MATH 213 - Lecture 1: Introduction to Differential Equations (DEs)

Lecture goals: To understand how some DEs are derived, the importance of boundary conditions/initial conditions and how to solve some simple DEs.

What are DEs and where do they come from?:

In 1687 Isaac Newton published the now famous equation $F = ma$ in his book *Philosophiae Naturalis Principia Mathematica* now commonly known as the *Principia Mathematica*. In $F = ma$, F denotes the net force being applied to an object, m denoted the mass of the object and a is the acceleration of the object. This simple principle gives rise to many DEs.

Example 1: DE for the motion of the height of a ball
Consider a ball of mass m being influenced by only the force of gravity. Use $F = ma$ to find equations for the vertical height of the ball.

$$ma = m\frac{d^2y}{dt^2}$$

$$-gm = m\frac{d^2y}{dt^2}$$

Example 2: Solving a simple DE
Solve the DE you found in Example 1 to find an expression for the height of the ball as a function of time.

$$-gm = m\frac{d^2y}{dt^2}$$ assume $m \neq 0$

$$-g = y''$$

$$\int y''\,dt = \int -g\,dt$$

$$y' = -gt + c_1$$

$$y = -\frac{g}{2}t^2 + c_1 t + c_2$$

Example 3: Initial Conditions (ICs)
The solution in Example 2 is not unique. Determine the extra information you need to find the exact height of the ball as a function of time.
This information is known as the **initial conditions** or more generally (and depending on context) as the **boundary conditions**.

$y(0) = c_2$ height
$y'(0) = c_1$ speed

Example 4: Constant Growth
Suppose you have 1 E. coli bacteria (in Minecraft) at time $t = 0$ and it is known that each E. coli continuously splits to produce 5 new bacterium (we allow for fractional numbers of E. coli).

a. Find a DE for the number of E. coli as a function of time, $f(t)$.

b. State the initial condition(s).

c. Solve the **Initial Value Boundary Problem** (IVBP) found in parts a-b.

a) $\frac{df}{dt} = 5f$

b) $f(0) = 1$

c) $\frac{df}{dt} = 5f$

$\frac{1}{f}\frac{df}{dt} = 5$

$\int \frac{1}{f}\,df = \int 5\,dt$

$\ln|f| + c_1 = 5t + c_2$

$\ln|f| = 5t + c_3$ $c_3 = c_2 - c_1$

$f = e^{5t + c_3}$ exponentials are always +ve

$f = e^{5t} \cdot e^{c_3}$

$f = ce^{5t}$ $c > 0$

$f(0) = 1$

$1 = ce^{5(0)}$

$1 = c$

$f = e^{5t}$

There is a slight problem with the previous model: In the real world, growth is not limitless!

But the number of E. coli in our previous model imply that there is a case of...

`memes/L1.png`

which does not exist!!
To correct for this we need to include factors that limit the population growth so that the population remains bounded over time!

Example 5: Limited growth
Consider the new model for our E. coli population:

$$\frac{df}{dt} = af - bf^2$$

where $a, b \in \mathbb{R}^+$ are constants.

a. Suppose that $0 < f(0) \ll \frac{a}{b}$.
   Without solving the above DE, find the "maximum" value for $f(t)$.

b. Solve the DE for $f(t)$.

a) $f'(t) = \frac{df}{dt}$

$0 = af - bf^2$

$0 = f(a - bf)$

$f = 0$ or $0 = a - bf$

$\downarrow$ min

$f = \frac{a}{b}$

$\downarrow$ max b/c $0 < f(0) \ll \frac{a}{b}$

b) if $0 < f < \frac{a}{b}$, $f' = af - bf^2 > 0$

$\frac{df}{dt} = af - bf^2$

$\int \frac{df}{af - bf^2} = \int dt$

$\int \frac{1}{f(a-bf)} df = \int dt$ $\xrightarrow{\text{partial fractions}}$ $\frac{1}{f(a-bf)} = \frac{A}{f} + \frac{B}{a-bf}$

Option 1:

$1 = A(a - bf) + Bf$

$1 = Aa + (B - Ab)f$

$\downarrow$ equate coeffs

$1 = Aa$ $\quad$ $0 = B - Ab$

$A = \frac{1}{a}$ $\quad$ $0 = B - \frac{1}{a}(b)$

$B = \frac{b}{a}$

Option 2 (better):

↳ look for singularities where we can divide by 0

$f = 0$:

$f\left(\frac{1}{f(a-bf)}\right) = \left(\frac{A}{f} + \frac{B}{a-bf}\right) f$

$\frac{1}{a-bf} = A + \frac{Bf}{a-bf}$

$\frac{1}{a} = A + 0$

$\boxed{A = \frac{1}{a}}$

$f = \frac{a}{b}$:

$(a-bf)\left(\frac{1}{f(a-bf)}\right) = \frac{A}{f} + \frac{B}{a-bf} \; (a-bf)$

$\frac{1}{f} = \frac{A(a-bf)}{f} + B$

$\frac{b}{a} = \frac{1}{a}(a-bf)(\frac{a}{b}) + B$

$\frac{b}{a} = \frac{1}{a}(a - b(\frac{a}{b}))(\frac{a}{b}) + B$

$\frac{b}{a} = \frac{1}{a}(a - a)(\frac{a}{b}) + B$

$\boxed{B = \frac{b}{a}}$

$\int \frac{1}{af} + \frac{b}{a(a-bf)} df = \int dt$

$\downarrow$ substitution

$$\boxed{\begin{aligned} u &= a - bf \\ \frac{du}{df} &= -b \\ df &= \frac{du}{-b} \end{aligned}}$$

$\int \frac{1}{af} df + \int \frac{b}{au} \left(\frac{du}{-b}\right) = \int dt$

$\frac{1}{a} \ln|f| + \frac{1}{a} \int \frac{1}{u} du = \int dt$

$\frac{1}{a} \ln|f| + \frac{1}{a} \ln|u| = \int dt$

$\frac{1}{a} (\ln|f| + \ln|a - bf|) = t + C_1$

$\frac{1}{a} \ln\left|\frac{f}{a-bf}\right| + C_2 = t + C_1$

$\ln\left|\frac{f}{a-bf}\right| = at + C_3 \longrightarrow C_3 = a(C_1 - C_2)$

$\frac{f}{a-bf} = e^{at + C_3}$

$\frac{f}{a-bf} = C_4 e^{at}$

$f = C_4 e^{at}(a - bf)$

$f = aC_4 e^{at} - bfC_4 e^{at}$

$f(1 + bC_4 e^{at}) = aC_4 e^{at}$

$f = \frac{aC_4 e^{at}}{1 + bC_4 e^{at}} \cdot \frac{\frac{1}{C_4} e^{-at}}{\frac{1}{C_4} e^{-at}}$

$f = \frac{a}{\frac{1}{C_4} e^{-at} + b}$

$f = \frac{a}{b} \cdot \frac{1}{\frac{1}{bC_4} e^{-at} + 1}$

$f = \frac{a}{b} \cdot \frac{1}{1 + Ce^{-at}}$ $\quad$ $C = \frac{1}{bC_4}$

$f(0) = 1$

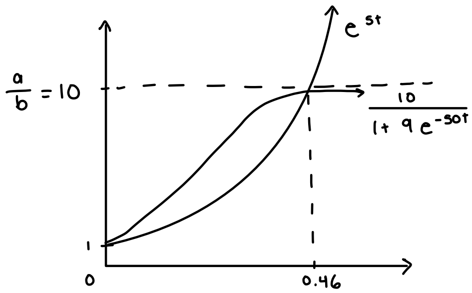
$$1 = \frac{a}{b} \cdot \frac{1}{1+c}$$

$$\frac{b}{a} = \frac{1}{1+c}$$

$$c = \frac{a}{b} - 1$$

$$f(t) = \frac{a}{b} \cdot \frac{1}{1+\left(\frac{a}{b}-1\right)e^{-at}}$$

If $a = 50$ + $b = 5$, the solns to ex 4 + ex 5:



Sometimes things change over space and time leading to a **Partial Differential Equation** (PDE).

Example 6: Linear Wave Equation

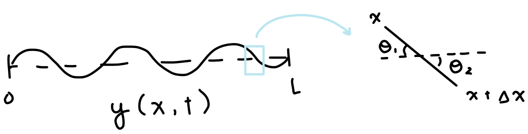
Consider a string of length L, constant density ρ and under uniform tension T.

a. Assuming the motion of the string is "small", find an equation for the perturbations, $y(x,t)$ of the string.

b. Show that $C_1 \sin\left(k\left(x - t\sqrt{\frac{T}{\rho}}\right)\right) + C_2 \sin\left(k\left(x + t\sqrt{\frac{T}{\rho}}\right)\right)$ solves the equation from part a.

c. Suppose our string is clamped at $x = 0, L$ so that $y(0,t) = y(L,t) = 0$. What limits do these boundary conditions impose of the solution given in part b?

a)



ρ: density

T: constant density

$ma = F$

$$\rho \Delta x \frac{\partial^2 y}{\partial t^2} = T\sin\theta_1 - T\sin\theta_2$$

$$\simeq T\tan\theta_1 - T\tan\theta_2$$

$$\simeq T \frac{\partial y}{\partial x}\Big|_x - T\frac{\partial y}{dx}\Big|_{x+\Delta x}$$

Hooke's Law

$$\rho \frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x} \longrightarrow f'(x) = \frac{f(x) - f(x+\Delta x)}{\Delta x}$$

b) $y_s = c_1 \sin(k(x - t\sqrt{\tfrac{T}{\rho}})) + c_2 \sin(k(x + t\sqrt{\tfrac{T}{\rho}}))$

$$\frac{\partial^2 y_s}{\partial t^2} = -c_1 (k\sqrt{\tfrac{T}{\rho}})^2 \sin(k(x - t\sqrt{\tfrac{T}{\rho}})) - c_2 (k^2 \tfrac{T}{\rho}) \sin(k(x + t\sqrt{\tfrac{T}{\rho}}))$$

$$\frac{\partial^2 y_s}{\partial x^2} = -c_1 k^2 \sin(k(x - t\sqrt{\tfrac{T}{\rho}})) - c_2 k^2 \sin(k(x + t\sqrt{\tfrac{T}{\rho}}))$$

$\therefore y_s$ solves PDE

c) $y_s(0,t) = 0$ + $y_s(L,t) = 0$

$$y_s = c_1 \sin(\tfrac{n\pi}{L}(x - t\sqrt{\tfrac{T}{\rho}})) + c_2 \sin(\tfrac{n\pi}{L}(x + t\sqrt{\tfrac{T}{\rho}}))$$

n = # nodes

MATH 213 - Lecture 2: Classifications of DEs and examples of linear DEs

Lecture goals: Understand how to classify DEs (in particular the different types of linear ODEs).

Classifying DEs:

Definition 1: Independent and Dependent Variables and Parameters

switch defns

The **independent variable(s)** of a DE are the unknown functions that we want to solve for i.e. $f(x)$, $y(x,t)$, etc.

The **dependent variable(s)** of a DE are the variable(s) that the independent variable(s) depend on i.e. x, t, etc.

A **parameter** is a term that is an unknown but is not an independent or dependent variable i.e. a, b, α, β etc.

Example 1

In the following DEs classify all the unknowns as an independent variable, dependent variable or a parameter:

a) $\frac{dy(t)}{dt} = ay(t)$

b) $\frac{d^2y(t)}{dt^2} = -g$

c) $\frac{d^2y(t)}{dt^2} = -g - \mu\frac{dy(t)}{dt}$

d) $\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2}$

	ind	dep	param
a	t	y	a
b	t	y	g
c	t	y	g, μ
d	x, t	u	c

Definition 2: Order of a DE
The **order** of a DE is the order of the highest derivative.

Example 2
Find the order for the following DEs

a) $\frac{dI}{dt} = R_0 I$.

b) $\frac{\partial^2}{\partial t^2}u(x,t) = \mu \frac{\partial^2}{\partial x^2}u(x,t)$.

c) $y'y^{(n)} + y^2 = 0$ where $n \in \{0, 1, 2, \ldots\}$ is some given number.

a) 1

b) 2

c) max (n, 1)

Definition 3: ODEs and PDEs

A DE is an **ordinary differential equation (ODE)** if it only contains ordinary derivatives (i.e. no partial derivatives).

A DE is a **partial differential equation (PDE)** if it contains at least one partial derivative of a independent variable.

Example 3

Classify the following DEs as an ODE or a PDE:

a) $\frac{dP}{dt} = aP(1-bP)(1+cP)$.

b) $\frac{\partial^2}{\partial t^2}u(x,t) = \mu \frac{\partial^2}{\partial x^2}u(x,t)$.

c) $y'y^{(n)} + y^2 = 0$ where $n \in \{0, 1, 2, \ldots\}$ is some given number.

a) ODE

b) PDE

c) ODE

Definition 4: Linear and nonlinear DEs

A DE that contains no products of terms involving the ~~in~~dependent variable(s) is called **linear**.

If a DE is not linear then it is **nonlinear**.

Example 4

Classify the following DEs as linear or nonlinear

a) $y'' = x^4$

b) $u_t + u_x = 0$

partial derivative wrt t

c) $yy'' = 0$

d) $u_t + uu_x = 0$

a) linear

b) linear

c) nonlinear

d) nonlinear

Definition 5: Homogeneous and Inhomogeneous: DEs
DE where every term depends on a dependent variable is called **homogeneous**.

A DE that is not homogeneous is called **inhomogeneous** or **nonhomogeneous**.

Example 5
Classify the following DEs as homogeneous or inhomogeneous:

a) $a(x)y'' + b(x)y' + c(x)y = 0$. b) $a(x)y'' + b(x)y' + c(x)y = f(x)$.

a) homogeneous

b) nonhomogeneous

$\frac{d}{dx} f = 0 \rightarrow$ soln is any constant

$\frac{d}{dx}(5+6) = 5\frac{d}{dx}(1) + 6\frac{d}{dx}(1)$

$= 0$

Linear homogeneous DEs have the property that if f_1 and f_2 both solve the DE then so does $af_1 + bf_2$ for all $a, b \in \mathbb{R}$.

This is the same property that was used to define linearity in MATH 115! i.e. a vector valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and for all $a, b, \in \mathbb{R}$, $f(a\vec{x} + b\vec{y}) = af(\vec{x}) + bf(\vec{y})$.

The difference is that we now study linear functions applied to the vector space of all sufficiently differentiable functions (e.g. $C^\infty(\mathbb{R})$) instead of vectors in $\mathbb{R}^n$ i.e. there are no matrix representations for linear functions.

This course focuses on linear ODEs of a particular form:

Theorem 1: Linear ODEs with Variable Coefficients
All DEs of the form

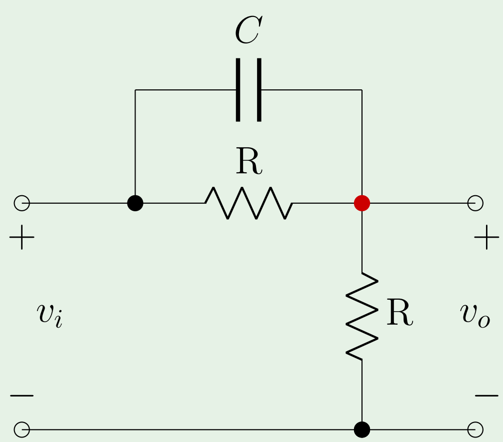
$$\frac{d^n}{dt^n}y(t) + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0(t)y(t) = f(t)$$

where $n \in \{0, 1, \ldots, n\}$, the $a_i(t)$ functions are real valued, but $y(t)$ and $f(t)$ can be complex valued are linear ODEs.

Here $f(t)$ is called the **forcing term**.

Equations of this form appear in many ECE/CS/SE problems.

Example 6: Circuit Example
Given the RC circuit:



Summing the currents flowing out of the upper-right node (red one) gives the DE:

$$\underbrace{\frac{v_o}{R}}_{\text{Bottom resistor}} + \underbrace{\frac{v_o - v_i}{R}}_{\text{Top Resistor}} + \underbrace{C\frac{d}{dt}(v_o - v_i)}_{\text{Capacitor}} = 0$$

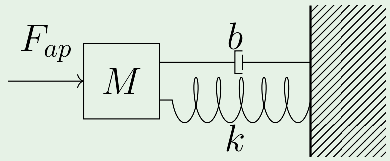
or by rearranging

$$\frac{d}{dt}v_o(t) + \frac{2}{Rc}v_o(t) = \underbrace{\frac{d}{dt}v_i(t) + \frac{1}{Rc}v_i(t)}_{f(t)}$$

This is a linear ODE with a forcing term that is determined by the input voltage.

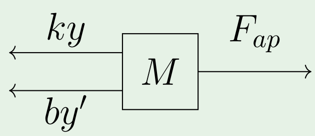
Example 7: Linear Harmonic Oscillator

Consider the spring mass system:



where F is an applied force, b is the coefficient of friction and k is the spring constant.

To find a DE we apply $F = ma$. Here is a force body diagram for the forces on the mass:



$F = \overset{m}{\cancel{M}} a$ then becomes

$$F_{ap} - by' - ky = my'' \qquad \text{or} \qquad y'' + \frac{b}{m}y' + \frac{k}{m}y = \frac{1}{m}F_{ap}$$

This is a linear ODE with a forcing term that is determined by F_{ap}.

MATH 213 - Lecture 3: Introduction to the Laplace Transform

Lecture goals: Understand the definition of the Laplace transform, the basic idea of frequency space and optionally the motivating linear algebra ideas behind the Laplace transform.

One cannot find exact solutions to many linear ODEs (more on that later), but when the $a_i(t)$'s in Theorem 1 from lecture 2 are constant and $f(t)$ is "sufficiently nice" we can!

Laplace Transform Motivation:

Definition 1: Differential operator
A **differential operator** is a special type of function (called a **functional**) that accepts a function and returns another function and only consists of taking derivatives, multiplying by functions, and adding other differential operators.

Example 1
The following are differential operators

- $\frac{d}{dx}$
- $\frac{d^2}{dx^2} + \frac{d}{dx}$
- $u\frac{\partial}{\partial x}$

Example 2: Differential operators
All ODEs are of the form

$$D(y(x)) = g(x)$$

where y is the independent variable, x is the dependent vairable, $g(x)$ is the forcing term and D is a differential operator.

Example 3: Connection to MATH 115
In MATH 115 you studied functionals called linear transformations!!!

We often asked you to solve equations of the from

$$A\vec{x} = \vec{b}$$

where $A \in \mathbb{R}^{n\times n}$ and $\vec{x}, \vec{b} \in \mathbb{R}^n$.

The linear function $f(\vec{x}) = A\vec{x}$ is a functional!

Notice that $A\vec{x} = \vec{b}$ looks quite similar to $Dy = g$ given that D is linear...
This is the basis of the Laplace transform approach to solving ODEs.

MATH 115 "review" time!!!!

$A\vec{x} = \lambda \underline{\vec{x}}$

Suppose that the $n \times n$ matrix A admits a basis of orthonormal eigenvectors.

Explicitly, there is a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that there are $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $A\vec{v}_i = \lambda_i v_i$ <u>and</u> $\vec{v}_i \cdot \vec{v}_j = 0$ when $i \neq j$ <u>and</u> $\vec{v}_i \cdot \vec{v}_i = 1$ for all $i \in \{1, \dots n\}$.

In this case we can solve the equation $A\vec{x} = \vec{b}$ by:

1. Writing $\vec{b}$ as $b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$.
2. Supposing that $\vec{x} = x_1\vec{v}_1 + \ldots + x_n\vec{v}_n$ (we can do this as $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis).
3. Noting that

$$\begin{aligned}
b_1\vec{v}_1 + \ldots + b_n\vec{v}_n &= \vec{b} && \text{def of } \vec{b} \\
&= A\vec{x} && \text{Eq. we want to solve} \\
&= A(x_1\vec{v}_1 + \ldots + x_n\vec{v}_n) && \text{def of } \vec{x} \\
&= x_1A\vec{v}_1 + \ldots + x_nA\vec{v}_n && A \text{ is a matrix} \\
&= x_1\lambda_1\vec{v}_1 + \ldots + x_n\lambda_n\vec{v}_n && \text{Def. of eigenvalue/vector}
\end{aligned}$$

4. Now we can solve for x_i by taking the dot product of both sides of the equation with $\vec{v}_i$. Explicitly

$$\begin{aligned}
b_1\vec{v}_1 + \ldots + b_n\vec{v}_n &= x_1\lambda_1\vec{v}_1 + \ldots + x_n\lambda_n\vec{v}_n && \text{above} \\
\vec{v}_i \cdot (b_1\vec{v}_1 + \ldots + b_n\vec{v}_n) &= \vec{v}_i \cdot (x_1\lambda_1\vec{v}_1 + \ldots + x_n\lambda_n\vec{v}_n) && \text{dot with } \vec{v}_i \\
0 + \ldots + 0 + b_i(v_i \cdot v_i) + 0 + \ldots 0 &= 0 + \ldots + 0 + \lambda_i x_i(v_i \cdot v_i) + 0 + \ldots 0 && \text{simplify} \\
b_i &= \lambda_i x_i && \text{simplify}
\end{aligned}$$

so $x_i = \frac{b_i}{\lambda_i}$ for all $i \in \{1, \ldots, n\}$.

5. Thus $\vec{x} = \frac{b_1}{\lambda_1}\vec{v}_1 + \ldots + \frac{b_n}{\lambda_n}\vec{v}_n$!

There is no need to invert the matrix if we know a orthogonal basis of eigenvectors!!!

Profound idea: To solve the DE $Dy = g$ why not find an orthogonal basis of **eigenfunctions** ("eigenvectors" but functions) of D and use the same process???

Problem: How do we find an orthonormal eigenbasis for D???

What does that even mean?

We don't but.... Luckily for us if D is "nice" then we can use the eigenfunctions of $\frac{d}{dx}$ then things tend to work out (because of integration by parts)...
more on that later.

Example 4
Find the eigenfunctions of the $\frac{d}{dx}$ operator. i.e. solve $\frac{d}{dx}f = \lambda f$ for all $\lambda \in \mathbb{R}$.

$f = ce^{\lambda x}$

$\{e^{\lambda x} \mid \lambda \in \mathbb{R}\}$

to solve. $Dy = g$

look for: $y = \int_{?_1}^{?_2} F(\lambda) e^{\lambda t} \, d\lambda$

While $\{e^{\lambda x} | \lambda \in \mathbb{R}\}$ does not form an orthonormal basis for the set of all differential functions (whatever that means...), it does form a basis for *something* **and** can be used to solve $\frac{d}{dx}y = f$. To do this:

1. The finite sum is replaced with an integral!
2. The dot product is replaced with a general **inner product** (more on that later).
3. The details for how to invert the system change...

It turns out that we can use this idea for all equations of the form $Dy = f$ where D is "sufficiently nice"!!!

In practice, this process turns the DE $Dy = f$ into an algebraic equation! Notice:

$$\underbrace{\frac{d}{dx}}_{\text{calculus}} y = \underbrace{\lambda y}_{Algebra}$$

Laplace Transform:

Definition 2: Laplace Transform
Given a function $f(t)$ the **Laplace transform** of $f(t)$ denoted by $F(s)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

given that the integral exists.

Here s is a complex valued scalar called the "**frequency**".

Question: Why is s called the frequency?

Example 5: Pure imaginary case
Suppose that $s = j\omega$ where $j = \sqrt{-1}$. In this case

$$\begin{aligned} e^{-st} &= e^{-j\omega t} \\ &= \cos(\omega t) - j\sin(\omega t) \end{aligned}$$

ω is the frequency of the sinusoidal waves!

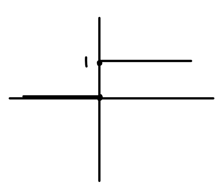
Example 6: Complex case
Suppose that $s = \sigma + j\omega$ where $j = \sqrt{-1}$. In this case

$$\begin{aligned} e^{-st} &= e^{-\sigma t - j\omega t} \\ &= e^{-\sigma t}e^{-j\omega t} \\ &= e^{-\sigma t}\left(\cos(\omega t) - j\sin(\omega t)\right) \end{aligned}$$

ω is still the frequency of the sinusoidal waves but the sigma controls the decay or growth of the amplitude of the waves

Definition 3: Unit Step Function or Heaviside Function
The **unit step function** or **heaviside function** is the function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & else \end{cases}$$



Example 7: $\mathcal{L}(u(t))$
compute the Laplace transform of the heaviside function and determine when it exists.

