c it's not associated w/ sinusoid & also function of ω
↳ $Z = R + jX$

- R is resistance
- X is reactance
- all Z, R, & X are measured in Ohms (Ω)

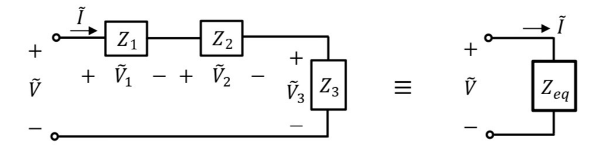
· admittance is defined as $Y = \frac{\tilde{I}}{\tilde{V}} = \frac{1}{Z}$

↳ $Y = G + Bj$

- G is conductance
- B is susceptance
- all Y, G, & B are measured in Siemens (S or Ω^{-1})

SERIES AND PARALLEL IMPEDANCES

· series impedances:



↳ $\tilde{V} = \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3$

$= Z_1\tilde{I} + Z_2\tilde{I} + Z_3\tilde{I}$

$= (Z_1 + Z_2 + Z_3)\tilde{I}$

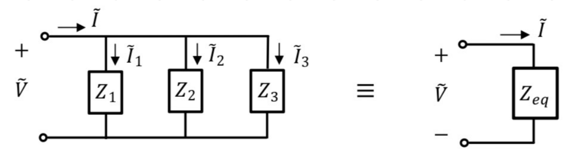
↳ $\tilde{V} = Z_{eq}\tilde{I}$

- equivalent impedance is $Z_{eq} = Z_1 + Z_2 + Z_3$

↳ voltage-division relations: $\tilde{V}_i = \dfrac{Z_i}{Z_1 + Z_2 + Z_3}\tilde{V}$

- $i = 1, 2, 3$

· parallel impedances:



↳ $\tilde{I} = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3$

$= \dfrac{\tilde{V}}{Z_1} + \dfrac{\tilde{V}}{Z_2} + \dfrac{\tilde{V}}{Z_3}$

$= \left(\dfrac{1}{Z_1} + \dfrac{1}{Z_2} + \dfrac{1}{Z_3}\right)\tilde{V}$

↳ $\tilde{I} = \dfrac{\tilde{V}}{Z_{eq}}$

- equivalent impedance is $\dfrac{1}{Z_{eq}} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2} + \dfrac{1}{Z_3}$ or $Y_{eq} = Y_1 + Y_2 + Y_3$

↳ current-division relations: $\tilde{I}_i = \dfrac{Y_i}{Y_1 + Y_2 + Y_3}\tilde{I}$

- $i = 1, 2, 3$

↳ special cases for 2 impedances in parallel:

- $Z_{eq} = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$
- $\tilde{I}_1 = \dfrac{Z_2}{Z_1 + Z_2}\tilde{I}$
- $\tilde{I}_2 = \dfrac{Z_1}{Z_1 + Z_2}\tilde{I}$

AC CIRCUIT ANALYSIS USING PHASORS AND IMPEDANCES

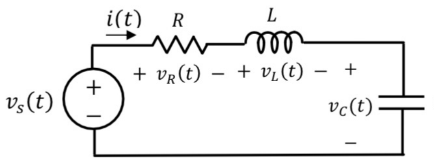
· procedure for AC circuit analysis:

1) convert circuit from time domain into phasor domain
   - replace all currents & voltages by their phasors
   - replace all elements by their impedances
2) solve circuit in phasor domain
   - similar to resistive circuits
3) convert result from phasor to time domain

· e.g. (1) Find $i(t)$, $v_R(t)$, $v_L(t)$, and $v_C(t)$.

(2) Draw phasor diagram.

Note: freq is $\omega = 500$



$R = 10\,\Omega$

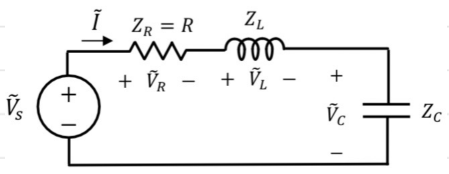
$L = 20\,mH$

$C = 100\,\mu F$

$v_s(t) = 20\cos(500t + 15°)$ (V)

SOLUTION

Phasor-domain circuit:



$\tilde{V}_s = 20\angle 15°$

$Z_L = j\omega L$

$= j(500)(20\ \ 10^{-3})$

$= j10\ \Omega$

$Z_C = -j\frac{1}{\omega C}$

$= -j\frac{1}{500(100\cdot 10^{-6})}$

$= -j20\ \Omega$

KVL: $\tilde{V}_s = \tilde{V}_R + \tilde{V}_L + \tilde{V}_C$

$= R\tilde{I} + Z_L\tilde{I} + Z_C\tilde{I}$

$= (R + Z_L + Z_C)\tilde{I}$

$= Z_{eq}\tilde{I}$

$Z_{eq} = R + Z_L + Z_C$

$= 10 + j10 - j20$

$= 10 - j10$

$= 10(1 - j)$

$= 10\sqrt{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$= 10\sqrt{2}\left(\cos(-45°) + j\sin(-45°)\right)$

$= 10\sqrt{2}\angle(-45°)$

$\tilde{I} = \frac{\tilde{V}_s}{Z_{eq}}$

$= \frac{20\angle 15°}{10\sqrt{2}\angle -45°}$

$= \sqrt{2}\angle 60°$

Convert to time domain:

$i(t) = \sqrt{2}\cos(500t + 60°)$ A

$\tilde{V}_R = R\tilde{I}$

$= 10\sqrt{2}\angle 60°$

$v_R(t) = 10\sqrt{2}\cos(500t + 60°)$ V

$\tilde{V}_L = Z_L\tilde{I}$

$= (j10)(\sqrt{2}\angle 60°)$

$= (10\angle 90°)(\sqrt{2}\angle 60°)$

$= 10\sqrt{2}\angle 150°$

$v_L(T) = 10\sqrt{2}\cos(500t + 150°)$ V

$\tilde{V}_C = Z_C\tilde{I}$

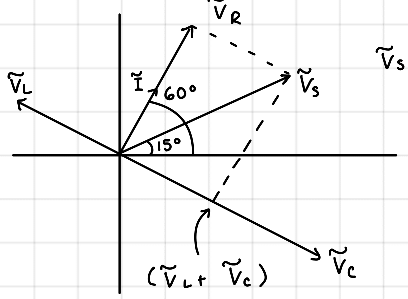
$= (-j20)(\sqrt{2}\angle 60°)$

$= (20\angle -90°)(\sqrt{2}\angle 60°)$

$= 20\sqrt{2}\angle -30°$

$v_C(T) = 20\sqrt{2}\cos(500t - 30°)$ V

Phasor diagram:



