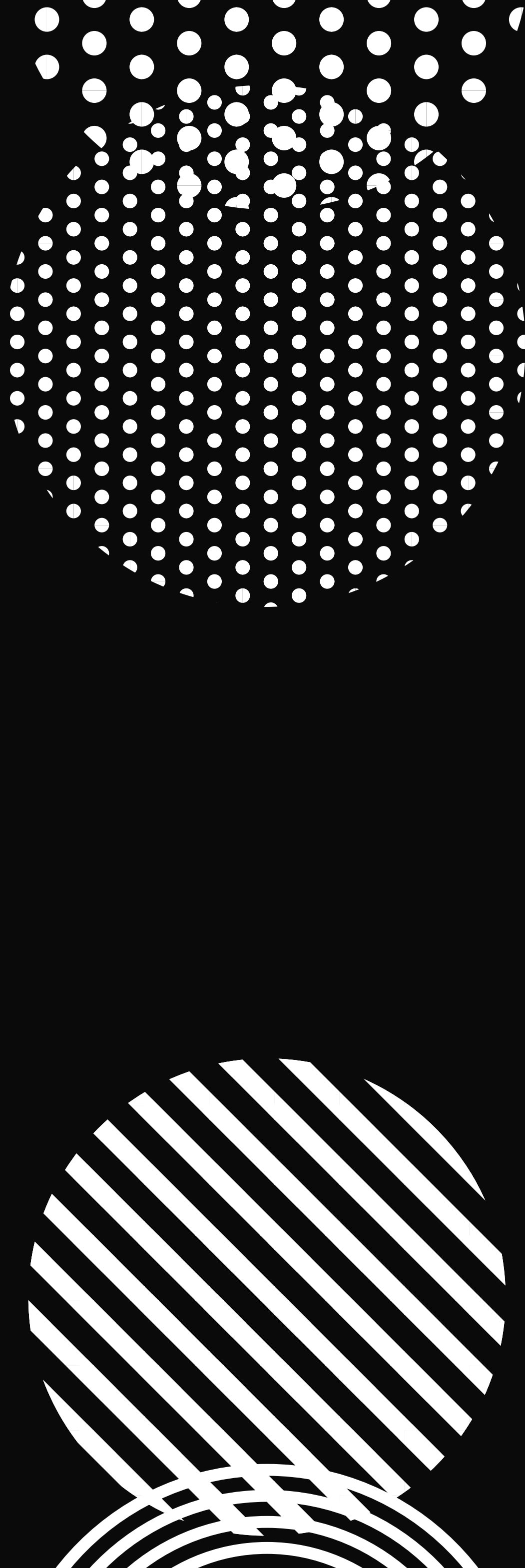


week 1

1.1 : REAL NUMBERS, FUNCTIONS, AND GRAPHS

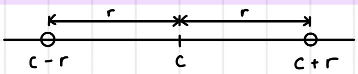
- real # is rep. by decimal or decimal expansion
  - ↳ 3 types of decimal expansion: finite, infinite repeating, & infinite non-repeating
  - ↳ set of real #s is denoted by $\mathbb{R}$
- set of rationals is denoted by $\mathbb{Q}$
  - ↳ rational: finite/infinite repeating decimal expansion
  - ↳ irrational: infinite non-repeating decimal expansion

- binomial expansion formula: $(a+b)^n = \sum_{p=0}^{n} \frac{n!}{(n-p)!\,p!} a^{n-p} b^p$
  - ↳ a term for each p going from 0 to n

- visualize real #s on line w/0 as origin
  - ↳ non-positive: $x \le 0$
  - ↳ non-negative: $x \ge 0$
  - ↳ absolute value of real # a: $|a| \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$

- distance btwn a & b is $|b-a|$
  - ↳ if decimal expansion of a & b agree to k places (right of decimal point), then $|b-a|$ is at most 10^{-k}
  - ↳ triangle inequality: $|a+b| \le |a| + |b|$
- infinite interval $(-\infty, \infty)$ is entire real line $\mathbb{R}$
- for an interval symmetric about value c, use absolute-value inequalities:

$|x-c| < r \Leftrightarrow c-r < x < c+r \Leftrightarrow x \in (c-r, c+r)$



  - ↳ e.g. $|x| < r \Leftrightarrow -r < x < r \Leftrightarrow x \in (-r, r)$
  - ↳ r is radius : $\frac{\max - \min}{2}$
  - ↳ c is midpoint/centre : $\frac{\max + \min}{2}$

NOTE: $\Leftrightarrow$ means "equivalent to" & $\Rightarrow$ means "implies"

- distance formula for $P_1 = (x_1, y_1)$ & $P_2 = (x_2, y_2)$: $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$
- equation of circle w/radius r centred at (a,b): $(x-a)^2 + (y-b)^2 = r^2$

- function f from set D to a set Y is rule that assigns each element x in D a unique element $y = f(x)$ in Y
  - ↳ $f: D \to Y$
  - ↳ x is independent variable
  - ↳ y is dependent variable
- a curve is a function only if it passes Vertical Line Test, which says every vertical line $x = a$ intersects the curve at most one point
- function is increasing on (a,b) if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$
- function is decreasing on (a,b) if $f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$
- monotonic: function is always either increasing/decreasing

- parity of functions means if they're even/odd
  - ↳ even: $f(x) = f(-x)$
    - symmetric about y-axis
  - ↳ odd: $f(-x) = -f(x)$
    - symmetric wrt origin

1.2: LINEAR AND QUADRATIC FUNCTIONS

- linear function: $f(x) = mx + b$
  - ↳ m: slope
  - ↳ b: y-int
  - ↳ slope-intercept form: $y = mx + b$
- slopes m_1 & m_2 are parallel when $m_1 = m_2$
- slopes m_1 & m_2 are perpendicular when $m_1 = -\frac{1}{m_2}$
- function is linear when $m = \frac{\Delta y}{\Delta x}$ is same for every possible interval
- general linear equation: $ax + by = c$
  - ↳ a & b are not both 0
- point-slope form: $y - b = m(x - a)$
  - ↳ slope m
  - ↳ point $P(a, b)$

- quadratic function: $f(x) = ax^2 + bx + c$, $a \neq 0$
  - ↳ discriminant $D = b^2 - 4ac$
    - $D < 0$: no real roots
    - $D = 0$: one real root
    - $D > 0$: 2 real roots
  - ↳ roots of $f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- completing the square gives vertex
  - ↳ $x^2 + bx + c$

$$= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$
$$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

1.3: THE BASIC CLASSES OF FUNCTIONS

- power function: $f(x) = x^m$, $m \in \mathbb{R}$
- polynomial is sum of multiples of power functions w/ exponents ≥ 0
  - ↳ $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
    - $a_0, a_1, \ldots, a_n$ are coefficients
    - degree of P is n ($a_n \neq 0$)
    - leading coefficient is a_n
    - domain of P is $x \in \mathbb{R}$
- rational function is quotient of 2 polynomials
  - ↳ $f(x) = \frac{P(x)}{Q(x)}$
  - ↳ domain is set of x such that $Q(x) \neq 0$
- algebraic function is sums, products, & quotients of roots of polynomials & rationals
- if not polynomial, rational, or algebraic, then function is transcendental
- exponential function: $f(x) = b^x$, $b > 0$, $b \neq 1$
  - ↳ inverse: logarithmic
- trignometric functions are built from $\sin x$ & $\cos x$

- composite function is defined by $f \circ g(x) = f(g(x))$
  - ↳ domain is set of values in D_g such that $g(x)$ lies in D_f
  - ↳ not commutative
- elementary function is taking sum, product, difference, quotient, & composition of basic functions
- piecewise-defined function is defining a function over 2+ distinct domains