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}\,dt \longrightarrow u(t)e^{-st} = \begin{cases} e^{-st} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

$\downarrow$ integral is 0

$$= \int_0^{\infty} e^{-st}\,dt \longrightarrow \text{Let } u = -st$$

$$du = -s\,dt$$

$$dt = -\frac{du}{s}$$

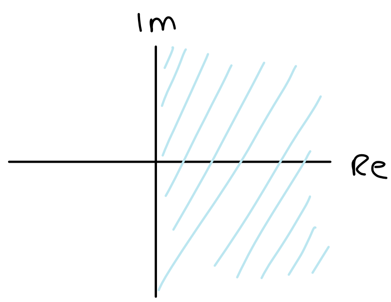
$$= \int_0^{\infty} e^{u}\left(-\frac{du}{s}\right)$$

$$= -\frac{1}{s}\int_0^{\infty} e^{u}\,du \longrightarrow \text{if sticking w/ } u\text{, make sure to change limits}$$

$$= -\frac{1}{s}e^{-st}\Big|_0^{\infty}$$

$$= -\frac{1}{s}\left(\lim_{t\to\infty} e^{-st} - 1\right)$$

$$= \begin{cases} -\frac{1}{s}(0-1) = \frac{1}{s}, & Re(s) > 0 \\ -\frac{1}{s}(\infty - 1) = \infty, & \text{else} \end{cases}$$



Region of Convergence (RoC)

Example 8
Compute the Laplace transform of

$$f(t) = e^{\alpha t}u(t) = \begin{cases} e^{\alpha t} & t \geq 0 \\ 0 & else \end{cases}$$

and determine when it exists.

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$

$$= \int_0^{\infty} e^{\alpha t} e^{-st}\,dt$$

$$= \int_0^{\infty} e^{-(s-\alpha)t}\,dt \longrightarrow \text{let } u = -(s-\alpha)t$$

$$du = -(s-\alpha)\,dt$$

$$dt = \frac{du}{-(s-\alpha)}$$

$$= \int_0^{\infty} e^{u}\left(\frac{du}{-(s-\alpha)}\right)$$

$$= -\frac{1}{s-\alpha}\int_0^{\infty} e^{u}\,du$$

$$= -\frac{1}{s-\alpha}\, e^{-(s-\alpha)t}\Big|_0^{\infty}$$

$$= -\frac{1}{s-\alpha}\left(\lim_{t\to\infty} e^{-(s-\alpha)t} - 1\right)$$

$$= \begin{cases} \frac{1}{s-\alpha}, & Re(s) > Re(\alpha) \\ \infty, & else \end{cases} \longrightarrow \begin{matrix} s-\alpha > 0 \\ s > \alpha \end{matrix}$$

Definition 4: Inverse Laplace Transform
Given a "nice" function $f(t)$ and its Laplace transform $F(s)$, we have

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

where σ is such the vertical line lies in the radius of convergence of the Laplace Transform. of $F(s)$

Note that

- The above is a contour integral in the complex plane!
- The above is "sum of exponentials" I talked about!!

We have not really taught you how to evaluate these... so we will use partial-fraction decomposition and look up the inverse transforms in a table.

Example 9
Suppose $F(s) = \frac{1}{s(s+10)}$ for $Re(s) > 10$. Find $f(t)$

$$F(s) = \frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10} \longrightarrow$$

$$= \frac{1}{10s} - \frac{1}{10}\left(\frac{1}{s+10}\right)$$

$s = 0$:

$$s\left(\frac{1}{s(s+10)}\right) = \left(\frac{A}{s} + \frac{B}{s+10}\right)s$$

$$\frac{1}{s+10} = A + \frac{Bs}{s+10}$$

$$\frac{1}{10} = A$$

$s = -10$:

$$(s+10)\left(\frac{1}{s(s+10)}\right) = \left(\frac{A}{s} + \frac{B}{s+10}\right)(s+10)$$

$$\frac{1}{s} = \frac{A(s+10)}{s} + B$$

$$\frac{1}{-10} = B$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{10}}{s} + \frac{-\frac{1}{10}}{s+10}\right\}$$

$$= \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$= \frac{1}{10}u(t) - \frac{1}{10}e^{-10t}u(t)$$

ex 7   ex 8

MATH 213 - Lecture 4: More Laplace Transforms and Properties of Laplace Transforms

Lecture goals: Be able to compute Laplace transforms from the definition, know what the one-sided or unilateral Laplace Transform is and understand some commonly used (and important) properties of the Laplace transform (and be able to prove them if asked).

More Examples:

Example 1
Compute the Laplace transform of $tu(t)$ and find the ROC.

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-st}\, dt$$

$$= \int_0^{\infty} t e^{-st}\, dt$$

$$\boxed{\begin{array}{ll} u = t & dv = e^{-st}dt \\ du = dt & v = -\frac{1}{s}e^{-st} \end{array}}$$

Integration by parts:

$(fg)' = f'g + fg'$

$\int (fg)'dt = \int f'g + fg'\, dt$

$fg = \int f'g + fg'\, dt$

$\int f'g = fg - \int fg'$

$\int u\,dv = uv - \int v\,du$

$$= -\frac{t}{s}e^{-st}\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{s}e^{-st}\, dt$$

$$= -\frac{t}{s}e^{-st}\Big|_0^{\infty} - \left(-\frac{1}{s}\right)^2 e^{-st}\Big|_0^{\infty}$$

$$= \underbrace{-\frac{t}{s}e^{-st}\Big|_0^{\infty}}_{-\frac{1}{s}\frac{t}{e^{st}}\Big|_0^{\infty} = -\frac{1}{s}(0-0)} - \underbrace{\frac{1}{s^2}e^{-st}\Big|_0^{\infty}}_{\frac{1}{s^2}\left(\lim_{t\to\infty} e^{-st} - 1\right) = -\frac{1}{s^2}}$$

$$= 0 - \left(-\frac{1}{s^2}\right) \quad Re(s) > 0$$

$$= \begin{cases} \frac{1}{s^2} & Re(s) > 0 \\ \infty & \text{else} \end{cases}$$

Often we care about functions $f(t)$ that are only defined for $t \geq 0$. There is a special transform for that

Definition 1: Unilateral Laplace Transform
The **Unilateral Laplace Transform** or **One-sided Laplace Transform** of a function $f(t)$ defined only for $t \geq 0$ is defined as

$$F(s) = \mathscr{L}\{f(t)\} = \int_{0+}^{\infty} f(t)e^{-st}dt.$$

Caution: If we use the symbol "$\mathscr{L}$" we mean the two sided transform unless otherwise stated.

Example 2
Compute the one-sided Laplace transform of $u(t-T)$ for $T > 0$ and find the ROC.

$$u(t-T) = \begin{cases} 1 & t \geq T \\ 0 & \text{else} \end{cases}$$

$$F(s) = \int_{0^+}^{\infty} u(t-T)\, e^{-st}\, dt$$

$$= \int_{T}^{\infty} e^{-st}\, dt$$

$$= \frac{e^{-st}}{-s}\Big|_{T}^{\infty}$$

$$= \lim_{t\to\infty} -\frac{e^{-st}}{s} - \left(\frac{-e^{-sT}}{s}\right)$$

$$= 0 - \left(-\frac{e^{-sT}}{s}\right) \quad \text{if } Re(s) > 0$$

$$= \frac{e^{-sT}}{s} \quad \text{if } Re(s) > 0$$

Example 3
Compute the Laplace transform of $\sin(\omega t)u(t)$ for $\omega \in \mathbb{R}$ and find the ROC.

Hint: Write sin as a sum of complex exponentials.

Option 1:

$$\mathcal{L}\{\sin(\omega t)u(t)\} = \int_{-\infty}^{\infty} \sin(\omega t)\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} \sin(\omega t)\, e^{-st}\, dt$$

integrate by parts

Option 2:

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$-\quad e^{-i\omega t} = \cos\omega t - i\sin\omega t$$

$$e^{i\omega t} - e^{-i\omega t} = 2i\sin\omega t$$

$$\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$\mathcal{L}\{\sin(\omega t)u(t)\} = \mathcal{L}\left\{\frac{e^{i\omega t} - e^{-i\omega t}}{2i} u(t)\right\}$$

$$= \frac{1}{2i}\mathcal{L}\{e^{i\omega t}u(t)\} - \frac{1}{2i}\mathcal{L}\{e^{-i\omega t}u(t)\}$$

$$= \frac{1}{2i}\frac{1}{s-i\omega} - \frac{1}{2i}\frac{1}{s+i\omega}, \quad Re(s) > 0$$ L3 ex 8 (will be given table)

$$= \frac{1}{2i}\left(\frac{s+i\omega}{s^2 - i^2\omega^2} - \frac{s-i\omega}{s^2 - i^2\omega^2}\right)$$

$$= \frac{1}{2i}\left(\frac{s+i\omega}{s^2+\omega^2} - \frac{s-i\omega}{s^2+\omega^2}\right)$$

$$= \frac{1}{2i}\left(\frac{2i\omega}{s^2+\omega^2}\right)$$

$$= \frac{\omega}{s^2+\omega^2}$$

Properties of the Laplace Transform:

Theorem 1
For any function $f(t)$, the one-sided Laplace transform will always converge given that there is some sufficiently large $s \in \mathbb{R}$ such that

$$\int_0^\infty f(t)e^{-st}dt$$

exists

Theorem 2: Laplace Transform is Linear
Suppose that $f(t)$ and $g(t)$ have Laplace transforms $F(s)$ and $G(s)$. Then for all $\alpha, \beta \in \mathbb{C}$

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

the the ROC is the intersection of the ROCs for $F(s)$ and $G(s)$.

$$A(a\vec{x} + b\vec{y}) = aA\vec{x} + bA\vec{y}$$

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \int_{-\infty}^{\infty} (\alpha f(t) + \beta g(t)) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \alpha f(t) e^{-st} + \beta g(t) e^{-st} dt$$

$$= \alpha \int_{-\infty}^{\infty} f(t) e^{-st} dt + \beta \int_{-\infty}^{\infty} g(t) e^{-st} dt$$

$$= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

Theorem 3: Time-Scaling
If $\mathscr{L}\{f(t)\} = F(s)$ then for $c > 0$, $\mathscr{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c})$

$$\mathscr{L}\{f(t)\} = \int_{-\infty}^{\infty} f(ct)\, e^{-st}\, dt \longrightarrow \text{Let } u = ct$$

$$du = c\,dt \qquad t = \frac{u}{c}$$

$$dt = \frac{du}{c}$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-s\frac{u}{c}}\, \frac{du}{c}$$

$$= \frac{1}{c}\int_{-\infty}^{\infty} f(u)\, e^{-(\frac{s}{c})u}\, du$$

$$= \frac{1}{c}\mathscr{L}\{f(t)\}\Big|_{s=\frac{s}{c}}$$

$$= \frac{1}{c}F\left(\frac{s}{c}\right)$$

Example 4

Use the fact that $\mathscr{L}\{\sin(t)u(t)\} = \dfrac{1}{s^2+1}$ to compute $\mathscr{L}\{\sin(\omega t)u(\omega t)\}$ for positive real ω without directly evaluating the integral.

$$\mathcal{L}\{\sin(\omega t)\,u(\omega t)\} = \frac{1}{\omega}\,\mathcal{L}\{\sin(t)u(t)\}\Big|_{s=\frac{s}{\omega}}$$

$$= \frac{1}{\omega}\,\frac{1}{\left(\frac{s}{\omega}\right)^2+1}$$

$$= \frac{1}{\omega}\,\frac{1}{\frac{s^2}{\omega^2}+1}$$

$$= \frac{1}{\omega}\left(\frac{\omega^2}{s^2+\omega^2}\right)$$

$$= \frac{\omega}{s^2+\omega^2} \qquad , \ \mathrm{Re}(s)>0$$

Theorem 4: Exponential Modulation
$\mathscr{L}\{e^{\alpha t}f(t)\} = F(s-\alpha)$.

Proof:

Let $F(s) = \mathscr{L}\{f(t)\}$

$$L\{e^{\alpha t}f(t)\} = \int_{-\infty}^{\infty} e^{\alpha t} f(t) e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} f(t) e^{-(s-\alpha)t}\, dt$$

$$= L\{f(t)\}\Big|_{s=s-\alpha}$$

$$= F(s-\alpha)$$

Example 5

Compute $\mathscr{L}\{e^{\alpha t}u(t)\}$ for $\alpha \in \mathbb{R}$ without directly evaluating the integral.

L3 ex7: $\mathcal{L}\{u(t)\} = \begin{cases} \frac{1}{s} & Re(s) > 0 \\ 0 & \text{else} \end{cases}$

$$\mathcal{L}\{e^{\alpha t}u(t)\} = \mathcal{L}\{u(t)\}\Big|_{s=s-\alpha}$$

$$= \begin{cases} \frac{1}{s-\alpha} & Re(s) > \alpha \\ 0 & \text{else} \end{cases} \longleftarrow \text{shift b/c } \; s-\alpha > 0, \; s > \alpha$$

Theorem 5: Time-Shifting
If $F(s) = \mathscr{L}\{f(t)u(t)\}$ and $g(t) = f(t-T)u(t-T)$ then

$$G(s) = e^{-sT}F(s).$$

Proof:

Let $F(s) = \mathcal{L}\{f(t)u(t)\}$ + $g(t) = f(t-T)u(t-T)$. ⟵ we use $u(t)$ to get one-sided integral

$$G(s) = \mathcal{L}\{g(t)\}$$

$$= \mathcal{L}\{f(t-T)u(t-T)\}$$

$$= \int_{-\infty}^{\infty} f(t-T)u(t-T)e^{-st}\,dt \longrightarrow \text{Let } r = t - T$$

$dr = dt$ $\quad t = r + T$

$$= \int_{-\infty}^{\infty} f(r)u(r)e^{-s(r+T)}\,dr$$

$$= \int_{-\infty}^{\infty} f(r)u(r)e^{-sr}e^{-sT}\,dr$$

$$= e^{-sT}\int_{-\infty}^{\infty} f(t)u(t)e^{-st}\,dt$$ change dummy var

$$= e^{-sT}\mathcal{L}\{f(t)u(t)\}$$

$$= e^{-sT}F(s)$$

Example 6
Evaluate $\mathscr{L}\{u(t-T)\}$ without directly evaluating the integral.

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{u(t-T)\} = e^{-sT}\mathcal{L}\{u(t)\}$$

$$= \frac{e^{-sT}}{s} \qquad , \quad Re(s) > 0$$

Theorem 6: Multiplication by t
If $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}\{tf(t)\} = -\frac{d}{ds}F(s)$.

Proof:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} \frac{d}{ds}\left(f(t)e^{-st}\right)dt$$

can swap only if eqn inside integral absolutely converges (it does b/c Laplace transformation exists)

$$= \int_{-\infty}^{\infty} f(t)\frac{d}{ds}\left(e^{-st}\right)dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-st}(-t)\,dt$$

$$= -\int_{-\infty}^{\infty} tf(t)e^{-st}\,dt$$

$$= -\mathcal{L}\{tf(t)\}$$

Example 7

Compute $\mathscr{L}\{tu(t)\}$ without directly computing the integral.

$$\mathcal{L}\{t\,u(t)\} = -\frac{d}{ds}\mathcal{L}\{u(t)\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s}\right)$$

$$= -\left(-\frac{1}{s^2}\right)$$

$$= \frac{1}{s^2}$$

Example 8: Foreshadowing

Use integration by parts to evaluate $\mathscr{L}\{f'(t)\}$ where we mean the one-sided transform for a "sufficiently nice" function $f(t)$.

$$\mathscr{L}\{f'(t)\} = \int_{0^+}^{\infty} f'(t)e^{-st}\,dt \longrightarrow$$

$u = e^{-st}$ $\qquad dv = f'(t)\,dt$

$du = -se^{-st}\,dt$ $\qquad v = f(t)$

$\int u\,dv = uv - \int v\,du$

$$= e^{-st}f(t)\Big|_{0^+}^{\infty} - \int_{0^+}^{\infty} f(t)(-se^{-st})\,dt$$

$$= \underbrace{\lim_{t\to\infty}\frac{f(t)}{e^{st}}}_{0} - e^{-s(0^+)}f(0^+) + s\int_{0^+}^{\infty} f(t)e^{-st}\,dt$$

$f(t)$ is nice b/c it doesn't grow faster than e^{st}

$$= -f(0^+) + s\mathscr{L}\{f(t)\}$$

Theorem 7: Laplace Transform of a Derivative/Integral
Let $f(t)$ be such that there is a real value α such that the integral

$$\int_{0^+}^{\infty} |f(t)|e^{-\alpha t}dt$$

converge **and** such that there exists a function $f'(t)$ such that for $t \geq 0$

$$f(t) = f(0^+) + \int_{0^+}^{\infty} f'(\tau)d\tau$$

and there exists a real value β such that

$$\int_{0^+}^{\infty} |f'(t)|e^{-\beta t}dt$$

converges. In this case

$$F(s) = \frac{1}{s}f(0^+) + \frac{1}{s}\mathscr{L}\{f'(t)\}$$

or in other-words

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0^+)$$

The proof for this is basically in the previous example. This theorem is how we will solve linear DEs!

MATH 213 - Lecture 5: More Laplace Transforms and Convolution

Lecture goals: Know how to use a Laplace Transform table and the properties of the Laplace transform to evaluate Laplace Transforms and "simple" inverse Laplace Transforms. Know what the convolution operator is and its connection to Laplace Transforms.

Motivation for why we care about Laplace:

Example 1
Use the Laplace Transform to solve the IVP $y' = \sin(t)$, $y(0) = 0$.

$$\mathcal{L}\{y'\} = \mathcal{L}\{\sin t\}$$

$$sY(s) - y(0) = \frac{1}{s^2+1} - 0$$

$$sY(s) = \frac{1}{s^2+1}$$

$$Y(s) = \frac{1}{s(s^2+1)}$$

partial frac

$$= \frac{1}{s} - \frac{s}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= 1 - \cos(t)$$

One-sided Laplace Table:

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\text{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
3. t^n	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\text{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\text{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\text{Re}(s) > \lvert\omega\rvert$
7. $\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$\text{Re}(s) > \lvert\omega\rvert$

Algebraic Properties:

$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Linearity
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$	Time Scaling
$e^{\alpha t} f(t)$	$F(s-\alpha)$	Exponential Modulation
$f(t-T)u(t-T)$	$e^{-sT}\mathscr{L}\{f(t)u(t)\}$	Time-Shifting
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Multiplication by t^n
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	

Example 2

Use the Laplace table to compute $\mathscr{L}^{-1}\left\{\frac{1}{s-3}+\frac{s}{s^2-4}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\bigg|_{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\}$$

$$= e^{3t} \cdot 1 + \mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\}$$

if heavyside fcn,
shift where it activates

$$= e^{3t} + \cosh(2t)$$

Example 3
Use the Laplace table to compute $\mathscr{L}\{e^2 t - \sin(4t) + 6t^3\}$

$\mathcal{L}\{e^2 t\} - \mathcal{L}\{\sin 4t\} + \mathcal{L}\{6t^3\}$

$= e^2 \mathcal{L}\{t\} - \frac{2}{s^2+2} + 6\mathcal{L}\{t^3\}$

$= \frac{e^2}{s^2} - \frac{4}{s^2+16} + \frac{6 \cdot 3!}{s^4}$

Convolution:

Before solving DEs we introduce one more super useful property of the Laplace Transform by first introducing convolution.

To motivate the continuous convolution operator we start with some discrete cases.

2D Image filtering:

Consider the image of a random internet cat:

Suppose we wanted to blur the image. We could do this by replacing each pixel with a weighted average of the nearby pixels

Image

$a_{1,1}$	$a_{2,1}$	$a_{3,1}$	$a_{4,1}$	$a_{5,1}$	$a_{6,1}$
$a_{1,2}$	$a_{2,2}$	$a_{3,2}$	$a_{4,2}$	$a_{5,2}$	$a_{6,2}$
$a_{1,3}$	$a_{2,3}$	$a_{3,3}$	$a_{4,3}$	$a_{5,3}$	$a_{6,3}$
$a_{1,4}$	$a_{2,4}$	$a_{3,4}$	$a_{4,4}$	$a_{5,4}$	$a_{6,4}$
$a_{1,5}$	$a_{2,5}$	$a_{3,5}$	$a_{4,5}$	$a_{5,5}$	$a_{6,5}$
$a_{1,6}$	$a_{2,6}$	$a_{3,6}$	$a_{4,6}$	$a_{5,6}$	$a_{6,6}$

$*$

Filter

$w_{1,1}$	$w_{2,1}$	$w_{3,1}$
$w_{1,2}$	$w_{2,2}$	$w_{3,2}$
$w_{1,3}$	$w_{2,3}$	$w_{3,3}$

$=$

Filtered Image

$b_{1,1}$	$b_{2,1}$	$b_{3,1}$	$b_{4,1}$	$b_{5,1}$	$b_{6,1}$
$b_{1,2}$	$b_{2,2}$	$b_{3,2}$	$b_{4,2}$	$b_{5,2}$	$b_{6,2}$
$b_{1,3}$	$b_{2,3}$	$b_{3,3}$	$b_{4,3}$	$b_{5,3}$	$b_{6,3}$
$b_{1,4}$	$b_{2,4}$	$b_{3,4}$	$\boxed{b_{4,4}}$	$b_{5,4}$	$b_{6,4}$
$b_{1,5}$	$b_{2,5}$	$b_{3,5}$	$b_{4,5}$	$b_{5,5}$	$b_{6,5}$
$b_{1,6}$	$b_{2,6}$	$b_{3,6}$	$b_{4,6}$	$b_{5,6}$	$b_{6,6}$

$$
\begin{aligned}
b_{4,4} &= a_{3,3}w_{1,1} + a_{4,3}w_{2,1} + a_{5,3}w_{3,1} \\
&\quad + a_{3,4}w_{1,2} + a_{4,4}w_{2,2} + a_{5,4}w_{3,2} \\
&\quad + a_{3,5}w_{1,3} + a_{4,5}w_{2,3} + a_{5,5}w_{3,3} \\
b_{i,j} &= \sum_{n=1}^{N} \sum_{m=1}^{M} a_{i-m,j-n} w_{m,n}
\end{aligned}
$$

This process of filtering the image is called **discrete convolution.** If we use a Gaussian filter matrix i.e. something of the form

$$
W = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}
$$

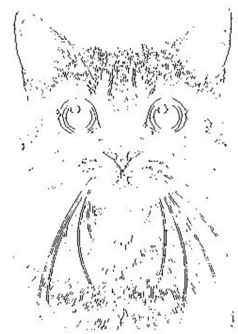
then the filtered cat image becomes

If we use a Sobel filter matrix i.e. something of the form

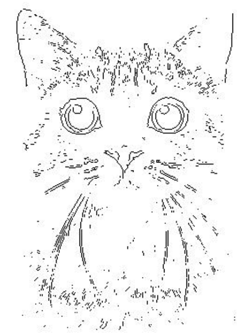
$$
\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}
$$

or the norm of the results then the aforementioned filtered cat images are summed becomes

or



or



Polynomial Multiplication:

Suppose that $p(x) = a_0 + a_1x + \ldots + a_mx^m$ and $q(x) = b_0 + b_1x + \ldots + b_nx^n$. The product would be

$$p(x)q(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \ldots + a_mb_na^{n+m}$$

Note that the coefficient of x^i is just $\sum_{\substack{j \text{ s.t.} \\ j\in\{0,\ldots,m\} \\ i-j\in\{0,\ldots,n\}}} a_jb_{i-j}$. This has the same functional form as the filtering example (but is in 1D) and is hence a 1D convolution of the vectors $[a_0, \ldots, a_m]$ and $[b_0, \ldots, b_n]$.

Continuous Convolution:

Motivated by the previous discrete applications we define a continuous version of the convolution.

Definition 1: Convolution
The **convolution** of functions $f(t)$ and $g(t)$, denoted by $(f * g)(t)$ is the integral

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

given that the integral converges.

Note: Similar to the discrete convolutions, this is used to apply filters and to do continuous analogues of the polynomial multiplication trick...