$\tilde{V}_s = \tilde{V}_R + \tilde{V}_L + \tilde{V}_C$ (vector sum)

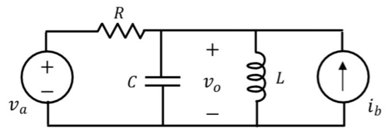
- in prev example, if $C = 200\mu F$, then:
  - ↳ $Z_C = -j\frac{1}{\omega C} = -j10 = -Z_L$
  - ↳ $Z_C + Z_L = 0$
  - ↳ $Z_{eq} = R + Z_L + Z_C = 10$

- combo of inductor & capacitor looks like short circuit
  - ↳ $\tilde{I} = \frac{\tilde{V}_s}{R} = \frac{20\angle 15°}{10} = 2\angle 15°$
  - ↳ $\tilde{V}_L + \tilde{V}_c = 0$ but:
    - $\tilde{V}_L = j\omega L \tilde{I}$
      $= j10(\sqrt{2}\angle 15°)$
      $\neq 0$
    - $\tilde{V}_c = -j\frac{1}{\omega L}\tilde{I}$
      $= -j10(2\angle 15°)$
      $\neq 0$
- series resonance happens when $Z_L + Z_c = 0$
  - ↳ $\omega L = \frac{1}{\omega C}$ or $\omega = \frac{1}{\sqrt{LC}}$
  - ↳ in time domain, $v_L(t)$ & $v_c(t)$ are sinusoids that are -ve of each other

chapter 9

NODAL ANALYSIS AND MESH ANALYSIS

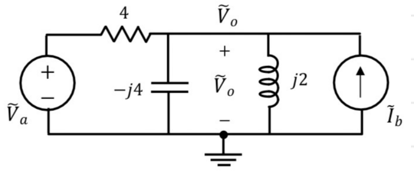
- procedures for AC circuits is same as for resistive DC circuits, except they're applied in phasor domain
- e.g. Find $v_o(t)$ in the circuit shown below, using nodal analysis.



$v_a = 2\cos(1000t + 30°)$
$i_b = \cos(1000t - 45°)$
$R = 4\,\Omega$
$C = 250\,\mu F$
$L = 2\,mH$

SOLUTION

In phasor domain:



↳ $\tilde{V}_a = 2\angle 30°$

↳ $\tilde{I}_b = 1\angle -45°$

↳ $Z_R = 4$

↳ $Z_C = -j\dfrac{1}{1000(250 \cdot 10^{-6})} = -j4$

↳ $Z_L = j(1000)(2 \cdot 10^{-3}) = j2$

KCL:

$$0 = \frac{\tilde{V}_o - \tilde{V}_a}{4} + \frac{\tilde{V}_o}{-j4} + \frac{\tilde{V}_o}{j2} - \tilde{I}_b$$

$$0 = j(\tilde{V}_o - \tilde{V}_a) - \tilde{V}_o + 2\tilde{V}_o - (j4)\tilde{I}_b$$

$$0 = j\tilde{V}_o - \tilde{V}_o + 2\tilde{V}_o - j\tilde{V}_a - (j4)\tilde{I}_b$$

$$0 = \tilde{V}_o(j+1) - j\tilde{V}_a - j4\tilde{I}_b$$

$$\tilde{V}_o(j+1) = j\tilde{V}_a + j4\tilde{I}_b$$

$$\tilde{V}_o(j+1) = (1\angle 90°)(2\angle 30°) + (4\angle 90°)(1\angle -45°)$$

$j+1$:
$r = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\quad \theta = \tan^{-1}\left(\frac{1}{1}\right) = 45°$

$$\tilde{V}_o(\sqrt{2}\angle 45°) = 2\angle 120° + 4\angle 45°$$

$$\tilde{V}_o(\sqrt{2}\angle 45°) = (-1 + j\sqrt{3}) + (2\sqrt{2} + j\,2\sqrt{2})$$

$$\tilde{V}_o(\sqrt{2}\angle 45°) \simeq 1.82843 + j4.56048$$

$$\tilde{V}_o \simeq \frac{4.91336\angle 68.1527°}{\sqrt{2}\angle 45°}$$

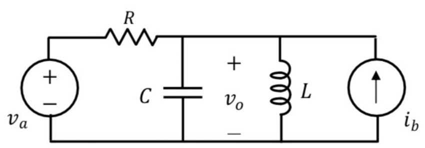
$$\simeq 3.47427\angle 23.1527°$$

$$\simeq 3.47\angle 23.15°$$

$$v_o(t) = 3.47\cos(1000t + 23.15°)$$

LINEARITY AND SUPERPOSITION

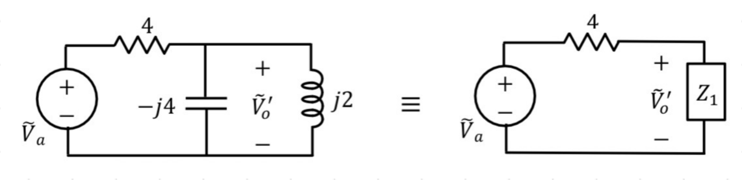
- when circuit has independent sources w/ diff freq, superposition must be used
  - ↳ apply superposition in time domain, solve each individual circuit in phasor domain, & add individual responses in time domain
- e.g. Find $v_o(t)$.