1.4: TRIGNOMETRIC FUNCTIONS

- 2 radian measures represent same angle if corresponding rotations differ by integer multiple of 2π
- angle θ subtends arc length θr on circle w/ radius r
  - ↳ $arc = \theta r$
- rad → deg: $\times \frac{180}{\pi}$
- deg → rad: $\times \frac{\pi}{180}$

- point corresponding to θ: $(r\cos\theta, r\sin\theta)$
  - ↳ $\sin(-\theta) = -\sin\theta$ (odd function)
  - ↳ $\cos(-\theta) = \cos\theta$ (even function)
- sine & cosine functions are periodic w/ period $T = 2\pi$
  - ↳ $y = \tan x$ & $y = \cot x$ have $T = \pi$
  - ↳ $y = \csc x$ & $y = \sec x$ have $T = 2\pi$

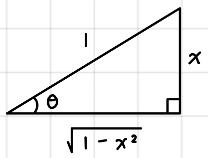
- Pythagorean: $\sin^2 x + \cos^2 x = 1$
  - ↳ $\tan^2 x + 1 = \sec^2 x$
  - ↳ $1 + \cot^2 x = \csc^2 x$
- complementary:
  - ↳ $\sin(\frac{\pi}{2} - x) = \cos x$
  - ↳ $\cos(\frac{\pi}{2} - x) = \sin x$
- addition:
  - ↳ $\sin(x+y) = \sin x\cos y + \cos x\sin y$
  - ↳ $\cos(x+y) = \cos x\cos y - \sin x\sin y$
  - ↳ $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$
- double-angle:
  - ↳ $\cos 2x = \cos^2 x - \sin^2 x$
    $= 1 - 2\sin^2 x$
    $= 2\cos^2 x - 1$
  - ↳ $\sin 2x = 2\sin x\cos x$
  - ↳ $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$
- half-angle:
  - ↳ $\sin^2(\frac{x}{2}) = \frac{1 - \cos x}{2}$
  - ↳ $\cos^2(\frac{x}{2}) = \frac{1 + \cos x}{2}$
- shift:
  - ↳ $\sin(x + \frac{\pi}{2}) = \cos x$
  - ↳ $\cos(x + \frac{\pi}{2}) = -\sin x$
- cosine law: $c^2 = a^2 + b^2 - 2ab\cos\theta$

1.5: INVERSE FUNCTIONS

- let f have domain D & range R; if there's function g w/ domain R such that $g(f(x)) = x$ for $x \in D$ & $f(g(x)) = x$ for $x \in R$, then f is invertible
  - ↳ g is inverse function & denoted as f^{-1}

- f can only have inverse when it's one-to-one
  - ↳ for all $a, b \in D$, if $a \neq b$ then $f(a) \neq f(b)$
  - ↳ i.e. injective
  - ↳ $f(x)$ is one-to-one only if horizontal line intersects graph of function in at most one point
    - i.e. Horizontal Line Test
- graph of f^{-1} is reflection of graph f through $y = x$

- $\theta = \sin^{-1} x$ is unique angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin\theta = x$
  - ↳ $\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$
  - ↳ $\sin^{-1}(\sin\theta) = \theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\theta = \cos^{-1} x$ is unique angle in $[0, \pi]$ such that $\cos\theta = x$
  - ↳ $\cos(\cos^{-1} x) = x$ for $-1 \leq x \leq 1$
  - ↳ $\cos^{-1}(\cos\theta) = \theta$ for $0 \leq \theta \leq \pi$
- $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$ & $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}$



↳ $\theta = \sin^{-1} x$

$x = \sin\theta$

- $\theta = \tan^{-1} x$ is unique angle in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan\theta = x$
- $\theta = \cot^{-1} x$ is unique angle in $(0, \pi)$ such that $\cot\theta = x$
- $\theta = \sec^{-1} x$ is unique angle in $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ such that $\sec\theta = x$
- $\theta = \csc^{-1} x$ is unique angle in $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ such that $\csc\theta = x$

week 2

PIECEWISE-DEFINED FUNCTIONS

- absolute value function, $|x|$, is defined as $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
  - ↳ can also be defined as $|x| = \sqrt{x^2} = (x^2)^{\frac{1}{2}}$
  - ↳ absolute value of function $f(x)$ is $|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$
- to solve absolute value inequalities:
  - ↳ identify intervals on which each argument of $|f(x)|$ won't change sign
  - ↳ replace w/either $f(x)$ or $-f(x)$
  - ↳ solve inequality
  - ↳ find intersection of interval being considered & solution to that inequality
  - ↳ combine results of each case for final answer

- Signum function assigns value to x based on its sign
  - ↳ $\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$
- ramp function describes straight line w/slope $c > 0$ which exists for $x \geq 0$
  - ↳ $r(x) = \begin{cases} 0 & \text{if } x < 0 \\ cx & \text{if } x \geq 0 \end{cases}$
- floor function $\lfloor x \rfloor$ is greatest integer $\leq x$; ceiling function $\lceil x \rceil$ is least integer $\geq x$
- fractional-part (sawtooth) function is defined by $\text{FRACPT}(x) = x - \lfloor x \rfloor$

HEAVISIDE FUNCTION

- Heaviside step function is $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$
- if there's signal function $f(t)$ that we want to activate at $t = 0$, multiply by $H(t)$
  - ↳ $f(t)H(t) = \begin{cases} 0 & \text{if } t < 0 \\ f(t) & \text{if } t \geq 0 \end{cases}$
  - ↳ acts as mathematical ON switch
- shifts in argument of $H(t)$ change at what t that signal is activated
  - ↳ $H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$
- piecewise function can be expressed as sum of functions multiplying Heaviside functions w/arguments shifted by diff amounts
  - ↳ start at leftmost section & use $H(t)$ to impose changes at each point where definition changes
  - ↳ can also define ON, OFF, & ON/OFF switches
    - ON : $f(t)H(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t) & \text{if } t \geq a \end{cases}$
    - OFF : $f(t)(1 - H(t-a)) = \begin{cases} f(t) & \text{if } t < a \\ 0 & \text{if } t \geq a \end{cases}$
    - ON/OFF : $f(t)(H(t-a) - H(t-b)) = \begin{cases} 0 & \text{if } t < a \\ f(t) & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b \end{cases}$
- for functions like $f(t) = \begin{cases} t & \text{if } t \leq a \\ -t & \text{if } t > a \end{cases}$ where endpoints belong to the left piece
  - ↳ OFF : $f(t)H(a-t) = \begin{cases} f(t) & \text{if } t \leq a \\ 0 & \text{if } t > a \end{cases}$
  - ↳ ON : $f(t)(1 - H(a-t)) = \begin{cases} 0 & \text{if } t \leq a \\ f(t) & \text{if } t > a \end{cases}$

- function is periodic if its values repeat at regular intervals over its domain
  - ↳ $f(x + nT) = f(x)$
    - T is period
    - n is any integer

PARTIAL FRACTION DECOMPOSITION

- rational function is ratio of polynomials
  - ↳ proper if deg of numerator is less than deg of denominator & improper otherwise
- any rational function can be written as sum of polynomial & proper rational function using polynomial long division
- any proper rational function can be expressed as sum of simpler rational functions where denominators are either linear or irreducibly quadratic

method of partial fractions:

- ↳ if improper, perform polynomial long division
- ↳ factor denominator into linear / irreducible quadratic factors
- ↳ predict form of decomposition based on denominator
  - linear factor, include term of form : $\frac{A}{c_1x+c_0}$, $A \in \mathbb{R}$
  - irreducible quadratic factor, include term of form: $\frac{Ax+B}{c_2x^2+c_1x+c_0}$, $A, B \in \mathbb{R}$
  - factor raised to n^{th} power, include n terms w/ numerators as given by previous 2 forms & denominators corresponding to each power from 1 to n
    - e.g. $\frac{a_3x^3+a_2x^2+a_1x+a_0}{(c_1x+c_0)^2(d_2x^2+d_1x+d_0)} = \frac{A}{(c_1x+c_0)} + \frac{B}{(c_1x+c_0)^2} + \frac{Cx+D}{d_2x^2+d_1x+d_0}$
- ↳ solve for unknown constants
  - bring both sides to common denominator
  - create system of equations

week 3

INVERSE TRIGNOMETRIC FUNCTIONS

- trignometric functions aren't one-to-one & don't have inverse
  - ↳ restrict domain
- arcsine function (i.e. $\sin^{-1}(x)$) has domain $[-1,1]$ & range $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  - ↳ $y = \sin^{-1}(x)$ is unique angle $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin y = x$
  - ↳ similarily, $y = \cos^{-1}(x)$ is unique angle $y \in [0, \pi]$ such that $\cos y = x$
    - domain $[-1,1]$
    - range $[0, \pi]$
- notation $\sin^{-1}(x)$ & $\cos^{-1}(x)$ aren't consistent w/standard notation for a function's inverse
  - ↳ $\sin(\sin^{-1}(x)) = x$ for all $x \in [-1,1]$
  - ↳ $\sin^{-1}(\sin(x)) = x$ only for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, although $\sin^{-1}(\sin(x))$ is defined for all $x \in \mathbb{R}$
  - ↳ $\cos(\cos^{-1}(x)) = x$ for all $x \in [-1,1]$
  - ↳ $\cos^{-1}(\cos(x)) = x$ only for $x \in [0, \pi]$, although $\cos^{-1}(\cos(x))$ is defined for all $x \in \mathbb{R}$
- $\tan^{-1}(x)$ has domain $x \in \mathbb{R}$ & range $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$
  - ↳ sometimes used as apx. to switching function
    - e.g. $y = \tan^{-1}(kx)$ for large k has graph that resembles a function that jumps from -1 to 1 at $x = 0$
- $y = \sec^{-1}(x)$ is unique angle $y \in [-\pi, \frac{\pi}{2}) \cup [0, \frac{\pi}{2})$ such that $\sec y = x$
  - ↳ domain $(-\infty, -1] \cup [1, +\infty)$
  - ↳ range $[-\pi, \frac{\pi}{2}) \cup [0, \frac{\pi}{2})$
- can usually avoid $\sec^{-1}(x)$, $\csc^{-1}(x)$, & $\cot^{-1}(x)$
  - ↳ $\sec^{-1}(y) = \cos^{-1}(\frac{1}{y})$
  - ↳ $\csc^{-1}(y) = \sin^{-1}(\frac{1}{y})$
  - ↳ $\cot^{-1}(y) = \tan^{-1}(\frac{1}{y})$

COMBINING SINE AND COSINE FUNCTIONS

- in many applications, input is of form $f(t) = B\sin(\omega t)$
  - ↳ B is amplitude
  - ↳ ω is angular frequency ($\omega = \frac{2\pi}{T}$)
- output is in form $g(t) = a\cos(\omega t) + b\sin(\omega t) = A\sin(\omega t + \alpha)$
  - ↳ A is different amplitude
  - ↳ α is phase shift
  - ↳ ω is same angular frequency
- general strategy when given $a\cos(\omega t) + b\sin(\omega t)$:
  - ↳ determine if $A\sin(\omega t + \alpha)$ or $A\cos(\omega t + \alpha)$ is wanted
  - ↳ $A\sin(\omega t + \alpha)$
    $= A\sin(\omega t)\cos(\alpha) + A\cos(\omega t)\sin(\alpha)$
  - ↳ $A\cos(\omega t + \alpha)$
    $= A\cos(\omega t)\cos(\alpha) - A\sin(\omega t)\sin(\alpha)$
  - ↳ equate coeff of $\sin(\omega t)$ & $\cos(\omega t)$
  - ↳ $A = \sqrt{a^2 + b^2}$
  - ↳ determine α via arctan by using sign of $\cos\alpha$ & $\sin\alpha$ to determine quadrant of angle α
    - use $\tan^{-1}\left(\frac{|\sin\alpha|}{|\cos\alpha|}\right)$ & shift angle to correct quadrant

LIMITS OF SEQUENCES

- sequence is an ordered list
- facts about sequences:
  - ↳ can be finite or infinite
    - typically consider infinite
  - ↳ typically begin at $n=0$ or $n=1$, but can be defined to start at any integer
  - ↳ to denote, write $\{a_n\}_{n=0}^{\infty}$ or $\{a_n\}$
  - ↳ some can be defined by formulas, others can't
  - ↳ when there's formula for a_n, it's called the general term
  - ↳ others may be defined recursively: one or more initial terms may be given & n^{th} term is computed in terms of preceeding terms using a formula
- infinite sequences classified into 2 categories by looking at trend in values when n increases
  - ↳ convergent: values tend twd particular # (aka the limit)
  - ↳ divergent: terms never settle down

formal definition of limit of sequences: a sequence a_n converges to limit L if, for any +ve # ε, there exists an integer N such that $n > N \Rightarrow |a_n - L| < \varepsilon$

- ↳ i.e. if limit exists, can make a_n as close to L by making n large enough, if able to do that no matter how small distance ε, limit must exist

week 4

LIMIT LAWS

- assuming that $\lim_{n\to\infty} a_n = a$ & $\lim_{n\to\infty} b_n = b$:
  - ↳ $\lim_{n\to\infty} c = c$
    - for any constant c
  - ↳ $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} a_n = a \pm b$
  - ↳ $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = ca$
    - for any constant c
  - ↳ $\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = ab$
  - ↳ $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{a}{b}$
    - provided $b \neq 0$
  - ↳ $\lim_{n\to\infty} \frac{1}{n^p} = 0$
    - for any $p > 0$
  - ↳ if $|r| < 1$, $\lim_{n\to\infty} r^n = 0$
  - ↳ if f is continuous, $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(a)$
- must assume that $\{a_n\}$ & $\{b_n\}$ converge to use limit laws
- Squeeze Theorem: if $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} c_n = L$, & $a_n \le b_n \le c_n$ for all n large enough, then $\lim_{n\to\infty} b_n = L$
- if limits have indeterminate forms, try rewriting general term of sequence
  - ↳ e.g. $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 1^∞

LIMITS OF FUNCTIONS

- all definitions & theorems for limits of sequences can be extended to limits of functions as $x \to \infty$ by replacing n w/ x
  - ↳ can be adapted to $x \to -\infty$ by:
    - in definition, replace $n > N$ by $-x > N$ (or $x < -N$)
    - if $|r| > 1$, then $\lim_{x\to-\infty} r^x = 0$
- limit of function at a point: limit of function $y = f(x)$ at point $x = a$ is equal to L if, as x approaches a, corresponding output values, f(x), approach L
  - ↳ $x \to a \Rightarrow f(x) \to L$
- one-sided limits may exist but overall limit may not
- vertical asymptote $x = a$ can be found if either one-sided limit at $x = a$ diverges
  - ↳ $\lim_{x\to a^-} f(x) = \pm\infty$ or
  - ↳ $\lim_{x\to a^+} f(x) = \pm\infty$
- $\lim_{x\to\pm\infty} f(x) = L$ when values of f(x) settle to L as x grows w/o bound
  - ↳ for any $r > 0$, $\lim_{x\to\infty} \frac{1}{x^r} = 0$
  - ↳ for $r > 0$ such that x^r is defined for all x, $\lim_{x\to-\infty} \frac{1}{x^r} = 0$
- $y = L$ is horizontal asymptote of f(x) if either is true:
  - ↳ $\lim_{x\to-\infty} f(x) = L$ or
  - ↳ $\lim_{x\to\infty} f(x) = L$