The second comment is useful for Laplace Transforms and for solving DEs!

Before we continue, we define the idea of an integral transform.

Definition 2: Integral Transform
An **integral transform** is a functional (function on a set of functions) that can be written in the form

$$\mathscr{I}\{f(t)\} = \int_{-\infty}^{\infty} f(t)K(t,s)dt.$$

The function $K(t,u)$ is called the **kernel** of the integral transform.

Example 4
The Laplace Transform is the integral transform with kernel $K(t,s) = e^{-st}$.

The convolution of f with some fixed function $g(t)$ is an integral transform with the kernel $K(t,s) = g(t-s)$.

Theorem 1: Convolution Properties

A. The convolution operator is commutative $(f * g)(t) = (g * f)(t)$.

B. If $f(t)$ and $g(t)$ are one-sided functions (i.e. $f(t) = g(t) = 0$ for $t < 0$) then
$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$
and hence the convolution is also one sided.

A) $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$ $\longrightarrow$ $u = t - \tau$ $\quad$ $\tau = t - u$

$du = -d\tau$

$d\tau = -du$

$= \int_{\infty}^{-\infty} f(t-u)g(u)(-du)$

$\tau \to \infty,\ u \to -\infty$

$\tau \to -\infty,\ u \to \infty$

$= \int_{-\infty}^{\infty} f(t-u)g(u)\,du$

$= \int_{-\infty}^{\infty} g(u)f(t-u)\,du$ $\longrightarrow$ u is dummy var

$= (g * f)(t)$

B) $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$

$= \int_{0}^{\infty} f(\tau)g(t-\tau)d\tau$ $\longrightarrow$ $f(\tau) = 0$ if $\tau < 0$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$\longrightarrow g(t-\tau) = 0$ if $t - \tau < 0$

$t < \tau$

Example 5

Convolve the function $f(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & else \end{cases}$ with $g(t) = e^{-t}u(t)$.

Plot the result.

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Case 1 : $0 \leq t \leq T$

$$(f * g)(t) = \int_0^t 1\, e^{-(t-\tau)}\, d\tau$$

$$= \int_0^t e^{-t+\tau}\, d\tau$$

$$= e^{-t} \int_0^t e^{\tau}\, d\tau$$

$$= e^{-t} \left(e^{\tau} \Big|_0^t \right)$$

$$= e^{-t} (e^{t} - 1)$$

$$= e^{0} - e^{-t}$$

$$= 1 - e^{-t}$$

Case 2: $t > T$

$$(f * g)(t) = \int_0^T 1 \cdot g(t-\tau)\, d\tau + \int_T^t 0 \cdot g(t-\tau)\, d\tau \longrightarrow$$ since when $t > T$, $f(t) = 0$ so we only integrate btwn $[0, T]$

$$= \int_0^T e^{-(t-\tau)}\, d\tau$$

$$= \int_0^T e^{-t+\tau}\, d\tau$$

$$= e^{-t} \int_0^T e^{\tau}\, d\tau$$

$$= e^{-t} \left(e^{\tau} \Big|_0^T \right)$$

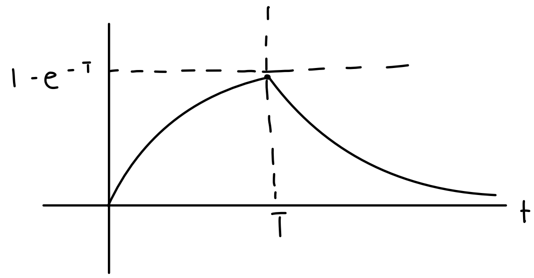
$$= e^{-t} (e^{\tau} - 1)$$

$$= e^{-t+T} - e^{-t}$$

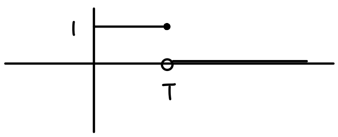
$$= e^{-(t-T)} (1 - e^{-T})$$

$$(f * g)(t) = \begin{cases} 1 - e^{-t}, & 0 \leq t \leq T \\ e^{-(t-T)}(1 - e^{-T}), & t > T \end{cases}$$

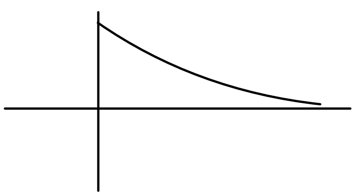
f * g



f



g



Theorem 2: Convolution Theorem
If there exist $\alpha, \beta \in \mathbb{R}$ such that the integrals

$$\int_{-\infty}^{\infty} |f(t)|e^{-\alpha t}dt \qquad \text{and} \qquad \int_{-\infty}^{\infty} |g(t)|e^{-\beta t}dt$$

converge then,

$$\mathscr{L}\{(f * g)(t)\} = F(s)G(s).$$

This theorem states that the Laplace Transform of a convolution is the product of the transforms! A direct result of this allows us to "quickly" compute inverse Laplace transforms:

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Note: for real valued functions f and g, the convolution is a real valued integral (i.e. not a contour integral in the complex plane)!!

Proof:

Suppose $\exists\ \alpha, \beta \in \mathbb{R}$ st $\int_{-\infty}^{\infty} |f(t)| e^{-\alpha t}\, dt$ + $\int_{-\infty}^{\infty} |g(t)| e^{-\beta t}\, dt$ converge.

Let $\delta = \max(\alpha, \beta)$.

We need to show:

1) $\mathscr{L}\{(f * g)(t)\}$ exists
2) $\mathscr{L}\{(f * g)(t)\} = F(s)G(s)$

1) Prove $\int_{-\infty}^{\infty} |(f*g)(t)| e^{-\delta t}\, dt < \infty$ so $\mathscr{L}\{(f*g)(t)\}$ exists.

$$\int_{-\infty}^{\infty} |(f*g)(t)| e^{-\delta t}\, dt$$

$$= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(u-\tau) g(\tau)\, d\tau \right| e^{-\delta u}\, du$$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau) g(\tau)|\, d\tau\, e^{-\delta u}\, du \qquad \longleftarrow \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)||g(\tau)|\, d\tau\, e^{-\delta u}\, du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)||g(\tau)|\, e^{-\delta(u-\tau)} e^{-\delta\tau}\, d\tau\, du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u-\tau)| e^{-\delta(u-\tau)} |g(\tau)| e^{-\delta\tau}\, du\, d\tau \qquad \longleftarrow \begin{array}{l} t = u-\tau \\ dt = du \end{array}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(t)| e^{-\delta t} |g(\tau)| e^{-\delta\tau}\, dt\, d\tau$$

$$= \left(\int_{-\infty}^{\infty} |f(t)| e^{-\delta t}\, dt\right)\left(\int_{-\infty}^{\infty} |g(\tau)| e^{-\delta\tau}\, d\tau\right)$$

$$< \infty$$

$\therefore\ \mathscr{L}\{(f*g)(t)\}$ exists.

2) $\mathcal{L}\{(f*g)(t)\} = \int_{-\infty}^{\infty} (f*g)(t) e^{-st}\, dt$

$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau \right) e^{-st}\, dt$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t-\tau) g(\tau) e^{-st}\, dt\, d\tau$

Fubini's thm

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) g(\tau) e^{-s(u+\tau)}\, du\, d\tau$

$u = t - \tau$ $\quad t = u + \tau$

$du = dt$

$= \left(\int_{-\infty}^{\infty} f(u) e^{-su}\, du \right) \left(\int_{-\infty}^{\infty} g(\tau) e^{-s\tau}\, d\tau \right.$

$= F(s)\, G(s)$

Example 6

Use the convolution theorem to compute $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\}$.

$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$

Let $F(s) = \frac{1}{s-1}$ + $G(s) = \frac{1}{s-2}$

From table : $\underbrace{f(t) = e^{t} \quad + \quad g(t) = e^{2t}}_{\text{one-sided}}$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t e^{\tau} e^{2(t-\tau)} \, d\tau$$

$$= e^{2t} \int_0^t e^{\tau - 2\tau} \, d\tau$$

$$= e^{2t} \int_0^t e^{-\tau} \, d\tau$$

$$= e^{2t} \left(-e^{-\tau}\right)\Big|_0^t$$

$$= e^{2t} \left(-e^{-t} - (-e^{0})\right)$$

$$= e^{2t}(-e^{-t} + 1)$$

$$= -e^{t} + e^{2t}$$

$$= e^{2t} - e^{t}$$

MATH 213 - Lecture 6: Solving DEs via Laplace Transforms

Basic idea:

Consider a Linear DE with <u>constant coefficients</u>

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

with the appropriate number of initial conditions

$$y(0) = y_0, \qquad y'(0) = y_1, \qquad \ldots, \qquad y^{n-1}(0) = y_{n-1}.$$

Taking the Laplace transform of both sides and using the linearity of the Laplace transform gives an equation of the form

$$(s^n + b_{n-1}s^{n-1} + \ldots + b_0 s)Y(s) + G(s, y_0, y_1, \ldots, y_{n-1}) = F(s)$$

In the above $Y(s)$ and $F(s)$ are the Laplace transforms of $y(t)$ and $f(t)$ respectively, G is a function of s and the initial conditions, and the $b_i s$ are coefficients that come from taking the Laplace transform of the derivative terms.

Note: One can find the exact form of this equation but I do not want you to memorize this formula so I am not writing the exact form.

We can solve the above for $Y(s)$ and then (in theory) compute the inverse Laplace transform to find $y(t)$.

Picture:

$$\begin{tikzcd}
\boxed{\text{Linear differential Eq (LDE)}} \arrow[r, "\mathscr{L}"] \arrow[d, "\text{Solve?}"'] & \boxed{\text{Algebraic equation (AE)}} \arrow[d, "\text{Solve!}"] \\
\boxed{\text{Solutionto LDE}} & \boxed{\text{Solution to AE}} \arrow[l, "\mathscr{L}^{-1}"']
\end{tikzcd}$$

One-sided Laplace Table:

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
3. t^n	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$
7. $\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$

Algebraic Properties:

$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Linearity
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$	Time Scaling
$e^{\alpha t} f(t)$	$F(s-\alpha)$	Exponential Modulation
$f(t-T)u(t-T)$	$e^{-sT}\mathscr{L}\{f(t)u(t)\}$	Time-Shifting
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Multiplication by t^n
$(f * g)(t)$	$F(s)G(s)$	Convolution Theorem
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	

Example 1

Solve $y'' = -g$, $y(0) = h_0$, $y'(0) = v_0$. Recall this is the model for a falling ball with initial height h_0 and initial velocity v_0 from the first lecture.
Plot the solution.

$$\mathcal{L}\{y''\} = \mathcal{L}\{-g\}$$ one-sided

$$s^2Y(s) - sY(0) - Y'(0) = -\frac{g}{s}$$

IC

$$s^2Y(s) - sh_0 - v_0 = -\frac{g}{s}$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s^2}\left(-\frac{g}{s} + sh_0 + v_0\right)$$

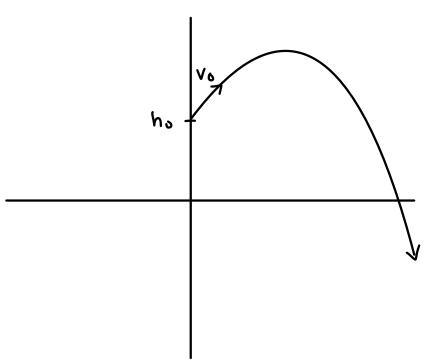
characteristic polynomial

$$= -\frac{g}{s^3} + \frac{h_0}{s} + \frac{v_0}{s^2}$$

Take $\mathcal{L}^{-1}$:

$$Y(t) = -g\left(\frac{1}{2!}t^2\right) + h_0(1) + v_0(t)$$

$$= -\frac{g}{2}t^2 + v_0t + h_0$$



Definition 1: Characteristic Polynomial

The **Characteristic Polynomial** of a linear differential equation with constant coefficients

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

The polynomial multiplied by $Y(s)$ after you take the Laplace transform. This polynomial will always be $p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$.

Example 2

Solve $y'' + y = e^{-2t}\sin(2t)$, $y(0) = 1$, $y'(0) = 0$ and plot the solution.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{e^{-2t}\sin 2t\}$$

$$s^2Y(s) - sY(0) - Y'(0) + Y(s) = \frac{2}{(s+2)^2+4}$$

$$s^2Y(s) + Y(s) = \frac{2}{(s+2)^2+4} + s\cdot(1) + 0$$

$$Y(s) = \underbrace{\frac{1}{s^2+1}}_{\text{char poly}}\left(\frac{2}{(s+2)^2+4} + s\right)$$

$$Y(s) = \frac{2}{(s^2+1)((s+2)^2+4)} + \underbrace{\frac{s}{s^2+1}}_{\cos t}$$

partial frac

$$\frac{2}{(s^2+1)((s+2)^2+4)} = \underbrace{\frac{As+B}{s^2+1}}_{A\cos t + B\sin t} + \underbrace{\frac{C(s+2)+2D}{(s+2)^2+4}}_{e^{-2t}(C\cos 2t + D\sin 2t)}$$

$s = j$ (root of s^2+1):

$$(s^2+1)\left(\frac{2}{(s^2+1)((s+2)^2+4)}\right) = \left(\frac{As+B}{s^2+1} + \frac{C(s+2)+2D}{(s+2)^2+4}\right)(s^2+1)$$

$$\frac{2}{(s+2)^2+4} = As+B + \frac{C(s+2)+2D}{(s+2)^2+4}(s^2+1)$$

$$\frac{2}{(j+2)^2+4} = Aj + B$$

$$\frac{2}{j^2+4j+4+4} = Aj + B$$

$$\frac{2}{4j+7} = Aj + B$$

$$2 = (Aj+B)(4j+7)$$

$$2 = 4Aj^2 + 7Aj + 4Bj + 7B$$
$$2 = -4A + 7Aj + 4Bj + 7B$$
$$2 = -4A + 7B + (7A+4B)j$$

$$2 = -4A + 7B$$
$$B = \frac{2+4A}{7}$$
$$B = \frac{2+4\left(-\frac{8}{65}\right)}{7}$$
$$B = \frac{14}{65}$$

$$0 = 7A + 4B$$
$$0 = 7A + 4\left(\frac{2+4A}{7}\right)$$
$$0 = 7A + \frac{8+16A}{7}$$
$$0 = \frac{49A + 8 + 16A}{7}$$
$$0 = 65A + 8$$
$$A = -\frac{8}{65}$$

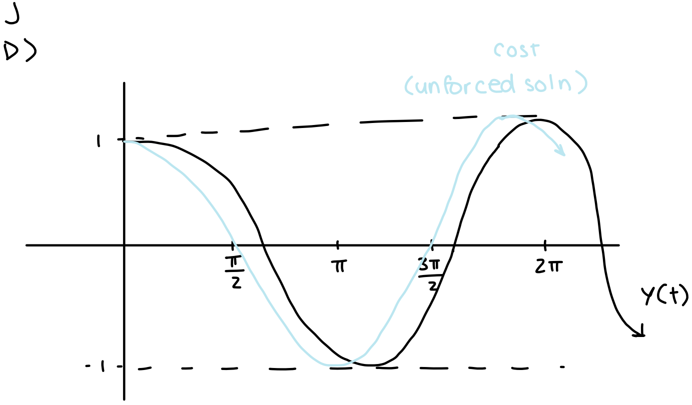
$$(s+2)^2 + 4 = 0$$
$$(s+2)^2 = -4$$
$$(s+2)^2 = 4j^2$$
$$s+2 = 2j$$
$$s = 2j - 2$$

$s = 2j - 2$:

$$((s+2)^2+4)\left(\frac{2}{(s^2+1)((s+2)^2+4)}\right) = \left(\frac{As+B}{s^2+1} + \frac{C(s+2)+2D}{(s+2)^2+4}\right)((s+2)^2+4)$$
$$\frac{2}{s^2+1} = \left(\frac{As+B}{s^2+1}\right)((s+2)^2+4) + C(s+2)+2D$$
$$\frac{2}{(2j-2)^2+1} = C(2j-2+2)+2D$$
$$\frac{2}{4j^2-8j+4+1} = 2Cj + 2D$$
$$\frac{2}{1-8j} = 2Cj + 2D$$
$$2 = (2Cj+2D)(1-8j)$$
$$2 = 2Cj - 16Cj^2 + 2D - 16Dj$$
$$2 = 16C + 2D + j(2C - 16D)$$

$$2 = 16C + 2D$$
$$D = \frac{2-16C}{2}$$
$$D = \frac{2-16\left(\frac{8}{65}\right)}{2}$$
$$D = \frac{1}{65}$$

$$0 = 2C - 16D$$
$$0 = 2C - 16\left(\frac{2-16C}{2}\right)$$
$$0 = 2C - 16 + 128C$$
$$16 = 130C$$
$$C = \frac{8}{65}$$



$$Y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= -\frac{8}{65}\cos t + \frac{14}{65}\sin t + e^{-2t}\left(\frac{8}{65}\cos 2t + \frac{1}{65}\sin 2t\right) + \cos t$$
$$= \frac{57}{65}\cos t + \frac{14}{65}\sin t + e^{-2t}\left(\frac{8}{65}\cos 2t + \frac{1}{65}\sin 2t\right)$$

Example 3
Solve $y'' + 9y = 2\sin(3t)$, $y(0) = 1$, $y'(0) = 0$ and plot the solution.

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{2\sin 3t\}$$

$$s^2 Y(s) - sY(0) - Y'(0) + 9Y(s) = 2\left(\frac{3}{s^2+9}\right)$$

$$Y(s)(s^2+9) = \frac{6}{s^2+9} + s + 0$$

$$Y(s) = \frac{1}{s^2+9}\left(\frac{6}{s^2+9} + s\right)$$

$$Y(s) = \frac{6}{(s^2+9)^2} + \underbrace{\frac{s}{s^2+9}}_{\cos 3t}$$

convolution

$$\frac{6}{(s^2+9)^2} = \left(\frac{2}{3}\right)\frac{3}{s^2+9} \cdot \frac{3}{s^2+9}$$

sin 3t

$$\mathcal{L}^{-1}\left\{\frac{6}{(s^2+9)^2}\right\} = \frac{2}{3}\sin 3t * \sin 3t$$

$$= \frac{2}{3}\int_0^t \sin(3\tau)\sin(3(t-\tau))\,d\tau \quad \longleftarrow \sin a \sin b = \tfrac{1}{2}(\cos(a-b) - \cos(a+b))$$

$$= \frac{1}{3}\int_0^t \cos(-3t+6\tau) - \cos(3t)\,d\tau$$

$$= \frac{1}{3}\left(\int_0^t \cos(-3t+6\tau)\tau - \int_0^t \cos 3t\,d\tau\right) \quad \longleftarrow u = -3t+6\tau$$

$$du = 6\,d\tau$$

$$d\tau = \frac{du}{6}$$

$$= \frac{1}{3}\left(\int_0^t \cos u\,\frac{du}{6} - \cos 3t\,(\tau)\Big|_0^t\right)$$

$$= \frac{1}{3}\left(\frac{1}{6}\sin(6\tau - 3t) - \tau\cos 3t\right)\Big|_0^t$$

$$= \frac{1}{3}\left(\frac{1}{6}\sin 3t - t\cos 3t - \frac{1}{6}\sin(-3t)\right)$$

$$= \frac{1}{3}\left(\frac{1}{6}\sin 3t - t\cos 3t + \frac{1}{6}\sin 3t\right) \quad \longleftarrow \sin(-x) = -\sin x$$

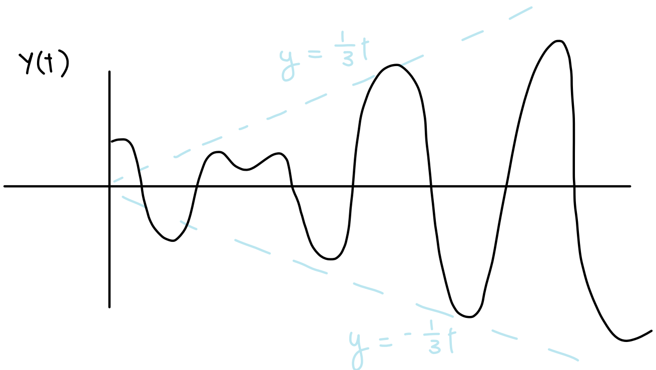
$$= \frac{1}{3}\left(\frac{1}{3}\sin 3t - t\cos 3t\right)$$

$$= \frac{1}{9}\sin 3t - \frac{t}{3}\cos 3t$$

$$Y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{9}\sin 3t \underbrace{- \frac{t}{3}\cos 3t}_{\text{unbounded}} + \cos 3t$$

bounded



$Y(t)$ blows up b/c $\frac{t}{3}\cos 3t$ came from $\mathcal{L}^{-1}\left\{\frac{6}{(s^2+9)^2}\right\}$:

$$\frac{6}{(s^2+9)^2} = \underbrace{\frac{1}{s^2+9}}_{\text{char poly}} \left(\frac{6}{s^2+9}\right)$$

$\mathcal{L}\{2\sin 3t\}$ is forcing term

↳ roots of s^2+9 (char poly) are roots of $\sin 3t$ (aka resonance)

Example 4

Solve $y^{(3)}+2y'=2\sin(3t)$, $y(0)=1$, $y'(0)=1$, $y''(0)=0$ and plot the solution.

$$\mathcal{L}\{y^{(3)}+2y'\}=\mathcal{L}\{2\sin 3t\}$$

$$s^3Y(s)-s^2Y(0)-sY'(0)-Y''(0)+2(sY(s)-Y(0))=\frac{6}{s^2+9}$$

$$(s^3+2s)Y(s)=\frac{6}{s^2+9}+s^2+s+0+2$$

$$Y(s)=\frac{1}{s^3+2s}\left(\frac{6}{s^2+9}+s^2+s+2\right)$$

$$=\frac{6}{s(s^2+2)(s^2+9)}+\frac{s^2}{s(s^2+2)}+\frac{s}{s(s^2+2)}+\frac{2}{s(s^2+2)}$$

$$=\underbrace{\frac{6}{s(s^2+2)(s^2+9)}}_{\text{PF}}+\underbrace{\frac{s}{s^2+2}}_{\cos(\sqrt{2}t)}+\underbrace{\frac{1}{s^2+2}}_{\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)}+\underbrace{\frac{2}{s(s^2+2)}}_{\text{PF}}$$

$$\frac{6}{s(s^2+2)(s^2+9)}=\frac{A}{s}+\frac{Bs+C}{s^2+2}+\frac{Ds+E}{s^2+9}$$

$s=0$:

$$s\left(\frac{6}{s(s^2+2)(s^2+9)}\right)=\left(\frac{A}{s}+\frac{Bs+C}{s^2+2}+\frac{Ds+E}{s^2+9}\right)s$$

$$\frac{6}{(s^2+2)(s^2+9)}=A+\left(\frac{Bs+C}{s^2+2}+\frac{Ds+E}{s^2+9}\right)s$$

$$\frac{6}{18}=A$$

$$A=\frac{1}{3}$$

$s=\sqrt{2}j$:

$$(s^2+2)\left(\frac{6}{s(s^2+2)(s^2+9)}\right)=\left(\frac{A}{s}+\frac{Bs+C}{s^2+2}+\frac{Ds+E}{s^2+9}\right)(s^2+2)$$

$$\frac{6}{s(s^2+9)}=Bs+C+\left(\frac{A}{s}+\frac{Ds+E}{s^2+9}\right)(s^2+2)$$

$$\frac{6}{\sqrt{2}j(-2+9)}=B(\sqrt{2}j+C)$$

$$\frac{6}{7\sqrt{2}j}=\sqrt{2}Bj+BC$$

$$6=14Bj^2+7\sqrt{2}BCj$$

$$6=-14B+7\sqrt{2}BCj$$

$6=-14B$ $\quad$ $0=7\sqrt{2}BC$

$B=-\frac{3}{7}$ $\quad$ $C=0$

$s=3j$:

$$(s^2+9)\left(\frac{6}{s(s^2+2)(s^2+9)}\right)=\left(\frac{A}{s}+\frac{Bs+C}{s^2+2}+\frac{Ds+E}{s^2+9}\right)(s^2+9)$$

$$\frac{6}{s(s^2+2)}=Ds+E+\left(\frac{A}{s}+\frac{Bs+C}{s^2+2}\right)(s^2+9)$$

$$\frac{6}{3j(-9+2)} = 3Dj + E$$

$$\frac{6}{-21j} = 3Dj + E$$

$$6 = -63Dj^2 + 21Ej$$

$$6 = 63D + 21Ej$$

$6 = 63D$ $\qquad$ $0 = 21E$

$D = \frac{2}{21}$ $\qquad$ $E = 0$

$$\frac{2}{s(s^2+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$$

$s = 0$:

$$s\left(\frac{2}{s(s^2+2)}\right) = \left(\frac{A}{s} + \frac{Bs+C}{s^2+2}\right)s$$

$$\frac{2}{s^2+2} = A + \left(\frac{Bs+C}{s^2+2}\right)s$$

$$\frac{2}{2} = A$$

$A = 1$

$s = 1$:

$$\frac{2}{1(1+2)} = \frac{1}{1} + \frac{B+C}{1+2}$$

$$\frac{2}{3} = 1 + \frac{B+C}{3}$$

$$-\frac{1}{3} = \frac{B+C}{3}$$

$$-1 = B + C$$

$$-1 - B = C$$

$$-1-(-1) = C$$

$C = 0$

$s = -1$:

$$\frac{2}{-1(1+2)} = \frac{1}{-1} + \frac{-B+C}{1+2}$$

$$-\frac{2}{3} = -1 + \frac{-B+C}{3}$$

$$\frac{1}{3} = \frac{-B+C}{3}$$

$$1 = -B + C$$

$$1 = -B - 1 - B$$

$$2 = -2B$$

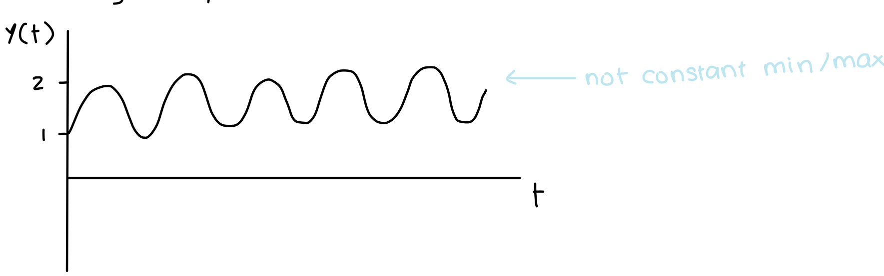
$B = -1$

$$Y(s) = \frac{1}{3s} - \frac{3}{7}\frac{s}{s^2+2} + \frac{2}{21}\frac{s}{s^2+9} + \frac{s}{s^2+2} + \frac{1}{s^2+2} + \frac{1}{s} - \frac{s}{s^2+2}$$

$$Y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{3} - \frac{3}{7}\cos(\sqrt{2}t) + \frac{2}{21}\cos(3t) + \cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t) + 1 - \cos(\sqrt{2}t)$$

$$= \frac{4}{3} - \frac{3}{7}\cos(\sqrt{2}t) + \frac{2}{21}\cos(3t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$$



Extensions of our method:

Our method also "works" for some Linear ODEs with non-constant coefficient. The problems are 1) evaluating the Laplace transform of $a(t)y^{(n)}(t)$ for some given function $a(t)$ and a unknown function $y^{(n)}$ and 2) we end up with another (but simpler) differential equation to solve. If $a(t) = t^n$ then this is particularly "nice"!