$v_a = 2\cos(1000t + 30°)$
$i_b = \cos(2000t - 45°)$
$R = 4\,\Omega$
$C = 250\,\mu F$
$L = 2\,mH$

SOLUTION

For v_a alone, $\omega_1 = 1000$:



$\tilde{V}_a = 2 \angle 30°$

$Z_L = j\omega L = j(1000)(2\ \ 10^{-3})$
$= j2$

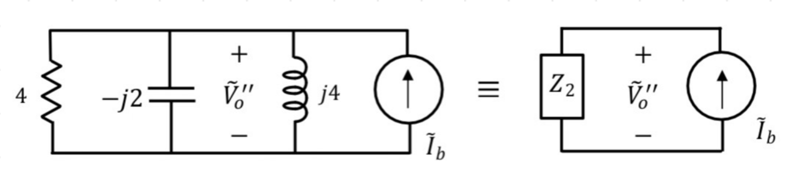
$Z_C = -j\frac{1}{\omega L} = -j\left(\frac{1}{1000 \cdot 250 \cdot 10^{-6}}\right)$
$= -j4$

$Z_1 = Z_L \parallel Z_C$
$= \frac{(j2)(-j4)}{j2 - j4}$
$= \frac{8}{-j2}$
$= \frac{4}{-j}$
$= j4$

Using voltage division: $\tilde{V}_{o1} = \tilde{V}_a\left(\frac{Z_1}{4 + Z_1}\right)$
$= (2\angle 30°)\left(\frac{j4}{4+j4}\right)$
$= (2\angle 30°)\left(\frac{4\angle 90°}{4\sqrt{2}\angle 45°}\right)$
$= (2\angle 30°)\left(\frac{\sqrt{2}}{2}\angle 45°\right)$
$= \sqrt{2}\angle 75°$

$v_{o1}(t) = \sqrt{2}\cos(1000t + 75°)$

For i_b alone, $\omega_2 = 2000$:



$\tilde{I}_b = 1\angle(-45°)$

$Z_L = j\omega L = j(2000)(2 \cdot 10^{-3})$
$= j4$

$Z_C = -j\frac{1}{\omega L} = -j\,\frac{1}{2000\ \ 250 \cdot 10^{-6}}$
$= -j2$

$\frac{1}{Z_2} = \frac{1}{4} + \frac{1}{Z_L} + \frac{1}{Z_C}$
$= \frac{1}{4} + \frac{1}{j4} + \frac{1}{-j2}$
$= \frac{1}{4} + j\left(-\frac{1}{4} + \frac{1}{2}\right)$
$= \frac{1}{4} + j\frac{1}{4}$
$= \frac{\sqrt{2}}{4}\angle 45°$

$Z_2 = 2\sqrt{2}(\angle -45°)$

$\tilde{V}_{o2} = Z_2 \tilde{I}_b$
$= (2\sqrt{2}(\angle -45°))(1\angle(-45°))$
$= 2\sqrt{2}(\angle -90°)$

$v_{o2}(t) = 2\sqrt{2}\cos(2000t - 90°)$

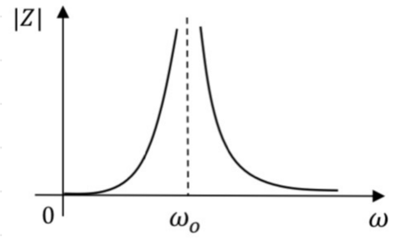
Using superposition in time domain, $v_o(t) = \sqrt{2}\cos(1000t + 75°) + 2\sqrt{2}\cos(2000t - 90°)$.

- superposition can be applied in phasor domain if all sources have same freq
- parallel resonance: parallel connection of inductor & capacitor has impedance

$Z = \dfrac{j\omega L \cdot \frac{1}{j\omega C}}{j\omega L - j\frac{1}{\omega C}} = \dfrac{L/C}{j(\omega L - \frac{1}{\omega C})}$

↳ magnitude is $|Z| = \dfrac{L/C}{|\omega L - \frac{1}{\omega C}|}$

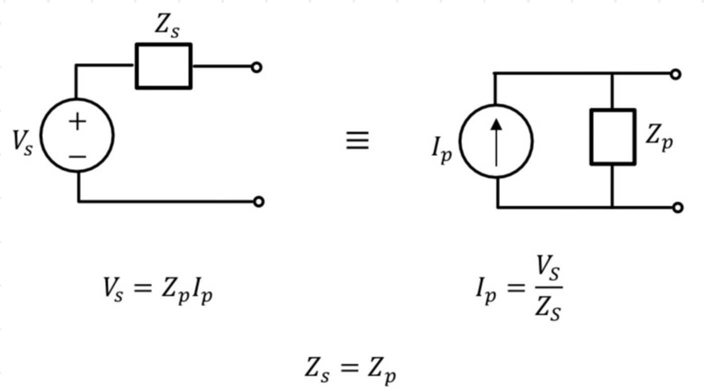
↳ plotting $|Z|$ as function of ω:



- $|Z| \to \infty$ (i.e. open circuit) when $\omega_0 L - \frac{1}{\omega_0 C} = 0$ or $\omega_0 = \frac{1}{\sqrt{LC}}$ (resonance freq)

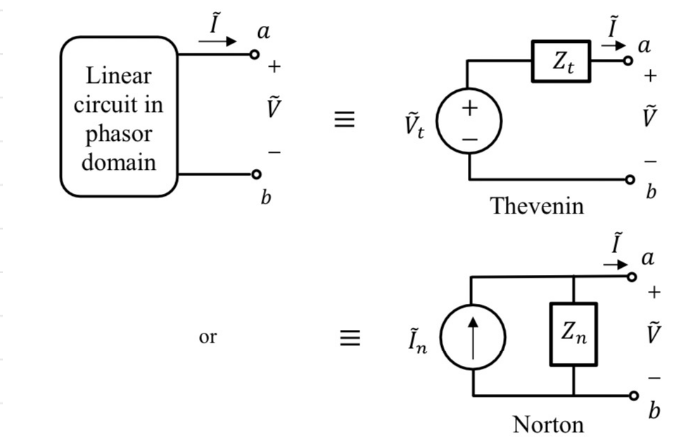
SOURCE TRANSFORMATION

- source transformation can be applied in phasor domain when all sources have same freq



THEVENIN AND NORTON EQUIVALENT CIRCUITS

- to apply Thevenin & Norton theorems to AC circuits in phasor domain, sources must have same freq
-