- if limits of f(x) & g(x) exist at $x = a$, c is constant, & n is +ve integer:
  - ↳ $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
  - ↳ $\lim_{x\to a} (c f(x)) = c \lim_{x\to a} f(x)$
  - ↳ $\lim_{x\to a} (f(x) \cdot g(x)) = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x))$

- $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
  - provided $\lim_{x\to a} g(x) \neq 0$
- $\lim_{x\to a} (f(x))^n = (\lim_{x\to a} f(x))^n$
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$
  - if n is even, assume $f(x) \geq 0$

- f is continuous at a when $\lim_{x\to a} f(x) = f(a)$
  - all polynomials, $\sqrt[n]{x}$, $\sin(x)$, $\cos(x)$, e^x, $\ln(x)$, $\frac{1}{?}$ $|x|$ are continuous on their domains
- Squeeze Theorem: if $f(x) \leq g(x) \leq h(x)$ when x is near a & $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$

- 3 important limits:
  - $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
    - cardinal sine function: $\text{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$
    - $\text{sinc}(x)$ is continuous for $x \in \mathbb{R}$
  - $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$
    - $\lim_{x\to 0} \frac{1-\cos(x)}{x}\left(\frac{1+\cos(x)}{1+\cos(x)}\right)$
      $= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$
      $= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right)$
      $= 1\left(\frac{0}{1+1}\right)$
      $= 0$
  - $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
    - $\frac{e^x-1}{x} \simeq 1 \Leftrightarrow e^x \simeq 1 + x \Leftrightarrow e \simeq (1+x)^{\frac{1}{x}}$
    - $e = \lim_{x\to 0} (1+x)^{\frac{1}{x}}$
    - $e = \lim_{n\to\infty} (1+\frac{1}{n})^n$

FORMAL DEFINITION OF LIMITS

- $\lim_{x\to a} f(x) = L$ means that for any $\varepsilon < 0$, there exists a number δ such that $0 < |x-a| < \delta \rightarrow |f(x) - L| < \varepsilon$
  - enforce $0 < |x-a|$ to omit when $x = a$
  - $f(x)$ can be made arbitrarily close to L by making x sufficiently close to a

week 5

CONTINUITY

- a function f is continuous at $x=a$ if:
  - ↳ $\lim_{x\to a} f(x)$ exists
  - ↳ $\lim_{x\to a} f(x) = f(a)$
- function is continuous on interval if it's continuous at every point in that interval
- when function is defined on closed interval domain, use one-sided limits to establish notion of continuity at endpoints of domain
  - ↳ continuous from left at $x=a$ if $\lim_{x\to a^-} f(x) = f(a)$
  - ↳ continuous from right at $x=b$ if $\lim_{x\to b^+} f(x) = f(b)$
- combos of continuous functions are continuous - if $f(x)$ & $g(x)$ are continuous on interval I & c is constant, following are also continuous on interval I:
  - ↳ $f(x) \pm g(x)$
  - ↳ $c\,f(x)$
  - ↳ $f(x) \cdot g(x)$
  - ↳ $\frac{f(x)}{g(x)}$
    - provided $g(a) \neq 0$ for all $a \in I$
- these standard functions are continuous on their domains:
  - ↳ polynomials
  - ↳ rationals
  - ↳ root functions
  - ↳ logs
  - ↳ trig functions
  - ↳ inverse trig functions
- if $g(x)$ is continuous at a & $f(x)$ is continuous at a, then $f \circ g$ is continuous at a
  - ↳ e.g. for what x-values is $h(x) = \ln(2+\sin x)$ continuous?

    SOLUTION

    Let $f(y) = \ln(y)$ & $g(x) = 2+\sin x$ so $h(x) = f\circ g$

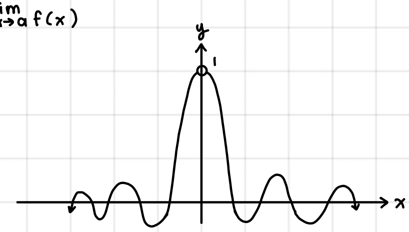
    $g(x)$ is continuous for $x\in\mathbb{R}$; $f(y)$ is continuous for $y\in\mathbb{R}$, $y>0$

    $-1 \leq \sin x \leq 1$

    $1 \leq 2+\sin x \leq 3$

    Since $g(x)$ is +ve for $x\in\mathbb{R}$, then $h(x) = f(g(x))$ is continuous for $x\in\mathbb{R}$.

- removable discontinuities can be removed/fixed by assigning value of function at point of discontinuity to be limit at that point
  - ↳ set $f(a)$ to be $\lim_{x\to a} f(x)$
  - ↳ e.g. $f(x) = \frac{\sin x}{x}$



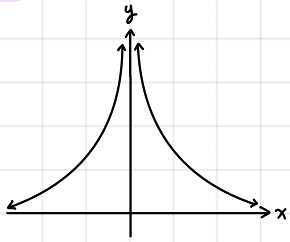
  - fix discontinuity to be $\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x}, & x\neq 0 \\ 1, & x=0 \end{cases}$
- jump discontinuities happen when behaviours to left & right of point yield differing expectations of value at that point
  - ↳ i.e. limit DNE at $x=a$ b/c $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

↳ e.g. Heaviside function is $H(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$

- $\lim_{t \to 0^-} H(t) = 0$ & $\lim_{t \to 0^+} H(t) = 1$
- $\lim_{t \to 0} H(t)$ DNE & there's a jump discontinuity at $t=0$

· infinite discontinuities occur when at least one side of limit approaches $\pm\infty$

↳ e.g. $f(x) = \frac{1}{x^2}$



- although we write $\lim_{x \to 0} \frac{1}{x^2} = \infty$, limit DNE as a real #

INTERMEDIATE VALUE THEOREM

· IVT: suppose $f(x)$ is continuous on closed interval $[a,b]$; let N be any # btwn $f(a)$ & $f(b)$; there exists input $x=c$ in $[a,b]$ such that $f(c) = N$

↳ i.e. function takes all values btwn function's values at endpoints

↳ e.g. if continuous function is -ve at left endpoint & +ve at right endpoint, must have root in that interval

· Bisection Algorithm estimates solution to $f(x) = 0$ for continuous function $f(x)$

1) apply IVT to isolate root within interval (x_1, x_2) noting signs of $f(x_1)$ & $f(x_2)$
2) bisect interval by labelling midpoint $x_3 = \frac{x_1 + x_2}{2}$; evaluate $f(x_3)$
3) use IVT to determine if root lies in (x_1, x_3) or (x_3, x_2)
4) repeat steps 2-3 until interval is small enough to generate an estimate to $f(x)=0$

· can use IVT to identify +ve/-ve intervals

↳ use single value in each interval btwn roots to test

· Extreme Value Theorem (EVT): suppose $f(x)$ is continuous on closed interval $[a,b]$; f always has max & min value on that interval

DEFINITION OF DERIVATIVE

· derivative of function $f(x)$ at point x_0 is $f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

↳ if limit DNE, f is not differentiable at x_0

↳ let $x_0 = x$ & $x_1 = x + h$, then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

- describes $f'(x)$ as function of x

↳ slope of tangent line to $f(x)$ at x_0 so equation of line is $y - f(x_0) = f'(x_0)(x - x_0)$