Example 5

Solve $y'' + 2ty' + y = 0$, $y(0) = 0$, $y'(0) = 1$ and plot the solution.

$$\mathcal{L}\{y'' + 2ty' + y\} = 0$$

$$s^2 Y(s) - sY(0) - Y'(0) + 2\mathcal{L}\{ty'\} + Y(s) = 0$$

$$\boxed{\begin{aligned} \mathcal{L}\{ty'\} &= -\frac{d}{ds}\mathcal{L}\{y'\} \\ &= -\frac{d}{ds}(sY(s) - Y(0)) \\ &= -(Y(s) + sY'(s)) \end{aligned}}$$

$$s^2 Y(s) - s(0) - 1 - 2(Y(s) + sY'(s)) + Y(s) = 0$$

$$Y'(s) + \frac{1-s^2}{2s} Y(s) = \frac{1}{2s}$$

Solve for $Y(s) =$ ______

Take $\mathcal{L}^{-1}\{Y(s)\}$

The coefficients need not be complex numbers, they can also be matrices. Consider a DE of the form

$$I\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

where a_i is a $n \times n$ matrix, I is the $n \times n$ identity matrix and the functions $y(t)$ $f(t)$ are n dimensional vectors.

Example 6: Minecraft Chickens
Suppose we have a population of chickens, $C(t)$ and eggs $E(t)$ and suppose that the eggs turn into chickens at a rate α, and the chickens lay eggs at a rate β derive a coupled system of DEs to model the populations of chickens and eggs.

$$\frac{d}{dt}(E(t)) = \beta C(t) - \alpha E(t)$$

eggs hatch

$$\frac{d}{dt}(C(t)) = \alpha E(t)$$

$$I\frac{d}{dt}\begin{bmatrix} E \\ C \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} E \\ C \end{bmatrix}$$

$$= \vec{0}$$

Looking ahead:

- Many applications of ECE/software engineering involve solving a system of linear DEs (i.e. multivariable systems).
- In control theory, we often want to find initial conditions so that we can "solve for $f(t)$". i.e. pick $f(t)$ and the ICs so that the solution to the DE does something we want.

MATH 213 - Lecture 7: Rational Functions, Poles, and Initial and Final Value Theorems

Lecture goals: Understand what a pole is and understand the initial and final value theorems.
After computing the Laplace transform of a function we usually end up with a function that is the ratio of polynomials. Such functions have a name.

Definition 1: Rational Function
A **rational function** is a function of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

In Math 119 we often simplified such functions with no regards to undefined values. As an example we might say

$$\frac{x}{x} = 1.$$

Formally at $x = 0$ this is not true since $\frac{0}{0}$ is undefined but we ignored that and did it anyways. We will make this idea a bit more formal to justify our use of this method for MATH 213.

Definition 2: Regularization
A function f is the **regularization** of a function g if

1. For all x in the domain of g, $f(x) = g(x)$.
2. For all x such that $\lim_{t\to x} g(\overset{t}{\cancel{x}})$ exists but $f(x)$ is undefined, $f(x) = \lim_{t\to x} g(\overset{t}{\cancel{x}})$.

Example 1
$\frac{s}{s^2+2s+2}$ is the regularization of $\frac{s(s+3)}{(s^2+2s+2)(s+3)}$ and also of $\frac{s}{s^2+2s+2}$.

Theorem 1
If $f(x)$ is the regularization of a function g that has a finite number of discontinuities then $\int f(x)dx = \int g(x)dx$.

When we write $\frac{x}{x} = 1$ we are redefining $=$ to be the equivalence relation "have the same regularization". We did this in MATH 119 because of Theorem 1 above and we will do in in MATH 213 for the same reason.

Back to MATH 213:

Definition 3: Finite Zeros and Finite Poles

A rational function $f(x) = \frac{P(x)}{Q(x)}$ that is its own ~~continuous continuation~~ regularization the roots of $P(x)$ are called the **(finite) zeros of** f and the roots of $Q(x)$ are called the **(finite) poles of** f.

Example 2

Find the finite poles and finite zeros of

$$F(s) = \frac{s}{s^2 + 2s + 2}.$$

Zero: $s = 0$

Poles:

$$s^2 + 2s + 2 = 0$$
$$s^2 + 2s + 1 - 1 + 2 = 0$$
$$(s+1)^2 + 1 = 0$$
$$(s+1)^2 = -1$$
$$s = -1 \pm j$$

Note that in the above example $\lim_{s\to\infty} F(s) = 0$.

Definition 4: Poles and Zeros at Infinity

Given a rational function $f(x) = \frac{P(x)}{Q(x)}$ has a **zero at infinity** if $\lim_{s\to\infty} \cancel{F}\,f(s) = 0$. $f(x)$ is said to have a **pole at infinity** if its reciprocal has a zero at infinity.

Example 3

$$F(s) = \frac{s}{s^2 + 2s + 2}$$

has a zero at infinity and $\frac{1}{F(s)}$ has a pole at infinity.

Convention 1

Unless otherwise stated when we talk about poles and zeros we mean the finite poles and finite zeros.

Definition 5: Proper and Strictly Proper Rational Functions
A rational function is **proper** if the degree of its numerator is less than or equal to that of its denominator. A function is **strictly proper** if the degree of the numerator is strictly less than that of its denominator.

Example 4

$$F(s) = \frac{s}{s^2 + 2s + 2}$$

is both proper and strictly proper.

We care about these ideas because the inverse Laplace transform is a complex valued integral and because of this, we can compute the inverse transform of $F(s)$ by **only** knowing information about the poles of $F(s)$.

This is closely related to how Green's theorem (MATH 119) can be use to compute closed line integrals of the vector field $\vec{F}(x, y)$ by examining the points where $\nabla \times \vec{F}$ is undefined.

While we will not learn how to directly compute inverse Laplace transforms using poles (google residue theorem if interested), we will cover two important results that are related to poles (and more later).

The initial value theorem:

Definition 6: Piecewise Continuous
A function $f(x)$ is **piecewise-continuous on a given finite interval** if

1. f has a finite number of discontinuities in that interval and
2. for each discontinuity x_0 both the left and right hand limits exits (note they can be different values).

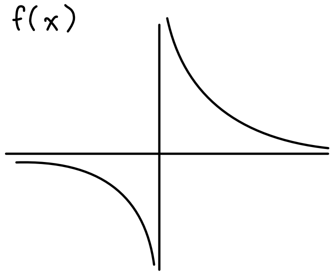
A function $g(x)$ is piecewise-continuous if it is piecewise-continuous on all finite intervals.

Example 5

Determine which (if any) of the following are piecewise-continuous:

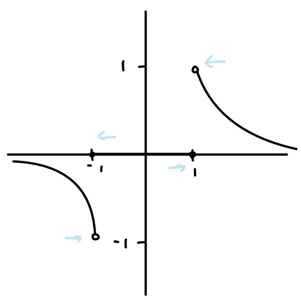
- $f(x) = 1/x$
- $g(x) = \begin{cases} 1/x & |x| > 1, \\ 0 & else \end{cases}$

f(x)



$\lim_{x \to 0\pm} f(x) = \pm\infty$ DNE so f is not PW-C

g(x)



g is PW-C

Theorem 2: Initial Value theorem
If $f(t)$ is piecewise-continuous and $\int_0^\infty |f(t)|e^{-\alpha t}$ converges for some $\alpha \in \mathbb{R}$ then

$$f(0^+) = \lim_{s\to+\infty} sF(s)$$

Understanding this theorem:

- The above gives us a way of computing the IC of $f(t)$ from $F(s)$ **without** computing the inverse Laplace transform and instead evaluating a limit in the complex plane.
- s is generally complex so the limit is in the complex plane **NOT** the real number line thus:
  - Re(s) tends to infinity
  - Im(s) can do anything (stay 0, oscillate, go to infinity, do random things, etc.)
- For many problems we can treat the limit as the standard limit you are used to working with.

Before seeing a proof we examine some examples

Example 6
For the following functions $F(s)$ compute $f(0^+)$ using the initial value theorem and verify the result using the inverse Laplace transform

A. $F(s) = e^{-sT}\frac{1}{s^2}$, $T > 0$

B. $F(s) = \frac{1}{s}$

C. $F(s) = \frac{s}{s^2+\omega^2}$

A) IVT:

$$f(0^+) = \lim_{s\to+\infty} s\frac{e^{-sT}}{s^2}$$

$$= \lim_{s\to+\infty} \frac{1}{se^{sT}}$$

$$= \lim_{s\to+\infty} \frac{1}{se^{Re(s)T}(\cos(Im(s)T) + j\sin(Im(s)T))}$$

$$= 0$$

Euler's formula:
$e^{j\theta} = \cos\theta + j\sin\theta$

$$f(t) = \mathcal{L}^{-1}\left\{e^{-sT}\frac{1}{s}\right\}$$

$$= (t-T)u(t-T)$$

B) $f(0^+) = \lim_{s \to +\infty} s \cdot \frac{1}{s}$

$= \lim_{s \to +\infty} 1$

$= 1$

$f(t) = \mathcal{L}^{-1} \{ \frac{1}{s} \}$

$= 1$

C) $f(0^+) = \lim_{s \to +\infty} s \cdot \frac{s}{s^2 + \omega^2}$

$= \lim_{s \to +\infty} \frac{s^2}{s^2 + \omega^2}$

$= 1$

$f(t) = \mathcal{L}^{-1} \{ \frac{s}{s^2 + \omega^2})$

$= \cos(\omega t)$

Proof of the initial value theorem:

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\int_0^\infty f(t)e^{-st}\,dt$$

$$= \lim_{s\to\infty} s\int_0^T f(t)e^{-st}\,dt + s\int_T^\infty f(t)e^{-st}\,dt \qquad T > 0$$

Note:

$$s\int_T^\infty f(t)e^{-st}\,dt = s\int_T^\infty f(t)e^{-\alpha t}e^{-(s-\alpha)t}\,dt$$

$$\le s\int_T^\infty |f(t)|\,\underset{\uparrow\ \text{always +ve}}{e^{-\alpha t}}\,|e^{-(s-\alpha)t}|\,dt$$

$$\le s\underbrace{|e^{-(s-\alpha)T}|}_{0}\underbrace{\int_T^\infty |f(t)|e^{-\alpha t}\,dt}_{\text{some constant } c} \longrightarrow \text{since } T \le t \text{, then } e^{-T} \ge e^{-t}$$

$$\lim_{s\to\infty} s\int_T^\infty f(t)e^{-st}\,dt = 0 \qquad \text{Squeeze thm}$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\int_0^T f(t)e^{-st}\,dt \longrightarrow f(t) = f(0) + \underbrace{f'(0)t + \frac{f''(0)}{2!}t^2 + \dots}_{O(1)}$$

$$= \lim_{s\to\infty} s\int_0^T (f(0) + O(1))e^{-st}\,dt$$

$$= \lim_{s\to\infty} \left(s f(0)\int_0^T e^{-st}\,dt + s\int_0^T O(t)e^{-st}\,dt\right)$$

$$= \lim_{s\to\infty} \left(s f(0)\,\frac{1-e^{-sT}}{s} + s\,O(1)\underbrace{\frac{-1}{s}e^{-sT}}_{0}\right)$$

$$= f(0)\lim_{s\to\infty}(1-e^{-sT}) + 0$$

$$= f(0)(1-0)$$

$$= f(0)$$

Example 7:
Prove the initial value theorem in the special case where $F(s)$ is a proper rational function.

The final value theorem:

Theorem 3: Final Value Theorem
For a function $F(s)$, if

- is a proper rational function
- has the property that all the poles have real parts that are strictly negative with the exception of a single pole (of order 1)[a] at $s = 0$

or if $F(s)$ is the product of a function satisfying the above conditions multiplied by a complex exponential e^{sT}, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

If the poles of a rational function do not satisfy the above condition, then $\lim_{t\to\infty} f(t)$ does not exist.

[a] i.e. $\frac{A}{s}$

Understanding this theorem:

- When it applies, the above gives us a way of computing the "final value" of $f(t)$ from $F(s)$ **without** computing the inverse Laplace transform and instead evaluating a limit in the complex plane.
- Again s is generally complex so the limit is in the complex plane **NOT** the real number line. Limits at finite points in the complex plane are a bit different from infinite limits.
  - Here the formal definition of a limit of a complex valued function $F(z)$ at $x_0 \in \mathbb{C}$ is the constant $L \in \mathbb{C}$ such that for every $\epsilon \in \mathbb{R}^+$ there is a $\delta \in \mathbb{R}^+$ such that if $|z - z_0| < \delta$ then $|F(z) - L| < \epsilon$ where $|\cdot|$ is the standard modulus function.
  - In practice for any problems of computing limits at 0, you can treat the limit as the standard limit you are used to working with.
- Connection to 2D limits from MATH 119.

– Provided that the limits exist, we have

$$\lim_{s\to 0} F(s) = \lim_{(a,b)\to(0,0)} \mathrm{Re}(F(a+bj)) + j \lim_{(a,b)\to(0,0)} \mathrm{Im}(F(a+bj)).$$

Before seeing a proof we examine some examples.

Example 8

For the following functions $F(s)$, use the final value theorem (if applicable) to determine the long time behaviour of the function $f(t)$. Compare this to the result you find from computing the inverse transform.

A. $F(s) = \frac{10}{5s+1}\frac{1}{s}$

B. $F(s) = \frac{1}{s}$

C. $F(s) = \frac{1}{s^2}$

D. $F(s) = \frac{s}{s^2+\omega^2}$, $\omega \in \mathbb{R}^+$

E. $F(s) = \frac{6}{(s^2+9)^2}$ ← From Ex 3 Lecture 6.

A) $\frac{\text{deg } 0}{\text{deg } 2}$ ✓

poles : $s = 0$, $s = -\frac{1}{5}$ ✓

↓ ok b/c $\frac{1}{s}$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \frac{10}{(5s+1)s}$$

$$= \lim_{s\to 0} \frac{10}{5s+1}$$

$$= 10$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{10}{5s^2+s}\right\} = 10 - 10e^{-\frac{t}{5}}$$ → so $\lim_{t\to\infty} f(t) = 10$

B) $\frac{\text{deg } 0}{\text{deg } 1}$ ✓

pole : $s = 0$ ✓

↓ ok b/c $\frac{1}{s}$

$$\lim_{t\to\infty} f(t) = \lim_{s\to\infty} s\left(\frac{1}{s}\right)$$

$$= 1$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$ → so $\lim_{t\to\infty} f(t) = 1$

C) pole : $s = 0$ w/ 2 poles

$\lim_{t\to\infty} f(t)$ DNE

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ $\longrightarrow$ so $\lim_{t\to\infty} f(t) = \infty$ (blows up)

D) poles: $s = \pm \omega j \longrightarrow Re(s) = 0$

$\lim_{t\to\infty} f(t)$ DNE

$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos(\omega t)$ $\longrightarrow$ so $\lim_{t\to\infty} f(t)$ oscillates

E) poles: $s = \pm 3j \longrightarrow Re(s) = 0$

$\lim_{t\to\infty} f(t)$ DNE

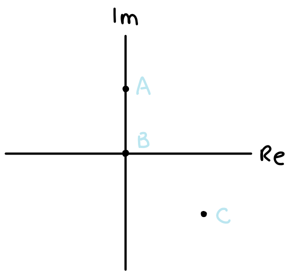
$$f(t) = \underbrace{\frac{1}{9}\sin 3t}_{\text{oscillates}} - \underbrace{\frac{t}{3}\cos(3t)}_{\text{blows up}}$$

Can't use FVT to show why things go wrong (no distinction btwn oscillation + blowing up)

Note: When the final value theorem does not apply, we do not know if the limit diverges or if it oscillates. Hence, we need more math to be able to ascertain if a solution is bounded or not (without computing the inverse transform)! See examples 8C and 8E for unstable systems and D for a stable but non-convergent system.

Proof of the final value theorem:

If F is proper rational fcn, but poles don't satisfy conditions :



A : pole on $s = aj$

$f(t) = c_1 \cos(at) + c_2 \sin(at) +$ stuff

$\lim_{t \to \infty} f(t)$ DNE

B : higher order pole at $s = 0$

$f(t) = A_n t^n + \ldots + A_0 +$ stuff

$\lim_{t \to \infty} f(t)$ DNE

C: pole at $s = a + bj$ w/ $a > 0$

$f(t) = e^{at} (c_1 \cos(at) + c_2 \sin(at)) +$ stuff

$\lim_{t \to \infty} f(t)$ DNE

If conditions on poles are satisfied:

$$F(s) = \frac{A}{s} + \sum_i \frac{B_{k_i} s^{k_i - 1} + \ldots + B_{k_i}}{(s - p_i)^{k_i}}$$

So, $f(t) = A +$ exponentially decaying terms. Then, $\lim_{t \to \infty} f(t) = A$. By Heaviside fcn, we have $A = \lim_{s \to 0} sF(s)$.

MATH 213 - Lecture 8: Zero-input, zero state response, and the delta function

Lecture goals: Understand how poles effect the inverse Laplace transform, what the zero-input and zero-state responses are. Know what the delta function is and its properties are.

Understanding Poles:

The final value theorem is useful but we can actually do a bit better so we will examine and analyze all the various cases of a function with a single pole.
Consider a function $F(s)$ with a single pole of (natural number greater than 1) order n at $a + bi$:

$$f(t) = \frac{1}{(s - (a + bi))^n}.$$

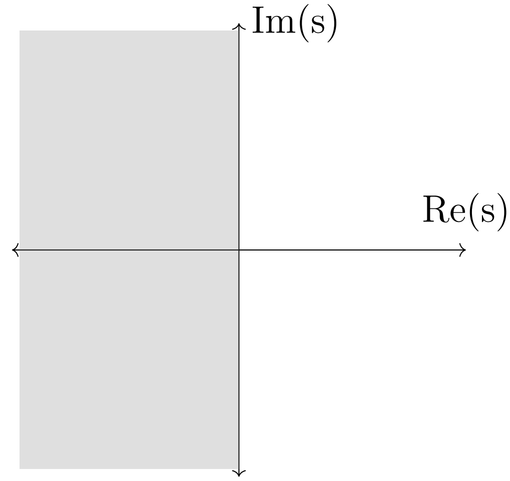
By the Fundamental Theorem of Algebra along with partial fraction decomposition **<u>all</u>** proper rational functions can be decomposed to a sum of functions of this form!

Taking the inverse Laplace transform gives

$$\begin{aligned}
f(t) &= \mathscr{L}^{-1}\{F(s)\} \\
&= \mathscr{L}^{-1}\left\{\frac{1}{(s - (a + bi))^n}\right\} \\
&= e^{(a+bi)t}\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} && \text{Exponential Modulation} \\
&= e^{(a+bi)t}\frac{t^{n-1}}{n!} && t^n \text{ transform}
\end{aligned}$$

Thus **all** functions that have a rational Laplace transform (the ones we have covered thus far), have an inverse transform of the form of a weighted sum of terms of the form of the above.
We hence analyze these functions via case analysis!

$a < 0, b \neq 0$: In this case the pole is in the region shown here (excluding the real and imaginary axes)



In these cases

$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = e^{at}\frac{t^{n-1}}{n!}(\cos(bt) + j\sin(bt))$$

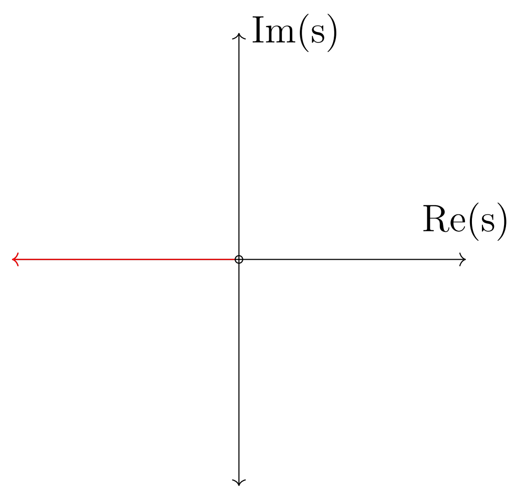
Since $a < 0$ and both t^{n-1} and the trig terms $\underbrace{\text{grow slower}}_{\text{don't grow}}$ than e^{at} decays,

$$\lim_{t\to\infty} f(t) = 0.$$

In this case the final value theorem applies and tells us that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{s}{(s-(a+bi))^n} = 0.$$

$a < 0, b = 0$: In this case the pole is in the region shown here (excluding the origin)



In these cases

$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = e^{at}\frac{t^{n-1}}{n!}$$

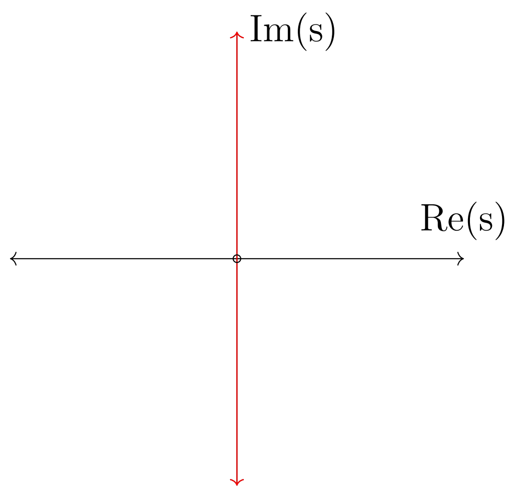
Since $a < 0$ and t^{n-1} grows slower than e^{at} decays,

$$\lim_{t\to\infty} f(t) = 0.$$

In this case the final value theorem applies and tells us that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{s}{(s-a)^n} = 0.$$

$a = 0$,$b \neq 0$: In this case the pole is in the region shown here (excluding the origin)



In these cases

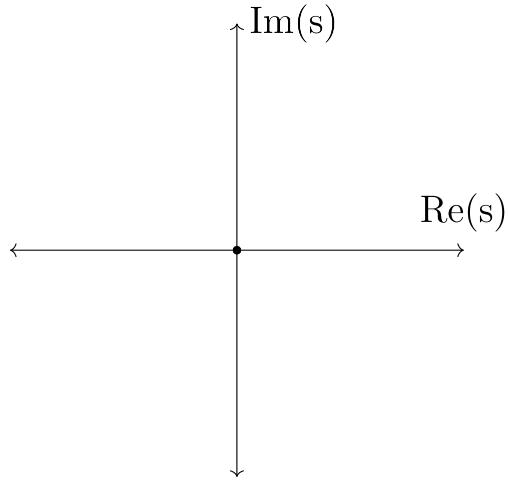
$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = \frac{t^{n-1}}{n!}(\cos(bt) + j\sin(bt)))$$

In these cases if $n = 1$ then we have bounded oscillations (and hence $\lim_{t\to\infty} f(t)$ DNE) or we have growing oscillations (and hence the limit also does not exist).