↳ $\tilde{V}_t = \tilde{V}_{oc}$

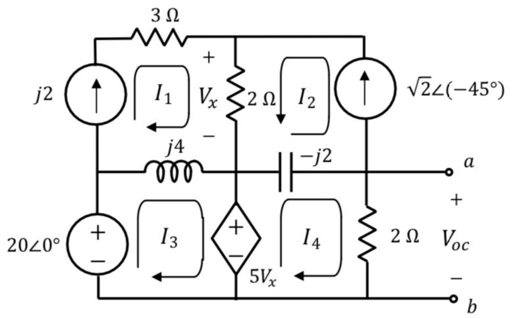
↳ $\tilde{I}_n = \tilde{I}_{sc}$

↳ $Z_t = Z_n = \dfrac{\tilde{V}_{oc}}{\tilde{I}_{sc}}$

↳ $Z_t = Z_n = Z_{eq}$
- equivalent impedance

↳ $Z_t = Z_n = \dfrac{\tilde{V}_s}{\tilde{I}_s}$
- external source
- use when circuit has only dependent source

· e.g Find Thevenin equivalent circuit between a and b?



SOLUTION

Use mesh analysis to find V_{oc}:

$I_1 = j2 = 2\angle 90°$

$I_2 = \sqrt{2}\angle(-45°) = 1 - j$

$V_x = 2(I_1 + I_2)$

$= 2(j2 + 1 - j)$

$= 2(1 + j)$

$= 2 + j2$

$= 2\sqrt{2}\angle 45°$

KVL around I_3:

$0 = -20 + j4(I_3 - I_1) + 5V_x$

$0 = -20 + j4I_3 - j4I_1 + 5V_x$

$j4I_3 = 20 + j4I_1 - 5V_x$

$j4I_3 = 20 + j4(j2) - 5(2 + j2)$

$j4I_3 = 20 - 8 - 10 - j10$

$j4I_3 = 2 - j10$

$I_3 = \frac{2 - j10}{j4}$

$\simeq \frac{2\sqrt{26}\angle -78.69°}{1\angle 90°}$

$\simeq 2\sqrt{26}\angle -168.69°$

KVL around I_4:

$0 = -5V_x - j2(I_2 + I_4) + 2I_4$

$0 = -5V_x - j2I_2 - j2I_4 + 2I_4$

$I_4(2 - j2) = 5V_x + j2I_2$

$I_4(2 - j2) = 5(2 + j2) + j2(1 - j)$

$I_4(2 - j2) = 10 + j10 + j2 + 2$

$I_4(2 - j2) = 12 + j12$

$I_4 = \frac{12\sqrt{2}\angle 45°}{2\sqrt{2}\angle -45°}$

$= 6\angle 90°$

$= j6$

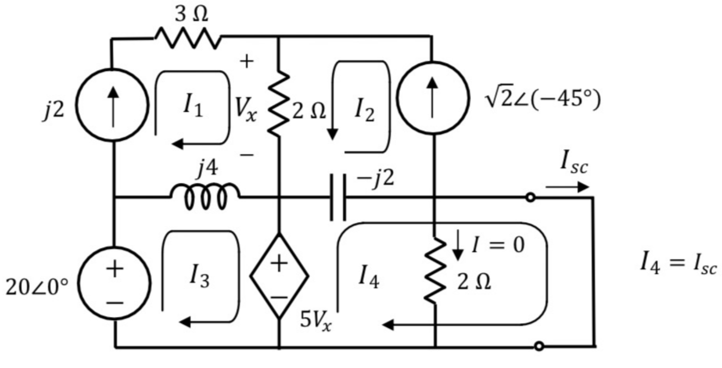
$V_{oc} = 2I_4$

$= 2(j6)$

$= j12$

$V_t = j12$

Find I_{sc}:



↳ I_1, I_2, I_3, & V_x are same as before

KVL around I_4:

$0 = -5V_x + (-j2)(I_4 + I_2)$

$0 = -5V_x - j2I_4 - j2I_2$

$j2I_4 = -5V_x - j2I_2$

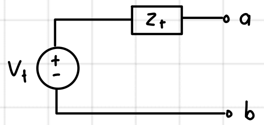
$j2I_4 = -5(2 + j2) - j2(1 - j)$

$j2I_4 = -10 - j10 - j2 - 2$

$$j2I_4 = -12 - j12$$
$$I_4 = \frac{12\sqrt{2}\angle 225°}{2\angle 90°}$$
$$= 6\sqrt{2}\angle 135°$$
$$Z_t = \frac{V_{oc}}{I_{sc}}$$
$$= \frac{12\angle 90°}{6\sqrt{2}\angle 135°}$$
$$= \sqrt{2}(\angle -45°)$$
$$= 1 - j \longrightarrow \text{capacitive since imaginary part is -ve}$$

> **NOTE**
> if circuit has sources w/diff freq, then each source has Thevenin / Norton equivalent that are entirely diff since their freq are diff
> ∘ can't combine circuits

Thevenin equivalent circuit:

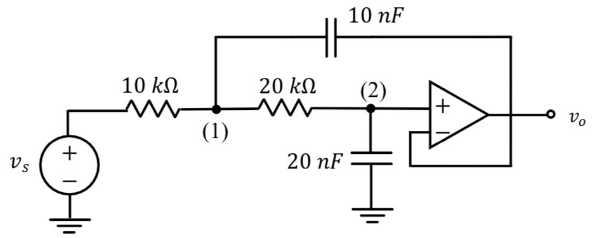


$V_t = j12$

$Z_t = 1 - j$

AC OP-AMP CIRCUITS

e.g. Given $v_s = 12\cos(5000t)$, find v_o assuming ideal op-amp.



