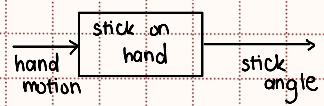
NOTES

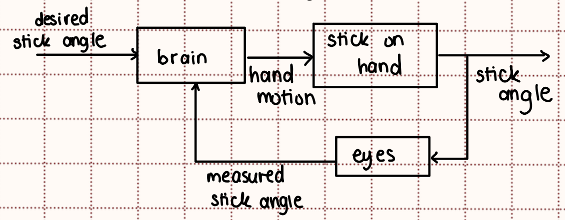
examples of control systems

date 09/06/2024

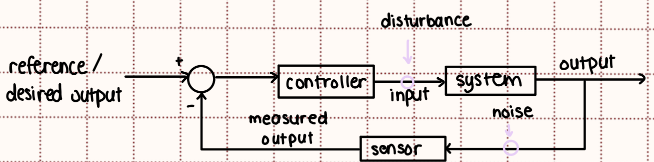
- e.g. balancing stick on hand
  - ↳ w/ eyes closed, open-loop system:



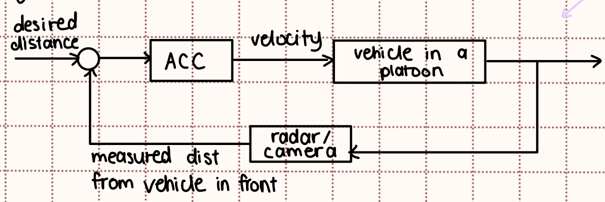
  - ↳ w/ eyes open, closed-loop system:

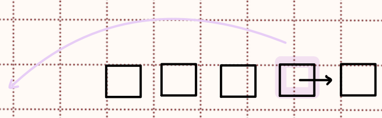


- negative feedback control (closed-loop):



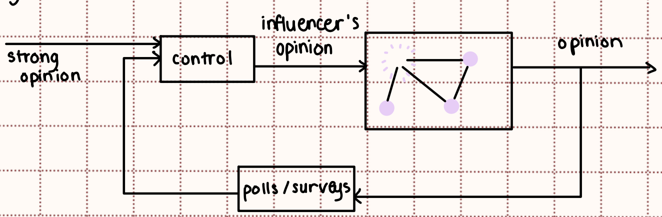
  - ↳ additional things that affect system:
    - environmental inputs (e.g. wind in room) ⟶ disturbances
    - max speed of controller
    - accurracy of sensor
  - ↳ negative b/c output is subtracted from input
- e.g. cruise control



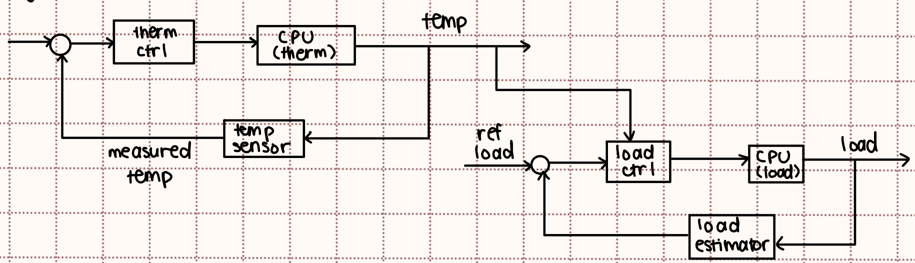


ACC - automatic cruise control

- e.g. social networks



  - ↳ example of network control
- e.g.



designing a controlled system

date 09/09/2024

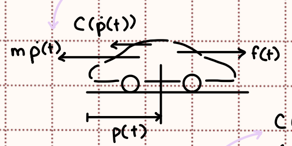
· steps to design controlled system:

1) study system
   - inputs + outputs
   - activators + sensors
2) define control specs
3) obtain a model of system
   - can do so using 1st principles or data
4) simplify model
   - e.g. linearization
5) design controller
6) simulate controlled system using model
   - use high-fidelity model for testing

iterate if necessary

· e.g.

pointing bwd b/c this is force proportional to acceleration + due to inertia, this is resistive force of acceleration



$p(t)$: position
$\dot{p}(t) = \frac{dp}{dt}$: speed
$\ddot{p}(t) = \frac{d^2p}{dt^2}$: acceleration

$C(\dot{p}(t))$ is simply fcn of $\dot{p}(t)$
(e.g. $C(\dot{p}(t)) = \dot{p}(t)^2$)

$$\underbrace{m\ddot{p}(t) + C(\dot{p}(t))}_{\text{output}} = \underbrace{f(t)}_{\text{input}}$$

$$f(t) \longrightarrow \boxed{\text{vehicle}} \xrightarrow{p(t)}$$

STATE SPACE FORM

· state of system is a collection of vars which describe evolution of system until current time

↳ operational defn: state $x(t)$ of system is collection of vars st if we know $x(t_0)$ + input signal $u(t)$, $t \in [t_0, t_1]$, then we can compute $x(t_1)$ + output $y(t_1)$

$$u(t) \longrightarrow \boxed{\begin{array}{c}\text{system}\\ x(t)\end{array}} \xrightarrow{y(t)}$$

· e.g. from prev example, going from differential eqn to state space

$$u(t) = f(t) \longrightarrow \boxed{\text{vehicle}} \xrightarrow{p(t) = y(t)}$$

↳ possible options for state:

$x(t) = p(t) \in \mathbb{R}$

$x(t) = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} \in \mathbb{R}^2$

$x(t) = \begin{bmatrix} p(t) \\ \dot{p}(t) \\ \ddot{p}(t) \end{bmatrix} \in \mathbb{R}^3$

state space form

date 09 / 11 / 2024

· e.g. from prev lec (initial value problem rephrased)

$m\ddot{p}(t) + C(\dot{p}(t)) = f(t)$

$\xrightarrow{f(t)} \boxed{\text{system}} \xrightarrow{p(t)}$

f(t) is input u(t) → acceleration in this

State dynamics for $x(t) = p(t)$:

$\dot{x}(t) = \dot{p}(t)$

$x(t_0) = p(t_0) = p_0$

$f(t),\ t \in [t_0, t_1]$

$x(t_1) = ?$

doesn't work b/c $x(t_1)$ is dependent on initial velocity

State dynamics for $x(t) = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix}$:

$\dot{x}(t) = \begin{bmatrix} \dot{p}(t) \\ \ddot{p}(t) \end{bmatrix} = \begin{bmatrix} \dot{p}(t) \\ -\frac{C(\dot{p}(t))}{m} + \frac{f(t)}{m} \end{bmatrix}$ → state space form of $m\ddot{p}(t) + C(\dot{p}(t)) = f(t)$

$x(t_0) = \begin{bmatrix} p(t_0) \\ \dot{p}(t_0) \end{bmatrix} = \begin{bmatrix} p_0 \\ v_0 \end{bmatrix}$

$f(t),\ t \in [t_0, t_1]$

$x(t_1) = \begin{bmatrix} p(t_1) \\ \dot{p}(t_1) \end{bmatrix}$ → this works

state space form: a way to rep systems of differential eqns

· in general, for $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, + $y(t) \in \mathbb{R}^p$, state space form of system:

↳ $\dot{x} = f(x, u)$

- $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

↳ $y = h(x, u)$

- $h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$

↳ general form of state-space eqn is:

- $\dot{x}(t) = Ax(t) + Bu(t)$
- $y(t) = Cx(t) + Du(t)$

A, B, C, + D are matrices that define how system evolves over time + how input affects state + output

↳ e.g. for above example

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ $\qquad \dot{x} = \begin{bmatrix} x_2 \\ -\frac{C(x_1)}{m} + \frac{n}{m} \end{bmatrix}$

$u \in \mathbb{R}$ $\qquad y = \underbrace{x_1}_{h(x,n)}$

$y \in \mathbb{R}$

n = 2
m = 1
p = 1

$\dot{x}_1 = x_2$
$\dot{x}_2 = -\frac{C(x_2)}{m} + \frac{u}{m}$
$y = x_1$

· to select correct state space form, can find state at t_1 given $x(t_0)$ + $f(t)$

· use linearization to apx system behaviour when it's close to equilibrium pt

↳ small perturbations on state + input

· $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is equilibrium point iff $f(\bar{x}, \bar{u}) = 0$, meaning rate of change of states is 0

↳ $\dot{x} = f(\bar{x}, \bar{u}) = 0$

$x(t_0) = \bar{x}$ → $x(t) = \bar{x}$ is solution of system

$u(t) = \bar{u},\ t \in [t_0, t_1]$

$\bar{x}, \bar{u}$ are equilibrium values

↳ $\dot{x} = f(x, u)$

$x(t_0) = \bar{x} + \delta\bar{x}$

$u(t) = \bar{u} + \delta u(t),\ t \in [t_0, t_1]$

for some small $\delta\bar{x}, \delta\bar{u}(t)$

$\delta x = x - \bar{x}$

$\delta u = u - \bar{u}$

$\delta\dot{x} = \dot{x} - \dot{\bar{x}}$

$= f(x, u) - 0$

around $\bar{x}, \bar{u}$ $= \cancel{f(\bar{x}, \bar{u})}^{0} + \underbrace{\frac{\partial f}{\partial x}(\bar{x}, \bar{u})}_{A}\underbrace{(x - \bar{x})}_{dx} + \underbrace{\frac{\partial f}{\partial u}(\bar{x}, \bar{u})}_{B}\underbrace{(u - \bar{u})}_{du}$ + hot (higher order terms)

$+ O(\|\delta\bar{u}\|^2) + O(\|\delta\bar{x}\|^2)$

date

(linearization)

$$\delta\dot{x} = A\,\delta x + B\,\delta u + \text{hot}$$

$$y = h(x, u)$$

$$= h(\bar{x}, \bar{u}) + \overbrace{\frac{\partial h}{\partial x}(\bar{x}, \bar{u})}^{C}\overbrace{(x - \bar{x})}^{dx} + \overbrace{\frac{\partial h}{\partial u}(\bar{x}, \bar{u})}^{D}\overbrace{(u - \bar{u})}^{du} + \text{hot}$$

$$\delta y = C\delta x + D\delta u + \text{hot} \leftarrow \text{letting } \delta y = y - h(\bar{x}, \bar{u})$$

(linearization)

↳ equilibrium doesn't imply states, output, + input are 0

e.g. Jacobian matrix

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_q}{\partial x_1} & \cdots & \frac{\partial f_q}{\partial x_n} \end{bmatrix}$$

A + B are Jacobian matrices that describe how state evolves wrt small changes in $x(t)$ + $u(t)$, respectively. C + D describe how output evolves.

Linearization

date 09/13/2024

· state space:

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$$

↳ $x \in \mathbb{R}^n$ → state

↳ $u \in \mathbb{R}^m$ → input

↳ $y \in \mathbb{R}^p$ → output

↳ $x(t) \equiv \bar{x}$, $u(t) \equiv \bar{u}$, $y(t) \equiv \bar{y} = h(\bar{x},\bar{u})$ is a soln

· equilibrium point is $f(\bar{x},\bar{u}) = 0$

↳ $(\bar{x},\bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$

↳ $x(t_0) = \bar{x} \in \mathbb{R}^n$

↳ $u(t) = \bar{u} \in \mathbb{R}^m$

· linearization of $\dot{x}$, y around $(\bar{x},\bar{u})$ using first order Taylor apx:

↳ $$\dot{x} = f(x,u) = \underbrace{f(\bar{x},\bar{u})}_{\in \mathbb{R}^n} + \underbrace{\frac{\partial f}{\partial x}(\bar{x},\bar{u})}_{\in \mathbb{R}^{n\times n}} \underbrace{(x-\bar{x})}_{\in \mathbb{R}^n} + \underbrace{\frac{\partial f}{\partial u}(\bar{x},\bar{u})}_{\in \mathbb{R}^{n\times m}} \underbrace{(u-\bar{u})}_{\in \mathbb{R}^m} + \text{hot}$$

$\delta x = x - \bar{x}$, $\delta u = u - b$

$\delta \dot{x} = \dot{x} - 0$

$= A\delta x + B\delta u + \text{hot}$

$$A = \frac{\partial f}{\partial x}(\bar{x},\bar{u}) \qquad B = \frac{\partial f}{\partial u}(\bar{x},\bar{u})$$

↳ $$y = h(x,u) = \underbrace{h(\bar{x},\bar{u})}_{\in \mathbb{R}^p} + \underbrace{\frac{\partial h}{\partial x}(\bar{x},\bar{u})}_{\in \mathbb{R}^{p\times n}} \underbrace{(x-\bar{x})}_{\in \mathbb{R}^n} + \underbrace{\frac{\partial h}{\partial u}(\bar{x},\bar{u})}_{\in \mathbb{R}^{p\times m}} \underbrace{(u-\bar{u})}_{\in \mathbb{R}^m} + \text{hot}$$

$\delta y = y - h(\bar{x},\bar{u})$

$= C\delta x + D\delta u + \text{hot}$

$$C = \frac{\partial h}{\partial x}(\bar{x},\bar{u}) \qquad D = \frac{\partial h}{\partial u}(\bar{x},\bar{u})$$

↳ to summarize, linearization of system is:

$$\begin{cases} \delta \dot{x} = A\delta x + B\delta u \\ \delta y = C\delta x + D\delta u \end{cases} \longrightarrow \text{around } (\bar{x},\bar{u})$$

∘ linear time invariant (LTI) system

Will omit dependence on time of state, input, + output vars when meaning is unambiguous. $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, + $y \in \mathbb{R}^p$ denotes vals of state, input, + output at time t.

Given non-linear system $\dot{x}(t) = f(x,u)$, it can be linearized at $(\bar{x},\bar{u})$ + rep by $\dot{x} \simeq f(\bar{x},\bar{u}) + \frac{\partial f}{\partial x}(\bar{x},\bar{u})(x-\bar{x}) + \frac{\partial f}{\partial u}(\bar{x},\bar{u})(u-\bar{u})$.

· e.g. epidemics

$$S \to I \to R$$

$S(t)$ = # susceptible individuals

$I(t)$ = infected

$R(t)$ = recovered

fractional so they all sum up to 1

Dynamics:

$\dot{S}(t) = -\beta I(t) S(t)$

$\dot{I}(t) = \beta I(t) S(t) - r I(t)$

$\dot{R}(t) = r I(t)$

$$\xrightarrow{\beta(t)} \boxed{} \xrightarrow{I(t)}$$

$$x(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$u(t) = \beta(t)$

date

$y = I(t)$

$$\begin{cases} \dot{x} = \begin{bmatrix} -u x_2 x_1 \\ u x_2 x_1 - r x_2 \\ r x_2 \end{bmatrix} = f(x,u) \\ y = x_2 = h(x,u) \end{cases}$$

Equilibrium:

$$\begin{bmatrix} -\bar{u}\bar{S}\bar{I} \\ \bar{u}\bar{S}\bar{I} - r\bar{I} \\ r\bar{I} \end{bmatrix} = 0$$

↳ 1 possible config: $\bar{S} = 100\%$, $\bar{I} = 0\%$, $\bar{R} = 0\%$, $\bar{u} = 0$

- i.e. if no one is infected, it'll stay that way even w/o protective measures ($\bar{u} = 0$)

$$A = \frac{\partial f}{\partial x}\Big|_{\substack{100\\0\\0\\0}}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_3} \\ \vdots & & \\ \frac{\partial f_3}{\partial x_1} & \cdots & \frac{\partial f_3}{\partial x_3} \end{bmatrix}\Big|_{\substack{100\\0\\0\\0}}$$

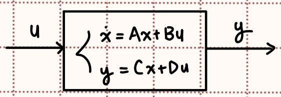
$$= \begin{bmatrix} -u x_2 & -u x_1 & 0 \\ u x_2 & u x_1 - r & 0 \\ 0 & r & 0 \end{bmatrix}\Big|_{\substack{100\\0\\0\\0}}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & r & 0 \end{bmatrix}$$

solving systems of differential equations

date 09/16/2024

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{matrix} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^p \end{matrix}$$



· transfer function of system is ratio of Laplace transform of its output over its input, assuming 0 initial conditions

↳ e.g.

$$G(s) = \frac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))} = \frac{Y(s)}{U(s)}$$ transfer fcn of SISO

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} sX(s) - \overset{0}{x(0^-)} = AX(s) + BU(s) & (1) \\ Y(s) = CX(s) + DU(s) & (2) \end{cases}$$

(1)

$$sIX(s) - AX(s) = BU(s)$$

$$\underbrace{(sI - A)}_{\in \mathbb{R}^{n \times n}} X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1} BU(s)$$ → can't inverse when $\det(sI - A) = 0$

↳ this happens when $s \in$ eigenvalue(A)

(2)

$$Y(s) = C(sI - A)^{-1} BU(s) + DU(s)$$

$$= (C(sI - A)^{-1} B + D) U(s)$$

$$\underbrace{G(s) = C(sI - A)^{-1} B + D}_{\in \mathbb{R}^{p \times m}}$$ ← transfer fcn of LTI

Eigenvalue (A) :

$Av = \lambda v$

$(A - \lambda I)v = 0$

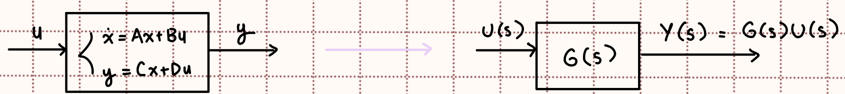
$\det(A - \lambda I) = 0$

* v is a vector $\forall$

$$G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1m}(s) \\ \vdots & & \vdots \\ G_{p1}(s) & \cdots & G_{pm}(s) \end{bmatrix}$$

$$\underbrace{Y(s)}_{\in \mathbb{R}^p} = \underbrace{G(s)}_{\in \mathbb{R}^{p \times m}} \underbrace{U(s)}_{\in \mathbb{R}^m}$$