$n > 1$.

In this case the final value theorem tells us the limit does not exist (since the poles have a real part of 0 and the imaginary part is also non-zero) but does not differentiate between bounded oscillations and unbounded oscillations.

$a = 0, b = 0$: In this case the pole is in the region shown here



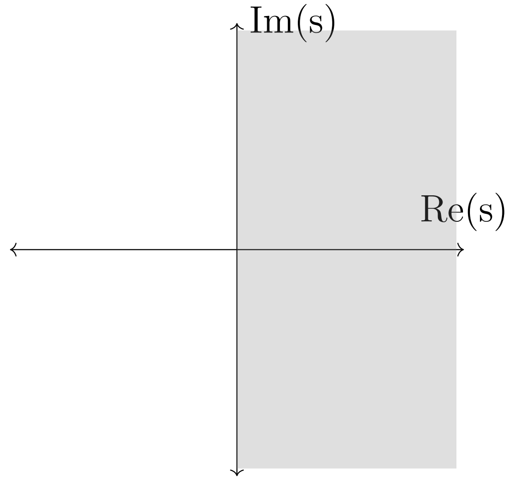
In these cases

$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = \frac{t^{n-1}}{n!}$$

In these cases when $n = 1$ then we have a constant solution (and hence $\lim_{t\to\infty} f(t) = 1$). Else the solution grows and the limit diverges.

The above is what the final value theorem tells us but it does not explicitly state that the function grows unbounded.

$a > 0, b \in \mathbb{R}$: In this case the pole is in the region shown here (including the real axis but excluding the imaginary axis)



In these cases

$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = e^{at}\frac{t^{n-1}}{n!}(\cos(bt) + j\sin(bt))$$

In these cases the exponential term diverges and hence regardless of the values of n or b the transform diverges. This is again what the final value theorem tells us but the details of oscillation or the addition of a power of t are lost.

In conclusion the location of the poles tells us about the behaviour of the solution at infinity (via the final value theorem) but also the initial values (via the initial value theorem).

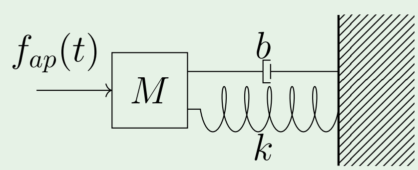
If only... there was a way to <u>control</u> where the poles are and to thus <u>control</u> the behaviour of the solution to a DE we care about...

Enter the ideas of the zero-input response, the zero-state responses and the transfer function.

To motivate the zero-input response, the zero-state response and the transfer function recall the Linear Harmonic Oscillator for Lecture 2:

Example 1: Linear Harmonic Oscillator

The DE that models the spring mass system:



where $f_{ap}(t)$ is an applied force divided by m, $\frac{b}{m}$ is the coefficient of friction and $\frac{k}{m}$ is the spring constant is

$$y'' + by' + ky = f_{ap}(t).$$

Use the Laplace transform method to solve this DE as far as possible (i.e. I am not telling you what $f_{ap}(t)$ is...).

The goal of this problem is to define some new terms "naturally"

$$\mathcal{L}\{y'' + by' + ky\} = \mathcal{L}\{f_{ap}(t)\}$$

$$s^2 Y(s) - sY(0) - Y'(0) + b(sY(s) - Y(0)) + kY(s) = F_{ap}(s)$$

$$Y(s)(s^2 + bs + k) = F_{ap}(s) + (s+b)Y(0) + Y'(0)$$

$$Y(s) = \underbrace{\boxed{\frac{1}{s^2+bs+k}} F_{ap}(s)}_{\substack{\text{zero-state response} \\ \text{(2nd term disappears} \\ \text{when init conds = 0)}}} + \underbrace{\frac{(s+b)Y(0) + Y'(0)}{s^2 + bs + k}}_{\substack{\text{zero-input response} \\ \text{(1st term disappears} \\ \text{when forcing term } F_{ap} = 0)}}$$

transfer fcn

Example 2: Primitive Control Example
Assuming that $y(0) = 1$ and $y'(0) = 0$ in the above example (i.e. the spring is held at $y(0) = 1$ with no initial velocity) find conditions on b, k and $F_{ap}(s)$ so that the solution to the DE approaches a finite value as $t \to \infty$.

ICs give :

$$Y(s) = \frac{F_{ap}(s)}{s^2 + bs + k} + \frac{s + b}{s^2 + bs + k}$$

We want $\lim_{t \to \infty} Y(t) = c$

↳ $F_{ap} = \frac{p(x)}{q(x)}$ where $\deg(p) < \deg(q)$

↳ poles of $Y(s)$ all have $Re(s) < 0$ or $s = 0$ w/ only 1 pole or zero-state + zero-input responses kill bad poles

$$s^2 + bs + k = 0 \Rightarrow s_{\pm} = \frac{-b \pm \sqrt{b^2 - 4k}}{2}$$

↳ need $Re(s_{\pm}) < 0$

↳ choose $F_{ap}(s) = -s - b$ so everything cancels to 0

Definition 1: Zero-State and Zero-Input Responses and the Transfer function

Given a linear DE with constant coefficients of the form

$$\frac{d^n}{dt^n}y + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y + \ldots + a_0 y = f(t)$$

along with the needed initial conditions $y(0)$, $y'(0)$, ..., $y^{(n-1)}(0)$, the Laplace transform of the equation will be of the form

$$Y(s) = \frac{F(s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + \frac{g(a_0, \ldots, a_{n-1}, y(0), \ldots, y^{(n-1)}(0), s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}$$

where g is some function of the ICs and coefficients.

- The **zero-input response** is $\frac{g(a_0,\ldots,a_{n-1},y(0),\ldots,y^{(n-1)}(0),s)}{s^n+a_{n-1}s^{n-1}+\ldots+a_0}$. This is the response of the system the DE models to the initial conditions (i.e. when the forcing term is 0).
- The **zero-state response** is $\frac{F(s)}{s^n+a_{n-1}s^{n-1}+\ldots+a_0}$. This is the response of the system the DE models to the forcing term (i.e. when the ICs are all 0).
- The **transfer function** is $\frac{1}{s^n+a_{n-1}s^{n-1}+\ldots+a_0}$. This function determines the zero-input and the effect of the forcing term.

To help understand the role of the transfer function in the relation between the zero input response and the zero-state response, we introduce the idea of **generalized functions**.

In 1927, Physicist Paul Dirac in his "The physical interpretation of the quantum dynamics" paper introduced the function now known as the "Dirac delta" function. ↑ Interesting read for anyone interested in the history of modern mathematics/-physics.

Definition 2: Dirac Delta Function

The $\delta(x)$ function is defined as the "function" (called a special function or a distribution[a]) that satisfies the properties that:

$$\delta(x) = 0 \text{ when } x \neq 0$$
$$\int_{-\infty}^{\infty} \delta(x) = 1$$

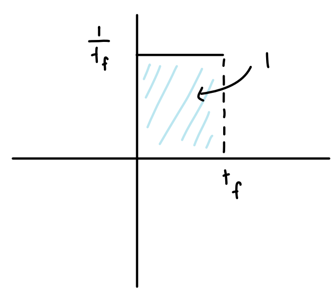
[a] Think probability distribution

Of course there is **<u>no</u>** classical function that satisfies the above conditions but we can think of δ as the limit of a collection of functions which we now explore.

Suppose you hit a nail with a hammer and do a total amount of work of 1 unit. If $f(t)$ is the amount of force applied over time then $1 = \int_0^{t_f} f(t)dt$.

If $f(t)$ is constant while in contact the the nail and 0 else, then $f(t) = \begin{cases} \frac{1}{t_f} & 0 \leq t \leq t_f \\ 0 & else \end{cases}$.

How long was the hammer in contact with the nail?



Not long! What if it was 0 seconds? In this case

$$f(t) = \begin{cases} \text{"}\infty\text{"} & t = 0 \\ 0 & else \end{cases}$$

and

$$\int_a^b f(t) = \begin{cases} 1, & a \leq 0 \leq b \\ 0 & else \end{cases}$$

We call this function $\delta(t)$. Mathematically one can make this definition formal but to do so requires a lot of mathematical background.

Theorem 1
If f is a "well-behaved" function defined at t then

$\delta(t-\tau) = \begin{cases} \infty & t=\tau \\ 0 & \text{else} \end{cases}$

$$\begin{aligned}\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty} f(t)\delta(t-\tau)d\tau \\ &= f(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau \\ &= f(t).\end{aligned}$$

The above is the main property of the delta function we need.

Example 3
Compute $\mathscr{L}\{\delta(t)\}$ and find the zero-state response of the harmonic oscillator (Ex 1) to the forcing term $f_{ap}(t) = \delta(t)$.

$$\begin{aligned}\mathscr{L}\{\delta(t)\} &= \int_0^{\infty} \delta(t) e^{-st} dt \\ &= \int_0^{\infty} e^{-st} \delta(t-0)\, dt \end{aligned}$$

$\longrightarrow$ from $\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$:
$\hookrightarrow \tau \to 0$

$$\begin{aligned} &= e^{-s(0)} \\ &= 1 \end{aligned}$$

$$Y(s) = \frac{F_{ap}(s)}{s^2+bs+k}$$

$$= \frac{1}{s^2+bs+k}$$ $\longleftarrow$ if $F_{ap}(t) = \delta(t)$

$$= T(s)$$ $\longleftarrow$ transfer fcn

In general if $Y(s) = \frac{F_{ap}(s)}{P(s)}$ where $F_{ap}(s)$ is the transform of the forcing term and $P(s)$ is the characteristic polynomial, then

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{F_{ap}(s)}{P(s)}\right\} \\ &= f_{ap}(t) * \mathscr{L}^{-1}\left\{\frac{1}{P(s)}\right\} \\ &= f_{ap}(t) * \mathscr{L}^{-1}\{T(s)\} \end{aligned}$$

where $T(s)$ is the system's transfer function.

Hence a system's response to any function $f_{ap}(t)$ is simply the convolution of $f_{ap}(t)$ with the systems response to the delta function – called the system's impulse response.

This is a theorem!

> **Theorem 2**
> The zero-state response of a linear DE is the convolution of the input (i.e. forcing term) with the systems impulse response. $\longrightarrow \mathscr{L}^{-1}\{T(s)\}$

For the proof, see the above comments.

With the above results in mind, the solution to any linear DE with constant coefficients can be written as

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\overbrace{\frac{F(s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}}^{\text{zero-state}} + \overbrace{\frac{g(a_0, \ldots, a_{n-1}, y(0), \ldots, y^{(n-1)}(0), s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}}^{\text{zero-input}}\right\} \\ &= f(t) * T(t) + \sum_{k=1}^{M} c_k t^{N_k - 1} e^{\lambda_k t} \end{aligned}$$

where T is the inverse Laplace transform of the transfer function, λ_k are the poles of the zero-input function, N_k is the order of the kth pole, c_k is the coefficient given by the partial fraction decomposition and M is the number of poles (which will always be finite because of the form of the second term).

MATH 213 - Lecture 9: delta function derivatives and intro to systems and signals

Lecture goals: Understand how to differentiate the delta function. Know what systems and signals are.

Derivatives of $\delta(t)$:

It turns out that it is useful to take the derivatives of $\delta(t)$.

In MATH 119 we told you that we can't differentiate functions at discontinuities so?

Recall that $\delta(t)$ is not a function in the classic sense but is instead a distribution, a collection of functions that have a common property when we take a limit[1]. Hence to differentiate such functions, we need to redefine what it is to be differentiable.

We will not do this formally but will instead play with delta until we magically arrive at the result.

We will start with the derivative of a simpler function.

[1] Definition for MATH 213. For a formal definition you can read wiki ← clickable.

Example 1: Derivative of the Heaviside function

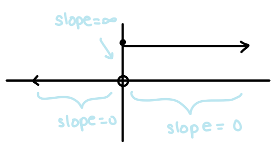
Find a "function" $f(t)$ such that $\int_{-\infty}^{t} f(t)dt = u(t)$ where $u(t)$ is the usual notation for the Heaviside function.

Such a function $f(t)$ would be the "derivative" of the Heaviside function since mechanically differentiating both sides of the previous equation under the assumption that $\lim_{t\to\infty} f(t) = Const.$ gives

$$f(t) = u'(t)$$

M1:

$$\int_{-\infty}^{t} f(x)\,dx = u(t)$$



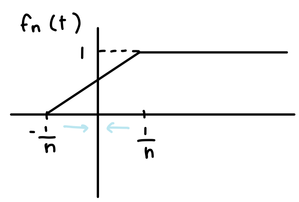
$$f(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$= \delta(x)$$

M2 (more formal):

$$u(t) = \lim_{n\to\infty} f_n(t)$$

$$u'(t) = \lim_{n\to\infty} f'_n(t)$$



as $n\to\infty$, $f_n(t) = u(t)$

$$f'_n(t) = \begin{cases} 0 & |t| > \frac{1}{n} \\ \text{undefined} & |t| = \frac{1}{n} \\ \frac{n}{2} & -\frac{1}{n} < t < \frac{1}{n} \end{cases}$$

$$\lim_{n\to\infty} f'_n(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

Example 2

Use the fact that $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$ for "nice" functions f to find an expression for $\int_{-\infty}^{\infty} f(t)\delta'(t)dt$ where $\delta'(t)$ is the derivative of $\delta(t)$.

This gives a definition for the derivative of $\delta(t)$ by defining what it does to a **test function** $f(t)$. Namely

$$\delta'[f] = \begin{cases} \text{what we find in this example} & t = 0 \\ 0 & else \end{cases}.$$

In the above

Notation 1

$\delta'[f]$ denotes $\int_{-\infty}^{t} f(t)\delta'(t)dt$ which is the effect of δ' on the test function f.

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt \longrightarrow$$

$u = f(t)$ $\quad dv = \delta'(t)dt$

$du = f'(t)\,dt$ $\quad v = \delta(t)$

$$= f(t)\delta(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\delta(t)\,dt$$

$$= 0 - f'(0)$$

$$= -f'(0)$$

$$d'[f] = \begin{cases} -f'(0) & t=0 \\ 0 & else \end{cases}$$

Graphically we can view $\delta'(x)$ by looking at some particular functions in its collection.

The Gaussian distribution centered at 0:

$$G(x;\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\dfrac{x^2}{2\sigma^2}}$$

is one of the collections of functions that is in $\delta(x)$. Explicitly as $\sigma \to 0^+$, the $G(x;\sigma)$ does what the delta function does when integrated:

$$\lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} G(x,\sigma) f(x) = f(0).$$

See matlab file for a numerical example of the above statement.

We thus can write

$$\delta(x) = \lim_{\sigma\to 0^+} G(x,\sigma)$$

where the limit is taken in the "weak" sense.

Now since $G(x;\sigma)$ acts like $\delta(x)$, $G_x(x;\sigma)$ will act like $\delta'(x)$:

$$\begin{aligned} \int_{-\infty}^{\infty} \delta'(x) dx &= \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} \frac{d}{dx} G(x;\sigma) dx \\ &= \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} \frac{-xG(x;\sigma)}{\sigma^2} dx \end{aligned}$$

See matlab file for a numerical example of the above statement.

We thus can write

$$\delta'(x) = \lim_{\sigma\to 0^+} G_x(x,\sigma)$$

where the limit is taken in the "weak" sense.

In general $\delta^{(n)}[f] = (-1)^n f^{(n)}(0)$.

Example 3
Compute the Laplace transform of $\delta'(t)$.

$$\mathcal{L}\{\delta'(t)\} = \int_{-\infty}^{\infty} \delta'(t) e^{-st}\, dt$$

$$= -\frac{d}{dt}\left(e^{-st}\right)\Big|_{t=0} \longrightarrow \text{ex } 2$$

$$= -(-se^{-st})\Big|_{t=0}$$

$$= (se^{-st})\Big|_{t=0}$$

$$= s$$

Updated One-sided Laplace Table:

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
3. t^n	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$
7. $\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$
8. $\delta^{(n)}(t)$	s^n	$\mathbb{C}$

Algebraic Properties:

$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Linearity
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$	Time Scaling
$e^{\alpha t} f(t)$	$F(s-\alpha)$	Exponential Modulation
$f(t-T)u(t-T)$	$e^{-sT}\mathscr{L}\{f(t)u(t)\}$	Time-Shifting
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Multiplication by t^n
$(f * g)(t)$	$F(s)G(s)$	Convolution Theorem
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	

Systems and Signals:

Given a linear differential equations and some specified initial conditions:

$$\frac{d^n}{dt^n}y(t)+a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t)+\ldots+a_0(t)y(t) = f(t), \quad y(0) = b_0, \ldots, y^{(n-1)}(0) = b_{n-1}.$$

There are two classic problems one considers in engineering

Definition 1: Analysis and Synthesis

- **Analysis:** Given a forcing term $f(t)$ solve for $y(t)$.
- **Synthesis:** Given a $y(t)$, or some conditions on $y(t)$, find a forcing term $f(t)$ such that $y(t)$ solves the differential equation.

We have thus far focused on the analysis problem, which is the easier of the two, but the synthesis problem is generally more useful in engineering design.

To solve the synthesis problem we need a good understanding of how $f(t)$ relates to $y(t)$. Think about the spring problem from L8 Ex 2 as an example.

Definition 2: Signals

A **signal** is a complex-valued function of an independent real variable t.

- t usually (but not always) represents time.
- If the domain of the function is $\mathbb{R}$ or some interval of $\mathbb{R}$[a] then the signal is said to be a **continuous-time (CT)** signal.
- If the domain of the function is $\mathbb{Z}$, $\mathbb{Q}$ or other discrete subset of $\mathbb{R}$ [b] then the signal is said to be a **discrete-time (DT)** signal.

[a] In general any uncountable subset of $\mathbb{R}$

[b] In general any countable subset of $\mathbb{R}$

Example 4

Physical variable such as voltages, currents, displacements, forces, etc. are continuous time.

Quantities such as those seen in economics (houses, money etc.), digital hardware, anything stored digitally are are discrete time.

> **Definition 3: Systems**
> Mathematically, a **system** is a map from a space of input signals to a space of output signals.

> **Notation 2**
> If S represents some system then $f \xrightarrow{S} y$ or $y(t) = (Sf)(t)$ or simply $y = Sf$ indicated that $y(t)$ is the **response** (solution) of the system S to the input signal $f(t)$.

The above is similar to systems of equations from 115 and the connection to matrices and the "input" and "output" vectors.

Explicitly, the system of equations

$$\begin{aligned} s_{11}f_1 + \ldots + s_{1n}f_n &= y_1 \\ &\vdots \\ s_{m1}f_1 + \ldots + s_{mn}f_n &= y_m \end{aligned}$$

can be written as

$$S\vec{f} = \vec{y}.$$

Here one can ask the analysis question of solving for $\vec{y}$ ("easy") or the synthesis question of solving for $\vec{f}$ ("hard") and generally wants to know the relation between the response and input.

Further, the matrix S is the map between the vector space of input functions (all valid $\vec{f}$s) and the vector space of output functions (all valid $\vec{y}$s).

> **Definition 4**
> A system S is continuous-time (CT) if both its input and output signals are CT.
> A system S is discrete-time (~~F~~DT) if both its input and output signals are DT.
> S is a hybrid system if ti contains both CT and DT signals.

MATH 213 - Lecture 10: systems and signals introduction continued

Lecture goals: Know what systems and signals are and understand the various properties we generally care about.

Connection between CT systems and DEs:

A DE represents a CT system only when for any signal in the set of input signals of the system, there is a signal in the output class that satisfied the DE.

Example 1

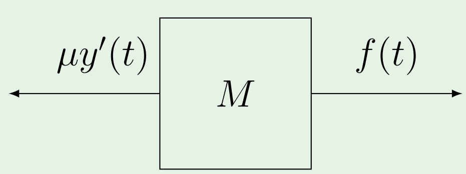
Consider the DE:

$$my''(t) = f(t) - \mu y'(t) \longrightarrow my'' + \mu y' = f$$

where $m, \mu > 0$ are constants and $f(t)$ is some positive valued function.

This DE comes from $ma = F$ applied to some object that experiences drag.

The force diagram is:



Given some forcing term $f(t)$, we can solve the DE to find $y(t)$.

Since this relation exists, we can say that this DE represents some system.

For a concrete example suppose that $f(t)$ is the function denoting the force that is applied to a car when you open the throttle and that μ denotes all drag forces (air resistance, friction, etc) that the car experiences.

In the above case, the DE is a representation of the CT system describing how the car moves based on the amount the accelerator is pressed.

From our work with DEs, if the initial conditions are all 0 then the solution to this DE is

$$y(t) = f(t) * \mathscr{L}^{-1}\left\{\frac{1}{ms^2 + \mu s}\right\} \quad \frac{A}{s} + \frac{B}{s + \ldots}$$

or assuming that $f(t)$ is on-sided

$$y(t) = \int_0^t f(t-\tau)\left(\frac{1}{\mu} - \frac{e^{-\frac{\mu}{m}\tau}}{\mu}\right) d\tau.$$

e.g.

$y^{(7)} + 2y^{(6)} + 8y = 0$

$T(s) = \frac{1}{s^7 + 2s^6 + 8}$

Thus for any particular parameters and any given forcing term, we can compute the motion of the car system.

To Lecture10_car.m!!

Connection between DT systems and difference equations:

DT systems are represented by difference equations. An example of a difference equation is the mine craft chicken model from assignment 1.

Here is another classic fun difference equation

$$x_{n+1} = rx_n(1 - x_n), \qquad r > 0$$

This models the normalized population of bunnies with the constraints that

- there is growth proportional to the population for small populations and
- if the population is close to the environmental limit, 1 in this model, then the population declines.

To Lecture10_magic.m!!
The previous two examples demonstrate that generally speaking we can explore the relation of the input and outputs for many systems by "simply" solving DEs, difference equations etc.

We now examine some more important properties of systems.

Definition 1: Memoryless vs Dynamic systems
A system is **memoryless**, if the instantaneous output value $y(t)$ depends only on the input value $f(t)$ at time t.

A system that is not memoryless is **dynamic**

Example 2

- An ideal amplifier system given by the equation
$$V_{out}(t) = kV_{in}(t)$$
is memoryless.
- An ideal resistor given by the equation
$$V(t) = Ri(t)$$
is memoryless.

In the previous two examples, the output signal $y(t)$ at some point t^* is determined by the input $f(t)$ evaluated at t^*

- The harmonic oscillator given by
$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad f(t) \neq 0 \text{ for } t > 0$$
is dynamic.