· as $f'(x)$ is function, can compute 2^{nd} derivative of f & so on

↳ e.g. if $s(t)$ describes displacement as function of time, $v(t) = s'(t)$ describes rate of change of displacement as function of time

week 7

NOTE: $(|g(x)|)' = \frac{g'(x) \cdot g(x)}{|g(x)|}$

DIFFERENTIATION FORMULAS

- power rule: $\frac{d}{dx} x^a = ax^{(a-1)}$
  - ↳ for all $a \in \mathbb{R}$
- derivative of constant: $\frac{d}{dx} c = 0$
  - ↳ for all $c \in \mathbb{R}$
- derivative of scalar multiple of function: $\frac{d}{dx} c f(x) = c \frac{d}{dx} f(x)$
  - ↳ $f(x)$ is differentiable function on interval I & c is real constant
- derivative of sum/difference: $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$
  - ↳ $f(x)$ & $g(x)$ are differentiable functions on interval I
- product rule: $\frac{d}{dx}[f(x)\ g(x)] = f'(x)g(x) + g'(x)f(x)$
  - ↳ $f(x)$ & $g(x)$ are differentiable functions on interval I
- quotient rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
  - ↳ $f(x)$ & $g(x)$ are differentiable functions on interval I
  - ↳ $g(x) \neq 0$
- derivatives of some transcendental functions:
  - ↳ $\frac{d}{dx} e^x = e^x$
  - ↳ $\frac{d}{dx} \sin x = \cos x$
  - ↳ $\frac{d}{dx} \cos x = -\sin x$
  - ↳ $\frac{d}{dx} b^x = b^x \ln(b)$ for $b > 0$
  - ↳ $\frac{d}{dx} \ln x = \frac{1}{x}$
- chain rule: $\frac{d}{dx}[f \circ g(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$
  - ↳ e.g. determine derivative of $f(x) = \frac{e^x}{3x+4}$

SOLUTION

$$f(x) = e^x(3x+4)^{-1}$$

$$\begin{aligned} f'(x) &= e^x(3x+4)^{-1} + (-1)(3x+4)^{-2}(3)(e^x) \\ &= \frac{e^x}{3x+4} - \frac{3e^x}{(3x+4)^2} \\ &= \frac{e^x(3x+4-3)}{(3x+4)^2} \\ &= \frac{e^x(3x+1)}{(3x+4)^2} \end{aligned}$$

IMPLICIT DIFFERENTIATION

- slopes of tangents to curves that aren't a single function are found through implicit differentiation
- to compute $\frac{dy}{dx}$:
  - ↳ take derivative wrt x of both sides
  - ↳ handle $\frac{d}{dx}$ of any y-term as chain rule (i.e. treat y as $y = y(x)$)
- e.g. compute derivative of $x^4 - 4xy^2 + y^4 = 1$

SOLUTION

$$\frac{d}{dx}(x^4 - 4xy^2 + y^4) = \frac{d}{dx}(1)$$

$$4x^3 - 4\left(y^2 + 2y\left(\frac{dy}{dx}\right)x\right) + 4y^3\left(\frac{dy}{dx}\right) = 0$$

$$4x^3 - 4y^2 - 8xy\left(\frac{dy}{dx}\right) + 4y^3\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}(4y^3 - 8xy) = 4y^2 - 4x^3$$

$$\frac{dy}{dx} = \frac{4(y^2 - x^3)}{4(y^3 - 2xy)}$$

$$\frac{dy}{dx} = \frac{y^2 - x^3}{y^3 - 2xy}$$

DERIVATIVES OF INVERSES

if f is invertible, write $y = f^{-1}(x)$ or $x = f(y)$

↳ $\frac{d}{dx}(x) = \frac{d}{dx}[f(y)]$

$1 = \frac{df}{dy}\frac{dy}{dx}$

$\frac{dy}{dx} = \left(\frac{df}{dy}\right)^{-1}$

↳ $\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}$

$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$

↳ $f^{-1\prime}(x)$ is reciprocal of $f'(y)$

↳ $\frac{d^2y}{dx^2} = -\left(\frac{dx}{dy}\right)^{-3} \cdot \frac{d^2x}{dy^2}$

- e.g. calculate derivative of $y = \ln(x)$

SOLUTION

$y = \ln(x) \longrightarrow x = e^y$

$\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$

$1 = e^y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{e^y}$

$= \frac{1}{e^{\ln x}}$ ⟵ sub in $y = \ln x$

$= \frac{1}{x}$

- derivatives of inverse trig functions:

↳ $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

↳ $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

↳ $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

↳ $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

LOGARITHMIC DIFFERENTIATION

- to differentiate function of form $y = f(x)^{g(x)}$:

↳ $\ln y = \ln(f(x)^{g(x)})$

$\ln y = g(x)\ln(f(x))$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}\, g(x)\ln(f(x))$

$\frac{1}{y}\frac{dy}{dx} = g'(x)\ln f(x) + \frac{1}{f(x)}f'(x)\,g(x)$

$\frac{dy}{dx} = y\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right)$

$\frac{dy}{dx} = f(x)^{g(x)}\left(g'(x)\ln f(x) + g(x)\left(\frac{f'(x)}{f(x)}\right)\right)$

week 8

THEOREMS

- differentiability implies continuity: if $f(x)$ is differentiable at a, then $f(x)$ is continuous at a
  - ↳ continuity does not imply differentiability
    - e.g. $f(x) = |x|$ is continuous but not differentiable at $x = 0$
- Mean Value Theorem (MVT): suppose that f is continuous on $[a,b]$ & differentiable on (a,b); then, there's a $c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$
  - ↳ i.e. IROC (tangent line) at c is equal (parallel) to AROC (secant line) over the interval $[a,b]$
  - ↳ when $f(a) = f(b)$, then $f'(c) = 0$
- L'Hôpital's Rule: given open interval I containing point $x = a$, if $f(x)$ & $g(x)$ are differentiable on I (except possibly at $x = a$) and either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ & $\lim_{x\to a} g(x) = \pm\infty$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
  - ↳ provided that $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists
  - ↳ can apply when $a = \pm\infty$
  - ↳ can be applied repeatedly as long as the limit yields an indeterminate form
- when faced w/indeterminate product, rewrite limit to apply l'Hôpital's Rule
  - ↳ e.g find $\lim_{t\to\infty} f(t) = te^{-t}$

SOLUTION

As $t \to \infty$, $t \to \infty$ and $e^{-t} \to 0$ so direct sub gives $\infty \cdot 0$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} te^{-t}$$

$$= \lim_{t\to\infty} t\left(\frac{1}{e^t}\right) \quad \longleftarrow \text{ using fact that } x = \frac{1}{\frac{1}{x}}$$

$$= \lim_{t\to\infty} \frac{t}{\frac{1}{e^{-t}}}$$

$$= \lim_{t\to\infty} \frac{t}{e^t}$$

$$= \lim_{t\to\infty} \frac{1}{e^t} \quad \longleftarrow \text{ direct sub gives } \frac{\infty}{\infty} \text{ so apply l'Hôpital's Rule}$$

$$= 0$$

- when faced w/indeterminate diff, rewrite limit to apply l'Hôpital's Rule

RELATED RATES

- to solve related rates problems, find expression that relates quantities & use chain rule to differentiate wrt time
- if 2 quantities are related by $y = f(x)$ & they're changing in time, then rates of change of 2 quantities are also related
  - ↳ $y(t) = f(x(t))$

$$\frac{d}{dt}(y(t)) = \frac{d}{dt}(f(x(t)))$$

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

  - $\frac{df}{dx}$ usually won't be constant so to find $\frac{dy}{dt}$, we need to know both $\frac{dx}{dt}$ & t
- e.g. air being pumped into a spherical balloon so volume is increasing at rate of $100\,cm^3/min$; how fast is radius increasing when diameter is $50\,cm$?