↳ using transfer fcn loses info abt state (i.e. RHS diagram is not equiv to LHS)



· to compute $y(t)$ response to $u(t)$:

1) $U(s) = \mathcal{L}(u(t))$

2) $Y(s) = G(s)U(s)$

3) $y(t) = \mathcal{L}^{-1}(Y(s))$

· for single input single output (SISO) systems, a transfer fcn $G(s)$ is real rational if $G(s) = \dfrac{b_m s^m + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$

where $a_i, b_j \in \mathbb{R}$

↳ $G(s)$ is proper if $\lim_{s \to \infty} G(s)$ exists

- $n \geq m$

↳ $G(s)$ is strictly proper if $\lim_{s \to \infty} G(s) = 0$

- $n > m$

↳ $p \in \mathbb{C}$ is a pole of $G(s)$ if $\lim_{s \to p} |G(s)| = \infty$

- root of denominator

↳ $z \in \mathbb{C}$ is zero of $G(s)$ if $\lim_{s \to z} |G(s)| = 0$

- root of numerator

transfer functions

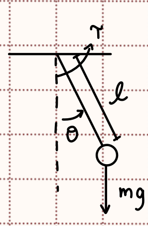
date 09/18/2024

- e.g. SISO system

$x = \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix} \in \mathbb{R}^2$

$u = \tau \in \mathbb{R}$

$y = \theta \in \mathbb{R}$



$\ddot\theta + \frac{g}{\ell}\theta = \tau$

$$\begin{cases} \dot x = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

linearized around $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\bar{u} = 0$

Transfer fcn.

$$G(s) = \frac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))}$$
$$= \frac{Y(s)}{U(s)}$$
$$= C(sI - A)^{-1}B + D$$

I denotes $n \times n$ identity matrix

$$= [1 \;\; 0]\left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$$

$$= [1 \;\; 0]\begin{bmatrix} s & -1 \\ \frac{g}{\ell} & s \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

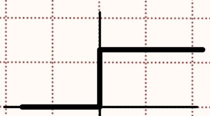
$$= [1 \;\; 0]\frac{\begin{bmatrix} s & -1 \\ \frac{g}{\ell} & s \end{bmatrix}}{s^2 + \frac{g}{\ell}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{[s \;\; 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2 + \frac{g}{\ell}}$$

$$= \frac{1}{s^2 + \frac{g}{\ell}}$$

$$U(s) \longrightarrow \boxed{\frac{1}{s^2 + \frac{g}{\ell}}} \longrightarrow Y(s)$$

$\tau = H(t) = u(t)$



Laplace transform of step fcn is $\frac{1}{s}$

$$y(t) = \mathcal{L}^{-1}(G(s)\,\mathcal{L}(u(t)))$$
$$= \mathcal{L}^{-1}\left(\frac{1}{s^2 + \frac{g}{\ell}} \cdot \frac{1}{s}\right)$$

use partial frac decomp $= \frac{A's + B'}{s^2 + \frac{g}{\ell}} + \frac{C'}{s} = \ldots$

$$G(s) = \frac{1}{s^2 + \frac{g}{\ell}}$$

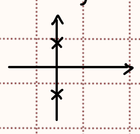
$b_0 = 1$, $b_i = 0$ for $i \geq 1$

$a_0 = \frac{g}{\ell}$, $a_1 = 0$, $a_i = 0$ for $i \geq 2$

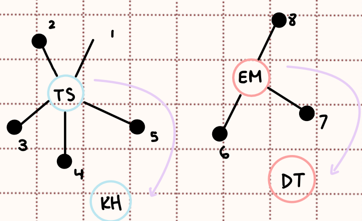
$m = 0$, $n = 2$ } strictly proper real rational

Zeros: none

Poles: $\pm j\sqrt{\frac{g}{\ell}}$



- e.g. MISO system: want to drive multi-input steering system to specific loc



$$\begin{cases} \dot x_{TS} = u_{TS} \\ \dot x_{EM} = u_{EM} \\ \dot x_i = x_{TS} - x_i, \; i \in \{1, \ldots, 5\} \\ \dot x_j = x_{EM} - x_j, \; j \in \{6, \ldots, 8\} \end{cases}$$

$u = \begin{bmatrix} u_{TS} \\ u_{EM} \end{bmatrix}$

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_{TS} \\ \dot{x}_{EM} \\ \dot{x}_1 \\ \vdots \\ \dot{x}_8 \end{bmatrix} = -\underbrace{L}_{\text{Laplacian matrix}} x + \begin{bmatrix} 1\;0 \\ 0\;1 \\ 0 \end{bmatrix} u \\ y = \frac{1}{N} \underbrace{\mathbb{1}^T}_{[1\ 1\ 1\ \dots\ 1\ 1]} x \end{cases}$$

- Laplacian transform of signal $f(t)$ is $F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t) e^{-st}\, dt$
  - ↳ useful Laplace transforms:

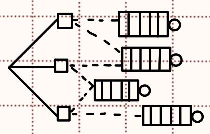
$f(t)$	$F(s)$
(impulse) $\delta(t)$	1
(step) $H(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

  - ↳ useful properties:
    - $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
    - $\mathcal{L}\{f(t-\tau)\} = e^{-\tau s} F(s)$
    - $\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$
    - $\mathcal{L}\{\frac{d}{dt} f(t)\} = sF(s) - f(0^-)$
    - $\mathcal{L}\{t f(t)\} = -\frac{d}{ds} F(s)$
    - $\mathcal{L}\{f(t) \underbrace{*}_{\text{convolution}} g(t)\} = F(s)G(s)$

inverse laplace to get time domain

date 09/20/2024

· e.g. queues



arrival rate, service rate

$q = u - s$

$x = q$

$\underbrace{\dot{x} = u - s}_{f(x,u)}$

$f(x,u) = u - s$

$\underbrace{f(\bar{x},\bar{u})}_{\text{equilibrium}} = 0 \longrightarrow \bar{u} - s = 0$, x = whatever

Choose $\bar{x} = 0$, $\bar{u} = s$:

$f(x,u) = f(\bar{x},\bar{u}) + \frac{\partial f}{\partial x}(\bar{x},\bar{u})(x - \bar{x}) + \frac{\partial f}{\partial u}(\bar{x},\bar{u})(u - \bar{u}) + \text{hot}$ — don't exist

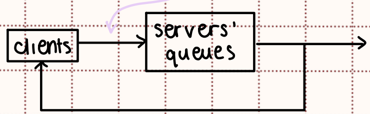
$= 0 + 0\ (x - 0) + 1\ (u - s)$

$= u - s$

$\delta\dot{x} = \delta u \longrightarrow$ single integrator

Dropping deltas, meaning that $u = \delta u + s$:

$\begin{cases} \dot{x} = u \\ y = x \end{cases}$



When we write $\dot{x} = u$, we actually mean $u = \delta u + s$ (dropping deltas, which is an offset)

Transfer fcn:

$\frac{Y(s)}{U(s)} = G(s) = C(sI - A)^{-1}B + D$

$= 1\,(s(1) - 0)^{-1}\,1 + 0$

$= \frac{1}{s}$

$Y(s) = \frac{1}{s}U(s)$

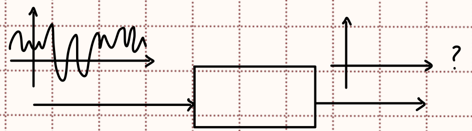
$y(t) = \int_0^t u(\tau)\,d\tau$

$q(t) = y(t)$

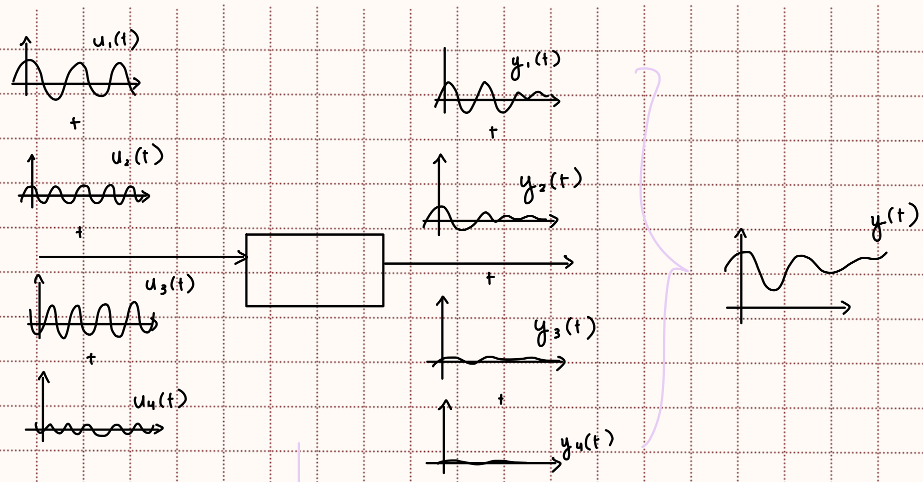
$= \int_0^t u(\tau)\,d\tau$ — undropping deltas : $\delta u = u - s$

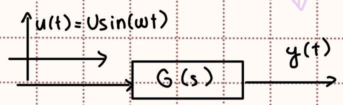
$= \int_0^t (\underbrace{u(\tau)}_{\text{arrival}} - \underbrace{s(\tau)}_{\text{service}})\,d\tau$

· e.g. steady-state response of system to sinusoidal



date





$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= \mathcal{L}^{-1}\{G(s)U(s)\}$$
$$= \mathcal{L}^{-1}\{G(s)\,\mathcal{L}\{u(t)\}\}$$
$$= \mathcal{L}^{-1}\left\{G(s)\frac{U\omega}{s^2+\omega^2}\right\}$$

assuming G(s) has real + distinct poles, p_i for $i = 1,\dots,n$

$$y(t) = \mathcal{L}^{-1}\left\{\underbrace{\sum_{i=1}^{n}\frac{A_i}{s-p_i}}_{\text{take inverse Laplace}} + \frac{Q}{s-j\omega} + \frac{\bar{Q}}{s+j\omega}\right\} \longrightarrow A_i, Q \text{ come from partial frac decomp}$$

$A_i e^{p_i t}$

As $t\to\infty$, $A_i e^{p_i t} \to 0$

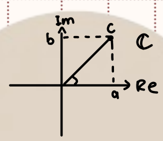
Assume $Re(p_i) < 0$:

$$y_{ss}(t) = \mathcal{L}^{-1}\left\{\frac{Q}{s-j\omega} + \frac{\bar{Q}}{s+j\omega}\right\}$$

steady-state output, meaning $t \to \infty$

solving for $Q, \bar{Q}$ $\longrightarrow$

$$Q = (s-j\omega)\,[Y(s)]_{s=j\omega}$$
$$= (s-j\omega)\left[G(s)\frac{U\omega}{s^2+\omega^2}\right]_{s=j\omega}$$
$$= (s-j\omega)\left[\frac{G(s)\,U\omega}{(s-j\omega)(s+j\omega)}\right]_{s=j\omega}$$
$$= G(j\omega)\frac{U\omega}{j\omega+j\omega}$$
$$= G(j\omega)\frac{U}{2j}$$
$$\bar{Q} = G(-j\omega)\frac{U}{-2j}$$
$$= \overline{G(j\omega)}\frac{U}{-2j}$$



$c = a + jb$

$Im(c) = |c| \sin\angle c$

$Re(c) = |c|\cos\angle c$

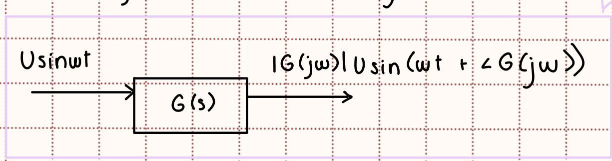
$$y_{ss}(t) = \mathcal{L}^{-1}\left\{\frac{G(j\omega)U}{2j}\frac{1}{s-j\omega} - \frac{\overline{G(j\omega)}\,U}{2j}\frac{1}{s+j\omega}\right\}$$
$$= \frac{G(j\omega)U}{2j}e^{j\omega t} - \frac{\overline{G(j\omega)}\,U}{2j}e^{-j\omega t}$$
$$= G(j\omega)\frac{U}{2j}(\cos\omega t + j\sin\omega t) - \overline{G(j\omega)}\frac{U}{2j}(\cos\omega t - j\sin\omega t)$$
$$= \frac{U}{2j}\left((G(j\omega) - \overline{G(j\omega)})\cos\omega t + j(G(j\omega) + \overline{G(j\omega)})\sin\omega t\right)$$
$$= \frac{U}{2j}\left(2j\,Im(G(j\omega))\cos\omega t + 2j\,Re(G(j\omega))\sin\omega t\right)$$
$$= U\left(|G(j\omega)|\sin\angle G(j\omega)\cos\omega t + |G(j\omega)|\cos\angle G(\omega t)\sin\omega t\right)$$
$$y_{ss}(t) = |G(j\omega)|\,U\sin(\omega t + \angle G(j\omega))$$

$a + jb - (a - jb) = 2jb$

$a + jb + a - jb = 2a$

$\sin(\alpha + \beta) =$

$\sin\alpha\cos\beta + \cos\alpha\sin\beta$



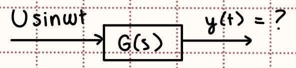
↳ for G(s) w/ poles p_i st $Re(p_i) < 0$

bode plots

date 09 / 23 / 2024

bode plots are graphical rep of frequency response of system

· e.g.

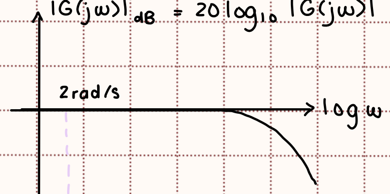


$u(t) = \sum_i U_i \sin\omega_i t$

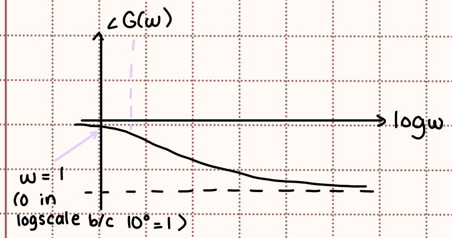
$y(t) = \sum_i y_i(t)$ ← response to u_i

Magnitude

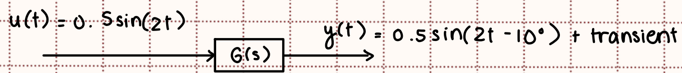
$|G(j\omega)|_{dB} = 20\log_{10}|G(j\omega)|$

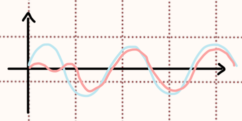


Phase ·



· e.g.





· e.g.

$$\begin{cases} \dot{x} = -10x + u \\ y = x \end{cases} \qquad x, u, y \in \mathbb{R}$$

$G(s) = C(sI - A)^{-1}B + D$

$= 1(s(1) + 10)^{-1}\, 1 + 0$

$= \frac{1}{s+10}$

Steady-state output signal to input:

$u(t) = 0.1\sin(0.2t)$

Frequency response ($\omega \in \mathbb{R}_+ \mapsto G(j\omega) \in \mathbb{C}$):

↳ magnitude

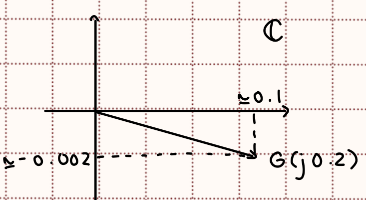
↳ phase

$G(j\omega) = \frac{1}{j\omega + 10}$

$G(j0.2) = \frac{1}{10 - j0.2}\left(\frac{10 + j0.2}{10 + j0.2}\right)$

$= \frac{10}{10^2 + 0.2^2} - j\frac{0.2}{10^2 + 0.2^2}$

$\approx 0.1 - j0.002$

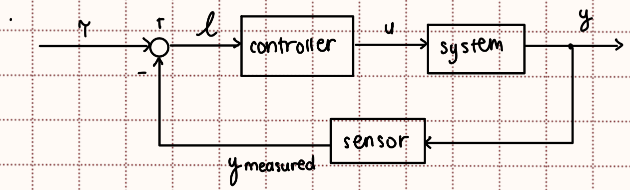


$y_{ss}(t) \approx 0.1(0.1)\sin(0.2t - 0.002)$

$|G(j\omega)|\ U\ \sin(\omega t + \angle G(j\omega))$

bode plot examples

date 09/25/2024



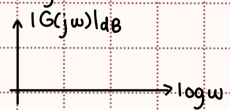
- mathematical models of systems:
  - ↳ differential eqns (in state space form)
    - time domain
  - ↳ transfer fcn
    - Laplace domain
  - ↳ frequency response
    - frequency domain

- $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$
  - ↳ $G(s) = \dfrac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))}$ ← w/o initial conditions (i.e. $x(t_0) = 0 \in \mathbb{R}^n$)