Here $y = \frac{1}{m}\int_0^t \left(\int_0^\tau f(u)du\right) d\tau$.

Most interesting problems involve systems that are dynamic.

Definition 2: Causality
A system A is **causal** if $y(t) = (Sf)(t)$ only depends on $f(\tau)$ for $\tau \leq t$.

Idea: the output signal y only depends the previous behaviour of $f(t)$ (i.e. not the future values of $f(t)$).

Definition 3: Causality (Alternate Definition)
A system A is **causal** if for all $t \geq 0$ whenever $f_1(\tau) = f_2(\tau)$ for all $\tau \leq t$ and $y_1(t) = (Sf_1)(t)$ and $y_2(t) = (Sf_2)(t)$ then $y_1(\tau) = y_2(\tau)$, for all $\tau \leq t$.

Example 3

- Memoryless systems are causal.

  $\tau = t$

- The harmonic oscillator given by

$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \quad f(t) \neq 0 \text{ for } t > 0$$

  is causal.

- The difference equation $\overset{f}{\cancel{y}}_k = f_{k+2}$ is noncausal.

- Real-time controllers are clausal.

- Most offline signal processing involves working with nonclausal systems.

  b/c 2^{nd} derivative is of higher deg

- The differential equation $y'(t) = f''(t) + f'(t) + f(t)$ for a known but arbitrary forcing term f is considered noncausal.

To see why the last example above is noncausal one needs to notice that $f''(t)$ is defined as a limit of a difference quotient and as such it depends on more future information than $y'(t)$ does.

Definition 4: Multivariable vs Scalar Systems
A system is **multivariable** if is has multiple inputs and/or outputs.

A system is **scalar** if it has a single input and a single output.

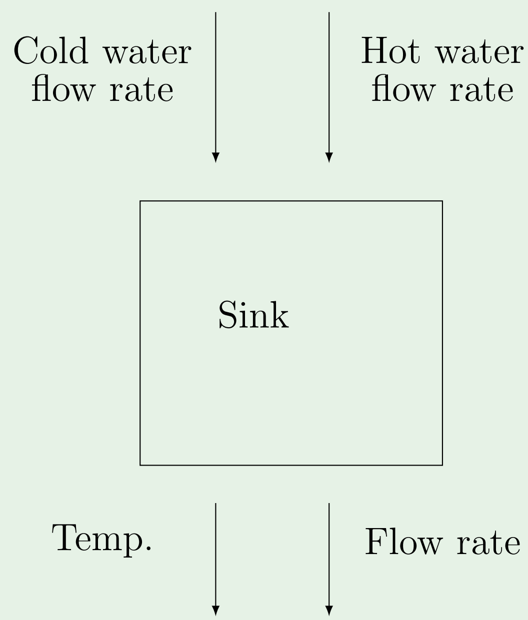
Example 4

- The system modelled by the IVP

$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad f(t) \neq 0 \text{ for } t > 0$$

  is scalar valued.

- Your sink is a multivariable system:



Multivariable systems are harder to control then scalar systems.

We will study scalar systems (but may examine multivariable systems a bit).

Definition 5: Linearity
A system S is **linear** if it has the property that if $f(t)$ and $g(t)$ are input signals and $\alpha, \beta \in \mathbb{C}$ then

$$S(\alpha f_1 + \beta f_2) = \alpha S(f_1) + \beta S(f_2).$$

Example 5
The following are linear systems

- The harmonic oscillator given by

$$my''(t) = f(\overset{0}{\cancel{t}}), \quad y(0) = 0, \quad y'(0) = 0, \qquad f(t) \overset{=}{\cancel{\neq}} 0 \text{ for } t > 0$$

- An ideal amplifier system given by the equation

$$V_{out}(t) = kV_{in}(t).$$

- The idealized system that models tsunami wave velocity in a deep ocean of depth D:

$$u_{tt} + \frac{g}{D} u_{xx} = 0$$

The following are nonlinear:

- The pendulum equation for a pendulum of length ℓ

$$m\ell^2 2\theta''(t) = -mg\ell \sin(\theta), \quad y(0) = y_0, \quad y'(0) = y_1$$

- The system with governing equation given by

$$y(t) = f(t) + 1$$

- The idealized system that models tsunami wave velocity near the shore:

$$u_t + uu_x = 0$$

Note if $\theta \ll 1$ then $\sin(\theta) \approx \theta$ and we can approximate the full pendulum by the linear oscillator:

$$m\ell^2 2\theta''(t) + mg\ell\theta = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

Linear systems are "easy" to study but tend to have "simpler dynamics" than nonlinear systems.

In this course we will focus on linear systems but “most” real world problems are nonlinear.

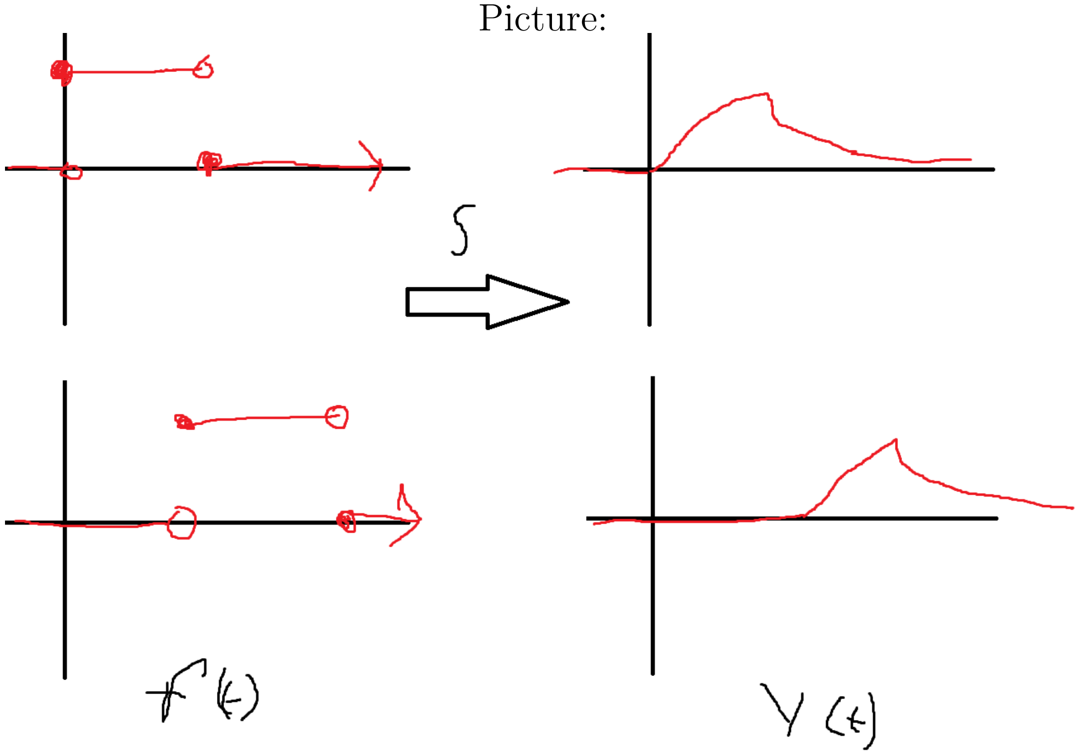
Let’s quickly watch a video of a double pendulum and then move on.

Definition 6: Time-invariance

A system is **time-invariant** if its response does not change with time. Explicitly if $f(t) \xrightarrow{S} y(t)$ then for all $T \in \mathbb{R}$, $f(t-T) \xrightarrow{S} y(t-T)$.

A system that is not time-invariant is called **time-~~in~~variant**.

Picture:



Example 6

- Consider the Harmonic oscillator

$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \quad f(t) \neq 0 \text{ for } t > 0.$$

The solution is

$$y(t) = \frac{1}{m} \int_{-\infty}^{t} \int_{-\infty}^{\tau} f(\theta) d\theta d\tau$$

.

If we shift the input signal by replacing $f(t)$ with $\tilde{f}(t-T)$ then the solution is

$$\begin{aligned}
\tilde{y}(t) &= \frac{1}{m} \int_{-\infty}^{t} \int_{-\infty}^{\tau} f(\theta - T) d\theta d\tau \\
&= \frac{1}{m} \int_{-\infty}^{t} \int_{-\infty}^{\tau - T} f(\theta) d\theta d\tau \\
&= \frac{1}{m} \int_{-\infty}^{t-T} \int_{-\infty}^{\tau} f(\theta) d\theta d\tau \\
&= y(t - T).
\end{aligned}$$

Hence system is time-invariant.

- Consider the ideal sampler system that takes a continuous signal $f(t)$ and outputs the discrete signal $y_k = f(k), k \in \mathbb{Z}$.

  If $f(t) = \cos(2\pi t)$ then $S(f) = y_k = 1$ for all $k \in \mathbb{Z}$.

  If $f(t) = \cos\left(2\pi t + \frac{\pi}{4}\right)$ or $\sin(2\pi t)$ then $S(f) = y_k = 0$ for all $k \in \mathbb{Z}$.

  Hence the ideal sampler is not time-invariant.

For the rest of this course we will focus on linear time invariant systems of LTSs.

These systems are often modelled by constant-coefficient ODES and hence our work on the Laplace transform can be used to study the system....

MATH 213 - Lecture 11: Midterm Review

Example 1

Let S be the system such that $(Sy)(t) = f(t)$ is described by

$$y^{(6)}(t) + y^{(4)}(t) + 6y(t) = f(t)$$

with all the needed ICs given by 0. Classify the system in terms of:

- Continuous time Vs discrete time (input + output are continuous)
- Memoryless vs dynamic $\int_0^t \rightarrow$ needs all inputs from $1 \rightarrow t$
- Causal vs non-causal
- Multivariable vs Scalar
- Linear vs nonlinear
- Time-invariant vs time-variant if $f(t-T) \rightarrow$ output is $Y(t-T)$

$+y^{(6)}(t) + y^{(4)}(t) + 6y(t) = f(t)$ is time-variant

Soln:

$$Y(t) = f(t) * \mathcal{L}^{-1}\left\{\frac{1}{s^6 + s^4 + 6}\right\}$$

$$= \int_0^t f(\tau)\, \mathcal{L}^{-1}\left\{\frac{1}{s^6+s^4+6}\right\}\Big|_{t-\tau} d\tau$$

Example 2
Compute $\mathcal{L}\{\cos(\omega t)u(t)\}$ from the definition of the Laplace transform. State the ROC with justification.

Difficult option:

$$\mathcal{L}\{\cos(\omega t)u(t)\} = \int_0^\infty \cos(\omega t)\, e^{-st}\, dt$$

Easier option:

$$e^{\omega jt} = \cos\omega t + j\sin\omega t$$
$$+\ e^{-\omega jt} = \cos\omega t - j\sin\omega t$$

$$e^{\omega jt} + e^{-\omega jt} = 2\cos\omega t$$

$$\cos\omega t = \frac{1}{2}\left(e^{\omega tj} + e^{-\omega tj}\right)$$

$$\mathcal{L}\{\cos(\omega t)u(t)\} = \frac{1}{2}\int_0^\infty \left(e^{\omega tj} + e^{-\omega tj}\right) e^{-st}\, dt$$
$$= \frac{1}{2}\left(\int_0^\infty e^{(\omega j - s)t} + e^{(-\omega j - s)t}\, dt\right)$$

$u = (\omega j - s)t$ $v = (-\omega j - s)t$

$du = (\omega j - s)\,dt$ $dv = (-\omega j - s)\,dt$

$dt = \dfrac{du}{\omega j - s}$ $dt = \dfrac{dv}{-\omega j - s}$

$$= \frac{1}{2}\left(\int_{t=0}^{t=\infty} e^u \frac{du}{\omega j - s} + \int_{t=0}^{t=\infty} e^v \frac{dv}{-\omega j - s}\right)$$
$$= \frac{1}{2}\left(\frac{1}{\omega j - s}\left(e^{(\omega j - s)t}\right)\Big|_0^\infty + \frac{1}{-\omega j - s}\left(e^{(-\omega j - s)t}\right)\Big|_0^\infty\right)$$
$$= \frac{1}{2}\left(\frac{1}{\omega j - s}\left(\lim_{t\to\infty} e^{(\omega j - s)t} - 1\right) - \frac{1}{\omega j + s}\left(\lim_{t\to\infty} e^{-(\omega j + s)t} - 1\right)\right)$$

0 when $Re(\omega j - s) < 0$, $Re(-s) < 0$, $Re(s) > 0$

0 when $Re(-\omega j - s) < 0$, $Re(-s) < 0$, $Re(s) > 0$

$$= \frac{1}{2}\left(\frac{1}{\omega j - s}(0 - 1) - \frac{1}{\omega j + s}(0 - 1)\right)$$
$$= \frac{1}{2}\left(\frac{-1}{\omega j - s} + \frac{1}{\omega j + s}\right)$$
$$= \frac{1}{2}\left(\frac{-\omega j - s + \omega j - s}{(\omega j - s)(\omega j + s)}\right)$$
$$= \frac{1}{2}\left(\frac{-2s}{\omega^2 j^2 - s^2}\right)$$
$$= \frac{-s}{-\omega^2 - s^2}$$
$$= \frac{s}{s^2 + \omega^2}$$

Example 3

Find the partial fraction decomposition of

$$F(s) = \frac{s^2+1}{(s-1)(s+2)(s-5)(s+7)(s-9)}.$$

You may leave the coefficients in an unsimplified form. i.e. things like $\frac{1}{8\cdot 9+2\cdot(-4)}$ are ok.

$$F(s) = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-5} + \frac{D}{s+7} + \frac{E}{s-9}$$

$s - 1 = 0$

$s = 1.$

Multiply both sides by $(s-1)$. Then, plug in $s=1$. By Heaviside cover-up method:

$$A = \frac{1^2+1}{(1+2)(1-5)(1+7)(1-9)}$$

$$= \frac{2}{3(-4)(8)(-8)}$$

$s+2=0$

$s = -2:$

$$B = \frac{(-2)^2+1}{(-2-1)(-2-5)(-2+7)(-2-9)}$$

$$= \frac{5}{(-3)(-7)(5)(-11)}$$

$s-5=0$

$s = 5.$

$$C = \frac{5^2+1}{(5-1)(5+2)(5+7)(5-9)}$$

$$= \frac{26}{4(7)(12)(-4)}$$

$s+7=0$

$s = -7:$

$$D = \frac{(-7)^2+1}{(-7-1)(-7+2)(-7-5)(-7-9)}$$

$$= \frac{50}{(-8)(-5)(-12)(-16)}$$

$s-9=0$

$s = 9:$

$$E = \frac{9^2+1}{(9-1)(9+2)(9-5)(9+7)}$$

$$= \frac{82}{(8)(11)(4)(16)}$$

Example 4
Solve the IVP:

$$y^{(4)} + y^{(3)} = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 1$$

$$s^4 Y(s) - \underbrace{s^3 Y(0)}_{0} - \underbrace{s^2 Y'(0)}_{0} - \underbrace{sY''(0)}_{0} - \underbrace{Y'''(0)}_{1} + s^3 Y(s) - \underbrace{s^2 Y(0)}_{0} - \underbrace{sY'(0)}_{0} - \underbrace{Y''(0)}_{0} = 0$$

$$Y(s)(s^4 + s^3) = 1$$

$$Y(s) = \frac{1}{s^4 + s^3}$$

$$= \frac{1}{s^3(s+1)}$$

Convolution:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{2!} t^2 * e^{-t}$$

exponential modulation

$$= \frac{1}{2}\int_0^t \tau^2 e^{-(t-\tau)}\, d\tau$$

$$= \frac{1}{2}\int_0^t \tau^2 e^{-t} e^{\tau}\, d\tau$$

$$= \frac{e^{-t}}{2}\int_0^t \tau^2 e^{\tau}\, d\tau$$

u	dv	$\pm$
τ^2	e^τ	+
2τ	e^τ	−
2	e^τ	+
0	e^τ	−
		+

$$y(t) = \frac{e^{-t}}{2}\left(e^{\tau}(\tau^2 - 2\tau + 2)\right)\Big|_0^t$$

$$= \frac{e^{-t}}{2}\left(e^t(t^2 - 2t + 2) - (2)\right)$$

$$= \frac{e^0}{2}(t^2 - 2t + 2) - e^{-t}$$

$$= \frac{1}{2}(t^2 - 2t + 2) - e^{-t}$$

Example 5

Solve the IVP:

$$y^{(4)} - y^{(2)} = t^2, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

You may leave the final solution as a convolution integral.

$$Y(s) = \frac{\mathcal{L}(t^2)}{s^4 - s^2}$$

$$= \frac{2!}{s^3 \cdot s^2(s^2-1)}$$

$$= \frac{2}{s^5(s^2-1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^5(s^2-1)}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} \longrightarrow \text{can also use sinh from table}$$

$$= \frac{2}{4!}t^4 * \mathcal{L}^{-1}\left\{\frac{-1}{2(s+1)} + \frac{1}{2(s-1)}\right\} \longrightarrow \frac{1}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

$$s = -1: \quad A = \frac{1}{-1-1} = -\frac{1}{2} \qquad s = 1: \quad B = \frac{1}{1+1} = \frac{1}{2}$$

$$= \frac{2}{4!}t^4 * \left(-\frac{1}{2}e^{-t} + \frac{1}{2}e^{t}\right)$$

$$= \frac{1}{4!}\int_0^t \tau^4\left(-e^{-(t-\tau)} + e^{t-\tau}\right)d\tau$$

$$= \frac{1}{4!}\int_0^t \tau^4\left(e^{t-\tau} - e^{-(t-\tau)}\right)d\tau$$

Example 6

Suppose that the system S has the transfer function

$$T(s) = \frac{1}{(s+2)(s+1)(s^2+1)}.$$

Determine if the zero-input response component of the solution is bounded for all possible initial conditions of the system.

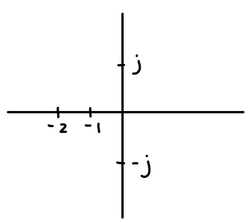
What conditions must be imposed on the poles of the forcing term to ensure that the zero-state response component of the solution is bounded.

DE.

$$y^{(4)}(t) + ay'''(t) + by''(t) + cy'(t) + dy(t) = f(t)$$

$$ZIR(s) = \frac{s^3 + us^2 + vs + w}{(s+2)(s+1)(s^2+1)} \longrightarrow \text{does } ZIR(t) \text{ stay bounded given poles}$$

$(s+j)(s-j)$



Only potential poles of $F(s)$ are $s = -2, -1, \pm j$. Since $Re(-2) < 0$ + $Re(-1) < 0$, then these make a bounded contribution. For $\pm j$, $Re(\pm j) = 0$. However, both poles are of order 1 so they're bounded as well. ZIR is bounded

$$ZSR(s) = \frac{F(s)}{(s+2)(s+1)(s^2+1)}$$

The poles of $F(s)$ must have $Re(s) \leq 0$ and if they're 0, they must be of order 1. They also can't be $\pm j$ b/c that would make the existing $\pm j$ poles be order 2.

Example 7

Write the inverse Laplace transform of

$$\frac{1}{(s^2+1)^3}$$

as an real valued integral.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right) * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= (\sin t * \sin t) * \sin t$$

$$= \left(\int_0^t \sin\tau \sin(t-\tau)\, d\tau\right) * \sin t$$

$$= \int_0^t \sin u \left(\int_0^{t-u} \sin\tau \sin(t-u-\tau)\, d\tau\right) du$$

MATH 213 - Lecture 11: Midterm Review

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- Continuous time Vs discrete time
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- Linear vs nonlinear
- Time-invariant vs time-variant

Example 2

Compute $\mathcal{L}\{\cos(\omega t)u(t)\}$ from the definition of the Laplace transform. State the ROC with justification.

$$\mathcal{L}\{\cos(\omega t)u(t)\} = \int_{-\infty}^{\infty} \cos(\omega t)u(t)e^{-st}\,dt$$

$$= \int_0^{\infty} \cos(\omega t)e^{-st}\,dt$$

$u = \cos(\omega t)$ $\quad dv = e^{-st}dt$

$du = -\omega\sin(\omega t)dt$ $\quad v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s}\cos(\omega t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{s}e^{-st}(-\omega\sin(\omega t))\,dt$$

$$= -\frac{1}{s}\cos(\omega t)e^{-st}\Big|_0^{\infty} - \frac{\omega}{s}\int_0^{\infty} e^{-st}(\sin(\omega t))\,dt$$

$u = \sin(\omega t)$ $\quad dv = e^{-st}dt$

$du = \omega\cos(\omega t)dt$ $\quad v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s}\cos(\omega t)e^{-st}\Big|_0^{\infty} - \frac{\omega}{s}\left(-\frac{1}{s}e^{-st}\sin(\omega t)\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{s}e^{-st}\omega\cos(\omega t)\,dt\right)$$

$$\int_0^{\infty}\cos(\omega t)e^{-st}\,dt = -\frac{1}{s}\cos(\omega t)e^{-st}\Big|_0^{\infty} - \frac{\omega}{s}\left(-\frac{1}{s}e^{-st}\sin(\omega t)\Big|_0^{\infty} + \frac{\omega}{s}\int_0^{\infty}e^{-st}\cos(\omega t)\,dt\right)$$

$$\int_0^{\infty}\cos(\omega t)e^{-st}\,dt = -\frac{1}{s}\cos(\omega t)e^{-st}\Big|_0^{\infty} + \frac{\omega}{s^2}e^{-st}\sin(\omega t)\Big|_0^{\infty} - \frac{\omega^2}{s^2}\int_0^{\infty}e^{-st}\cos(\omega t)\,dt$$

$$\left(1+\frac{\omega^2}{s^2}\right)\int_0^{\infty}\cos(\omega t)e^{-st}\,dt = \frac{1}{s}\left(\frac{\omega}{s}e^{-st}\sin(\omega t) - \cos(\omega t)e^{-st}\right)\Big|_0^{\infty}$$

$$\int_0^{\infty}\cos(\omega t)e^{-st}\,dt = \frac{1}{\frac{s^2+\omega^2}{s^2}}\left(\frac{1}{s}\right)\left(\lim_{t\to\infty}\left(\frac{\omega}{s}e^{-st}\sin(\omega t) - e^{-st}\cos(\omega t)\right) - \left(\frac{\omega}{s}e^{0}\sin(0) - e^{0}\cos(0)\right)\right)$$

$$= \frac{s^2}{s^2+\omega^2}\left(\frac{1}{s}\right)\left(0 - \left(\frac{1}{s}(0) - 1\right)\right)$$

$$= \frac{s}{s^2+\omega^2}(1)$$

$$\mathcal{L}\{\cos(\omega t)u(t)\} = \frac{s}{s^2+\omega^2}, \quad Re(s) < 0$$

$$e^{-\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$+ \quad e^{\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-\omega t} + e^{\omega t} = 2\cos(\omega t)$$

$$\cos(\omega t) = \frac{1}{2}\left(e^{-\omega t} + e^{\omega t}\right)$$

$$\mathcal{L}\{\cos(\omega t)u(t)\} = \int_0^\infty \frac{1}{2}\left(e^{-\omega t} + e^{\omega t}\right)e^{-st}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{t(-\omega-s)} + e^{t(\omega-s)}\,dt$$

$$= \frac{1}{2}\left(\int_0^\infty e^{t(-\omega-s)}\,dt + \int_0^\infty e^{t(\omega-s)}\,dt\right)$$

$$u = t(-\omega-s) \qquad\qquad v = t(\omega-s)$$

$$du = (-\omega-s)dt \qquad\qquad dv = (\omega-s)\,dt$$

$$dt = \frac{du}{-\omega-s} \qquad\qquad dt = \frac{dv}{\omega-s}$$

$$= \frac{1}{2}\left(\int_0^\infty e^u \frac{du}{-\omega-s} + \int_0^\infty e^v \frac{dv}{\omega-s}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{\omega+s}\left(e^{t(-\omega-s)}\right)\Big|_0^\infty + \frac{1}{\omega-s}\left(e^{t(\omega-s)}\right)\Big|_0^\infty\right)$$

$$= \frac{1}{2}\left(-\frac{1}{\omega+s}\left(\lim_{t\to\infty} e^{t(-\omega-s)} - e^0\right) + \frac{1}{\omega-s}\left(\lim_{t\to\infty} e^{t(\omega-s)} - e^0\right)\right)$$

$$= \frac{1}{2}\left(-\frac{1}{\omega+s}(0-1) + \frac{1}{\omega-s}(0-1)\right)$$

$$= \frac{1}{2}\left(\frac{1}{\omega+s} - \frac{1}{\omega-s}\right)$$

$$= \frac{1}{2}\left(\frac{\omega-s-\omega-s}{\omega^2-s^2}\right)$$

$$= \frac{1}{2}\left(\frac{-2s}{\omega^2-s^2}\right)$$

$$= \frac{1}{2}\left(\frac{2s}{s^2-\omega^2}\right)$$

$$= \frac{s}{s^2-\omega^2} \quad , \ Re(s) < 0$$

Example 3

Find the partial fraction decomposition of

$$F(s) = \frac{s^2+1}{(s-1)(s+2)(s-5)(s+7)(s-9)}.$$

You may leave the coefficients in an unsimplified form. i.e. things like $\frac{1}{8\cdot 9+2\cdot(-4)}$ are ok.