SOLUTION

$\frac{dV}{dt} = 100 \frac{cm^3}{min}$

$r = 25\,cm$

$\frac{dr}{dt} = ?$

$$V = \frac{4\pi}{3}r^3$$

$$\frac{dV}{dt} = \frac{4\pi}{3}(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$100 = 4\pi(25)^2 \frac{dr}{dt}$$

$$100 = 2500\pi \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{25\pi}$$

The radius is increasing at $\frac{1}{25\pi}$ cm/min.

TANGENT LINE APPROXIMATION

- some transcendental functions are hard to evaluate except at certain points
  - ↳ e.g. for $f(x) = \sqrt{x}$, what's $f(16.08)$?
- linearization of $f(x)$ at $x=a$: tangent line to $f(x)$ at $x=a$ is linear apx / linearization & denote it by: $L_a(x) = f(a) + f'(a)(x-a)$
  - ↳ in general, as distance from x to a increases, potential error in using $L_a(x)$ to apx $f(x)$ also increases
  - ↳ if graph is more curved near a, greater potential error

DIFFERENTIALS

- apx are often described in terms of infinitesimal amounts or differences called differentials
- if $f(x)$ is differentiable, then differential dx is independent variable & differential df is defined by $df = f'(x)dx$
  - ↳ linear apx using differentials: $\Delta y = \frac{dy}{dx} \Delta x$

week 9

GLOBAL AND LOCAL EXTREMA

- suppose that $f(x)$ is a continuous function defined on closed interval $I = [a, b]$
  - ↳ if $c \in I$ is a point such that $f(c) \geq f(x)$ for all $x \in I$, then $f(c)$ is global / absolute maximum of $f(x)$ on I
    - functions can have multiple global maxima over given interval (e.g. $f(x) = \sin x$)
  - ↳ if $c \in I$ is a point such that $f(c) \leq f(x)$ for all $x \in I$, then $f(c)$ is global/absolute minimum of $f(x)$ on I
  - ↳ if $c \in I$ is a point such that $f(c) \geq f(x)$ for all x near c, then $f(c)$ is local maximum of $f(x)$ on I
  - ↳ if $c \in I$ is a point such that $f(c) \leq f(x)$ for all x near c, then $f(c)$ is local minimum of $f(x)$ on I
- maxima & minima are collectively called extrema
- Extreme Value Theorem (EVT): a continuous function defined on a bounded closed interval always achieves both a global max & min value

for function $f(x)$ defined on interval $I = [a, b]$, a value $c \in I$ is a critical point of $f(x)$ if either $f'(c) = 0$ or $f'(c)$ DNE

Fermat's Theorem: local extrema can only occur at critical points or at endpoints of interval
  - ↳ global extrema must be local extrema so theorem applies as well
- if $f(x)$ on closed interval $I = [a, b]$ has global extremum at point $c \in I$, then one of following must be true:
  - ↳ $f'(c) = 0$
  - ↳ $f'(c)$ DNE
  - ↳ c is an endpoint of I
    - i.e. $c = a$ or $c = b$
- let $f(x)$ be continuous function defined over closed interval $I = [a, b]$; global extrema can be determined through closed interval method:
  1) determine derivative $f'(x)$
  2) determine all critical points that lie within I
  3) evaluate $f(x)$ at critical points & at endpoints
  4) compare all output values
     - largest is global max
     - smallest is global min

APPLICATION TO CURVE SKETCHING

- function $f(x)$ is strictly increasing on an interval I if $f'(x) > 0$ for all $x \in I$
- function $f(x)$ is strictly decreasing on an interval I if $f'(x) < 0$ for all $x \in I$
- if function $f(x)$ is either only increasing / decreasing on interval I, $f(x)$ is monotonic on I
- use first derivative test to determine whether critical point $x = c$ of function $f(x)$ is local extremum or not
  - ↳ if sign of $f'(x)$ doesn't change upon crossing c, then c is neither local max nor local min
  - ↳ if $f'(x) > 0$ for $x < c$ & $f'(x) < 0$ for $x > c$, then c is local max
  - ↳ if $f'(x) < 0$ for $x < c$ & $f'(x) > 0$ for $x > c$, then c is local min
- intervals of increase/decrease will typically have local extrema as endpoints

- second derivative $f''(x)$ is ROC of ROC of values of $f(x)$
  - ↳ e.g. if $s(t)$ describes displacement of object at time t, then $s'(t) = v(t)$ describes its velocity & $s''(t) = v'(t) = a(t)$ describes its acceleration
- can use second derivative $f''(x)$ to determine concavity:
  - ↳ if $f''(x) > 0$ over interval I, then $f(x)$ is concave up on I
  - ↳ if $f''(x) < 0$ over interval I, then $f(x)$ is concave down on I
- for function $f(x)$ defined on domain D, $c \in D$ is point of inflection (POI) of $f(x)$ if:
  - ↳ $f''(c) = 0$ or $f''(c)$ DNE &
  - ↳ $f''(x)$ switches signs when crossing c
- second derivative test can be used to classify critical points
  - ↳ if $f'(c) = 0$ & $f''(c) < 0$, $f(x)$ has local max at $x = c$
  - ↳ if $f'(c) = 0$ & $f''(c) > 0$, $f(x)$ has local min at $x = c$

week 10

AREA UNDER A CURVE

- sigma notation is used to denote adding a sequence of #s
  - ↳ general form is $a_m + a_{m+1} + \dots + a_n = \sum_{i=m}^{n} a_i$
    - i is index
    - m is lower limit
    - n is upper limit
  - ↳ e.g. $\sum_{i=2}^{5} i^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54$
- summation properties:
  - ↳ distributivity: for any $c \in \mathbb{R}$, $\sum_{i=m}^{n} c a_i = c \sum_{i=m}^{n} a_i$
  - ↳ associativity pt 1: $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$
  - ↳ associativity pt 2: for $m < n < p$, $\sum_{i=m}^{p} a_i = \sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i$
- summation formulas:
  - ↳ $\sum_{i=1}^{n} 1 = n$
  - ↳ $\sum_{i=1}^{n} c = cn$
  - ↳ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
  - ↳ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
  - ↳ $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

- let $f(x)$ be a function defined over an interval $[a,b]$; to apx area of region under graph $y = f(x)$ & above x-axis, set up Riemann sum using n rectangles
  - ↳ each rectangle has width $\Delta x = \frac{b-a}{n}$
  - ↳ to take right hand sum, we use location of right endpoint of i^{th} rectangle
    - this point is $x_i = a + i\Delta x$ so height of rectangle is $f(x_i)$
  - ↳ add up all areas to determine total area R_n which gives apx of area under curve
- exact area under curve is limit of Riemann sum when # of rectangles infinitely increase
  - ↳ $A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left[\sum_{i=1}^{n} f(x_i) \cdot \Delta x \right]$

DEFINITE INTEGRALS

- given a function $f(x)$, area under curve $y = f(x)$ over interval is given by a definite integral:
  $\int_a^b f(x)\,dx = \lim_{n \to \infty} \left[\sum_{i=1}^{n} f(\tilde{x}_i) \cdot \Delta x \right]$
  - ↳ $\Delta x = \frac{b-a}{n}$
  - ↳ $x_i = a + i\Delta x$
  - ↳ each $\tilde{x}_i \in [x_{i-1}, x_i]$
- definite integral is signed area bounded btwn graph & x-axis
  - ↳ when $f(x) < 0$, -ve contribution to integral
- definite integral properties:
  - ↳ $\int_a^a f(x)\,dx = 0$
  - ↳ $\int_a^b 1\,dx = \int_a^b dx = b - a$
  - ↳ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
  - ↳ $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$
    - $c \in \mathbb{R}$
  - ↳ $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
  - ↳ $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