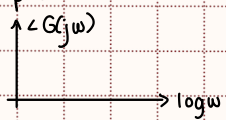
    $= C(sI - A)^{-1}B + D$

  $G(s)|_{s=j\omega} = G(j\omega)$

- bode plots are graphical rep of $G(j\omega)$
  - magnitude

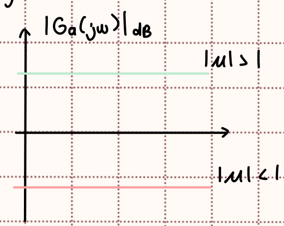


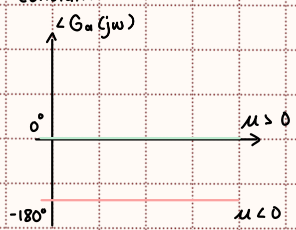
  - phase

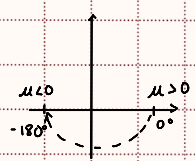


- $$G(s) = \frac{\mu}{s^p} \frac{\Pi_i (1 + T_i s)}{\Pi_i (1 + \tau_i s)} \frac{\Pi_i \left(1 + \frac{2\mathcal{E}_i}{\alpha_{ni}} s + \frac{s^2}{\alpha_{ni}^2}\right)}{\Pi_i \left(1 + \frac{2\mathcal{E}_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}\right)}$$
  - ↳ μ : gain
  - ↳ T_i, τ_i : time constraints for zeroes / poles
  - ↳ p : # poles − # zeroes
  - ↳ $\mathcal{E}_i, e_i$: damping of zeroes/poles (response)
  - ↳ α_{ni}, ω_{ni} : natural freq of zeroes/poles (response)

- e.g. $G_a(s) = \mu$ → μ is constant







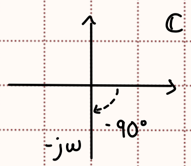
date

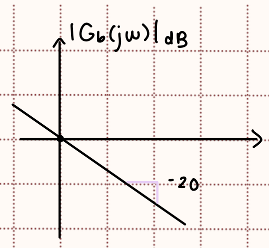
· e.g. $G_b(s) = \frac{1}{s}$

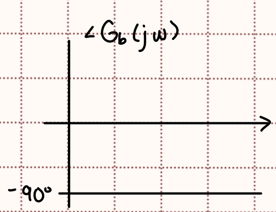
$$G_b(j\omega) = \frac{1}{j\omega}\left(\frac{j}{j}\right) = -\frac{j}{\omega}$$

$$|G_b(j\omega)|_{dB} = 20\log_{10}\left|-\frac{j}{\omega}\right| = 20\log_{10}\left|\frac{1}{\omega}\right| = -20\log_{10}|\omega|$$

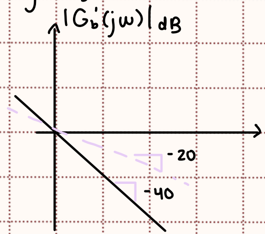
$$\angle G_b(j\omega) = \angle\frac{1}{j\omega} = \angle -\frac{j}{\omega} = -90°$$

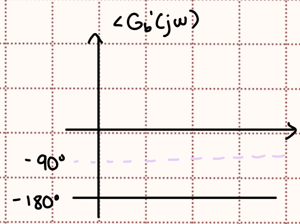






· e.g. $G_b(s) = \frac{1}{s^2} = \frac{1}{s}\left(\frac{1}{s}\right)$

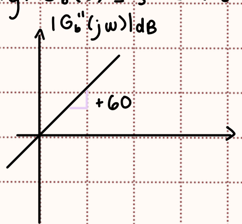


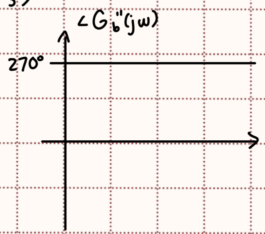


--- $\frac{1}{s}$

— $\frac{1}{s^2}$

· e.g. $G_b''(s) = s^3 = \left(\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}\right)^{-1}$



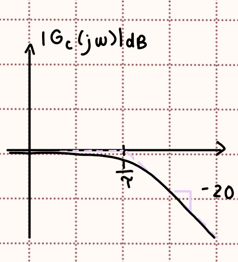


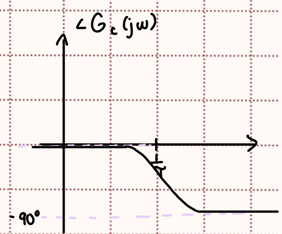
· e.g. $G_c(s) = \frac{1}{1+\tau s}$

$$|G_c(j\omega)|_{dB} = 20\log_{10}\left|\frac{1}{1+j\omega\tau}\right| = -20\log_{10}|1+j\omega\tau| = -20\log_{10}\sqrt{1+\omega^2\tau^2}$$

$$\approx \begin{cases} 0 & \omega \ll \frac{1}{\tau} \\ -20\log_{10}\omega\tau & \omega \gg \frac{1}{\tau} \end{cases}$$

$$\approx \begin{cases} 0 & \omega \ll \frac{1}{\tau} \\ -20\log_{10}\tau \underbrace{-20\log_{10}\omega}_{\text{slope of } -20} & \omega \gg \frac{1}{\tau} \end{cases}$$





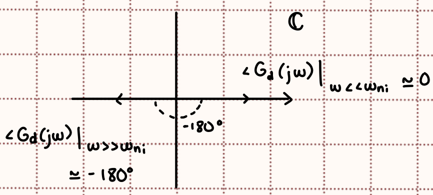
more bode plots

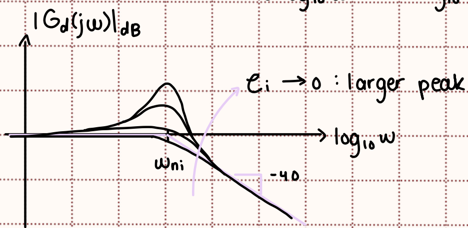
date

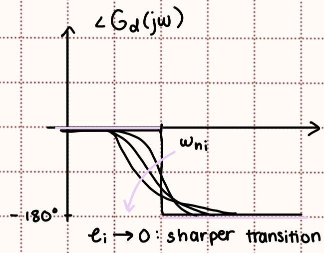
- e.g. $G_d(s) = \dfrac{1}{1 + \frac{2\zeta_i}{\omega_{ni}} s + \frac{s^2}{\omega_{ni}^2}}$

$$|G_d(j\omega)|_{dB} = -20\log_{10}\left|1 + \frac{2\zeta_i}{\omega_{ni}} j\omega - \frac{\omega^2}{\omega_{ni}^2}\right|$$
$$= -20\log_{10}\left|1 - \frac{\omega^2}{\omega_{ni}^2} + j\frac{2\zeta_i\omega}{\omega_{ni}}\right|$$
$$= -20\log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_{ni}^2}\right)^2 + \left(\frac{2\zeta_i\omega}{\omega_{ni}}\right)^2}$$
$$\approx \begin{cases} 0 & \omega \ll \omega_{ni} \\ -20\log_{10}\frac{\omega^2}{\omega_{ni}^2} & \omega \gg \omega_{ni} \end{cases}$$
$$= -20\log_{10}\omega^2 + 20\log_{10}\omega_{ni}^2$$
$$= -40\log_{10}\omega^2 + 20\log_{10}\omega_{ni}^2$$

$$\angle G_d(j\omega) = \angle\left(1 - \frac{\omega^2}{\omega_{ni}^2} + j\frac{2\zeta_i\omega}{\omega_{ni}}\right)$$

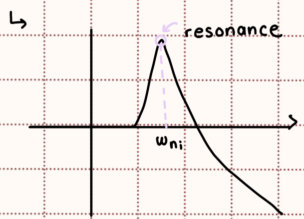






No peak in amplitude plot when $\zeta \geq \frac{\sqrt{2}}{2}$

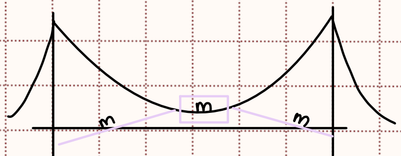
↳



- resonance freq: $\omega_r = \sqrt{1+\zeta^2}\ \omega_n$

- e.g. $G(s) = \dfrac{1}{ms^2 + \sigma s + k}$

$$= \frac{1}{k\left(1 + \frac{\sigma s}{k} + \frac{s^2}{k/m}\right)}$$
$$= \underbrace{\frac{1}{k}}_{\mu} \cdot \frac{1}{1 + \frac{\sigma}{k}s + \frac{s^2}{k/m}}$$



$$\omega_n = \sqrt{\frac{k}{m}}$$
$$\frac{2\zeta}{\omega_n} = \frac{\sigma}{k}$$
$$\zeta = \frac{1}{2}\frac{\sigma}{k}\sqrt{\frac{k}{m}}$$
$$= \frac{1}{2}\frac{\sigma}{\sqrt{km}}$$

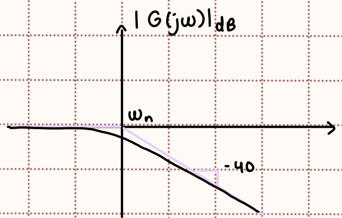
Giving numerical vals:

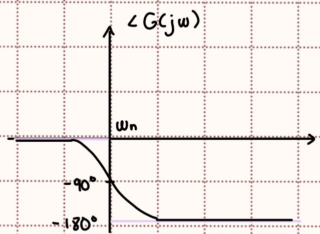
$m = 1$, $k = 1$, $\sigma = 2$ → $\omega_n = 1$ → $\log_{10}\omega_n = \log_{10} 1 = 0$

$\zeta = \frac{1}{2}\left(\frac{2}{1}\right) = 1$

$\mu = 1$

$$G(s) = \frac{1}{s^2 + 2s + 1}$$
$$= \frac{1}{(s+1)^2}$$



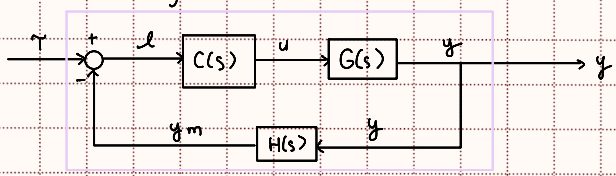


stability

date 09/30/2024

- $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$ $\quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

$G(j\omega)$ $\quad G(s) = C(sI-A)^{-1}B + D$



- $\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$ is
  - ↳ stable if $\forall x_0 \in \mathbb{R}^n$, $x(t) = e^{At}x_0$ is bounded
  - ↳ asymptotically stable if, in addition, $\lim_{t\to\infty} x(t) = 0$
  - ↳ unstable if not stable, meaning $\exists x_0 \in \mathbb{R}^n$ st $x(t)$ is not bounded

x is bounded if $\exists\, 0 < M < \infty$ st $\|x(t)\| < M \quad \forall t$

- $e^{at} = 1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots$
  - ↳ Taylor expansion
- $e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$

  $\in \mathbb{R}^{n\times n}$
- $x(t) = e^{At}x_0$

$$\dot{x} = \frac{d}{dt}e^{At}x_0$$
$$= \left(0 + A + 2\frac{A^2t}{2} + 3\frac{A^3t^2}{3\cdot 2} + \dots\right)x_0$$
$$= A\underbrace{\left(I + At + \frac{A^2t^2}{2} + \dots\right)}_{e^{At}}x_0$$

$$\dot{x} = Ax(t)$$

- $x(t)|_{t_0} = e^{At_0}x_0$

$$= e^{0}x_0 \qquad t_0 = 0$$
$$= Ix_0$$

↳ for $t_0 \neq 0$, $x(t) = e^{A(t-t_0)}x_0$

- thm: $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ is asymptotically stable $\iff Re(eig(A)) < 0$

- $G(s)$ is bounded input bounded output (BIBO) stable if $\forall$ bounded input signals u, output signal y is bounded
  - ↳ can't check all possible inputs
  - ↳ use thm instead: $G(s)$ is BIBO stable $\iff Re(poles(G(s))) < 0$
- thm: if $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(0) = x_0 \end{cases}$ is asymptotically stable $\Rightarrow G(s) = C(sI-A)^{-1}B + D$ is BIBO stable

asymptotic stability

date 10/02/2024

- asymptotic stability

  $\forall x_0 \in \mathbb{R}^n$, $x(t) = e^{At} x_0$ is bounded

  $x(t) \xrightarrow{t\to\infty} 0$

  $\Updownarrow$

  $Re(eig(A)) < 0$

$\Longrightarrow$

BIBO stability

$\forall$ bounded input signals (u), corresponding output signal (y) is bounded

$\Updownarrow$

$Re(poles(G)) < 0$

$G(s) = C(sI - A)^{-1}B + D$

- $G(s) = C(sI - A)^{-1} B + D$

  $= C \dfrac{adj(sI - A)}{det(sI - A)} B + D$

  ↳ $det(s^* I - A) \Leftrightarrow s^* \in eig(A)$

  ↳ $poles(G) \subseteq eig(A)$

asymptotically stable $\Rightarrow$ BIBO

$\not\Leftarrow$

- e.g. pendulum w/ friction

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -b \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [1 \ \ 0]\, x \end{cases}$$

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, $u = \dfrac{\tau}{m\ell^2}$, $y = \theta$

$det(\lambda I - A)$

$= det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ \frac{g}{\ell} & b \end{bmatrix}\right)$

$= det\left(\begin{bmatrix} \lambda & -1 \\ \frac{g}{\ell} & \lambda + b \end{bmatrix}\right)$

$= \lambda^2 + b\lambda + \frac{g}{\ell}$

$= 0$

$\lambda_{1,2} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{g}{\ell}} \longrightarrow Re(\lambda_{1,2}) < 0$

Pendulum w/ friction is asymp stable.

↓ implies BIBO stability

$G(s) = C(sI - A)^{-1} B + D$

$= [1 \ 0] \begin{bmatrix} s & -1 \\ \frac{g}{\ell} & s+b \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0$

$= [1 \ \ 0] \dfrac{\begin{bmatrix} s+b & 1 \\ -\frac{g}{\ell} & s \end{bmatrix}}{s^2 + sb + \frac{g}{\ell}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \dfrac{1}{s^2 + sb + \frac{g}{\ell}} [s+b \ \ \ 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \dfrac{1}{s^2 + sb + \frac{g}{\ell}}$

$poles(G) = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{g}{\ell}} < 0$

Asymp stable:

$\forall \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \end{bmatrix}$, $\begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} = e^{\begin{bmatrix} 0 & 1 \\ -g/\ell & -b \end{bmatrix} t} \begin{bmatrix} \theta_0 \\ \theta_0 \end{bmatrix}$ is bounded + as $t \to \infty$, $\longrightarrow 0$.

$\Downarrow$

BIBO stable:

$\forall$ bounds $\tau(t)$, corresponding $\theta(t)$ is bounded.

- e.g.

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = [1 \ \ 0]\, x \end{cases}$$

Not asymp stable b/c $eig(A) = -1, 2$ $(Re(2) > 0)$.

$G(s) = [1 \ \ 0] \begin{bmatrix} s+1 & 0 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$

$$= \frac{[1 \quad 0]}{(s+1)(s-2)} \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

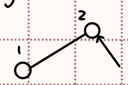
$$= \frac{s-2}{(s+1)(s-2)}$$

$$= \frac{1}{s+1}$$

BIBO stable:

poles(G) = -1

· e.g. network



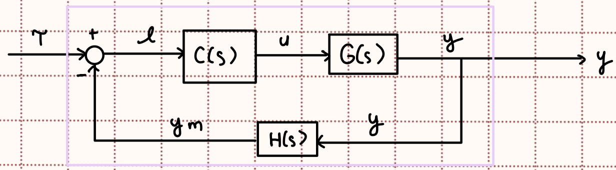
$$\begin{cases} \dot{x}_1 = x_2 - x_1 \\ \dot{x}_2 = x_1 - x_2 + u \\ y = x_1 - x_2 \end{cases} \longrightarrow \begin{cases} \dot{x} = \overbrace{-L}^{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \\ y = [1 \quad -1]\, x \end{cases}$$

Not asymp stable b/c A has linearly dependent rows → not full rank → null space is non-empty → at least 1 eig(A) = 0.