SOLUTION

$$C_1 = 10\,nF = -j\frac{1}{\omega C_1}$$
$$= -j\frac{1}{5000(10\cdot 10^{-9})}$$
$$= -j20\ k\Omega$$

$$C_2 = 20\,nF = -j\frac{1}{\omega C_2}$$
$$= -j\frac{1}{5000(20\cdot 10^{-9})}$$
$$= -j10\ k\Omega$$

Phasor domain:

$$V_s = 12\angle 0° = 12$$

$$V_2 = V_o$$
$$0 = \frac{V_1 - V_s}{10} + \frac{V_1 - V_2}{20} + \frac{V_1 - V_o}{-j20}$$
$$0 = \frac{V_1}{10} - \frac{V_s}{10} + \frac{V_1}{20} - \frac{V_o}{20} + \frac{V_1}{-j20} + \frac{V_o}{j20}$$
$$0 = V_1\left(\frac{3}{20} + \frac{j}{20}\right) - \frac{V_s}{10} + V_o\left(-\frac{1}{20} - \frac{j}{20}\right)$$
$$\frac{V_s}{10} = \frac{V_1}{20}(3+j) + \frac{V_o}{20}(-1-j)$$
$$2V_s = V_1(3+j) - V_o(1+j) \quad (1)$$

$$0 = \frac{V_2 - V_1}{20} + \frac{V_2}{-j10}$$
$$0 = \frac{V_o}{20} - \frac{V_1}{20} + \frac{V_o j}{10}$$
$$V_1 = V_o(1 + j2) \quad (2)$$

Sub (2) into (1):

$$2V_s = V_o(1+j2)(3+j) - V_o(1+j)$$
$$2V_s = V_o(3 + j + j6 + j^2 2) - V_o(1+j)$$
$$2V_s = V_o(1 + j7) - V_o(1+j)$$
$$2V_s = V_o(1 + j7 - 1 - j)$$
$$24\angle 0° = V_o(j6)$$
$$V_o = \frac{24\angle 0°}{6\angle 90°}$$
$$V_o = 4\angle -90°$$

Time domain: $v_o = 4\cos(5000t - 90°) = 4\sin(5000t)$

chapter 10

AVERAGE POWER AND RMS VALUE

- when applying periodic voltage $v(t)$ w/ period T to resistor R:
  - ↳ instantaneous absorbed power is $p(t) = v(t)i(t) = \frac{v^2(t)}{R}$
  - ↳ avg power is $P_{av} = \frac{1}{T}\int_0^T p(t)\,dt$
- Root-Mean-Square (rms) value of $v(t)$ is $V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$
  - ↳ constant value
- avg power is $P_{av} = \frac{V_{rms}^2}{R}$, which is also a constant
- RMS value V_{rms} is aka effective value V_{eff} since $P_{av} = \frac{V_{rms}^2}{R}$ is similar to DC power $P = \frac{V_{dc}^2}{R}$ w/ $V_{dc} = V_{rms}$
- periodic current $i(t)$ w/period T has RMS value $I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$
  - ↳ $P_{av} = RI_{rms}^2$

NOTE
- $v(t)$ is periodic & not necessarily a sinusoid

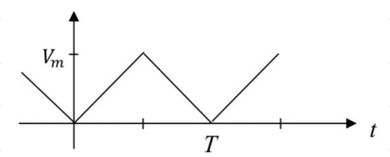
- to determine RMS value for sinusoid:
  - ↳ $v(t) = V_m\cos(\omega t + \theta)$
  - ↳ $V_{rms} = \sqrt{\frac{1}{T}\int_0^T V_m^2\cos^2(\omega t+\theta)\,dt}$

$$= \left(\frac{V_m^2}{T}\int_0^T \frac{1+\cos(2\omega t+2\theta)}{2}\,dt\right)^{1/2}$$
$$= \left(\frac{V_m^2}{2T}\left[t + \frac{1}{2\omega}(\sin(2\omega t+2\theta))\right]_0^T\right)^{1/2}$$
$$= \left(\frac{V_m^2}{2T}\left(T + \frac{1}{2\omega}\sin(2\omega T+2\theta) - \frac{1}{2\omega}\sin(2\theta)\right)\right)^{1/2}$$

Since $T = \frac{2\pi}{\omega} \rightarrow 2\pi = \omega T$

$$V_{rms} = \left(\frac{V_m^2}{2T}\left(T + \frac{1}{2\omega}\sin(4\pi+2\theta) - \frac{1}{2\omega}\sin(2\theta)\right)\right)^{1/2}$$
$$= \left(\frac{V_m^2}{2T}\left(T + \frac{1}{2\omega}\sin(2\theta) - \frac{1}{2\omega}\sin(2\theta)\right)\right)^{1/2}$$
$$= \left(\frac{V_m^2}{2T}\cdot T\right)^{1/2}$$
$$= \sqrt{\frac{V_m^2}{2}}$$
$$= \frac{V_m}{\sqrt{2}}$$

- ↳ $V_{rms} = \frac{V_m}{\sqrt{2}}$ for sinusoid
- current $i(t) = I_m\cos(\omega t+\theta)$ has RMS value $I_{rms} = \frac{I_m}{\sqrt{2}}$
- for triangular voltage, $V_{rms} = \frac{V_m}{\sqrt{3}}$
  - ↳

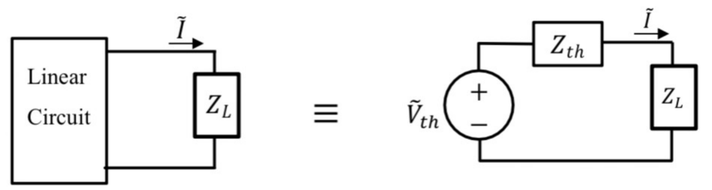


- let $v(t) = V_{m1}\cos(\omega_1 t+\theta_1) + V_{m2}\cos(\omega_2 t+\theta_2)$ w/ $V_{rms1} = \frac{V_{m1}}{\sqrt{2}}$ & $V_{rms2} = \frac{V_{m2}}{\sqrt{2}}$
  - ↳ to find RMS value for sum of sinusoids:
    - if $\omega_1 \neq \omega_2$, $V_{rms} = \sqrt{V_{rms1}^2 + V_{rms2}^2}$
    - if $\omega_1 = \omega_2$, $V_{rms} = \sqrt{\frac{1}{2}V_{m1}^2 + \frac{1}{2}V_{m2}^2 + V_{m1}V_{m2}\cos(\theta_1-\theta_2)}$ (combine into 1 sinusoid in phasor domain first)
- RMS value of V_{dc} isn't $\frac{V_{dc}}{\sqrt{2}}$
  - ↳ $V_{rms} = V_{dc}$