NOTE: Comparison Theorem states if f & g are integrable & $g(x) \le f(x)$ for x in $[a,b]$, then $\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx$

NOTE: on interval $[-a,a]$, if function is even: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ & if function is odd: $\int_{-a}^{a} f(x)\,dx = 0$

FUNDAMENTAL THEOREM OF CALCULUS

- Fundamental Theorem of Calculus Part 1 (FTC 1): given function f that's continuous over interval $[a,b]$, define another function $A(x)$ so that for all $x \in [a,b]$, $A(x) = \int_a^x f(t)dt$; thus, on interval (a,b), $A'(x) = f(x)$
  - ↳ $A(x)$ is antiderivative of $f(x)$
  - ↳ i.e. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$
- general antiderivative $F(x)$ is class of functions that differ only by an additive constant
  - ↳ $F(x) = A(x) + C$
    - ◦ C is some constant
- Fundamental Theorem of Calculus Part 2 (FTC 2): given function f that's continuous over interval $[a,b]$, $\int_a^b f(x)\,dx = F(b) - F(a)$
  - ↳ $F(x)$ is antiderivative of $f(x)$
  - ↳ other notations are $F(b) - F(a) = F(x)\Big|_a^b = [F(x)]_a^b$
- to differentiate integrals w/ variable limits of the form $\int_{g(x)}^{h(x)} f(t)dt$:

SOLUTION

Let $F(u) = \int_0^u f(t)dt$.

$$\int_{g(x)}^{h(x)} f(t)dt = F(h(x)) - F(g(x)) \longleftarrow \text{by FTC 2}$$

$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt &= \frac{d}{dx}(F(h(x)) - F(g(x))) \\ &= \frac{dF}{dh}\cdot\frac{dh}{dx} - \frac{dF}{dg}\cdot\frac{dg}{dx} \\ &= f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)\end{aligned}$$

week 11

INDEFINITE INTEGRALS

- indefinite integral of a function $f(x)$, denoted by $\int f(x)\,dx$, is collection of all possible antiderivatives
  - ↳ if antiderivative $F(x)$ is known, $\int f(x)\,dx = F(x) + C$ or $\int f(x)\,dx = \int_a^x f(t)\,dt$
  - ↳ $+C$ is very important
- indefinite integral of integrand $f(x)$ can be used to determine value of any definite integral w/integrand $f(x)$
- list of indefinite integrals:
  - ↳ $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
    - $n \neq -1$
  - ↳ $\int \frac{1}{x}\,dx = \ln|x| + C$
    - for $\ln x$, $x > 0$
  - ↳ $\int e^x\,dx = e^x + C$
  - ↳ $\int a^x\,dx = \frac{a^x}{\ln a} + C$
  - ↳ $\int \cos x\,dx = \sin x + C$
  - ↳ $\int \sin x\,dx = -\cos x + C$
  - ↳ $\int \sec^2 x\,dx = \tan x + C$
  - ↳ $\int \sec x \tan x = \sec x + C$
  - ↳ $\int \cosh x\,dx = \sinh x + C$
  - ↳ $\int \sinh x\,dx = \cosh x + C$
  - ↳ $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$
  - ↳ $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
    - $\int \frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1}x + C$ so $\sin^{-1}x$ & $\cos^{-1}x$ differ only by a constant: $\sin^{-1}x + \cos^{-1}x = K$; let $x = 0$ so $K = \frac{\pi}{2}$ & we get identity $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
  - ↳ $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$

INTEGRATION BY SUBSTITUTION

- given a differentiable function $u = g(x)$, then $\int f'(g(x)) \cdot g'(x)\,dx = \int f'(u)\,du$

given a function $u = g(x)$ which is differentiable for all $x \in [a,b]$,
$\int_{x=a}^{b} f'(g(x)) \cdot g'(x)\,dx = \int_{u=g(a)}^{g(b)} f'(u)\,du$

- ↳ change limits of integration to reflect any substitution
- ↳ don't need to change back to original variable

INTEGRATION BY PARTS

- given $u = u(x)$ & $v = v(x)$ w/ $du = u'(x)dx$ & $dv = v'(x)\,dx$, then $\int u\,dv = uv - \int v\,du$
- e.g. evaluate $\int x\cos(x)\,dx$

SOLUTION

Let $u = x$ & $dv = \cos x\,dx$

$\frac{du}{dx} = 1$ $\qquad$ $\frac{dv}{dx} = \cos x$

$du = dx$ $\qquad$ $v = \sin x$

$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$= x\sin x + \cos x + C$

reduction formula: $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

- repeat applying this formula to evaluate integral for any n

- given $u = u(x)$ & $v = v(x)$ w/ $du = u'(x)dx$ & $dv = v'(x)dx$, then $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

week 12

TRIGNOMETRIC SUBSTITUTIONS

- certain integrands containing $\sqrt{a^2-x^2}$ or $\sqrt{x^2 \pm a^2}$ can't be resolved w/ u-sub of $u = a^2 - x^2$ or $u = x^2 \pm a^2$
  - ↳ use inverse substitutions & assume $x = g(t)$ so $dx = g'(t)\,dt$
    - $\int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$
- for 3 cases, use trignometric subs w/ identities $\sin^2\theta + \cos^2\theta = 1$ & $\tan^2\theta + 1 = \sec^2\theta$

1) $\sqrt{x^2+a^2}$

Let $x = a\tan\theta$

$\frac{dx}{d\theta} = a\sec^2\theta$

$dx = a\sec^2\theta\,d\theta$

$$\begin{aligned}&\sqrt{x^2+a^2}\\ &= \sqrt{a^2\tan^2\theta + a^2}\\ &= a\sqrt{\tan^2\theta+1}\\ &= a|\sec\theta|\\ &= a\sec\theta\end{aligned}$$

⟵ $\theta = \tan^{-1}\left(\frac{x}{a}\right)$ so $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; on this interval, $\sec\theta > 0$ so omit absolute value

2) $\sqrt{x^2-a^2}$

Let $x = a\sec\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$dx = a\sec\theta\tan\theta\,d\theta$

$$\begin{aligned}&\sqrt{x^2-a^2}\\ &= \sqrt{a^2\sec^2\theta - a^2}\\ &= a\sqrt{\sec^2\theta - 1}\\ &= a|\tan\theta|\\ &= a\tan\theta\end{aligned}$$

⟵ $\theta = \sec^{-1}\left(\frac{x}{a}\right)$ so $\theta \in \left[-\pi, -\frac{\pi}{2}\right) \cup \left[0, \frac{\pi}{2}\right)$; on these intervals, $\tan\theta > 0$ so omit absolute value

3) $\sqrt{a^2-x^2}$

Let $x = a\sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$dx = a\cos\theta\,d\theta$

$$\begin{aligned}&\sqrt{a^2-x^2}\\ &= \sqrt{a^2 - a^2\sin^2\theta}\\ &= a\sqrt{1-\sin^2\theta}\\ &= a|\cos\theta|\\ &= a\cos\theta\end{aligned}$$

⟵ $\theta = \sin^{-1}\left(\frac{x}{a}\right)$ so $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; on this interval, $\cos\theta > 0$ so omit absolute value