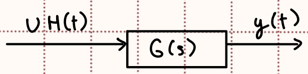
performance metrics

date 10/04/2024

- once a system is stable, talk abt performance

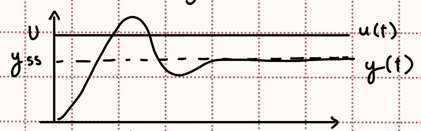


- performance metrics:



↳ steady state gain : $G(0) = \frac{y_{ss}}{U}$

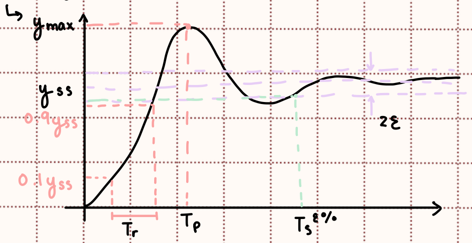
$$G(s) = \frac{\mu}{s^{\rho}} \frac{\pi(\;)}{\pi(\;)} \underbrace{\frac{\pi(\quad)}{\pi(\quad)}}_{s=0 \to 1}$$



$$y_{ss} = \lim_{t \to \infty} y(t)$$
$$= \lim_{s \to 0} s\, Y(s)$$
$$= \lim_{s \to 0} s\, G(s)\, U(s)$$
$$= \lim_{s \to 0} s\, G(s) \frac{U}{s}$$

$$y_{ss} = G(0)\, U$$

↳



↳ rise time : T_r

- time to go for 1st time from $0.1y_{ss}$ to $0.9y_{ss}$

↳ peak time : T_p

- time to reach y_{max}

↳ overshoot : OS%

- $\frac{y_{max} - y_{ss}}{y_{ss}} \cdot 100\%$

↳ settling time at ε% : $T_s^{\varepsilon\%}$

- time required to remain confined within ε% of y_{ss}

- first order systems:

$G(s) = \frac{\mu}{1+\tau s}$ → $\tau > 0,\ \mu > 0$

$$y(t) = \mathcal{L}^{-1}\{G(s)\,\mathcal{L}(H(t))\}$$
$$= \mathcal{L}^{-1}\left\{\frac{\mu}{1+\tau s} \cdot \frac{1}{s}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{A}{1+\tau s} + \frac{B}{s}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{\mu}{s} - \frac{\mu\tau}{1+\tau s}\right\}$$
$$= \mu\left(1 - e^{-\frac{t}{\tau}}\right)$$ → $t \geq 0$

$$A = \frac{\mu}{(1+\tau s)s}(1+\tau s)\Big|_{s=-\frac{1}{\tau}} = -\mu\tau$$

$$B = \frac{\mu}{(1+\tau s)s}(s)\Big|_{s=0} = \mu$$

calculating performance

date 10/07/2024

1) steady state gain
2) rise time
3) peak time
4) overshoot
5) settling time at $\varepsilon\%$

- first-order system: $G(s) = \frac{\mu}{1+\tau s}$

↳ $\tau > 0$, $\mu > 0$

↳ $y(t) = \mathcal{L}^{-1}\{G(s)\mathcal{L}\{H(t)\}\}$

$= \mu(1 - e^{-\frac{t}{\tau}})$ → $t \geq 0$

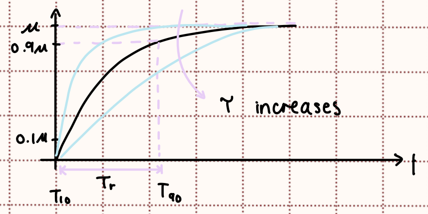
① $G(0) = \mu$

② $\mu(1 - e^{-\frac{T_{10}}{\tau}}) = 0.1\mu$

$T_{10} = -\tau \ln 0.9 =$

$T_{90} = -\tau \ln 0.1 =$

$T_r = T_{90} - T_{10} \simeq 2\tau$



③ + ④ : not defined

⑤ $\mu(1 - e^{-\frac{T_s^{\varepsilon\%}}{\tau}}) = (1-\varepsilon)\mu$

$T_s^{\varepsilon\%} = -\tau \ln \varepsilon$

- $\varepsilon = 0.05 \simeq 3\tau$
- $\varepsilon = 0.03 \simeq 4\tau$
- $\varepsilon = 0.01 \simeq 5\tau$

- second-order system: $G(s) = \frac{\mu\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

↳ $\omega_n > 0$

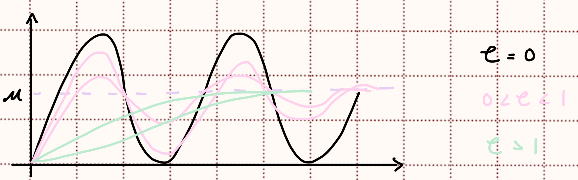
Poles:

$p^2 + 2\zeta\omega_n p + \omega_n^2 = 0$

$p_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$ → $\zeta \geq 0$ for stability

$y(t) = \mathcal{L}^{-1}\{G(s)\mathcal{L}\{H(t)\}\}$

$= \mu(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} (\sin(\sqrt{1-\zeta^2}\,\omega_n t + \arccos(\zeta))))$ → $t \geq 0$



- if $\zeta > 1$, behaviour is similar to 1st order system

$G(s) = \frac{\mu}{(1+\tau_1 s)(1+\tau_2 s)}$

↳ when $\zeta > 1$, then $\tau_1 \ll \tau_2$

- $p_1 = -\frac{1}{\tau_1}$
- $p_2 = -\frac{1}{\tau_2}$
- p_2 will dominate b/c it's closer to Im axis

① $G(0) = \mu$ → $\zeta > 0$

④ $OS\% = 100\, e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$

⑤ $T_s^{\varepsilon\%} \simeq -\frac{\ln\varepsilon}{\zeta\omega_n}$ → $0 < \zeta < 1$

higher-order systems

date 10/09/2024

· performance:

↳ 1^{st} order : $G(s) = \frac{u}{1+\Upsilon s}$

↳ 2^{nd} order: $G(s) = \frac{u\,\omega_n^2}{s^2 + \underbrace{2\epsilon\omega_n}_{\text{inversely proportional to settling time}} s + \omega_n^2}$

· e.g. 3rd order system

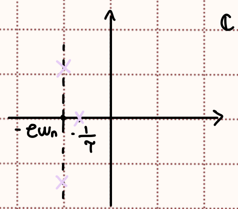
$$G(s) = \underbrace{\frac{1}{1+\Upsilon s}}_{\text{transient behaviour depends on } \Upsilon} \; \underbrace{\frac{1}{s^2+2\epsilon\omega_n + \omega_n^2}}_{\text{transient behaviour depends on } \epsilon\omega_n}$$

Settling time:

↳ $\propto \Upsilon$ for 1^{st} order component

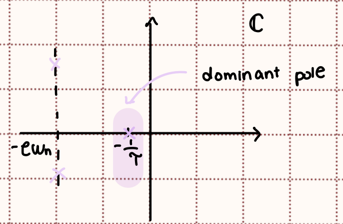
↳ $\propto \frac{1}{\epsilon\omega_n}$ for 2^{nd} order component

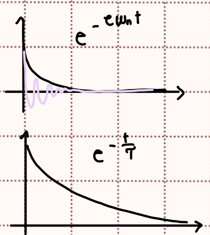
Case 1:



↳ not much analysis can be done by looking at separate components

Case 2:

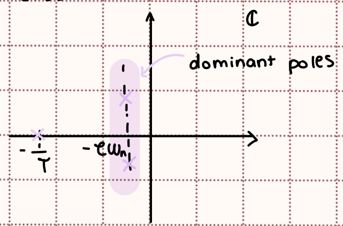


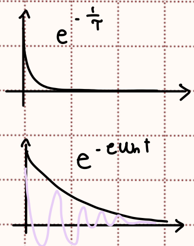


↳ $G(s) \simeq G_1(s)$

- can use 1^{st} order system to apx b/c $-\frac{1}{\Upsilon}$ pole dominates

Case 3:



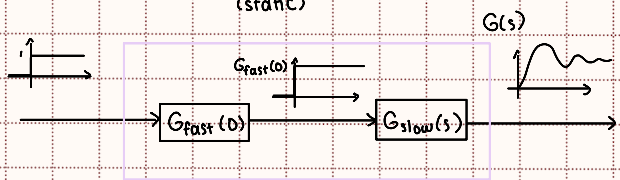


↳ $G(s) \simeq G_2(s)$

- can use 2^{nd} order system to apx

· in general, $G(s) = G_{fast}(s)\, G_{slow}(s)$

$$G(s) \simeq \underbrace{G_{fast}(0)}_{\text{constant (static)}} \; \underbrace{G_{slow}(s)}_{\text{dynamic}}$$



If $G(s) = G_{fast}(s)\, G_{slow}(s)$, then $G(s) \simeq G_{fast}(0)\, G_{slow}(s)$. Since G_{fast} decays a lot faster, we can just use its steady-state value. Using FVT $\lim_{t\to\infty} g_{fast}(t) = \lim_{s\to 0} s\, G_{fast}(s)$

system identification

date 10/11/2024

· steps for system identification:

$$u(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

1) Assume structure of model

$$G(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

$$G(s) = \frac{Y(s)}{U(s)}$$

$$\downarrow$$

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_1 y'(t) + a_0 y(t)$$
$$= b_m u^{(m)}(t) + \dots + b_1 u'(t) + b_0$$

2) Supply input signal $u(t)$ + read output signal $y(t)$

For any t_k:

$$y^{(n)}(t_k) = b_m u^{(m)}(t_k) + \dots + b_1 u'(t_k) + b_0 u(t_k) + a_{n-1}(-y^{(n-1)}(t_k)) + \dots + a_1(-y'(t_k)) + a_0(-y(t_k))$$

$$= [u^{(m)}(t_k), \dots, u'(t_k), u(t_k), -y^{(n-1)}(t_k), \dots, -y'(t_k), -y(t_k)] \begin{bmatrix} b_m \\ \vdots \\ b_1 \\ b_0 \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}$$

$$= d^T(t_k)\theta$$

3) For $k = 1, \dots, N$ and $N >> n+m+1$:

$$\underbrace{\begin{bmatrix} y^{(n)}(t_1) \\ y^{(n)}(t_2) \\ \vdots \\ y^{(n)}(t_N) \end{bmatrix}}_{Y \in \mathbb{R}^N} = \underbrace{\begin{bmatrix} — d^T(t_1) — \\ — d^T(t_2) — \\ \vdots \\ — d^T(t_N) — \end{bmatrix}}_{D \in \mathbb{R}^{N \times (n+m+1)}} \underbrace{\theta}_{\in \mathbb{R}^{n+m+1}}$$

To minimize θ using $\|Y - D\theta\|^2$:

$$\frac{d}{d\theta}\|Y - D\theta\|^2 = 0$$

$$2(Y - D\theta^*)^T D = 0 \longrightarrow \theta^* \text{ is solution we're looking for}$$

$$Y^T D - \theta^{*T} D^T D = 0$$

$$D^T Y - D^T D\theta^* = 0$$

$$\underbrace{D^T D}_{\in \mathbb{R}^{(n+m+1)\times(n+m+1)}}\theta^* = \underbrace{D^T Y}_{\in \mathbb{R}^{n+m+1}}$$

$$\theta^* = (D^T D)^{-1} D^T Y$$

$$= D^+ Y$$

$$= \begin{bmatrix} b_m^* \\ b_{m-1}^* \\ \vdots \\ b_1^* \\ b_0^* \\ a_{n-1}^* \\ \vdots \\ a_1^* \\ a_0^* \end{bmatrix} \longleftarrow \text{coeffs of best transfer fcn that'll map input to output}$$

↳ above allows for continuous time model identification from sample data

$$\frac{d}{dx}\|x\|^2$$
$$= \frac{d}{dx} x^T x$$
$$= x^T + x^T$$
$$= 2x^T$$

$$(\sqrt{x_1^2 + \dots + x_n^2})^2$$
$$= \left[\frac{\partial}{\partial x_1}\|x\|^2 \cdots \frac{\partial}{\partial x_n}\|x\|^2\right]$$
$$= [2x_1 \ \cdots \ 2x_n]$$
$$= 2x^T$$

date

e.g. $G(s) = \frac{10(0.1s+1)}{(10s+1)(s^2+s+1)}$

Zeroes:

$0.1s+1 = 0$

$s = -10 \rightarrow \log 10 = 1$

Poles:

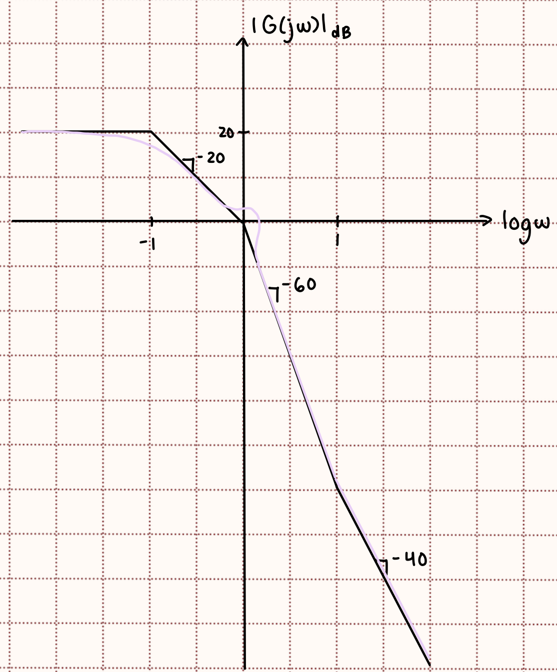
$10s+1 = 0$ $\qquad$ $s^2+s+1 = s^2 + 2\zeta\omega_n s + \omega_n^2$

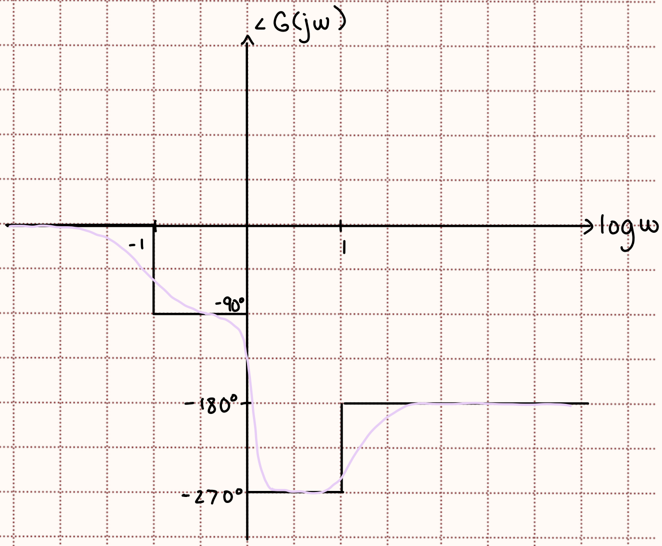
$s = -0.1 \downarrow \log 0.1 = -1$ $\qquad$ $\omega_n = 1 \downarrow \log 1 = 0$ $\qquad$ $1 = 2\zeta\omega_n$, $\frac{1}{2} = \zeta$

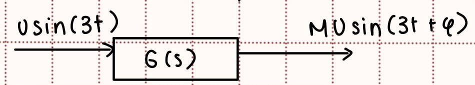
Start amp:

$|G(j\omega)|_{dB} = 20\log 10$

$= 20$







e.g. $\begin{cases} \dot{p}x = v\cos\theta \\ \dot{p}y = v\sin\theta \\ \dot{\theta} = \omega \end{cases}$

$x = \begin{bmatrix} px \\ py \\ \theta \end{bmatrix}, \quad u = \begin{bmatrix} v \\ \omega \end{bmatrix}, \quad y = \begin{bmatrix} px \\ py \end{bmatrix}$

$\dot{x} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} = \begin{bmatrix} u_1\cos x_3 \\ u_1 \sin x_3 \\ u_2 \end{bmatrix} = f(x,u)$ $\qquad$ $y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = h(x,u)$

$f(\bar{x},\bar{u}) = 0$ $\qquad$ $h(\bar{x},\bar{u}) = 0$

$\begin{bmatrix} u_1\cos x_3 \\ u_1\sin x_3 \\ u_2 \end{bmatrix} = 0$ $\qquad$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\bar{u}_1 = 0, \quad \bar{u}_2 = 0$

$\bar{x}_1, \bar{x}_2, \bar{x}_3$ are free

Let's pick $\bar{x} = [1 \quad -2 \quad \frac{\pi}{2}]^T$, $\bar{u} = [0 \quad 0]^T$

$\dot{x} = f(\bar{x},\bar{u}) + \frac{\partial f}{\partial x}(\bar{x},\bar{u})(x-\bar{x}) + \frac{\partial f}{\partial u}(\bar{x},\bar{u})(u-\bar{u}) + \text{hot}$

$= 0 + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_3} \\ \vdots & & \vdots \\ \frac{\partial f_3}{\partial x_1} & \cdots & \frac{\partial f_3}{\partial x_3} \end{bmatrix}(\bar{x},\bar{u})(x-\bar{x}) + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \end{bmatrix}(\bar{x},\bar{u})(u-\bar{u}) + \text{hot}$

date

$$= \begin{bmatrix} 0 & 0 & -\bar{u}_1 \sin\bar{x}_3 \\ 0 & 0 & \bar{u}_1\cos\bar{x}_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - (-2) \\ x_3 - \frac{\pi}{2} \end{bmatrix} + \begin{bmatrix} \cos\bar{x}_3 & 0 \\ \sin\bar{x}_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 - 0 \\ u_2 - 0 \end{bmatrix} + hot$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 + 2 \\ x_3 - \frac{\pi}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + hot$$

$\delta x = x - \bar{x}$

$\delta u = u - \bar{u}$

$\delta y = y - h(\bar{x}, \bar{u})$

$$\begin{cases} \delta\dot{x} \approx [0]\,\delta x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \delta u \\ \delta y = y - \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ \quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x + 0\,\delta u \end{cases}$$

$$y = h(\bar{x}, \bar{u}) + \frac{\partial h}{\partial x}(\bar{x}, \bar{u})(x - \bar{x}) + \frac{\partial h}{\partial u}(\bar{x}, \bar{u})(u - \bar{u}) + hot$$

$$= \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \cdots & \frac{\partial h_2}{\partial x_3} \end{bmatrix} (\bar{x}, \bar{u})(x - \bar{x}) + \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} \\ \frac{\partial h_2}{\partial u_1} & \frac{\partial h_2}{\partial u_2} \end{bmatrix} (\bar{x}, \bar{u})(u - \bar{u}) + hot$$

$$= \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} (x - \bar{x}) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u - \bar{u}) + hot$$

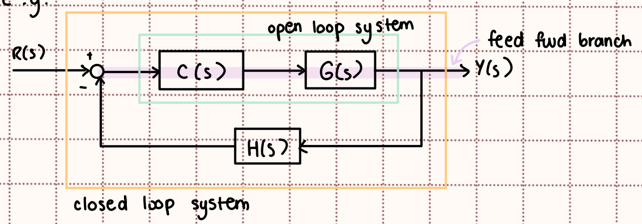
$$\delta y \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x + 0\,\delta u$$

Not asymptotically stable b/c eigenvalues are all 0. Not BIBO stable since not asymptotically stable.

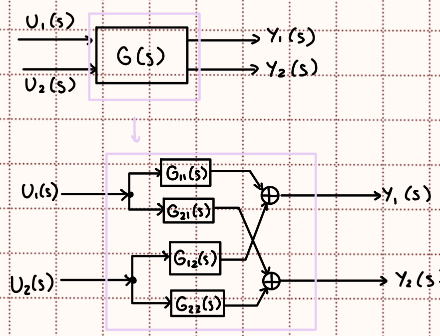
Block diagrams

date 10/23/2024

· e.g.