MAXIMUM AVERAGE POWER TRANSFER

- load Z_L is connected to given AC linear circuit & find Z_L st it absorbs max avg power
  - ↳ replace circuit w/ Thevenin equivalent $\tilde{V}_{th}$ & $Z_{th} = R_{th} + jX_{th}$

↳



- case 1: load is complex impedance $Z_L = R_L + jX_L$
  - ↳ max avg power is absorbed when $R_L = R_{th}$ & $X_L = -X_{th}$
    - $Z_L = Z_{th}^*$
  - ↳ if Z_{th} is capacitive, then Z_L is inductive & vice versa
  - ↳ $P_{max} = \frac{|\tilde{V}_{th}|^2}{8R_{th}} = \frac{V_{rms}^2}{4R_{th}}$
- case 2: load is real
  - ↳ i.e $Z_L = R_L$
  - ↳ $R_L = |Z_{th}| = \sqrt{R_{th}^2 + X_{th}^2}$
  - ↳ $P_{max} = \frac{1}{2}|\tilde{I}|^2 R_L = I_{rms}^2 R_L$

ACTIVE POWER, REACTIVE POWER, AND POWER FACTOR

- for general element w/ $T = \frac{2\pi}{\omega}$ as time period.
  - ↳ $v(t) = V_m \cos(\omega t + \theta_v)$
  - ↳ $i(t) = I_m \cos(\omega t + \theta_i)$
- take $i(t)$ as time-ref, subtract θ_i from both to give:
  - ↳ $v(t) = V_m \cos(\omega t + \theta_v - \theta_i)$
  - ↳ $i(t) = I_m \cos(\omega t)$
- instantaneous power is $p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v - \theta_i)\cos(\omega t)$
  - ↳ using trig identities, $p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)\cos(2\omega t) - \frac{1}{2}V_m I_m \sin(\theta_v - \theta_i)\sin(2\omega t)$
- active (real) power is $P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$
  - ↳ $V_{rms} = \frac{V_m}{\sqrt{2}}$
  - ↳ $I_{rms} = \frac{I_m}{\sqrt{2}}$
  - ↳ P is constant
  - ↳ rep portion of power that's transformed from electric to non-electric energy (e.g. heat)
- reactive power is $Q = \frac{1}{2}V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$
  - ↳ Q is constant
  - ↳ rep portion of power that's not transformed into non-electric energy but exchanged btwn circuit elements
- rewrite $p(t)$ using P & Q: $p(t) = P + P\cos(2\omega t) - Q\sin(2\omega t)$
  - ↳ avg power is $P_{av} = \frac{1}{T}\int_0^T p(t)\,dt$ where T = period
    - integral of P is P
    - integrals of last 2 terms are 0 since integration is over complete period
  - ↳ average power P_{av} = active/real power P
    - i.e. $P_{av} = P$
- power factor is $pf = \cos(\theta_v - \theta_i)$
  - ↳ $P = V_{rms} I_{rms} \cdot pf$
  - ↳ since $0 \le |pf| \le 1$, power factor rep percentage of product $V_{rms} I_{rms}$ that's real/active power
  - ↳ pf depends only on angles
- for power for resistor, $\theta_v = \theta_i$ & $pf = \cos 0 = 1$
  - ↳ $P = \frac{1}{2}V_m I_m = V_{rms} I_{rms}$
  - ↳ since $V_m = R I_m$ & $V_{rms} = R I_{rms}$, $P = \frac{1}{2}R I_m^2 = R I_{rms}^2 = \frac{1}{2}\frac{V_m^2}{R} = \frac{V_{rms}^2}{R}$

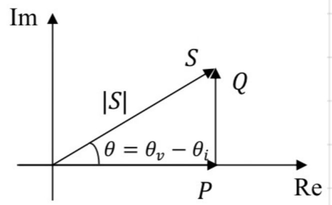
- since $\sin 0 = 0$, reactive power is $Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = 0$
    - $Q = 0$ for resistors always
- for power for inductor, $\theta_v - \theta_i = 90°$ & $pf = \cos 90° = 0$
  - $P = 0$
  - since $\sin(\theta_v - \theta_i) = \sin 90° = 1$, then $Q = \frac{1}{2} V_m I_m = V_{rms} I_{rms}$
    - always +ve
  - since $\tilde{V} = j\omega L \tilde{I}$, $V_m = \omega L I_m$, & $V_{rms} = \omega L I_{rms}$, $Q = \frac{1}{2}\omega L I_m^2 = \omega L I_{rms}^2 = \frac{1}{2}\frac{V_m^2}{\omega L} = \frac{V_{rms}^2}{\omega L}$
  - inductor stores energy in 1 half cycle & releases it in other half
- for power for capacitor, $\theta_v - \theta_i = -90°$ & $pf = \cos(-90°) = 0$
  - $P = 0$
  - since $\sin(\theta_v - \theta_i) = \sin(-90°) = -1$, then $Q = -\frac{1}{2} V_m I_m = -V_{rms} I_{rms}$
    - always -ve
  - since $\tilde{V} = -j\frac{1}{\omega C}\tilde{I}$, $V_m = \frac{1}{\omega C} I_m$, & $V_{rms} = \frac{1}{\omega C} I_{rms}$, $Q = -\frac{1}{2}\frac{I_m^2}{\omega C} = -\frac{I_{rms}^2}{\omega C} = -\frac{1}{2}\omega C V_m^2 = -\omega C V_{rms}^2$
  - conductor doesn't dissipate P & only exchanges Q w/other circuit elements