INTEGRATION OF RATIONAL FUNCTIONS

- for general rational functions, perform long division if necessary & each part of partial fraction decomposition is integrated separately
- basic formulas:
  - ↳ $\int \frac{dx}{b+x} = \ln|b+x| + C$
  - ↳ $\int \frac{dx}{b-x} = -\ln|b-x| + C$
  - ↳ $\int \frac{dx}{(b+x)^n} = -\frac{1}{n-1}\frac{1}{(b+x)^{n-1}} + C$
    - $n \neq 1$
  - ↳ $\int \frac{dx}{(b-x)^n} = \frac{1}{n-1}\frac{1}{(b-x)^{n-1}} + C$
    - $n \neq 1$
  - ↳ $\int \frac{dx}{b^2+x^2} = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + C$
  - ↳ $\int \frac{dx}{b^2-x^2} = \frac{1}{2b}\ln\left|\frac{b+x}{b-x}\right| + C$
- 3 more general cases:
  - ↳ $\frac{a}{x^2+bx+c}$ is a constant over an arbitrary quadratic
    - denominator will be factorisable (use partial fractions) or irreducibly quadratic (use $\tan^{-1}$)
  - ↳ $\frac{ax+b}{x^2+cx+d}$ is an arbitrary linear function over an arbitrary quadratic
    - may require u-subs, partial fractions, or $\tan^{-1}$

↳ $\frac{a}{(x^2+b^2)^n}$ is a constant over an irreducible quadratic raised to the n^{th} power

- may require trig subs (e.g. $x = b\tan\theta$)

AREA BETWEEN CURVES

· area btwn curves $y = f(x)$ & $y = g(x)$ in the interval $x \in [a,b]$ is $A = \int_a^b |f(x) - g(x)|\,dx$

↳ if $f(x) \geq g(x)$ for some values of x & $f(x) \leq g(x)$ for other values of x, must break interval $[a,b]$ into subintervals in which only one of the inequalities apply

· area btwn $g(y)$ & $f(y)$ in the interval $y \in [c,d]$ is $A = \int_c^d (f(y) - g(y))\,dy$

MEAN VALUES OF FUNCTIONS

· mean value of continuous function f on $[a,b]$ is $m.v.(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$

· root mean square value of continuous function f on $[a,b]$ is $r.m.s.(f) = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$

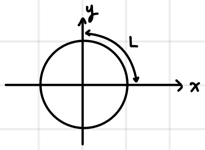
↳ will always give value at least as large as mean of absolute value of f

week 13

LENGTH OF CURVES

- arc length $= \int_a^b \sqrt{1+(f'(x))^2}\,dx$
  - ↳ if $x = g(y)$ instead, $L = \int_c^d \sqrt{1+(g'(y))^2}\,dy$
- e.g. determine circumference of circle w/ radius R

SOLUTION



Quarter-arc of circle in QI is graph of $f(x) = \sqrt{R^2 - x^2}$.

$$f(x) = \sqrt{R^2 - x^2}$$
$$f'(x) = \tfrac{1}{2}(R^2 - x^2)^{-\frac{1}{2}}(-2x)$$
$$= \frac{-x}{\sqrt{R^2 - x^2}}$$

$$\sqrt{1+(f'(x))^2} = \sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2} = \sqrt{1+\frac{x^2}{R^2-x^2}} = \sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}} = \frac{R}{\sqrt{R^2-x^2}}$$

$$L = \int_0^R \sqrt{1+(f'(x))^2}\,dx$$
$$= \int_0^R \frac{R}{\sqrt{R^2-x^2}}\,dx \longrightarrow \text{Let } x = R\sin\theta \longrightarrow \sin\theta = \frac{x}{R}$$
$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{R}{\sqrt{R^2 - R^2\sin^2\theta}}(R\cos\theta\,d\theta) \qquad \frac{dx}{d\theta} = R\cos\theta \qquad \theta = \sin^{-1}\left(\frac{x}{R}\right)$$
$$= \int_0^{\frac{\pi}{2}} \frac{R^2\cos\theta}{R\sqrt{1-\sin^2\theta}}\,d\theta \qquad dx = R\cos\theta\,d\theta \qquad \text{When } x=0:\ \theta = \sin^{-1}\left(\frac{0}{R}\right) = 0$$
$$= \int_0^{\frac{\pi}{2}} \frac{R\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} R\,d\theta \qquad \text{When } x = R:\ \theta = \sin^{-1}\left(\frac{R}{R}\right) = \frac{\pi}{2}$$
$$= [R\theta]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{2}R - 0$$
$$= \frac{\pi}{2}R$$

$$\text{Circumference} = 4L = 4\left(\frac{\pi}{2}R\right) = 2\pi R$$

VOLUMES OF SOLIDS OF REVOLUTION

- volume enclosed by curve $y = p(x)$ rotated abt x-axis btwn $x=0$ & $x=L$ is $V = \int_{x=0}^{L} A(x)\,dx$
  - ↳ $A(x) = \pi(p(x))^2$
- e.g. derive formula for volume of sphere

SOLUTION

Rotate $y = p(x) = \sqrt{r^2 - x^2}$ abt x-axis.

$$V = \int_{-r}^{r} A(x)\,dx$$
$$= \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$
$$= \pi\left[r^2x - \tfrac{1}{3}x^3\right]_{-r}^{r}$$
$$= \pi\left(r^2(r) - \tfrac{1}{3}r^3 - (r^2(-r) - \tfrac{1}{3}(-r)^3)\right)$$
$$= \pi\left(r^3 - \tfrac{1}{3}r^3 - (-r^3 + \tfrac{1}{3}r^3)\right)$$
$$= \pi\left(\tfrac{2}{3}r^3 - (-\tfrac{2}{3}r^3)\right)$$
$$= \tfrac{4}{3}\pi r^3$$

- when calculating volume using washers instead of disks: $A(x) = A_{outer}(x) - A_{inner}(x)$
  - $A(x) = \pi((p_{outer}(x))^2 - (p_{inner}(x))^2)$
- volume using cylindrical shell method is $V = \int_{r_{min}}^{r_{max}} 2\pi x h(x)\,dx$
  - ↳ $h(x)$ is function of radius
- e.g. calculate volume by rotating area enclosed by $y = x^3 + x^2$, $y=0$, $x=0$, & $x=1$ abt y-axis

SOLUTION

$h(x) = x^3 + x^2$

$r_{min} = 0$

$r_{max} = 1$

$V = \int_0^1 2\pi x(x^3 + x^2)\,dx$

$= 2\pi \int_0^1 x^4 + x^3\,dx$

$= 2\pi \left[\frac{1}{5}x^5 + \frac{1}{4}x^4\right]_0^1$

$= 2\pi \left(\frac{1}{5} + \frac{1}{4}\right)$

$= 2\pi \left(\frac{9}{20}\right)$

$= \frac{9}{10}\pi$

NOTE: $f(x) = \frac{1}{x}$ marks boundary btwn converging & diverging power functions

IMPROPER INTEGRALS

- can define definite integrals over infinite intervals
  - ↳ $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$
  - ↳ $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$
  - ↳ combine above definitions to get $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$
- if improper integral has finite value, it's convergent
  - ↳ otherwise, it's divergent (i.e. limit is infinite or DNE)
  - ↳ e.g. determine if $\int_1^{\infty} \frac{1}{x}\,dx$ is convergent / divergent

SOLUTION

$\int_1^{\infty} \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$= \lim_{t\to\infty} [\ln|x|]_1^t$

$= \lim_{t\to\infty} [\ln t - \ln 1]$

$= \lim_{t\to\infty} [\ln t]$

$= \infty$

The integral is divergent.

- for improper integral $\int_1^{\infty} \frac{1}{x^p}\,dx$:
  - ↳ convergent for $p > 1$
  - ↳ divergent for $p \leq 1$

- if $f(x)$ is continuous except at $x = b$, then $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$
- if $f(x)$ is continuous except at $x = a$, then $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$
- when discontinuity is inside interval of integration (i.e. discontinuity is at $x = c$ & $a < c < b$), then combine above definitions to get $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- for integral $\int_0^1 \frac{1}{x^p}\,dx$:
  - ↳ convergent for $p < 1$
  - ↳ divergent for $p \geq 1$