· e.g project

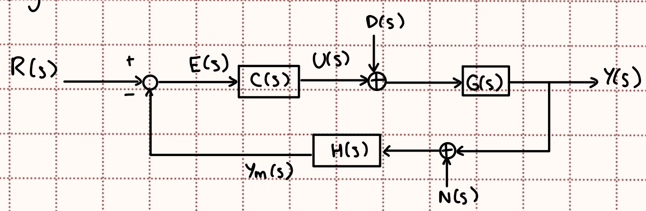


$Y_1(s) = G_{11}(s)U_1(s) + G_{12}(s)U_2(s)$

$Y_2(s) = G_{21}(s)U_1(s) + G_{22}(s)U_2(s)$

$$Y(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} = G(s)U(s)$$

· e.g.



Inputs: $\begin{bmatrix} R(s) \\ D(s) \\ N(s) \end{bmatrix}$ Outputs: $\begin{bmatrix} Y(s) \\ U(s) \\ E(s) \end{bmatrix}$

$$\begin{bmatrix} Y(s) \\ U(s) \\ E(s) \end{bmatrix} = \begin{bmatrix} \cdot & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} R(s) \\ D(s) \\ N(s) \end{bmatrix}$$

↳ effect of R(s) on Y(s) → $\frac{GC}{1+GCH}$

↳ effect of D(s) on Y(s) → $\frac{G}{1+GCH}$

↳ effect of N(s) on Y(s) → $-\frac{GCH}{1+GCH}$

↳ effect of R(s) on U(s)

↳ effect of R(s) on E(s)

$$\begin{aligned} Y(s) &= G(s)(U(s) + D(s)) \\ &= G(s)C(s)E(s) + G(s)D(s) \\ &= G(s)C(s)(R(s) - Y_m(s)) + G(s)D(s) \\ &= G(s)C(s)R(s) - G(s)C(s)H(s)(Y(s) + N(s)) + G(s)D(s) \\ &= G(s)C(s)R(s) - G(s)C(s)H(s)Y(s) - G(s)C(s)H(s)N(s) + G(s)D(s) \end{aligned}$$

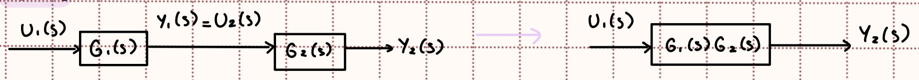
$$(1 + G(s)C(s)H(s))Y(s) = G(s)C(s)R(s) - G(s)C(s)H(s)N(s) + G(s)D(s)$$

$$Y(s) = \frac{G(s)C(s)}{1+G(s)C(s)H(s)}R(s) + \frac{G(s)}{1+G(s)C(s)H(s)}D(s) - \frac{G(s)C(s)H(s)}{1+G(s)C(s)H(s)}N(s)$$

date

· interconnections :

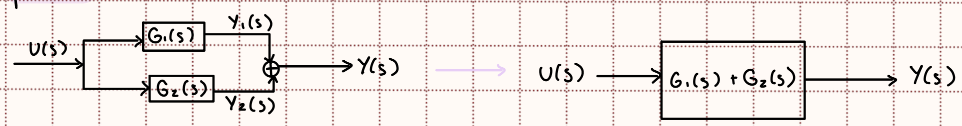
↳ series .



$$\frac{Y_2(s)}{U_1(s)} = \frac{Y_2(s)}{U_2(s)} \cdot \frac{Y_1(s)}{U_1(s)}$$

$$\frac{Y(s)}{U(s)} = G_1(s)\,G_2(s)$$

↳ parallel :



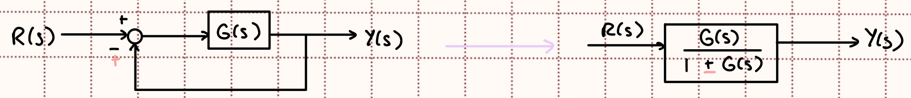
$$Y(s) = Y_1(s) + Y_2(s)$$

$$= G_1(s)U(s) + G_2(s)U(s)$$

$$= (G_1(s) + G_2(s))U(s)$$

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$

↳ feedback :



$$Y(s) = G(s)\,E(s)$$

$$Y(s) = G(s)\,(R(s) - Y(s))$$

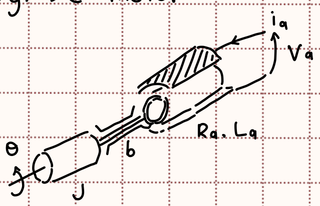
$$Y(s) = G(s)\,R(s) - G(s)\,Y(s)$$

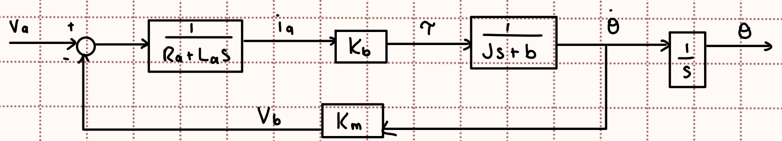
$$Y(s)(1 + G(s)) = G(s)R(s)$$

$$Y(s) = \frac{G(s)R(s)}{1 + G(s)}$$

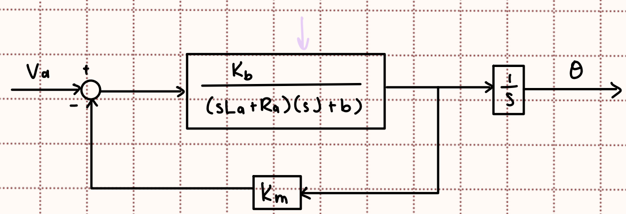
$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

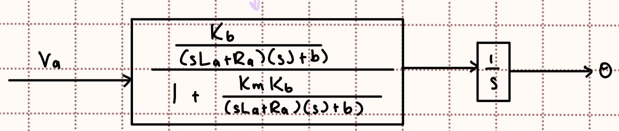
· e.g. DC motor





$$s\theta(s) = \dot{\theta}(s)$$
$$\theta(s) = \tfrac{1}{s}\dot{\theta}(s)$$





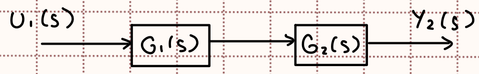
date

$$V_a \longrightarrow \boxed{\dfrac{K_b}{\left((sL_a+R_a)(sJ+b)+K_aK_b\right)s}} \longrightarrow \theta$$

interconnected systems stability

date 10/25/2024

· e.g.



$G_1(s) = \dfrac{N_1(s)}{D_1(s)}$ — numerator / denominator

$G_2(s) = \dfrac{N_2(s)}{D_2(s)}$

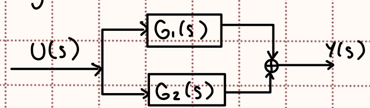
$\dfrac{Y_2(s)}{U_1(s)} = G(s) = G_1(s)\,G_2(s)$

$= \dfrac{N_1(s)N_2(s)}{D_1(s)D_2(s)}$

G_1	G_2	G
stable	stable	stable
stable	not stable	not stable
not stable	stable	not stable
not stable	not stable	not stable

assuming no cancellations

· e.g.



$G(s) = \dfrac{Y(s)}{U(s)}$

$= G_1(s) + G_2(s)$

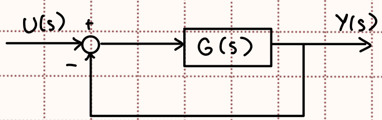
$= \dfrac{N_1(s)}{D_1(s)} + \dfrac{N_2(s)}{D_2(s)}$

$= \dfrac{N_1(s)D_2(s) + N_2(s)D_1(s)}{D_1(s)D_2(s)}$

G_1	G_2	G
s	s	s
s	¬s	¬s
¬s	s	¬s
¬s	¬s	¬s

assuming no cancellations

· e.g.



$F(s) = \dfrac{Y(s)}{U(s)}$

$= \dfrac{G(s)}{1+G(s)}$

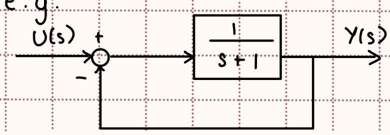
$= \dfrac{\frac{N(s)}{D(s)}}{1+\frac{N(s)}{D(s)}}$

$= \dfrac{N(s)}{D(s)+N(s)}$

G	F
s	?
¬s	

set $D(s)+N(s)=0$ to find poles, then determine stability

↳ e.g.



date

$$F(s) = \frac{\frac{1}{s+1}}{1+\frac{1}{s+1}}$$

$$= \frac{1}{s+1}\left(\frac{s+1}{s+1+1}\right)$$

$$= \frac{1}{s+2}$$

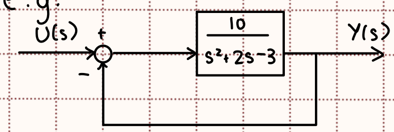
Poles:

$s+2=0$

$s=-2$

Since $Re(s)<0$, system is BIBO stable.

↳ e.g.



Poles:

$0 = s^2+2s-3$

$0 = (s+3)(s-1)$

$s=-3,\ s=1$

Since $Re(s=1)>0$, open-loop system isn't BIBO stable.

$$F(s) = \frac{10}{10+s^2+2s-3}$$

$$= \frac{10}{s^2+2s+7}$$

Poles:

$s^2+2s+7=0$

$$s = \frac{-2\pm\sqrt{2^2-4(1)(7)}}{2(1)}$$

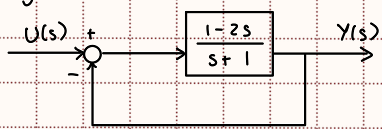
$$= -1\pm\sqrt{4-28}$$

$$= -1\pm\sqrt{-24}$$

$$= -1\pm j2\sqrt{6}$$

Since $Re(s)<0$, closed-loop system is BIBO stable.

↳ e.g.



Open-loop system is stable.

$$F(s) = \frac{1-2s}{1-2s+s+1}$$

$$= \frac{1-2s}{2-s}$$

$$= \frac{2s-1}{s-2}$$

Poles:

$s-2=0$

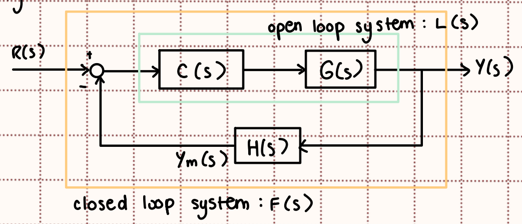
$s=2$

Since $Re(s)>0$, closed-loop system is unstable.

Routh - Hurwitz

date 10/28/2024

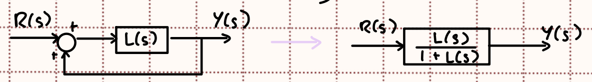
- e.g.



↳ $L(s) = C(s)G(s)$

↳ $F(s) = \frac{L(s)}{1+L(s)H(s)}$

- apx H(s) by 1 by assuming measuring is fast + precise



↳ if L(s) is stable, F(s) can be stable / unstable

↳ if L(s) is unstable, F(s) can be stable / unstable

- Routh - Hurwitz criterion figures out properties of $1 + L(s)$
  - ↳ since closed-loop system transfer fcn is $F(s) = \frac{L(s)}{1+L(s)}$, can tell us smth abt stability
  - ↳ design algo to check whether roots of polynomial are in LHS of complex plane ($\mathbb{C}^-$) w/o computing them
  - ↳ consider polynomial : $\pi(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$, $a_i \in \mathbb{R}$
    - $\pi(s)$ is Hurwitz if all roots are in $\mathbb{C}^-$
    - CLS stable $\iff$ $1 + L(s)$ is Hurwitz
    - $\pi(s) = (s-\lambda_1)(s-\lambda_2)\dots(s-\lambda_r)(s-\mu_1)(s-\bar{\mu}_1)\cdots(s-\mu_p)(s-\bar{\mu}_p)$
      - → r real roots $\lambda_1, \dots, \lambda_r$
      - → p complex conjugate pairs of roots $\mu_1, \bar{\mu}_1, \dots, \mu_p, \bar{\mu}_p$
      - → $\pi(s)$ Hurwitz $\iff$ $\lambda_i < 0$ for $i \le r$, $Re(\mu_i) < 0$ for $i \le p$
        - $(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_r)$ is polynomial w/ all +ve coeffs
        - $(s-\mu_i)(s-\bar{\mu}_i) = s^2 - (\mu_i + \bar{\mu}_i)s + \mu_i\bar{\mu}_i$

$$= s^2 \underbrace{- 2\underbrace{Re(\mu_i)}_{<0}}_{>0} s + \underbrace{(\mu_i)^2}_{>0}$$

      - → $\pi(s)$ Hurwitz $\Rightarrow$ $\pi(s)$ has all +ve coeffs

e.g. $s^4 + 3s^3 - 2s^2 + 5s + 6$

↳ Hurwitz ?

No b/c $a_2 < 0$

- e.g. $s^3 + 4s + 6$

↳ Hurwitz?

No b/c $a_2 = 0$

- e.g. $s^3 + 5s^2 + 9s + 1$

↳ Hurwitz ?

Don't know b/c all +ve coeffs isn't sufficient

- Routh's algo using $\pi(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$

1) build following table:

date

s^n	1	a_{n-2}	a_{n-4}	...
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	...
$\vdots$	$r_{2,0}$	$r_{2,1}$	$r_{2,2}$	...
	$r_{3,0}$	$r_{3,1}$	$r_{3,2}$	...
s^2	$\vdots$			
s^1	$r_{n-1,0}$			
s^0	$r_{n,0}$			

$$r_{2,0} = -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$r_{2,1} = -\frac{1}{a_{n-1}} \begin{vmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

same

always 1^{st} col   next col

$$r_{3,0} = -\frac{1}{r_{2,0}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ r_{2,0} & r_{2,1} \end{vmatrix}$$

$$r_{3,1} = -\frac{1}{r_{2,0}} \begin{vmatrix} a_{n-1} & a_{n-5} \\ r_{2,0} & r_{2,2} \end{vmatrix}$$

↳ stop along each row when we get 0

↳ terminate when get 0 in 1^{st} col

2) thm (Routh - Hurwitz criterion):

↳ $\pi(s)$ is Hurwitz $\Leftrightarrow$ all elmts in Routh arr (1^{st} col of Routh table) have same sign

↳ if Routh array has no 0s, then:

- # sign changes = # bad roots (i.e. non -ve real part)
- $\nexists$ roots on Im axis (i.e. all bad roots have +ve real parts)

· e.g. generic 2^{nd}-order polynomial: $\pi(s) = a_2 s^2 + a_1 s + a_0$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 1$

1)

s^2	1	a_0	0
s^1	a_1	0	0
s^0	a_0	0	

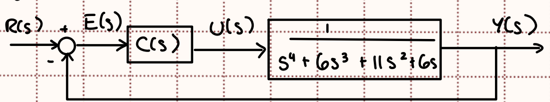
$$-\frac{1}{a_1}\begin{vmatrix} 1 & a_0 \\ a_1 & 0 \end{vmatrix} = -\frac{1}{a_1}(1(0) - a_1 a_0)$$
$$= -\frac{1}{a_1}(-a_1 a_0)$$
$$= a_0$$

$$-\frac{1}{a_1}\begin{vmatrix} 1 & 0 \\ a_1 & 0 \end{vmatrix} = -\frac{1}{a_1}(a_1(0) - 1(0))$$
$$= 0$$

$s^2 + a_1 s + a_0$ has roots w/ -ve real parts $\Leftrightarrow$ $a_1, a_0 > 0$ (i.e. no sign changes in Routh arr)

↳ $G(s) = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$ is BIBO stable $\Leftrightarrow$ $a_1, a_0 > 0$

· e.g. 1^{st} closed loop control design



Open loop:

$s^4 + 6s^3 + 11s^2 + 6s = 0$

↳ Hurwitz?