COMPLEX POWER AND APPARENT POWER

- complex power for general element is $S = \frac{1}{2}\tilde{V}\tilde{I}^* = \tilde{V}_{rms}\tilde{I}_{rms}^*$
  - $\tilde{V}_{rms} = V_{rms}\angle\theta_v$ & $\tilde{I}_{rms} = I_{rms}\angle\theta_i$ are RMS phasors
    - same as regular phasors except for scalar division of magnitudes of $\sqrt{2}$
    - $\tilde{V}_{rms} = \frac{\tilde{V}}{\sqrt{2}}$
    - $\tilde{I}_{rm} = \frac{\tilde{I}}{\sqrt{2}}$
- complex power can be related to active & reactive power
  - $S = \tilde{V}_{rms}\tilde{I}^*_{rms}$

$$= (V_{rms}e^{j\theta_v})(I_{rms}e^{-j\theta_i})$$
$$= V_{rms} I_{rms} e^{j(\theta_v - \theta_i)}$$
$$= V_{rms} I_{rms}\cos(\theta_v - \theta_i) + j V_{rms} I_{rms}\sin(\theta_v - \theta_i)$$

$$S = P + jQ$$

- apparent power is $P_{app} = V_{rms} I_{rms}$
  - $P_{app} = |S| = \sqrt{P^2 + Q^2}$
- convention for units of power:
  - active P: watt (W)
  - reactive Q: volt-ampere-reactive (VAR)
  - complex S: volt-ampere (VA)
  - apparent $|S|$: volt-ampere (VA)
- power triangle used to compute all kinds of power
  - 

  - angle of S is $\theta = \theta_v - \theta_i$
  - $\frac{Q}{P} = \frac{\sin(\theta_v - \theta_i)}{\cos(\theta_v - \theta_i)}$

$$\frac{Q}{P} = \tan(\theta_v - \theta_i)$$

$$\theta_v - \theta_i = \tan^{-1}\left(\frac{Q}{P}\right) = \cos^{-1}(pf)$$

- convention for pf:
  - pf lagging means $\tilde{I}$ lags $\tilde{V}$
    - Q +ve (i.e. inductive)

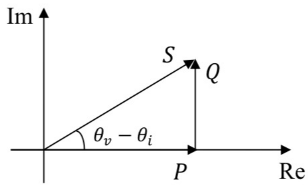
NOTE

- pf leading means $\theta_v - \theta_i$ is -ve
- pf lagging means $\theta_v - \theta_i$ is +ve

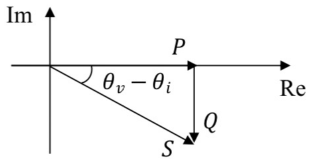
↳ pf leading means $\tilde{I}$ leads $\tilde{V}$
- Q -ve (i.e. capacitive)

· in power calc for impedance, complex power is $S = \tilde{V}_{rms}\tilde{I}^*_{rms}$ & impedance is $Z = R + jX$
↳ $V_{rms} = Z\tilde{I}_{rms}$
↳ $S = Z\tilde{I}_{rms}\tilde{I}^*_{rms}$
$= (R+jX)I^2_{rms}$
$= RI^2_{rms} + jXI^2_{rms}$
$S = P + jQ$
- $P = RI^2_{rms}$ (+ve for $R \geq 0$)
- $Q = XI^2_{rms}$ (+ve for inductive Z & -ve for capacitive Z)

· power triangles for impedance $Z = R + jX$:



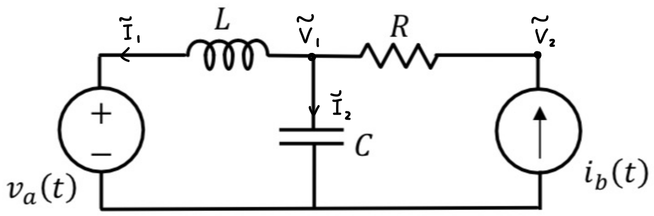
(Inductive Z)



(Capacitive Z)

↳ $\tan(\theta_v - \theta_i) = \frac{Q}{P} = \frac{X}{R}$

· e.g. Find the *active* and *reactive* power for every element.



$R = 1\,\Omega$

$L = \frac{1}{100}\,H$

$C = \frac{1}{50}\,F$

$v_a(t) = \cos(100t - 90°)\quad (V)$

$i_b(t) = \sqrt{2}\cos(100t + 45°)\quad (A)$

SOLUTION

In phasor domain:

$\tilde{V}_a = 1\angle{-90°} = -j$

$\tilde{I}_b = \sqrt{2}\angle 45° = 1 + j$

$Z_L = j\omega L = j(100)(\frac{1}{100}) = j$

$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{100(1/50)} = -j\frac{1}{2}$

Compute currents & voltages using nodal analysis:

$0 = \frac{\tilde{V}_1 - \tilde{V}_a}{Z_L} + \frac{\tilde{V}_1}{Z_C} - \tilde{I}_b$

$\tilde{I}_b = \frac{\tilde{V}_1 + j}{j} + \frac{\tilde{V}_1}{\frac{1}{2j}}$

$1 + j = -j\tilde{V}_1 + 1 + 2j\tilde{V}_1$

$j = j\tilde{V}_1$

$\tilde{V}_1 = 1 = 1\angle 0°$

$\tilde{I}_1 = \frac{\tilde{V}_1 - \tilde{V}_a}{j}$
$= \frac{1-(-j)}{j}$
$= -j + 1$

$\tilde{I}_1 = 1 - j = \sqrt{2}\angle{-45°}$

$\tilde{I}_2 = \frac{\tilde{V}_1}{Z_C}$
$= \frac{1}{\frac{1}{2j}}$
$\tilde{I}_2 = 2j = 2\angle 90°$

$\tilde{I}_b = \frac{\tilde{V}_2 - \tilde{V}_1}{R}$
$1 + j = \frac{\tilde{V}_2 - 1}{1}$
$\tilde{V}_2 = 2 + j \simeq \sqrt{5}\angle 26.565°$

Absorbed power for $\tilde{V}_a$ (PSC):

$P_a = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\cos(-90° - (-45°))$

$= \frac{1}{\sqrt{2}} \cos(-45°)$

$P_a = \frac{1}{2}$ W (actually absorbed)

$Q_a = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$

$= \frac{1}{\sqrt{2}} \sin(-45°)$

$Q_a = -\frac{1}{2}$ VAR (capacitive reactive power)