$$F(s) = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-5} + \frac{D}{s+7} + \frac{E}{s-9}$$

Heaviside coverup method:

$s-1=0$

$s=1$:

$$A = \frac{s^2+1}{(s+2)(s-5)(s+7)(s-9)} = \frac{1^2+1}{(1+2)(1-5)(1+7)(1-9)} = \frac{2}{3(-4)(8)(-8)}$$

$s+2=0$

$s=-2$:

$$B = \frac{s^2+1}{(s-1)(s-5)(s+7)(s-9)} = \frac{(-2)^2+1}{(-2-1)(-2-5)(-2+7)(-2-9)} = \frac{5}{(-3)(-7)(5)(-11)}$$

$s-5=0$

$s=5$:

$$C = \frac{s^2+1}{(s-1)(s+2)(s+7)(s-9)} = \frac{5^2+1}{(5-1)(5+2)(5+7)(5-9)} = \frac{26}{(4)(7)(12)(-4)}$$

$s+7=0$

$s=-7$:

$$D = \frac{s^2+1}{(s-1)(s+2)(s-5)(s-9)} = \frac{(-7)^2+1}{(-7-1)(-7+2)(-7-5)(-7-9)} = \frac{50}{(-8)(-5)(-12)(-16)}$$

$s-9=0$

$s=9$:

$$E = \frac{s^2+1}{(s-1)(s+2)(s-5)(s+7)} = \frac{9^2+1}{(9-1)(9+2)(9-5)(9+7)} = \frac{82}{(8)(11)(4)(16)}$$

Example 4

Solve the IVP:

$$y^{(4)} + y^{(3)} = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 1$$

$$\mathcal{L}\{y^{(4)} + y^{(3)}\} = \mathcal{L}\{0\}$$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) + s^3Y(s) - s^2y(0) - sy'(0) - y''(0) = 0$$

$$s^4Y(s) - 0 - 0 - 0 - 1 + s^3Y(s) - 0 - 0 - 0 = 0$$

$$Y(s)(s^4 + s^3) = 1$$

$$Y(s) = \frac{1}{s^4+s^3}$$

$$Y(s) = \frac{1}{s^3(s+1)}$$

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^3} * \frac{1}{s+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\
&= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} * e^{-t} \\
&= \frac{1}{2}(t^2) * e^{-t} \\
&= \int_0^t \frac{1}{2}\tau^2 \, e^{-(t-\tau)} \, d\tau \\
&= \frac{1}{2}\int_0^t \tau^2 e^{-t+\tau} \, d\tau \\
&= \frac{e^{-t}}{2}\int_0^t \tau^2 e^{\tau} \, d\tau \\
&= \frac{e^{-t}}{2}\left(\tau^2e^{\tau} - 2\tau e^{\tau} + 2e^{\tau}\right)\Big|_0^t \\
&= \frac{e^{-t}}{2}\left(e^{\tau}(\tau^2 - 2\tau + 2)\right)\Big|_0^t \\
&= \frac{e^{-t}}{2}\left(e^{t}(t^2 - 2t + 2) - e^{0}(2)\right) \\
&= \frac{e^{-t}}{2}\left(e^{t}(t^2 - 2t + 2) - 2\right) \\
&= \frac{1}{2}(t^2 - 2t + 2) - e^{-t} \\
&= \frac{t^2}{2} - t + 1 - e^{-t}
\end{aligned}$$

$\int u\,dv = uv - \int v\,du$

u	dv	$+/-$
τ^2	e^{τ}	$+$
2τ	e^{τ}	$-$
2	e^{τ}	$+$
0	e^{τ}	$-$
		$+$

Example 5

Solve the IVP:

$$y^{(4)} - y^{(2)} = t^2, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

You may leave the final solution as a convolution integral.

$$\mathcal{L}\{y^{(4)} - y^{(2)}\} = \mathcal{L}\{t^2\}$$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - (s^2Y(s) - sy(0) - y'(0)) = \frac{2!}{s^3}$$

$$s^4Y(s) - 0 - 0 - 0 - 0 - (s^2Y(s) - 0 - 0) = \frac{2}{s^3}$$

$$Y(s)(s^4 - s^2) = \frac{2}{s^3}$$

$$Y(s) = \frac{1}{s^4 - s^2}\left(\frac{2}{s^3}\right)$$

$$= \frac{2}{s^3(s^2(s^2-1))}$$

$$= \frac{2}{s^5(s+1)(s-1)}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)(s-1)}\right\}$$

$$= 2\left(\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}\right)$$

$$= 2\left(\frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} * e^{-t} * e^{t}\right)$$

$$= 2\left(\frac{1}{4!}t^4 * e^{-t} * e^{t}\right)$$

$$= 2\left(\frac{t^4}{4!} * \int_0^t e^{-\tau} e^{(t-\tau)}\, d\tau\right)$$

$$= 2\left(\frac{t^4}{4!} * \int_0^t e^{-\tau+t-\tau}\, d\tau\right)$$

$$= 2\left(\frac{t^4}{4!} * \int_0^t e^{t-2\tau}\, d\tau\right)$$

$$= 2\left(\frac{t^4}{4!} * e^{t}\int_0^t e^{-2\tau}\, d\tau\right)$$

$$= 2\left(\frac{t^4}{4!} * e^{t}\left(-\frac{1}{2}e^{-2\tau}\right)\Big|_0^t\right)$$

$$= 2\left(\frac{t^4}{4!} * -\frac{e^{t}}{2}\left(e^{-2t} - e^{0}\right)\right)$$

$$= 2\left(\frac{t^4}{4!} * -\frac{1}{2}\left(e^{-t} - e^{t}\right)\right)$$

$$= 2\left(\frac{t^4}{4!} * \frac{1}{2}\left(e^{t} - e^{-t}\right)\right)$$

$$= 2\int_0^t \frac{\tau^4}{4!} \cdot \frac{1}{2}\left(e^{t-\tau} - e^{-(t-\tau)}\right) d\tau$$

$$= \frac{1}{4!}\int_0^t \tau^4\left(e^{t-\tau} - e^{-(t-\tau)}\right) d\tau$$

Example 6
Suppose that the system S has the transfer function

$$T(s) = \frac{1}{(s+2)(s+1)(s^2+1)}.$$

Determine if the zero-input response component of the solution is bounded for all possible initial conditions of the system.

What conditions must be imposed on the poles of the forcing term to ensure that the zero-state response component of the solution is bounded.

Poles:

$s=-2$, $s=-1$, $s=\pm j$

$Re(-2)<0$ + $Re(-1)<0$ so these poles have bounded contribution. $Re(\pm j)=0$ but poles are order 1 so also bounded.

$$ZIR = \frac{as^3+bs^2+cs+d}{(s+2)(s+1)(s^2+1)}$$

Thus, ZIR is bounded $\forall$ ICs.

$$ZSR = \frac{F(s)}{(s+2)(s+1)(s^2+1)}$$

$F(s)$ have poles w/ $Re(s)<0$ or if $Re(s)=0$, they cannot be $\pm j$ + also be of order 1.

Example 7

Write the inverse Laplace transform of

$$\frac{1}{(s^2+1)^3}$$

as an real valued integral.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \sin t * \sin t * \sin t$$

$$= \int_0^t \sin\tau \sin(t-\tau)\, d\tau \quad * \sin t$$

$$= \int_0^t \sin(u) \int_0^{t-u} \sin\tau \sin(t-u-\tau)\, d\tau \; du$$

MATH 213 - Lecture 12: Fun with Foreshadowing

In the synthesis problem we are given $y(t)$ and want to find $f(t)$. Later we will cover how to do this in mathematical detail but... how do we in principle do this??? We will look at an example!

Example 1
Consider the DE

$$my'' = f(t) - \mu y'(t)$$

that comes from $ma = F$ applied to a car with mass m and drag coefficient mu.

Solve the DE for $y(t)$ given $f(t)$. You can leave your solution as a convolution integral.

Solution: The solution is

$$y(t) = \int_0^t f(\tau)\mathscr{L}^{-1}\left\{\frac{1}{ms^2+\mu s}\right\}\bigg|_{t-\tau} d\tau$$

To compute the inverse Laplace transform note that

$$\begin{aligned}\frac{1}{ms^2+\mu s} &= \frac{1}{s\,(ms+\mu)}\\ &= \frac{1}{\mu s} - \frac{m}{\mu(ms+\mu)}\\ &= \frac{1}{\mu s} - \frac{1}{\mu\left(s+\frac{\mu}{m}\right)}\end{aligned}$$

Hence,

$$\mathscr{L}^{-1}\left\{\frac{1}{ms^2+\mu s}\right\} = \frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu t}{m}}.$$

Combining this with the previous result gives

$$y(t) = \int_0^t f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right) d\tau$$

Now suppose that we want to find $f(t)$ such that $y(t)$ is given by a particular provided function. In the next few examples, we will derive a P controller that is commonly used to solve this problem in real time.

Example 2
Suppose someone tells you that $f(t)$ is a forcing term that might result in the $y(t)$ you want. Find the expression for the error between the system's response to this forcing term and and actual function $\tilde{y}(t)$ that you wanted.

The response of the system to $f(t)$ is given by

$$y(t) = \int_0^t f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right)d\tau$$

and hence the error is

$$\tilde{y}(t) - \int_0^t f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right)d\tau$$

Example 3
Formulate an optimization problem that if solved, would solve the synthesis problem of finding $f(t)$ so that the system response is some given $\tilde{y}(t)$.

If the error is 0 for all values of t, then we found the solution to the synthesis problem!

We thus want to pick $f(t)$ (ideally that is smooth) such that the error is as close to 0 as possible. Hence we want to solve the optimization problem

$$\min_{f(t)}\left|\tilde{y}(t) - \int_0^t f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right)d\tau\right|$$

In the above there are a few different ways to define what we mean by min. Two common ones are

- Minimize the maximum value over all t i.e. the L_∞ norm
- Minimize the average value i.e. the L_1 norm

Example 4
In many real world problems in computing, time is discrete and one knows $y(\tau)$ and $f(\tau)$ for $\tau \leq T$.

In these cases for the synthesis problem, one wants to pick $f(T)$ so that $y(T+\Delta t)$ is approximately $\tilde{y}(T+\Delta t)$ where Δt is the discreet increment in time where we know $y(t)$.

Use the error term above to find an expression for $f(T)$ that will approximately minimize the error at $T+\Delta t$.

The (signed) error at $T+\Delta t$ is given by

$$\begin{aligned} E(T+\Delta t) &= \tilde{y}(T+\Delta t) - \int_0^{T+\Delta t} f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right) d\tau \\ &= E(T) + \tilde{y}(T+\Delta t) - \tilde{y}(T) - \int_T^{T+\Delta t} f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right) d\tau \end{aligned}$$

Assuming $\tilde{y}(t)$ is continuous and Δt is small, we can write

$$\tilde{y}(T+\Delta t) - \tilde{y}(T) \approx 0$$

and

$$\int_T^{T+\Delta t} f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right) d\tau \approx f(T)\int_T^{T+\Delta t} \frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}} d\tau = k_p f(T)$$

where k_p is some number (formally it is the integral but we don't want to compute it in practice...). Thus

$$E(T+\Delta t) \approx E(T) - k_p f(T).$$

If we want $E(T+\Delta t) = 0$ then we can pick $f(T) \approx \frac{E(T)}{k_p}$ which we will write as $f(T) = k_p E(T)$ where k_p is a non-zero constant.

Example 5
Use the previous results to write an algorithm that takes a function $\tilde{y}(t)$ and computes $f(t)$ that produces a $y(t)$ that approximates $\tilde{y}(t)$.

Here is some psudo code

```
Define y_want(t), kp, the error at t=a and
the convolution kernal, G, for the given DE.
for t=0:Dt:b
        f(t+Dt) = kp*E(t)
        y(t+Dt) = num_int(f(u)*G(t-u),u,0,t+Dt)
        E(t+Dt)= y_want(T+Dt)-y(t+Dt)
end
```

In the above num_int is some numerical method that integrates.
Lecture12_basic_PID.m implements a method that uses the above psudo code with extra controls for the integral and derivative of the error.

MATH 213 - Lecture 13: Systems Input and Responses

Lecture goals: Understand the impulse response of an LTI, the convolution property of LTIs and the response of an ~~LIT~~ LTI to exponential inputs.

Recap of some DE results:

Recall that for a linear DE with constant coefficients the zero state response is

$$Y(s) = H(s)F(s)$$

where $H(s)$ is the transfer function of the DE and $F(s)$ is the Laplace transform of the forcing term.

If the forcing term is the delta impulse function (i.e. $f(t) = \delta(t)$), then $F(s) = 1$ and thus

$$Y(s) = H(t) \cdot 1 = H(t).$$

Thus we also call

$$h(t) = \mathscr{L}^{-1}\{H(s)\}$$

the **impulse response of the DE**.

Note that this is simply the zero-state response to a unit impulse.

Thus when we write

$$Y(s) = H(s)F(s)$$

the transfer function, $H(s)$, is always just the Laplace transform of the impulse response. Finally by the convolution theorem the zero-state solution is simply

$$y(t) = (h * f)(t).$$

Impulse response of a LTI:

The above holds for Linear DEs with constant coefficients but what about more general LTIs. In general these might not be able to be modelled by a DE... so how do we find their responses to general inputs.

$\downarrow$

e.g. $A\vec{f} = \vec{y}$

Theorem 1

linear time-invariant

The response of an LTI system is the convolution of the input with the system's impulse response.

Proving this in the CT case is a bit messy so we will show that the result holds for the DT case and then make the connection to the CT case.

discrete time

continuous time

First we need to define some DT analogues of CT objects we have worked with.

Definition 1: Kronecker delta

The **Kronecker delta** function $\delta[t]$ is defined as the function from $\mathbb{Z}$ to $\{0, 1\}$ given by

$$\delta[t] = \begin{cases} 1 & t = 0 \\ 0 & \text{else} \end{cases}$$

Theorem 2

For all functions $f : \mathbb{Z} \to \mathbb{C}$

$$f[t] = \sum_{\tau=-\infty}^{\infty} f[\tau]\delta[\tau - t]$$

$$f[t] = \begin{cases} \vdots & \\ f[-1] & t=-1 \\ f[0] & t=0 \\ f[1] & t=1 \\ \vdots & \end{cases} = \sum_{\tau=-\infty}^{\infty} f[\tau] \underbrace{\delta[\tau - t]}_{\substack{\text{only 1 when } \tau = t \\ + \text{ 0 otherwise}}}$$

Theorem 2 shows how to decompose every function f into a linear combination of impulse functions (i.e. $\{\delta[\tau - t] | \tau \in \mathbb{Z}\}$ is a basis for all DT functions.)

Proof of Th 1:

don't know what this looks like

Let $\delta[t-\tau] \xrightarrow{S} h[t,\tau]$

For any $f: \mathbb{Z} \to \mathbb{C}$, $f[t] = \sum_{\tau=-\infty}^{\infty} f[\tau]\delta[t-\tau]$.

System response to f:

$$\begin{aligned} y[t] &= S(f[t]) \\ &= S\left(\sum_{\tau=-\infty}^{\infty} f[\tau]\delta[t-\tau]\right) \\ &= \sum_{\tau=-\infty}^{\infty} f[\tau]\, S(\delta(t-\tau]) \qquad \text{S is linear + f[t] is constant wrt } \tau \\ &= \sum_{\tau=-\infty}^{\infty} f[\tau]\, h[t,\tau] \end{aligned}$$

This isn't a convolution b/c we need $h[t,\tau] = $ "$h[t-\tau]$". Use time invariance to prove:

$\delta[t] \xrightarrow{S} h[t,0]$ so $\delta[t-\tau] \xrightarrow{S} h[t-\tau,0]$ b/c time invariance

Since $\delta[t-\tau] \xrightarrow{S} h[t,\tau]$, then $h[t,\tau] = h[t-\tau,0]$. Call $h[t-\tau,0] \to h[t-\tau]$.

$$\begin{aligned} y[t] &= \sum_{\tau=-\infty}^{\infty} f[\tau]\, h[t,\tau] \\ &= \sum_{\tau=-\infty}^{\infty} f[\tau]\, h[t-\tau] \\ &= (f * h)[t] \end{aligned}$$

Thus, thm 1 holds for DT LTIs.

For CT LTIs.

If $f \xrightarrow{S} y$, then $y(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$.

$h(t)$ is system's impulse response.

To prove, do same computations w/ CT fcns + S.

Theorem 3: More explicit version of Th 1

Given the LTI

$$\xrightarrow{f(t)} \boxed{\text{LTI}} \xrightarrow{y(t)}$$

$$y(t) = (h * f)(t)$$

where $h(t)$ is the **impulse response of the system**.

Theorem 4: LTI response to an exponential

If $S : f \to y$ is a LTI then for any $s \in \mathbb{C}$

$$e^{st} \to H(s)e^{st}$$

where $H(s)$ is the LT of the system's impulse response and is called the **system's transfer function**.

Note this means that complex exponentials are the eigenfunctions of LTIs and the transfer function tells you the eigenvalues of the system! (like with DEs)

Proof:

We only know system is LTI + nothing else.

Suppose $S: f \to y$ is LTI + suppose for some $s \in \mathbb{C}$, $\boxed{e^{st} \xrightarrow{S} y(t)}$.

S is time-invariant so:

$$e^{s(t-T)} \xrightarrow{S} y(t-T)$$

$$e^{s(t-T)} = e^{-sT} e^{st} \xrightarrow{S} e^{-sT} y(t)$$ linearity

Thus, $\forall T \in \mathbb{R}$, $y(t-T) = e^{-sT} y(t)$.

If we pick $T = t$, then $y(0) = e^{-st} y(t)$ or $y(t) = y(0)e^{st}$.

Hence, $e^{st} \xrightarrow{S} y(0)e^{st}$ or $\underbrace{S(e^{st}) = y(0)e^{st}}_{\text{eigenfcn eqn}}$

By thm 3:

$$\begin{aligned} y(t) &= (h * e^{st})(t) \\ &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} \, d\tau \\ &= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, d\tau}_{y(0) = H(s)} \end{aligned}$$

eigenvalues

TLDR; In general the response of a LTI system to an input $f(t)$ is given by

$$y(t) = (h * f)(t)$$

and since the eigenfunctions of LTIs are complex exponentials, we can decompose $f(t)$ into the eigenbasis of complex exponentials (like we did with DEs).

In the frequency domain we have

$$Y(s) = H(s)F(s).$$

Hence, if we want to analyze the system we can replace convolution with multiplication and simply analyze the system based on its eigenfunction decomposition, poles, etc. (like we did with DEs).

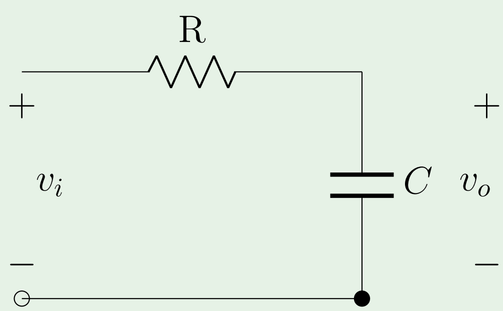
MATH 213 - Lecture 14: Transfer functions, Responses and the standard first order system

Lecture goals: Understand the connection between our previous material for DEs and systems. Know what the standard 1st order system is

If a system is able to be modelled by a linear DE with constant coefficients, i.e. $D(y(t)) = f(t)$, then the transfer function can be obtained from the differential equation.

Example 1: RC circuit

Find the transfer function for the system $S : v_i \to v_o$ given by the circuit



Verify that the transfer function is the system response of the impulse response.

$$RC\frac{dv_o}{dt} + v_o = v_i \quad , \quad v_o(0) = 0$$

$$\mathcal{L}\{RC\frac{dv_o}{dt} + v_o\} = \mathcal{L}\{v_i\}$$

$$RC[s\bar{V}_o(s) - v_o(0^+)] + \bar{V}_o(s) = \bar{V}_i(s)$$

$$\bar{V}_o(s)(RCs + 1) = \bar{V}_i(s)$$

$$\bar{V}_o(s) = \bar{V}_i(s) \cdot \boxed{\frac{1}{RCs+1}} = H(s)$$

transfer fcn

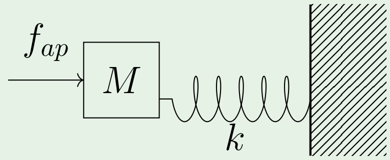
Set $v_i(t) = \delta(t)$:

$$v_i(t) = \delta(t) \Rightarrow \bar{V}_i(s) = 1$$

$$\bar{V}_o(s) = \frac{1}{RCs+1} = H(s)$$

Example 2: Linear Harmonic Oscillator

Consider the system $S : f_{ap} \to y$ given by the undampened linear harmonic oscillator



- Find the impulse response.
- Use the impulse response to find the general form of the system's response to a general input $f_a p(t)$

DE: $my'' + ky = f_{ap}(t)$

ZSR: $Y(s) = \dfrac{F_{ap}(s)}{ms^2 + k}$ so $H(s) = \dfrac{1}{ms^2 + k}$

Impulse response:

$$h(t) = \mathcal{L}^{-1}\left\{ \frac{1}{ms^2 + k} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{m}\left(\frac{1}{s^2 + \frac{k}{m}} \right) \right\}$$

$$= \frac{1}{m}\left(\sqrt{\frac{m}{k}}\right) \mathcal{L}^{-1}\left\{ \frac{\sqrt{\frac{k}{m}}}{s^2 + \frac{k}{m}} \right\}$$

$$= \frac{1}{\sqrt{km}} \sin\left(\sqrt{\frac{k}{m}}\, t \right)$$

In general

$$Y(s) = F_{ap}(s) = H(s)$$

$$y(t) = \mathcal{L}^{-1}\{F_{ap}(s)H(s)\}$$

$$= f_{ap}(t) * h(t)$$

A few things to note:

- In general $\cancel{F_a p(s)}$ can have poles of its own and thus the system response to will reflect the poles of both the transfer function, $H(s)$, and the LT of the forcing term $\cancel{F_a p(s)}$.
  $f_{ap}(t)$
- The effects of the transfer function are present for any input so it is particularly important to understand the effects of the poles (and zeros) of $H(s)$.
- We will mostly look at the cases where the input is a unit impulse, $\delta(t)$, or a unit step, $u(t)$. Recall from A3 Q3 that for a second order DE these terms allow us to effectively set the initial condition at 0^+ of for the system.
  - The response to the unit impulse is $Y(s) = H(s)$.
  - The response to the unit step impulse is $Y(s) = \frac{1}{s} H(s)$. – The step response is the integral of the impulse response (subtle connection to L12...)

    In the "real world" it is often easier to physically generate a unit step function than a unity impulse so point 2 above gives us a nice way to compute transfer functions.

Understanding general system responses:

Recall that all polynomials with real valued coefficients can be factored into a product of linear and quadratic terms. Hence if we want to understand the system response of any system, it is sufficient to understand how first and second order linear systems respond. We will thus explore the standard 1st and 2nd order systems in detail. All other ~~LITs~~ can be studied by taking a linear combination of the results of the standard systems. LTIs

Definition 1: Standard 1st order system

The standard 1st order system has the transfer function

$$H(s) = \frac{\kappa}{s\tau + 1}$$

where $\kappa, \tau > 0$. κ is called the **DC gain** and τ is called the **time constant.**

Example 3

1) Find and plot the impulse response of the standard 1st order system.