No b/c $a_0 = 0$

↳ root at 0

Closed loop: $C(s) = K$

$U(s) = K E(s)$

↳ proportional controller b/c U is proportional to E

$$F(s) = \frac{L(s)}{1 + L(s)}$$
$$= \frac{C(s)\,G(s)}{1 + C(s)\,G(s)}$$

$$1 + C(s)\,G(s) = 1 + \frac{K}{s^4 + 6s^3 + 11s^2 + 6s}$$

$$0 = \frac{\boxed{s^4+6s^3+11s^2+6s+K}^{\pi(s)}}{s^4+6s^3+11s^2+6s}$$

Want $\pi(s)$ to be Hurwitz $\iff$ denom of CL transfer fcn has roots in $\mathbb{C}^-$ (i.e. CLS is stable)

Routh algo:

1)

s^4	1	11	K	0
s^3	6	6	0	0
s^2	10	K	0	
s^1	$6-\frac{3}{5}K$	0		
s^0	K	0		

right col is 0 vector

$-\frac{1}{6}\begin{vmatrix} 1 & 11 \\ 6 & 6 \end{vmatrix} = -\frac{1}{6}(6-66) = 10$

$-\frac{1}{6}\begin{vmatrix} 1 & K \\ 6 & 0 \end{vmatrix} = -\frac{1}{6}(0-6K) = K$

$-\frac{1}{6}\begin{vmatrix} 1 & 0 \\ 6 & 0 \end{vmatrix} = 0$

$-\frac{1}{10}\begin{vmatrix} 6 & 6 \\ 10 & K \end{vmatrix} = -\frac{1}{10}(6K-60) = 6-\frac{3}{5}K$

$-\frac{1}{6-\frac{3}{5}K}\begin{vmatrix} 10 & K \\ 6-\frac{3}{5}K & 0 \end{vmatrix} = -\frac{1}{6-\frac{3}{5}K}\left(0-(6-\frac{3}{5}K)K\right) = K$

2) Routh arr: $\begin{bmatrix} 1 \\ 6 \\ 10 \\ 6-\frac{3}{5}K \\ K \end{bmatrix}$

$\pi(s)$ Hurwitz $\iff \begin{cases} 6-\frac{3}{5}K > 0 \\ K > 0 \end{cases}$

$\iff \begin{cases} K < 10 \\ K > 0 \end{cases}$

$\iff 0 < K < 10$

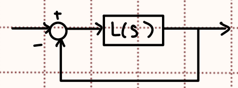
$F(s) = \frac{Y(s)}{R(s)}$ is stable w/ $C(s) = K \iff 0 < K < 10$.

nyquist criterion

date 11/1/2024

· closed-loop stability criteria :

↳ Routh-Hurwitz



$F(s) = \frac{L(s)}{1+L(s)}$

∘ $1 + L(s)$ is Hurwitz ⟺ CLS stable

↳ Nyquist criterion

· fcn that maps from complex to complex is $f: \mathbb{C} \to \mathbb{C}$ $(f: x \to f(x))$

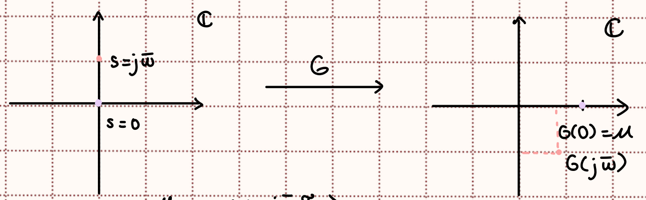
↳ e.g. transfer fcn

$G: \mathbb{C} \to \mathbb{C}$

$: s \to G(s)$

↳ e.g. $G(s) = \frac{\mu}{1+\tau s}$

∘ $\mu, \tau > 0$



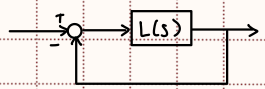
• $G(j\bar{\omega}) = \frac{\mu}{1+j\bar{\omega}\tau}\left(\frac{1-j\bar{\omega}\tau}{1-j\bar{\omega}\tau}\right)$

$= \frac{\mu - j\mu\bar{\omega}\tau}{1+\bar{\omega}^2\tau^2}$

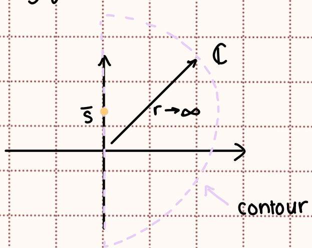
$= \underbrace{\frac{\mu}{1+\bar{\omega}^2\tau^2}}_{>0} - j\underbrace{\frac{\mu\bar{\omega}\tau}{1+\bar{\omega}^2\tau^2}}_{>0}$ → +ve Re part , -ve Im part

· Nyquist criterion needs :

↳ open loop transfer fcn $L(s)$

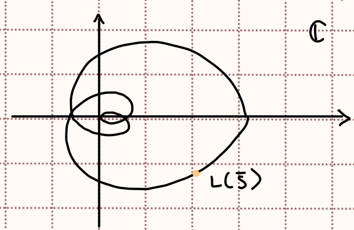


↳ Nyquist contour



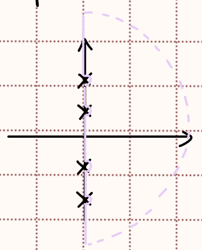
∘ r is radius

· Nyquist plot is img of Nyquist contour through $L(s)$



date

↳ if $L(s)$ has poles p_i st $Re(p_i) = 0$, Nyquist contour does infinitesimal adjustments around those poles

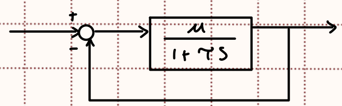


↳ $L(-j\omega) = \overline{L(j\omega)}$

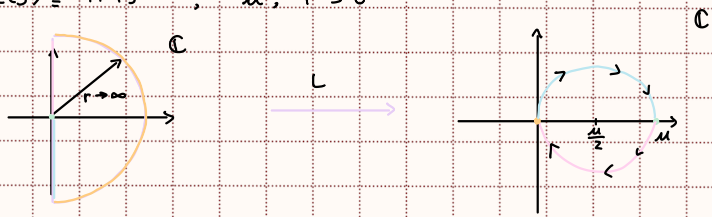
- symmetric wrt real line

↳ if $L(s)$ is strictly proper $\Rightarrow \lim_{|\omega| \to \infty} L(j\omega) = 0$

· e.g.



$L(s) = \frac{\mu}{1+\tau s}$, $\mu, \tau > 0$



$$L(j\omega) = \frac{\mu}{1+j\omega\tau}\left(\frac{1-j\omega\tau}{1-j\omega\tau}\right)$$
$$= \underbrace{\frac{\mu}{1+\omega^2\tau^2}}_{>0} - j\,\frac{\mu\omega\tau}{1+\omega^2\tau^2}$$

$$Im(L(j\omega)) \begin{cases} <0 , & \omega > 0 \\ >0 , & \omega < 0 \end{cases}$$

- $|L(s)| \xrightarrow{|s| \to \infty} 0$

· Nyquist criterion :

↳ let P be # poles of $L(s)$ w/ +ve real part

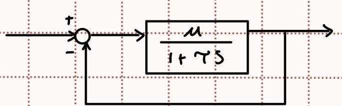
↳ let N be # loops Nyquist plot makes around the part $-1 \in \mathbb{C}$

- >0 if CCW
- <0 if CW
- undefined if Nyquist plot goes through -1

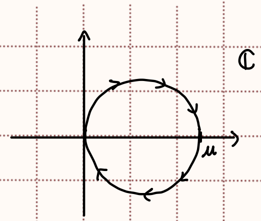
↳ CLS is stable $\iff N = P$

- if N is undefined, CLS may be stable or unstable
- if N is well defined + $N \neq P$, CLS is unstable
- $P - N$ = # poles of CLS that have +ve real part

· e.g.



$L(s) = \frac{\mu}{1+\tau s}$, $\mu, \tau > 0$



P = # open loop +ve poles = 0 ⟶ $p_1 = -\frac{1}{\tau} < 0$

N = # loops of Nyquist plot around -1 = 0

By Nyquist criterion, CLS is stable since N = P.

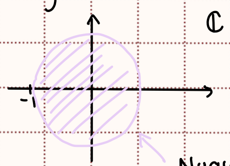
To check:

$$F(s) = \frac{L(s)}{1+L(s)} = \frac{\mu}{1+\tau s+\mu} = \frac{\mu}{1+\mu+\tau s}$$

$$\underbrace{p_1 = -\frac{1+\mu}{\tau}}_{<0}$$

· corollary: if $L(s)$ stable (i.e $P=0$), then:

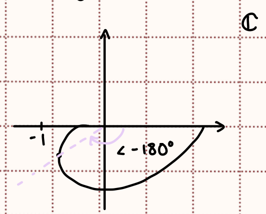
↳ $|L(j\omega)| < 1 \;\; \forall \omega \Rightarrow$ CLS stable



Nyquist plot contained in this disk

- N = 0

↳ $|\angle L(j\omega)| < 180° \;\; \forall \omega \Rightarrow$ CLS stable

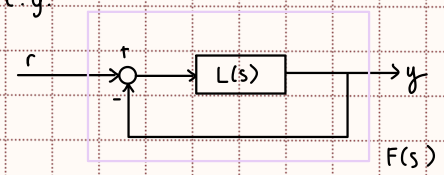


- N = 0

Bode criterion

date 11 / 1 / 2024

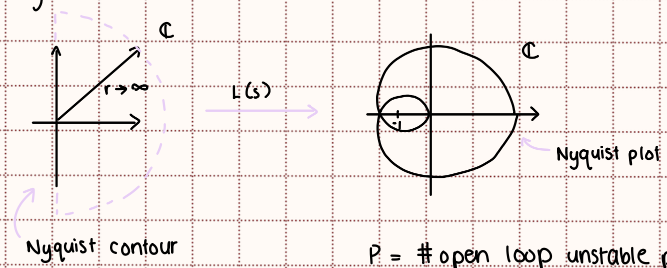
- e.g.



$F(s) = \dfrac{L(s)}{1 + L(s)}$

↳ Hurwitz $\iff$ CLS stable

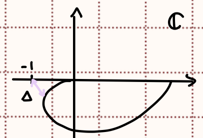
- e.g.



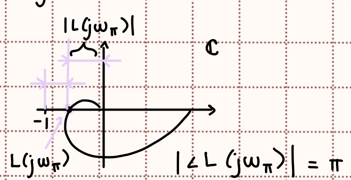
P = # open loop unstable poles

N = # loops of Nyquist plot around -1

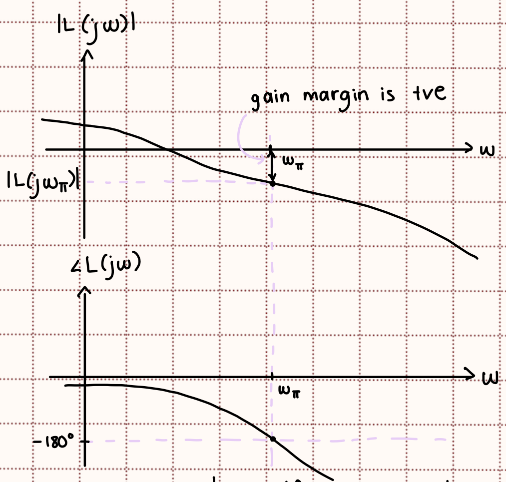
- Bode criterion



↳ 1st way: analyzing gain margin, which is amount of gain that can be increased before system becomes unstable

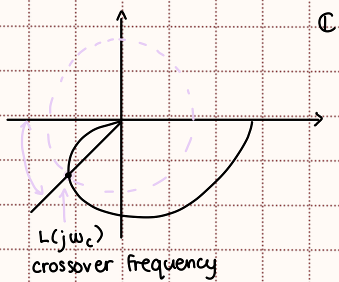


- L(s) stable
- Nyquist plot of L(s) intersects -ve real axis only once

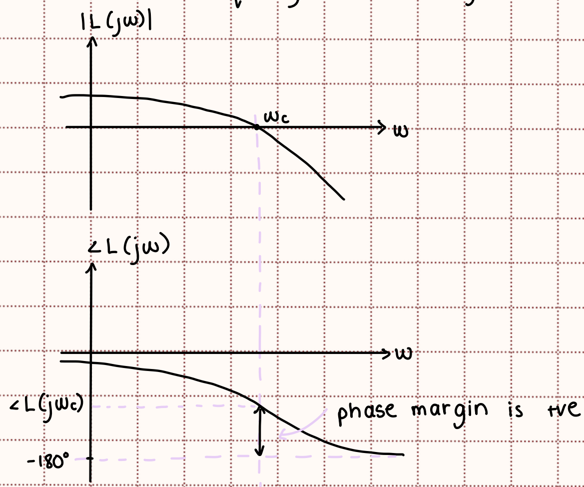


- gain margin $\begin{cases} \text{+ve if } |L(j\omega_\pi)|_{dB} < 0_{dB} \rightarrow \text{good} \\ \text{-ve if } |L(j\omega_\pi)|_{dB} > 0_{dB} \rightarrow \text{bad b/c already unstable} \end{cases}$

↳ 2^{nd} way: analyzing phase gain, which is amount of additional phase lag that can be tolerated before system becomes unstable



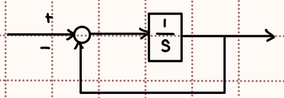
- $L(s)$ stable
- Nyquist plot of $L(s)$ crosses unit circle only once, from outside to inside
- crossover frequency is when $|L(j\omega_c)| = 1$ or $|L(j\omega_c)|_{dB} = 0$

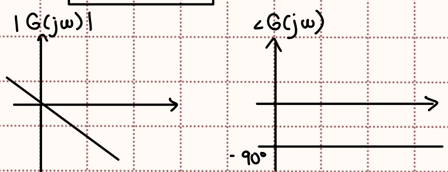


- phase margin $\begin{cases} +ve & \text{if } 180° - |\angle L(j\omega_c)| > 0 \\ -ve & \text{if } 180° - |\angle L(j\omega_c)| < 0 \end{cases}$ → good / → bad

· e.g. single integrator

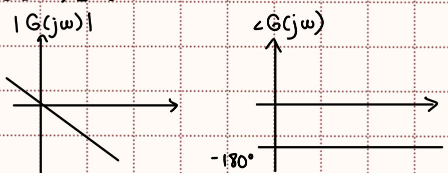
$G_1(s) = \frac{1}{s}$



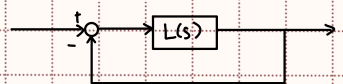


· e.g. double integrator

$G_2(s) = \frac{1}{s^2}$



· Bode criterion



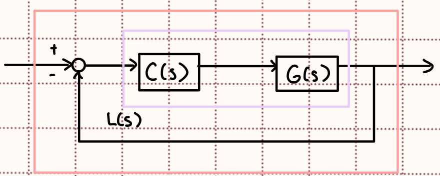
↳ L(s) has +ve gain $\mu > 0$, phase margin $\varphi_m > 0$, + gain margin $K_m > 0$, then
$F(s) = \frac{L(s)}{1 + L(s)}$ is stable

↳ can't use when a pole has +ve real part

stability analysis using bode

date 11/4/2024

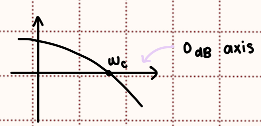
- to determine stability of system:
  - ↳ Routh-Hurwitz
  - ↳ Nyquist
  - ↳ Bode



F(s)

↳ $L(s) = \frac{\mu}{s^p} \frac{\Pi_i(\ )\,\Pi_i(\ )}{\Pi_j(\ )\,\Pi_j(\ )}$

- Bode criterion :
  - ↳ if :
    - $L(s)$ has no poles w/+ve real parts
    - $|L(j\omega)|_{dB}$ crosses 0_{dB} axis only once, from above to below

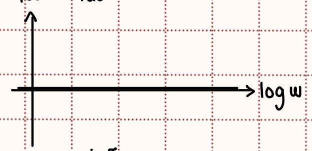


↳ then $\mu > 0$, $\varphi_m > 0 \iff F(s) = \frac{L(s)}{1+L(s)}$ is stable
- μ is gain margin
- φ_m is phase margin

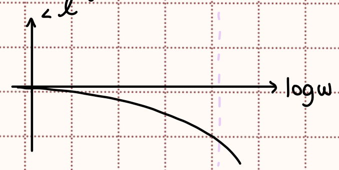
- loop shaping : translate control specs of closed-loop system into constraints of Bode plots of open-loop system

1) stability of F(s)
   - ↳ $K_m > 0$, $\varphi_m > 0$
   - ↳ $y(t) \rightarrow \boxed{e^{-\tau s}} \rightarrow y(t-\tau)$
     delay

$|e^{-j\omega\tau}|_{dB}$

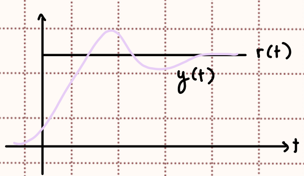


$\angle e^{-j\omega\tau}$



2) robust stability :
   - ↳ high K_m, high φ_m

3) steady-state gain :



$F(0) \rightarrow 1$ (want ratio of $\frac{Y(s)}{R(s)} = 1$)

$F(s) = \frac{L(s)}{1+L(s)}$

date

$$\lim_{s\to 0} F(s) = \lim_{s\to 0} \frac{\mu/s^{\rho}}{1+\mu/s^{\rho}}$$

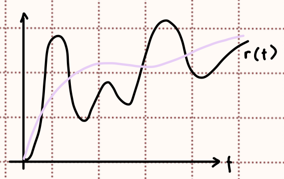
$$= \lim_{s\to 0} \frac{\mu}{\mu+s^{\rho}}$$

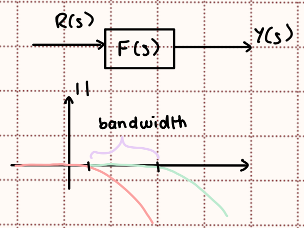
$$= \begin{cases} \frac{\mu}{\mu+1} & , \rho = 0 \\ 1 & , \rho > 0 \end{cases}$$

more specifications for stability

date 11/6/2024

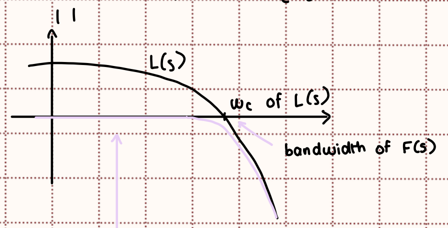
4) dynamic performance :





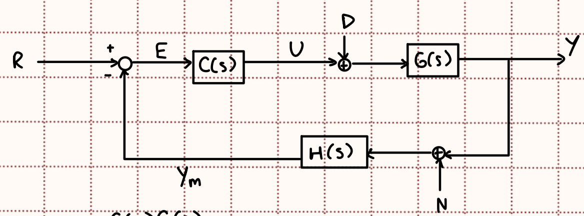
$F(s) = \frac{L(s)}{1+L(s)}$

$F(s) \simeq \begin{cases} 1, & \omega \ll \omega_c \text{ of } L(s) \\ L(s), & \omega \gg \omega_c \text{ of } L(s) \end{cases}$



F(s) is apx using above piecewise

↳ lower bound on ω_c of L(s)



$Y(s) = \frac{C(s)G(s)}{1+C(s)G(s)} R(s) + \frac{G(s)}{1+C(s)G(s)} D(s) - \frac{C(s)G(s)}{1+C(s)G(s)} N(s)$

↳ F(s)

↳ disturbance signals contain bw freq

↳ noise signals contain high freq

5) reject disturbance : want $\left| \frac{G(j\omega)}{1+C(j\omega)G(j\omega)} \right|$ to be low for $\underbrace{\omega \text{ contained in } D}_{\text{low freq}}$

6) attenuate noise : want $\left| \frac{C(j\omega)G(j\omega)}{1+C(j\omega)G(j\omega)} \right|$ to be low for $\underbrace{\omega \text{ contained in } N}_{\text{high freq}}$

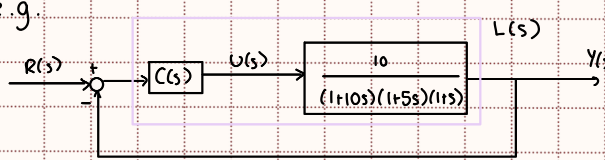
↳ upper bound on ω_c of L(s)

7) in $L(s) = C(s)G(s)$, C(s) must be proper in order to be realizable in state space form

↳ slope of $|L(j\omega)| \leq$ slope of $|G(j\omega)|$ as $\omega \to \infty$

↳ $\left. \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \right|_{|s| \to \infty} = \frac{b_m s^m}{s^n}$

e.g.