NOTE

- P_a & Q_a can also be computed from $S_a = \tilde{V}_{rms} \tilde{I}^*_{rms} = P_a + jQ_a$
  - ↳ $S_a = \frac{1}{\sqrt{2}} \angle -90° \cdot \frac{\sqrt{2}}{\sqrt{2}} \angle 45°$
  - $= \frac{1}{\sqrt{2}} \angle -45°$
  - $= \frac{1}{2} - j\frac{1}{2}$

Supplied power for $\tilde{I}_b$ (not PSC):

$S_b = \tilde{V}_{2rms} \tilde{I}^*_{brms}$

$P_b + jQ_b = \left(\frac{\sqrt{5}}{\sqrt{2}} \angle 26.565°\right)\left(\frac{\sqrt{2}}{\sqrt{2}} \angle -45°\right)$

$= \frac{\sqrt{5}}{\sqrt{2}} \angle -18.435°$

$= \frac{3}{2} - j\frac{1}{2}$

$P_b = \frac{3}{2}$ W (acc supplied)

$Q_b = -\frac{1}{2}$ VAR (capacitive reactive power)

Power for R:

$P_R = V_{rms} I_{brms}$

$= R I^2_{rms}$

$= 1\left(\frac{\sqrt{2}}{\sqrt{2}}\right)^2$

$= 1$

$P_R = 1$ W

$Q_R = 0$

Power for L:

$P_L = 0$

$\tilde{V}_L = \tilde{V}_1 - \tilde{V}_a$ $\qquad \tilde{I}_L = \tilde{I}_1$

$= 1 - (-j)$ $\qquad \tilde{I}_L = 1 - j = \sqrt{2} \angle -45°$

$\tilde{V}_L = 1 + j = \sqrt{2} \angle 45°$

$Q_L = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$

$= \frac{\sqrt{2}}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) \sin(45° + 45°)$

$Q_L = 1$ VAR

NOTE

- simpler method:
  - $Q_L = \omega L I^2_{rms}$
  - $= 100\left(\frac{1}{100}\right)\left(\frac{\sqrt{2}}{\sqrt{2}}\right)^2$
  - $= 1$ VAR

Power for C:

$P_c = 0$

$Q_c = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right) \sin(0° - 90°)$

$Q_c = -1$ VAR

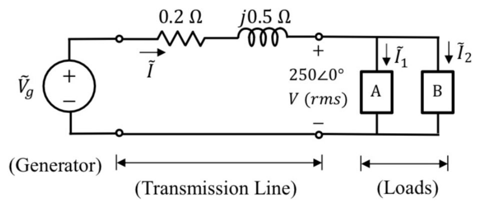
NOTE

- simpler methods:
  - ↳ $Q_c = -\frac{1}{\omega C} I^2_{rms}$
  - $= -\frac{1}{100\left(\frac{1}{50}\right)}\left(\frac{2}{\sqrt{2}}\right)^2$
  - $= -1$ VAR
  - ↳ $Q_c = -\omega C V^2_{rms}$
  - $= -1$ VAR

conservation of power applies to P, Q, & S

↳ $\Sigma P = 0$, $\Sigma Q = 0$, & $\Sigma S = 0$

- e.g. Two loads A and B are connected to a generator through a transmission line as shown. The voltage at the loads must be maintained at 250 V(rms).



- Load A absorbs 8 kW at $pf_1 = 0.8$ *leading*.
- Load B absorbs 20 kVA at $pf_2 = 0.6$ *lagging*.

SOLUTION

a) Computing pf for combined 2 loads:

Load A:

$P_A = 8$ kW

$pf_1 = \cos(\theta_v - \theta_i) = 0.8$

$\theta_v - \theta_i = -36.87°$ (-ve since pf_1 is leading)

$\frac{Q_A}{P_A} = \tan(\theta_v - \theta_i)$

$Q_A = 8k \tan(-36.87°)$

$= -6k$ VAR

$S_A = 8 - j6$ kVA

Load B:

$P_{app} = |S_B| = V_{rms} I_{rms} = 20$ kVA

$\theta_v - \theta_i = \cos^{-1}(pf_2) = 53.13°$ (+ve since pf_2 is lagging)

$P_B = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$

$= |S_B| pf_2$

$= 20k(0.6)$

$= 12$ kW

$Q_B = |S_B| \sin(\theta_v - \theta_i)$

$= 20k \sin 53.13°$

$= 16k$ VAR

$S_B = 12 + j16$ kVA

$S_{AB} = S_A + S_B$

$= 8 - j6 + 12 + j16$

$= 20 + j10$

$= 10\sqrt{5} \angle 26.565°$ kVA

$pf = \cos 26.565°$

$pf = 0.894$ (lagging)

b) Avg power loss due to transmission line:

$S = \tilde{V}_{rms} \tilde{I}^*_{rms}$

$\tilde{I}^*_{rms} = \frac{S}{\tilde{V}_{rms}}$

$= \frac{20 + j10}{250}$

$= 0.08 + j0.04$ kA

$\tilde{I}_{rms} = 80 - j40$ A

$= 40\sqrt{5} \angle -26.565°$ A

$P_{line} = R|\tilde{I}_{rms}|^2$

$= 0.2(40\sqrt{5})^2$

$= 1600$ W

$P_{line} = 1.6$ kW (power lost)

NOTE

- only consider power loss from resistance since inductor just exchanges energy w/ other circuit elements

c) Generated active & reactive powers for source:

$Q_{line} = X|\tilde{I}_{rms}|^2$

$= 0.5(40\sqrt{5})^2$

$= 4000$ VAR

$= 4$ kVAR

$P_g = P_{line} + P_A + P_B$

$= 1.6 + 20$

$= 21.6$ kW

$Q_g = Q_{line} + Q_A + Q_B$

$= 4 + 10$

$= 14$ kVAR

$S_g = 21.6 + j14$ kVA

$= 25.74 \angle 32.95°$ kVA

d) Generator voltage $\tilde{V}_g$:

$S_g = \tilde{V}_{grms} \tilde{I}^*_{grms}$

$\tilde{V}_{grms} = \frac{25.74 \angle 32.95°}{40\sqrt{5} \angle 26.565°}$

$= 0.286214 \angle 6.385°$ kV

$= 286.214 \angle 6.385°$ V