Specs :

1) $|e_\infty| \leq 0.1$ in response to step reference

date

2) $\omega_c \geq 0.2$

3) $\varphi_m \geq 60°$

To satisfy 1:

↳ choose $C(s) = \mu$:

$L(s) = \frac{10\mu}{(1+10s)(1+5s)(1+s)}$

$F(0) = \frac{10\mu}{1+10\mu} \geq 0.9 \longrightarrow \mu \geq 0.9$

↳ choose $C(s) = \frac{H}{s}$:

$F(0) = 1$ for any H

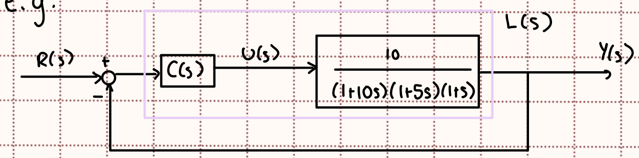
· turning specs from time domain behaviour of $F(s)$ to freq domain behaviour of $L(s)$:

Time domain spec of $F(s)$	Freq domain spec of $L(s)$
Stability	· +ve gain margin $k_m > 0$ · +ve phase margin $\varphi_m > 0$ · no cancellations in computation of $L(s) = C(s)G(s)$
Robust stability	higher $k_m, \varphi_m \rightarrow$ more robust · upper bound on ω_c ↳ time delay τ contributes to $-\omega_c\tau$ phase shift at ω_c
Static performance ↳ $\lim_{s \to s} s\frac{F(s)}{s} = F(0) = \frac{\mu}{s^p + \mu}$ ↳ want $F(0)$ as close to 1 as possible ↳ $\begin{cases} \text{higher } \mu, & p = 0 \\ 1, & p > 0 \end{cases}$	· high magnitude at low freqs · $\lvert L(j\omega)\rvert \xrightarrow{\omega \to 0} \infty$ so $F(0) = 1$ ↳ i.e. 0 steady-state error
Dynamic performance ↳ track fast reference trajectories, not just regulate to constant refs	· lower bound on ω_c ↳ as $\lvert F(j\omega)\rvert \simeq 1$ for $\omega < \omega_c$, while $\omega > \omega_c$ are attenuated
Disturbance rejection ↳ $\frac{Y(s)}{D(s)} = \frac{G(s)}{1 + C(s)G(s)}$ must attenuate freqs characterizing disturbance (typically at low freqs)	· $\lvert L(j\omega)\rvert \gg 1$ for range of ω characterizing disturbance (typically low freqs) ↳ i.e. lower bound on ω_c
Noise attenuation ↳ $\frac{Y(s)}{N(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}$ must attenuate freqs characterizing noise (typically at high freqs)	· $\lvert L(j\omega)\rvert \ll 1$ for range of ω characterizing noise (typically low freqs) ↳ i.e. upper bound on ω_c
Realizability of controller ↳ $C(s)$ must be proper	· slope of $\lvert L(j\omega)\rvert \leq$ slope of $\lvert G(j\omega)\rvert$ as $\omega \to \infty$

designing a controller

date 11/8/2024

· e.g.



Specs:

1) $|e_\infty| \leq 0.1$ in response to step reference
   - static precision using steady-state error
2) $\omega_c \geq 0.2$
   - speed
3) $\varphi_m \geq 60°$
   - stability

1) $C(s) = \frac{\mu}{s^\rho}$

If $\rho > 0$, $|e_\infty| = 0$, $F(0) = 1$

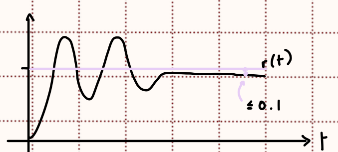
If $\rho = 0$ + μ high, $F(0) \xrightarrow{\mu \to \infty} 1$

$$F(0) = \frac{L(0)}{1+L(0)} = \frac{C(0)G(0)}{1+C(0)G(0)} = \frac{\mu 10}{1+\mu 10} \geq 0.9$$

$$\downarrow$$

$$\mu \geq 0.9$$

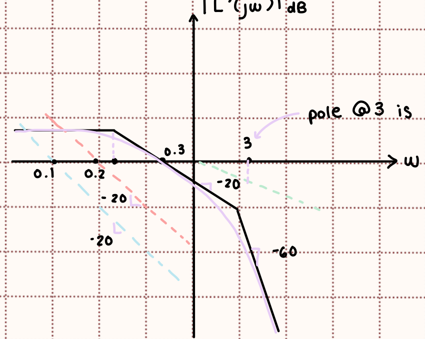
To make a system realizable, where controller $C(s) = \frac{L^*(s)}{G(s)}$, must have same degree on numerator + denominator. To do this, add poles to C(s).



Choose $\mu = 1$.

2) $L'(s) = 1 \frac{10}{(1+10s)(1+5s)(1+s)}$ → $L'(s)$ is $L(s)$ w/ chosen $\mu = 1$

Since we want $\omega_c \geq 0.2$, choose $\omega_c = 0.3$ → pole @ 0.3

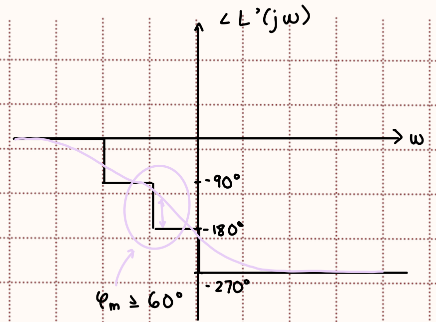


pole @3 is a dec away from 0.3 so there's no effect on phase plot around 0.3

To make C(s) proper, need 2 more poles other than 0.3 to cancel out $(1+10s)(1+5s)(1+s)$.

Choose 1st-order pole @ 0.3 + 2nd-order pole @ 3.

3) Since poles @ 3 are far away enough from 0.3, phase margin is close to 90°.



$$L^*(s) = \frac{10}{(1+\frac{s}{0.03})(1+\frac{s}{3})^2} = C(s)$$

$$C(s) = \frac{L^*(s)}{G(s)}$$

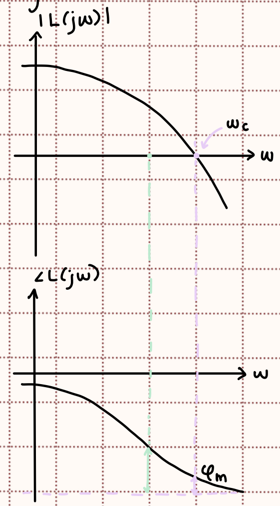
$$= \frac{(1+10s)(1+5s)(1+s)}{(1+\frac{s}{0.03})(1+\frac{s}{3})^2}$$

- to make system faster, cancel out slow poles of system in controller transfer fcn

controller examples

date 11/11/2024

e.g.



$L(s) \to kL(s)$, $k>1$

- what happens when we choose smaller w_c

- e.g. lead controller (inc stability + speed by increasing phase margin)

$$C(s) = \mu \frac{1+Ts}{1+\alpha Ts}$$

↳ $\mu > 0$

↳ $T > 0$

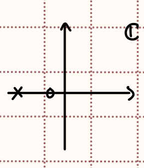
↳ $0 < \alpha < 1$

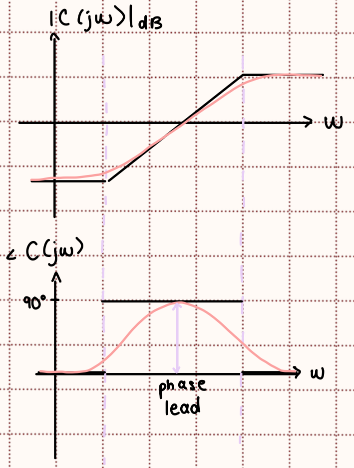
Zeros:

$z = -\frac{1}{T}$

Poles:

$p = -\frac{1}{\alpha T}$



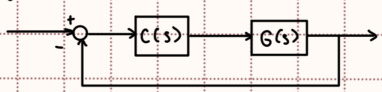


↳ high-freq gain is $\frac{\mu}{\alpha}$

↳ good val of α is 0.1

- $\simeq 55°$ phase lead

↳ e.g. $G(s) = \frac{\mu \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$



↳ as $\alpha \to 0$, we have proportional derivative controller:

date

$C(s) = \mu(1+Ts)$

$\frac{U(s)}{E(s)} = C(s)$

$= \mu + \mu Ts$

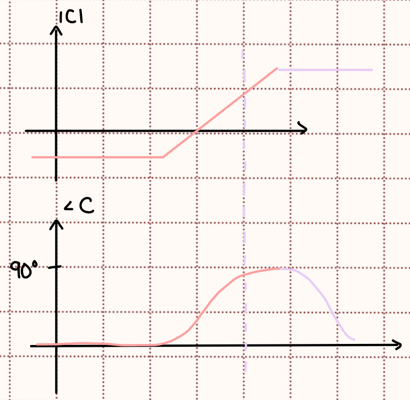
$U(s) = \mu E(s) + \mu Ts E(s)$

$\underbrace{u(t)}_{\text{control action}} = \underbrace{\mu e(t)}_{\text{proportional to error}} + \underbrace{\mu T \frac{de}{dt}(t)}_{\text{proportional to time derivative of error}}$

Apx @ time t_k:

$u(t_k) = \mu e(t_k) + \mu T \underbrace{\frac{e(t_k) - e(t_{k-1})}{t_k - t_{k-1}}}_{\text{apx of future error}}$ $\frac{e(t_{k+1}) - e(t_k)}{t_{k+1} - t_{k-1}}$ using past error

$C(s) = \mu \frac{1+Ts}{1+\alpha Ts}$



- can add any poles @ higher freq + not have effect on ω_c + φ_m

- e.g. lag controller (reduce steady-state error by increasing $L(0)$)
  - ↳ add zero + pole a decade before ω_c so that there's minimal effect on phase + ω_c

$C(s) = \mu \frac{1+Ts}{1+\alpha s}$

↳ $\mu > 0$

↳ $T > 0$

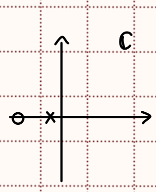
↳ $\alpha > 1$

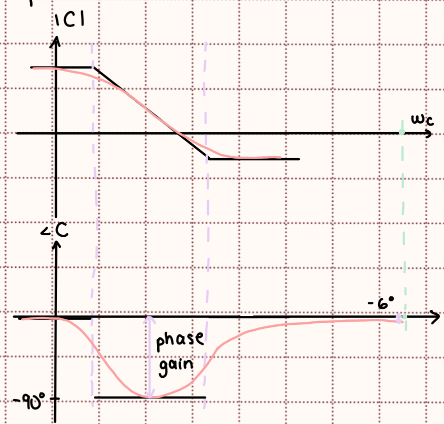
Zeros:

$z = -\frac{1}{T}$

Poles:

$p = -\frac{1}{\alpha T}$





date

↳ $10 \frac{1}{T} < \omega_c$

$\circ \simeq -6^\circ$ @ ω_c

↳ when $\mu = \alpha$, high freq behaviour is unchanged

↳ when $\mu = 1$, lower ω_c so higher φ_m

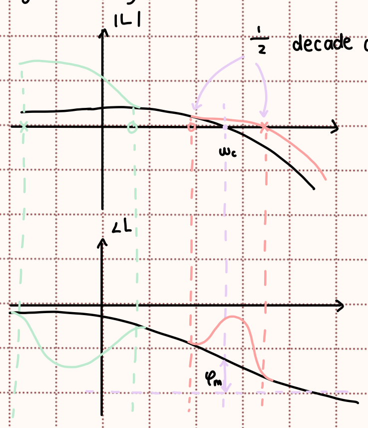
↳ as $\alpha \to \infty$, keeping $\frac{\mu}{\alpha} = C$

lead-lag and pid

date 11/13/2024

- e.g. lead-lag

$\frac{1}{2}$ decade away from ω_c $\left(\omega_c = \frac{1}{\sqrt{\alpha}\,T}\right)$



- τ of closed-loop poles $\simeq \frac{\varphi_m^{\circ}}{100}$

- e.g in lag controller, move 1 pole to 0

$C(s) = \mu \frac{1+Ts}{1+\alpha Ts}, \quad \alpha > 1$

$\alpha \to \infty \quad \left(p = -\frac{1}{\alpha T} \to 0\right)$
$\frac{\mu}{\alpha} \to K > 0$

$\Bigg\}\; C(s) = \frac{\mu}{\alpha} \frac{(1+Ts)}{\left(\frac{1}{\alpha}+Ts\right)} \xrightarrow[\frac{\mu}{\alpha}=K]{\alpha\to\infty} K\frac{1+Ts}{Ts}$

$C(s) = K + \frac{K}{T}\frac{1}{s}$

$\frac{U(s)}{E(s)} = C(s)$

$U(s) = KE(s) + \frac{K}{T}\frac{1}{s}E(s)$

$\downarrow \mathcal{L}^{-1}$

$u(t) = \underbrace{Ke(t)}_{\text{proportional to } e(t)} + \underbrace{\frac{K}{T}\int_0^t e(\tau)\,d\tau}_{\text{proportional to } \int e(t)}$ → PI (proportional integral controller)

↳ realizable b/c $C(s)$ is proper

lead controller is realizable proportional derivative (PD) controller + lag controller w/ $p=0$ is PI

so lead-lag controller is realizable PID

↳ PID(s)

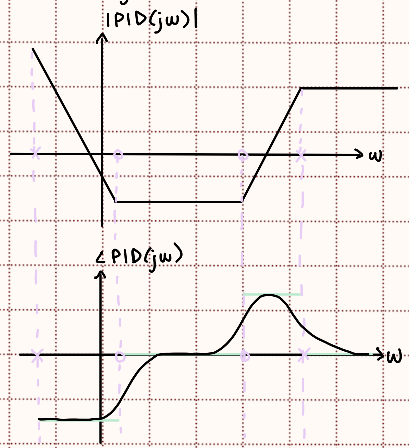
$= K_p E(s) + \frac{K_D\, s E(s)}{1+\tau_D s} + \frac{K_I}{s}E(s)$

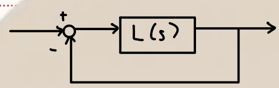
$-\frac{1}{\tau_D}$ is pole @ high freq

root locus

date 11/15/2024

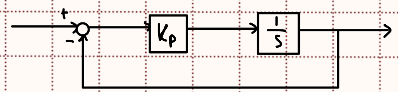
- root locus: graphical rep of possible locations of CLS's poles as controller gain K changes
  - ↳ root: roots of denom of F(s)
  - ↳ locus: geometric locus



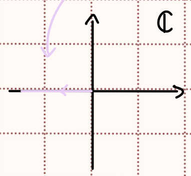


Root locus for loop gain is set of all points of complex plane which are roots of $1 + L(s)$ for some $\mu \in (-\infty, 0) \cup (0, \infty)$.

- e.g.



poles will fall on this line as we vary K_p



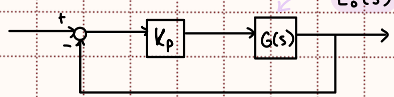
$L(s) = \frac{K_p}{s}$

$F(s) = \frac{L(s)}{1 + L(s)}$

$= \frac{K_p}{K_p + s}$ → $p = -K_p$

- find roots of denom of F(s) as fcn of $K_p \in (0, \infty)$

$L_o(s)$ is modified $G(s)$ (e.g. $L_o(s) = C_o(s)G(s)$), which is first stage of $L(s)$



$F(s) = \frac{K_p G(s)}{1 + K_p G(s)}$

↳ to get root locus for +ve controller gain, find roots of $1 + K_p G(s) = 1 + K_p L_o(s)$ as K_p goes from $0 \to \infty$

$1 + K_p L_o(s) = 0$

$L_o(s) = -\frac{1}{K_p}$

$\frac{N_o(s)}{D_o(s)} = -\frac{1}{K_p}$ → $\begin{cases} \left|\frac{N_o(s)}{D_o(s)}\right| = \frac{1}{K_p} \\ \angle \frac{N_o(s)}{D_o(s)} = (2x+1)\pi, \ x \in \mathbb{Z} \end{cases}$ → since $-\frac{1}{K_p}$ is -ve, phase is odd multiple of π

To solve for phase:

$\angle N_o(s) - \angle D_o(s) = (2x+1)\pi$

$\sum_{i=1}^{m} \angle (s - z_i) - \sum_{i=1}^{n} (s - p_i) = (2x+1)\pi$ → z_i, p_i being open-loop zeroes + poles

- rules to plot root locus for $\mu > 0$:

Always start from form $1 + K_p G(s) = 0$.

1) root locus has n branches (n is deg of denom if G(s) is proper)
2) branches originate from open-loop poles + go to zeroes
   ↳ $1 + K_p \frac{N_o(s)}{D_o(s)} = 0$

   $D_o(s) + K_p N_o(s) = 0$ → since we start at $K_p = 0$
3) $\begin{cases} m \text{ branches go to open-loop zeroes} \\ n-m \text{ branches go to } \infty \end{cases}$

   ↳ $1 + K_p \frac{N_o(s)}{D_o(s)} = 0$

date

$D_o(s) + K_p N_o(s) = 0$

$\frac{D_o(s)}{K_p} + N_o(s) = 0$

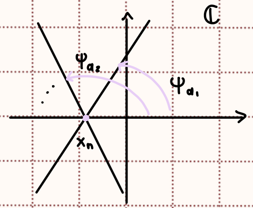
$\downarrow K_p \to \infty$

$N_o(s) = 0$

4) asymptotes of n-m branches going to ∞

$x_n = \frac{1}{n-m}(\Sigma p_i - \Sigma z_i)$ $\longrightarrow$ x_n is point on real axis where asymptotes intersect

$\varphi_{a_k} = \frac{2y+1}{n-m}\pi$, $y = 0, \dots, n-m-1$



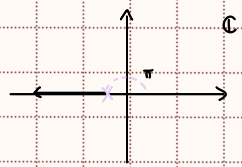
↳ x_n is centroid of extra poles

↳ φ_{a_k} is angle of asymptotes

1 line $\to \infty$:

$\varphi_{a_0} = \frac{2(0)+1}{1}\pi$

$= \pi$

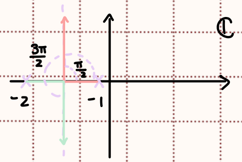


2 lines $\to \infty$:

$\varphi_{a_0} = \frac{2(0)+1}{2}\pi = \frac{\pi}{2}$

$\varphi_{a_1} = \frac{2(1)+1}{2}\pi = \frac{3\pi}{2}$

$x_n = \frac{-2-1-0}{2} = -1.5$



5) root locus contains points on real axis which are on left of an odd # of zeroes/poles

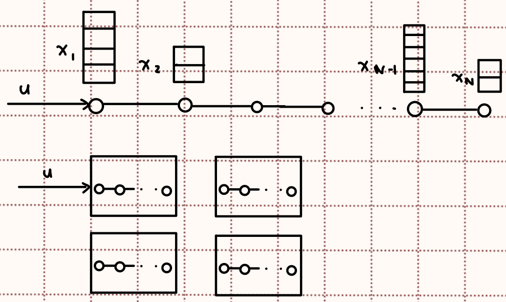
6) root locus is symmetric wrt real axis

↳ complex conjugate poles move as pairs

designing a controller

date 11/18/2024

· e.g. sequential msg queues

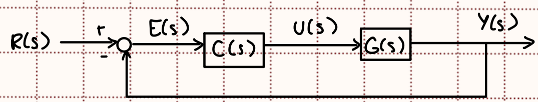


$$\begin{cases} \dot{x}_1 = x_2 - x_1 + \mu x_1 + u \\ \dot{x}_2 = x_1 - x_2 + x_3 - x_2 + \mu x_2 \\ \vdots \\ \dot{x}_N = x_{N-1} - x_N + \mu x_N \\ y = \sum_{i=1}^{N} \frac{1}{N} x_i \end{cases}$$

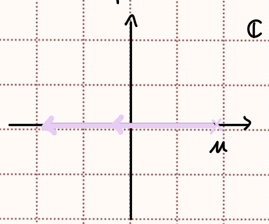
μ = arrival - service rate

$$= \begin{cases} \dot{x} = \begin{bmatrix} -1+\mu & 1 & 0 & \cdots & 0 \\ 1 & -2+\mu & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -1+\mu \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix} x \end{cases}$$

$$G(s) = C(sI - A)^{-1}B + D$$
$$= \frac{1}{N} \frac{1}{s-\mu} \longrightarrow \text{unstable OLS if } \mu > 0$$



$C(s) = K_p$



1) 1 branch $(n = 1)$

3) $\begin{cases} 0 \text{ branches} \rightarrow 0 \\ 1 \text{ branch} \rightarrow \infty \end{cases}$

4) $\varphi_{a_1} = \frac{2y+1}{1-0} \pi$

$y = 0$

$\varphi_{a_1} = \pi$

5) root locus contains points on real axis which are on left of an odd # of zeroes/poles

Want $T_s^{1\%} \leq 5s$:

$T_s^{1\%} = 5\tau \leq 5$

$\frac{5}{-p} \leq 5$

date

$-p \geq 1$

$p \leq -1$

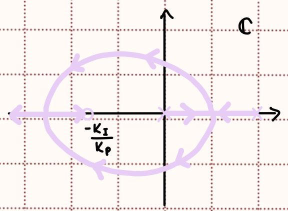
Want 0% e_{ss} so intro integrator:

$C(s) = K_p + \frac{K_I}{s}$

$= K_p \left(1 + \frac{K_I}{K_p} \frac{1}{s}\right)$

↳ choose $K_I = 1$

↳ K_p to be chosen



1) 2 branches

3) 1 branch → zero
1 branch → ∞

4) $\varphi_{a_1} = \frac{2y+1}{1-0}\pi$

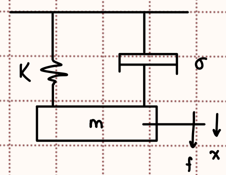
$y = 0$

$= \pi$

using root locus to design controller

date 11/20/2024

· e.g.



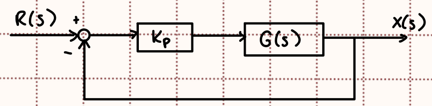
$m = 1$

$\sigma = 3$

$K = 2$

$G(s) = \frac{X(s)}{F(s)}$

$= \frac{1}{ms^2 + \sigma s + K}$



Important performance metrics:

· $OS\% = 100 e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$

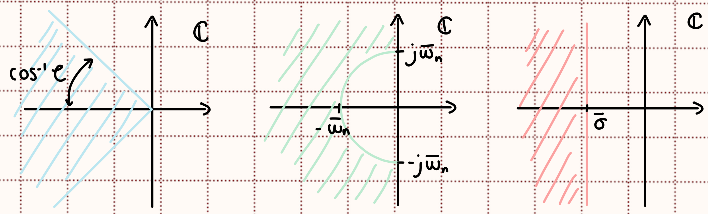
· peak time $= \frac{1}{\omega_n\sqrt{1-\xi^2}}$

· oscillation frequency $= \omega_n\sqrt{1-\xi^2}$

· settling time $= -\frac{\ln 0.01\xi}{\xi\omega_n} = -\frac{\ln 0.01\,\xi}{\xi\omega_n}$

· phase margin $\simeq 100\xi$

· crossover frequency $\simeq \omega_n$



$\xi \geq \bar{\xi}$ (OS%, settling time, φ_m)

$\omega_n \geq \bar{\omega}_n$ (rise time, settling time, ω_c)

$-\sigma \leq -\bar{\sigma}$ (settling time)

to satisfy all 3 inequalities, find intersection of all 3 regions

↳ ξ is damping of dominant poles
  - controls OS%, peak time, frequency of oscillations, settling time, + phase margin

↳ ω_n is natural frequency of dominant poles
  - controls peak time, frequency of oscillations, settling time, + crossover frequency

↳ $\sigma = \xi\omega_n$ is real part of dominant poles
  - controls settling time

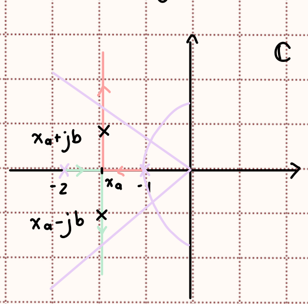
↳ since poles of CLS transfer fcn are on LHP, system will always be stable

$G(s) = \frac{1}{ms^2 + \sigma s + K}$

$= \frac{1}{s^2 + 3s + 2}$ → $s^2 + 3s + 2 = 0$

$(s+2)(s+1) = 0$

$p_1 = -2,\ p_2 = -1$



1) 2 branches

2) $p_{1,2} = -1, -2$

3) 2 branches → ∞

4) $x_a = \frac{1}{n-m}\left(\sum_i p_i - \sum_i z_i\right)$

$= \frac{1}{2}(-2-1)$

$= -\frac{3}{2}$

$\varphi_{a_k} = \frac{1}{n-m}(2k+1)\pi \quad , k = 0, 1$

$\varphi_{a_1} = \frac{\pi}{2}$

$\varphi_{a_2} = \frac{3}{2}\pi$

Specifications:

↳ $OS\% \leq 25\% \rightarrow \xi$

$$100 \cdot e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \le 25$$

$$-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} \le \ln 0.25$$

↳ $T_r \le 5s \rightarrow \zeta, \omega_n$

↳ $T_s^{1\%} \le 10s \rightarrow \zeta, \omega_n$ $(\zeta\omega_n = \sigma)$

$$\frac{4}{\zeta\omega_n} \le 10s$$

$$\zeta\omega_n > 0.4$$

$$-\bar{\sigma} \le 0.4$$

To find K_p, choose 2 poles $x_a + jb$.

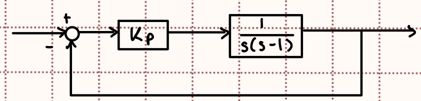
$1 + K_p \frac{N(s)}{D(s)} = 0$ ← $N(s)$, $D(s)$ are num + denom of $G(s)$

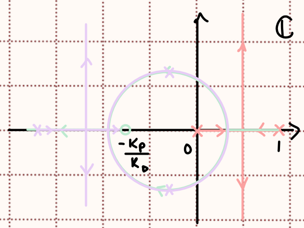
$$\left|\frac{N(s)}{D(s)}\right| = \left|-\frac{1}{K_p}\right|$$

$K_p = \left|\frac{D(s)}{N(s)}\right|$ ← evaluate at $x_a + jb$

e.g.

$$G(s) = \frac{1}{s(s-1)}$$





$C(s) = K_p$

$C(s) = K_p + \underbrace{K_D s}_{\text{derivative}}$

$C(s) = \frac{K_p + K_D s}{1 + \tau s}$

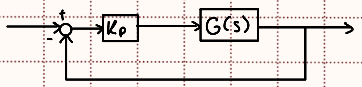
τ very small

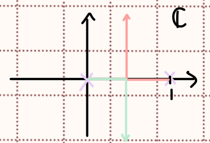
state feedback controller

date 11/23/2024

· e.g.

$G(s) = \frac{1}{s(s-1)}$

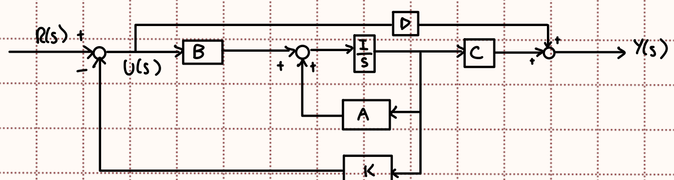




root locus for $L(s) = \frac{\mu}{s(s-1)}$

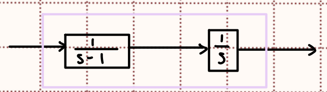
↳ static feedback is not sufficient to stabilize system since there's no μ st CLS is stable

↳ in these cases, static state feedback might be soln



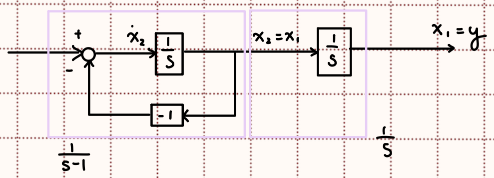
- to build state feedback controller, need space state rep of transfer fcn
- given $G(s) = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$, controllable canonical state form is:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \\ y = [b_0 \; b_1 \; \cdots \; b_{n-1}]\, x \end{cases}$$



$G(s) = \frac{1}{s^2 - s}$

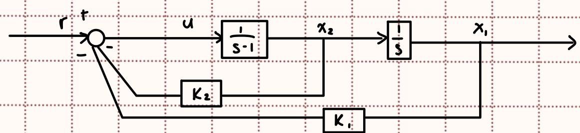
~



Setting $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \\ y = \underbrace{[1 \;\; 0]}_{C}\, x \end{cases}$$

$G(s) = C(sI - A)^{-1} B$



↳ state feedback is $u = -K_1 x_1 - K_2 x_2 + r$

$u = -Kx + r$, $K^T \in \mathbb{R}^n$

↳ controlling system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ w/ controller $u = -Kx + r$ gives:

date

$$\begin{cases} \dot{x} = Ax + B(-Kx + r) \\ y = Cx \end{cases}$$

$$= \begin{cases} \dot{x} = (A - BK)x + Br \\ y = Cx \end{cases}$$

↳ state feedback controlled CLS behaves according to eigenvalues of $A - BK$ (denom of transfer fcn)

· thm (pole placement via state feedback): given system in controllable canonical form + n complex #s $\lambda_1^*, \dots, \lambda_n^*$ (either real or pairs of complex conjugate #s), $\exists K^T \in \mathbb{R}^n$ st eigenvalues of $A - BK$ are $\lambda_1^*, \dots, \lambda_n^*$ ♡

↳ proof:

Let $K = [K_1 \; K_2 \; \cdots \; K_n]$. Given matrices A + B, eigenvalues of A - BK are roots of:

$$|\lambda I - (A - BK)|$$

$$= \left| \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \ddots & \\ & & & \lambda \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} [K_1 \; K_2 \; \cdots \; K_n] \right) \right|$$

$$= \left| \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \ddots & \\ & & & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & 1 \\ -a_0 - K_1 & -a_1 - K_2 & \cdots & & -a_{n-1} - K_n \end{bmatrix} \right|$$

$$= \lambda^n + (a_{n-1} + K_n)\lambda^{n-1} + \dots + (a_1 + K_2)\lambda + (a_0 + K_1)$$

Given $\lambda_1^*, \dots, \lambda_n^*$, desired characteristic polynomial of $A - BK$ is $P(\lambda) = (\lambda - \lambda_1^*) \cdots (\lambda - \lambda_n^*)$

Choose K st $\begin{cases} a_{n-1} + K_n = c_{n-1} \\ \vdots \\ a_1 + K_2 = c_1 \\ a_0 + K_1 = c_0 \end{cases}$ $\quad P(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$

· e.g.

$$G(s) = \frac{1}{s(s-1)} = \frac{1}{s^2 - s}$$

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [1 \;\; 0]\, x \end{cases}$$

$u = -Kx + r$

Let's choose $\lambda_1^* = -1$, $\lambda_2^* = -2$ to be desired poles.

$$P(\lambda) = (\lambda + 1)(\lambda + 2)$$

$$= \lambda^2 + \underbrace{3}_{c_1}\lambda + \underbrace{2}_{c_0}$$

$$K_1 = c_0 - a_0 \qquad\qquad K_2 = c_1 - a_1$$
$$= 2 - 0 \qquad\qquad = 3 - (-1)$$
$$= 2 \qquad\qquad = 4$$

State feedback control law to stabilize system is $u = -2x_1 - 4x_2 + r$

$$u = -2x_1 - 4x_2 + r$$

$$= \underbrace{-2y - 4\dot{y}}_{\text{PD controller}} + r$$

· to implement state feedback controller, need access to state

· not always easy to design position of poles to satisfy time / freq domain specs

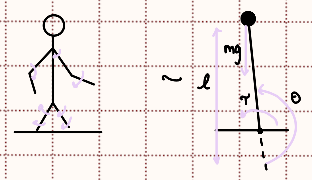
· state feedback control design process is same for MIMO systems

· while there are techniques to obtain estimate of state from input + output measurements, there might be uncontrollable states

static feedback controller design

date 11/25/2024

e.g.



$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$u = \tau$

$y = \theta$

$$\text{eig}\left(\begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix}\right)$$

$$0 = \det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix}\right)$$

$$0 = \det\left(\begin{bmatrix} \lambda & -1 \\ -g/\ell & \lambda \end{bmatrix}\right)$$

$$0 = \lambda^2 - \frac{g}{\ell}$$

$$\lambda = \pm\sqrt{\frac{g}{\ell}}$$

Let's design controller st CLS:

↳ is stable

↳ has $\leq 10\%$ OS%

↳ has $\leq 1\,s$ $T_s^{2\%}$

Use state feedback controller: $u = -Kx + r$

eig$(A - BK)$:

↳ -ve real part

↳ $100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \leq 10$

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} \geq -\ln 0.1$$

Choose $\zeta = 0.7$

↳ $\frac{4}{\zeta\omega_n} \leq 1$

$$\omega_n \geq \frac{4}{0.7}$$

$$\omega_n \geq 5.6$$

Choose $\omega_n = 6$

$$\lambda^*_{1,2} = \zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n$$

$$= -4.2 \pm j(0.7 \cdot 6)$$

$$= -4.2 \pm j4.2$$

$$p^*(\lambda) = (\lambda - \lambda_1^*)(\lambda - \lambda_2^*)$$

$$= \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$$

$$= \lambda^2 + \underbrace{8.4}_{c_1}\lambda + \underbrace{36}_{c_0}$$

$$\begin{cases} K_1 = c_0 - a_0 = 36 - \frac{g}{\ell} \\ K_2 = c_1 - a_1 = 8.4 - 0 \end{cases}$$

$$u = -\left(36 + \frac{g}{\ell}\right)x_1 - 8.4x_2 + r$$