2) Determine if the standard first order system is causal.

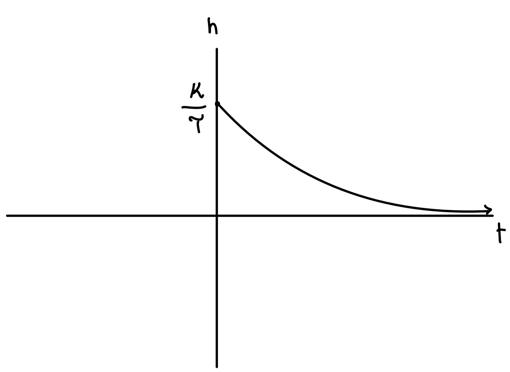
3) Explore the effect of τ on the system's impulse response.
What does the location of the pole tell you about the speed that the system's impulse response decays?

4) Find and plot the system's step response.

5) Explore the effect of κ on the system's step response.
Use the final value theorem to compute the long term behaviour of the system's step response.

1)
$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{\kappa}{s\tau+1}\right\}$$
$$= \kappa\,\mathcal{L}^{-1}\left\{\frac{1}{\tau}\left(\frac{1}{s+\frac{1}{\tau}}\right)\right\}$$
$$= \frac{\kappa}{\tau}e^{-\frac{t}{\tau}}u(t)$$

Plot:



2) General soln: $y(t) = h(t) * f(t)$

$$= \int_0^t f(\tau)\,h(t-\tau)\,d\tau$$

t+1 would mean it's non-causal

causal b/c $\tau \le t$

3) τ changes $h(0) = \frac{K}{\tau}$

τ changes "growth rate" $\longrightarrow$ pole at $s = -\frac{1}{\tau}$

↳ τ inc $\rightarrow$ decays slower

↳ τ dec $\rightarrow$ decays faster

this is why we call τ the "time constant"

4) $Y(s) = H(s) \cdot F(s)$ $\longleftarrow$ $f(t) = u(t)$

$\mathcal{L}\{f(t)\} = \frac{1}{s}$

$$= H(s) \cdot \frac{1}{s}$$

$$= \frac{K}{s(s\tau+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{K}{s(s\tau+1)}\right\} \xrightarrow{\text{PF}} \frac{K}{s(s\tau+1)} = \frac{A}{s} + \frac{B}{s\tau+1}$$

$$= \mathcal{L}^{-1}\left\{\frac{K}{s} - \frac{K\tau}{s\tau+1}\right\}$$

$$= K - \tau\left(\frac{K}{\tau} e^{-\frac{t}{\tau}}\right)$$

$$= K\left(1 - e^{-\frac{t}{\tau}}\right) u(t)$$

Heaviside coverup:

$s = 0$:

$$\frac{K}{s\tau+1} = A$$

$$A = K$$

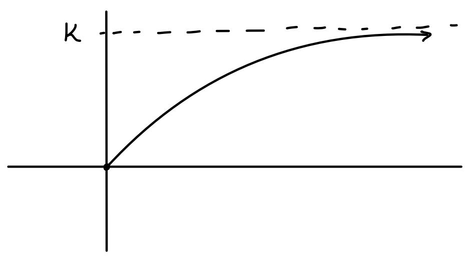
$s\tau + 1 = 0$

$s = -\frac{1}{\tau}$:

$$\frac{K}{s} = B$$

$$\frac{K}{-\frac{1}{\tau}} = B$$

$$B = -K\tau$$



↳ K inc $\rightarrow \lim_{t\to\infty} y(t)$ inc

↳ K dec $\rightarrow \lim_{t\to\infty} y(t)$ dec

5) $Y(s)$ is proper rational fcn.

Poles of $Y(s)$: $s = 0$, $s = -\frac{1}{\tau}$

order 1 $Re(s) < 0$

$$\text{FVT}: \lim_{t\to\infty} y(t) = \lim_{s\to 0^+} sY(s)$$

$$= \lim_{s\to 0^+} s\left(\frac{K}{s(s\tau+1)}\right)$$

$$= \lim_{s\to 0^+} \frac{K}{s\tau+1}$$

$$= \lim_{s\to 0^+} H(s)$$

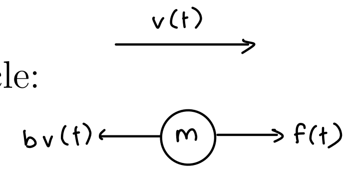
$$= H(0)$$

$$= K \longleftarrow \text{DC gain}$$

MATH 213 - Lecture 15: Basic Control Theory - Cruise Control

Lecture goals: Understand the basics of closed control loops, what proportional and integral control systems are, their benefits/limitations and how to set them up to control a system.

Consider the simple system that models the velocity of a vehicle:



$$mv'(t) + bv(t) = f(t)$$

where $v(t)$ is the velocity of the vehicle, m is the mass, b is the coefficient of drag and $f(t)$ is the velocity forcing term which is a combination of the effects of the engine/brakes as well as the effects of the road (hills etc).

It is nice to split up the input into terms we can control (brake/gas) and terms we can't (road conditions):

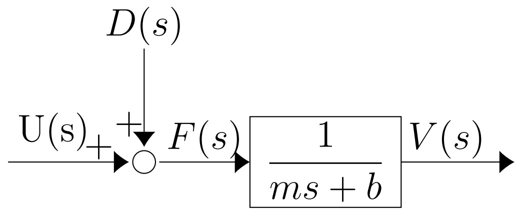
$$f(t) = \underbrace{u(t)}_{\text{Control input}} + \underbrace{d(t)}_{\text{Disturbance input}}$$

Goal: Find $u(t)$ so that the vehicle velocity $v(t)$ follows some given reference input, $r(t)$.

Solution: We use a "controller" to adjust $u(t)$ to obtain our desired velocity. We do this is frequency space.

We examine two options.

Open loop controller:



Here:

- $U(s)$ is the Laplace transform of the control input.
- $D(s)$ is the Laplace transform of the disturbance input.
- $F(s)$ is the Laplace transform of the total forcing term.
- $V(s)$ is the Laplace transform of the resulting velocity.

These controllers only work well if we know the disturbance the system undergoes, if the system always needs a constant known output or if the user can make adjustments as needed.

Some applications include washer/dryers for clothing, toasters, watering systems, step motors etc but open loop systems do not work well for our application.

We want $R(s)$.

$V(s) = \left(\frac{1}{ms+b}\right)(U(s) + D(s))$

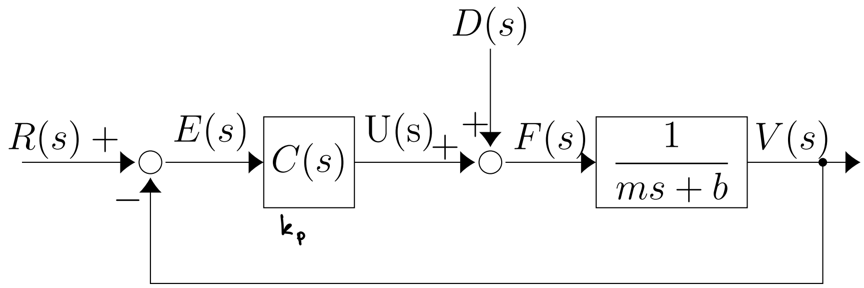
Solve $(V(s) = R(s))$

$R(s) = \left(\frac{1}{ms+b}\right)(U(s) + D(s))$ for $U(s)$

$U(s) = (ms+b)R(s) - D(s)$

$u(t) = \mathcal{L}^{-1}\{(ms+b)R(s)\} - d(t)$

Closed loop controller:



Here:

- $U(s)$, $D(s)$, $F(S)$ and $V(s)$ remain unchanged from before but...

- We added a controller with transfer function $C(\overset{s}{\cancel{S}})$ to dynamically adjust $U(s)$ based on feedback.

- $R(s)$ is the Laplace transform of the **desired** velocity function $v(t)$.

- $E(s)$ is the Laplace transform of the error of our velocity $r(t) - v(t)$. This is the input into the control system!

New Goal: Find a transfer function $C(s)$ such that $E(s) = 0$ and hence we have the desired velocity i.e. $V(s) = R(s)$. as $t \to \infty$

Proportional control:

Idea: Let $u(t) = k_p e(t)$ for $k_p \in \mathbb{R}_{>0}$. In this case

- if $e(t) > 0$ then we "hit the gas" to speed up,
- if $e(t) < 0$ we "hit the breaks" to slow down and
- if $e(t) = 0$ we do nothing.

We will explore how this works in the case where $D(s) = 0$.

If we use this controller then $U(s) = \underbrace{k_p}_{\text{Transfer function } C(s)} E(s)$.

We have

$$V(s) = \frac{1}{ms+b}F(s) \qquad \text{System response to } f(t) \text{ is } H(s)F(s).$$

$$= \frac{1}{ms+b}U(s) \qquad \text{System input is } F(s) = D(s) + U(s) \text{ and } D(s) = 0$$

$$= \frac{1}{ms+b}k_pE(s) \qquad \text{Controller output is } U(s) = k_pE(s)$$

$$= \frac{1}{ms+b}k_p(R(s) - V(s)) \qquad E(s) = R(s) - V(s)$$

Solve for $V(s)$:

$$\left(1 + \frac{1}{ms+b}k_p\right)V(s) = \frac{1}{ms+b}k_pR(s)$$

$$V(s) = \frac{\frac{k_p}{ms+b}}{1+\frac{k_p}{ms+b}}R(s)$$

Let's write this in standard form:

$$\begin{aligned}
V(s) &= \frac{\frac{k_p}{ms+b}}{1+\frac{k_p}{ms+b}}R(s) \\
&= \frac{k_p}{\left(1+\frac{k_p}{ms+b}\right)(ms+b)}R(s) \\
&= \frac{k_p}{ms+b+k_p}R(s) \\
&= \frac{k_p/(b+k_p)}{[m/(b+k_p)]s+1}R(s)
\end{aligned}$$

The transfer function for the controlled system is

$$\frac{k_p/(b+k_p)}{[m/(b+k_p)]s+1}$$

which is the transfer function for a first order system with a DC gain of

$$\kappa = \frac{k_p}{b+k_p}$$

and a time constant of

$$\tau = \frac{m}{b + k_p}.$$

This transfer function looks similar to the transfer function for the uncontrolled car:

$$\frac{k_p/(b+k_p)}{m/(b+k_p)s+1} \qquad \text{vs} \qquad \frac{1/b}{(m/b)s+1}$$

but we can now adjust the DC gain and time constants by changing k_p.

We will call this transfer function for the controlled system

$$H_{RV}(s) = \frac{k_p/(b+k_p)}{m/(b+k_p)s+1}$$

RV because it controls based on R and V.

$$R(s) \longrightarrow \boxed{H_{RV}} \longrightarrow V(s)$$

If we wanted to account for a disturbance term, we can simply find a transfer function $H_{DV}(s)$ such that $V(s) = H_{DV}(s)D(s)$.

By linear superposition (all the systems are linear) the net response would be

$$V(s) = H_{RV}(s)R(s) + H_{DV}(s)D(s).$$

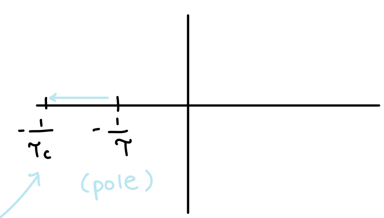
Now, we want to pick k_p appropriately so that the system does what we want.

Changing k_p lets us

- Adjust the time constant:

(open loop)

Uncontrolled τ	Controlled τ
$\frac{m}{b}$	$\frac{m}{b+k_p}$



For larger values of k_p, τ becomes smaller.

faster decay

Recall that a smaller τ leads to faster decay to the “final” value of the system (which always exists for the standard first order system).

- Adjust the DC gain:

(open loop) Uncontrolled κ	Controlled κ
$\frac{1}{b}$	$\frac{k_p}{b+k_p}$

by changing k_p we can make κ obtain any value between 0 and 1.

Example 1

Suppose we want to use a proportional controller to set a speed of 50 (i.e. $r(t) = 50$) in a vehicle with a mass of m, a coefficient of drag of b and no outside disturbances $d(t) = 0$.

1) Find the function $V(S)$ for the controlled system in terms of the k_p constant.

2) Use the FVT to explore and analyze the effects that changing k_p has on the "final" velocity.

$t \to \infty$

3) Find $v(t)$ and plot $v(t)$ for several values of k_p to explore this effect in the time domain.

1) $V(s) = H_{RV}(s)\, R(s)$

$$= \left(\frac{\frac{k_p}{b+k_p}}{\left(\frac{m}{b+k_p}\right)s+1} \right) \left(\frac{50}{s} \right)$$

$R(s) = \mathcal{L}\{r(t)\}$

$= \mathcal{L}\{50\}$

$= \frac{50}{s}$

2) $V(s)$ is proper rational fcn so FVT applies.

Poles:

$s = 0$ w/order 1

$$s = -\frac{1}{\frac{m}{b+k_p}}$$

$$= -\frac{b+k_p}{m} < 0$$

$Re(s) < 0$

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0^+} sV(s)$$

$$= \lim_{s\to 0^+} s\left(\frac{\frac{k_p}{b+k_p}}{\left(\frac{m}{b+k_p}\right)s+1}\right)\left(\frac{50}{s}\right)$$

$$= \lim_{s\to 0^+} 50\left(\frac{\frac{k_p}{b+k_p}}{\left(\frac{m}{b+k_p}\right)s+1}\right)$$

$$= 50\cdot\frac{k_p}{b+k_p}$$

As $k_p \to \infty$, "$v(\infty)$" $\to 50$

3) $v(t) = \mathcal{L}^{-1}\{V(s)\}$

$$= \mathcal{L}^{-1}\left\{\left(\frac{\frac{k_p}{b+k_p}}{\left(\frac{m}{b+k_p}\right)s+1}\right)\left(\frac{50}{s}\right)\right\} \xrightarrow{PF} \frac{A}{s} + \frac{B}{\left(\frac{m}{b+k_p}\right)s+1}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{50k_p}{b+k_p}}{s} + \frac{50k_p}{b+k_p}\left(-\frac{m}{b+k_p}\right)\left(\frac{1}{\left(\frac{m}{b+k_p}\right)s+1}\right)\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{50k_p}{b+k_p}\left(\frac{1}{s} - \left(\frac{m}{b+k_p}\right)\left(\frac{1}{\left(\frac{m}{b+k_p}\right)s+1}\right)\right)\right\}$$

$$= \frac{50k_p}{b+k_p}\left(1 - e^{-\left(\frac{b+k_p}{m}\right)t}\right)$$

To see the effects of k_p in the time domain run Lecture15_pcontroller.m

Major limitation: If $e(t)$ is ever 0 then $u(t) = k_p r(t) = 0$.

Thus in the absence of any disturbance to the system (i.e. $d(t) = 0$), the input into the system $f(t)$ also vanishes.

Thus there is no system input and hence $v(t)$ will drop (since $b > 0$).

In general p controllers, can never achieve the perfect asymptotic velocity!

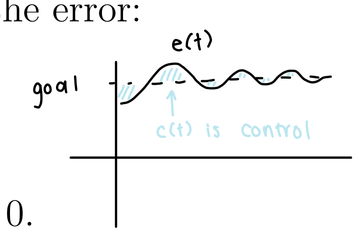
While in this example, we can get as close to the desired velocity as we want, in for higher order systems p controllers tend to overshoot and potentially become unstable for large values of k_p.

Hence we introduce

Integral controllers:

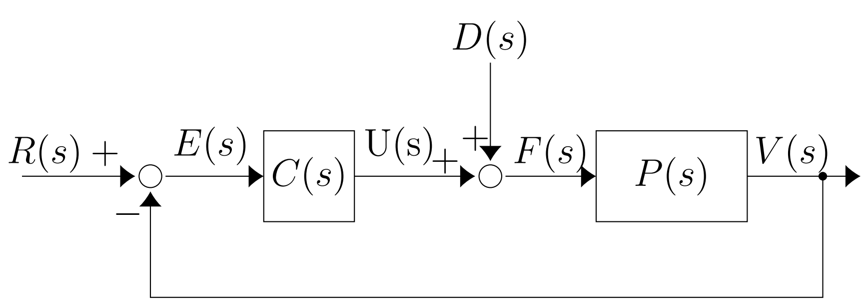
Let the error term be proportional to the integral of the error:

$$c(t) = k_i \int_0^t e(\tau) d\tau$$



This will go to zero exactly when the average error is 0.

It can hence (in principle) bring the asymptotic error exactly to 0!
We hence repeat the same analysis to this new controller.



The transfer function for the integral term is $C(s) = \frac{k_i}{s}$ (Hw 4 Q1).
Hence

$$\begin{aligned} V(s) &= \underbrace{P(s)}_{\text{Car transfer function}} \underbrace{C(s)}_{\text{Controller transfer function}} E(s) \\ &= P(s)C(s)(R(s) - V(S)) \end{aligned}$$

so

$$V(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} R(s).$$

Thus the new controlled system has a transfer function of

$$\begin{aligned} H(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} &= \frac{\frac{1}{ms+b} \frac{k_i}{s}}{1 + \frac{1}{ms+b} \frac{k_i}{s}} \\ &= \frac{k_i/m}{s^2 + \frac{b}{m}s + \frac{k_i}{m}} \qquad \leftarrow \text{Do some Algebra} \end{aligned}$$

This is the transfer function for a second order system so we will delay its full analysis until we cover the standard second order system in the next lecture but the DC gain is

$$H(0) = \frac{k_i/m}{0^2 + 0 \cdot \frac{b}{m} + \frac{k_i}{m}} = 1. \qquad \text{i.e. "}V(\infty) = R(\infty)\text{"}$$

Now let's play with this system a bit more.

Example 2

Suppose we want to use an integral controller to set a speed of 50 (i.e. $r(t) = 50$) in a vehicle with a mass of m, a coefficient of drag of b and no outside disturbances $d(t) = 0$.

1) Find the function $V(s)$ for the controlled system in terms of the k_i constant.

2) Use the FVT to explore and analyze the effects that changing k_i has on the "final" velocity.

3) Find $v(t)$ and plot $v(t)$ for several values of k_i to explore this effect in the time domain.

1) $V(s) = H(s)\,R(s)$

$$= \frac{\frac{k_i}{m}}{s^2+\left(\frac{b}{m}\right)s+\frac{k_i}{m}}\left(\frac{50}{s}\right)$$

2) Poles:

$s = 0$ w/order 1

$$s = \frac{-\frac{b}{m} \pm \sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k_i}{m}}}{2}$$

$$= -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2-4k_i m}$$

↳ if $b^2-4k_i m<0$, poles are complex w/ $Re(s)<0$

↳ if $-b+\sqrt{b^2-4k_i m} < 0$, poles have $Re(s)<0$

↳ if $-b+\sqrt{b^2-4k_i m} \geq 0$, would be unstable

- however:

$$\sqrt{b^2-4k_i m} \geq b$$

$$-4k_i m < 0$$

$$\underbrace{k_i < 0 \quad \text{or} \quad m < 0}_{\text{doesn't happen}}$$

FVT:

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0^+} sV(s)$$

$$= \lim_{s\to 0^+} s\,\frac{\frac{k_i}{m}}{s^2+\left(\frac{b}{m}\right)s+\frac{k_i}{m}}\left(\frac{50}{s}\right)$$

$$= \lim_{s\to 0^+} 50\left(\frac{\frac{k_i}{m}}{s^2+\left(\frac{b}{m}\right)s+\frac{k_i}{m}}\right)$$

$$= 50$$

$v(\infty) = R(\infty)\ \forall\ k_i > 0$

3) $$V(s) = \frac{\frac{k_i}{m}}{s^2+\left(\frac{b}{m}\right)s+\frac{k_i}{m}}\left(\frac{50}{s}\right)$$

$$= \frac{A}{s} + \frac{B}{s-s_+} + \frac{C}{s-s_-}$$ (poles can be sometimes complex or real so use $e^{(a+bj)}$)

↓ coverup method

$$= \frac{50}{s} + \frac{A_+}{s-s_+} + \frac{A_-}{s-s_-}$$ only if $s_+ \neq s_-$

$$S_{\pm} = \frac{-b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4k_i m}$$

$$A_{\pm} = \frac{25\left(\mp b - \sqrt{b^2 - 4k_i m}\right)}{\sqrt{b^2 - 4k_i m}}$$

$$v(t) = 50 + \underbrace{A_+ e^{s_+ t} + A_- e^{s_- t}}$$

$$= \begin{cases} e^{-\#_1 t}\left(A\sin(\#_2 t) + B\cos(\#_2 t)\right) & \text{if } b^2 - 4k_i m < 0 \\ e^{-\#_1 t} + e^{-\#_2 t} & \text{if } b^2 - 4k_i m > 0 \\ e^{-\#_1 t} + te^{-\#_1 t} & \text{if } b^2 - 4k_i m = 0 \end{cases}$$

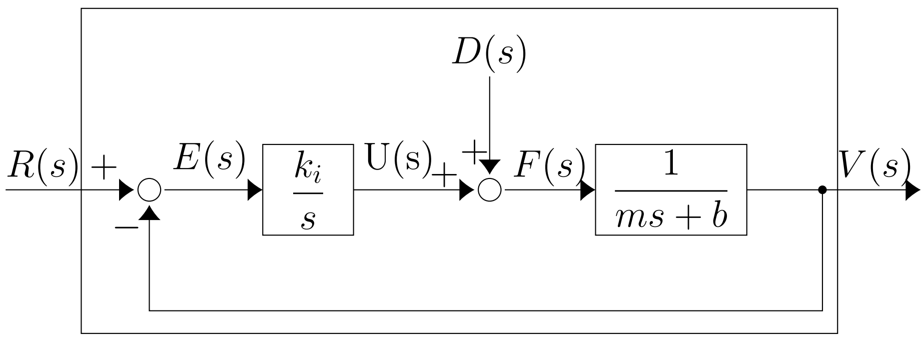
To see the effects of k_p in the time domain run Lecture15_pcontroller.m
Major Limitations:

- Since we adjust based on the integral the error, integral controllers do strongly quickly adjust to changes. i.e. they "adjust the average"
- The above leads to the potential to overshooting the goal velocity. This is also im part because the solutions generally with come in conjugate pairs and hence the solution will have sinusoidal terms that will decay to 0.

MATH 213 - Lecture 16: Standard 2nd order system, PI controllers and extra poles

Lecture goals: Understand the basics of the standard second order system, Know what a PI controller is.

The system diagram for the integrally controlled car cruise control problem from last lecture



and we found that the transfer function for the controlled system (the big box) is

$$H(s) = \frac{k_i/m}{s^2 + \frac{b}{m}s + \frac{k_i}{m}}.$$

Systems with transfer functions of this form are common so we will introduce and analyze the standard second order system

Definition 1: Standard second order system
The standard second order system has a transfer function given by

$$H(s) = \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}$$

where $\omega > 0$ and $\xi \geq 0$.

Examples of these systems include cruise control with integral control, the harmonic oscillator (with or without damping), and the RLC circuit.

We will now analyze the behaviour of the possible behaviours of the standard second order system. By doing this, we validate the analysis of the cruse control system with an integral controller from the last lecture.

The quadratic formula tells us that the poles of $H(s)$ are

$$s_{\pm} = -\xi\omega \pm \omega\sqrt{\xi^2 - 1}.$$

There are hence three different forms of the solution based on where the poles lie.

Case 1: Two real distinct pole. In this case $\sqrt{\xi^2 - 1}$ is real **and** greater than 0 then there are two distinct real valued poles.

Since $\xi > 0$ for the standard second order system, this happens when

$$\xi^2 - 1 > 0 \qquad \text{or} \qquad \xi > 1.$$

In this case we decompose $H(s)$ via PF as

$$H(s) = \frac{A_+}{s - s_+} + \frac{A_-}{s - s_-}$$

$\longrightarrow$ $s_{\pm}$ must be -ve so poles have $Re(s) < 0$ + system doesn't blow up

for some $A_+, A_- \in \mathbb{R}$.

Thus in this case the standard second order system can be decomposed as a sum of two first order systems!!

To show that these are well behaved first order systems (i.e. remain bounded for bounded inputs), first note that for $\xi > 1$,

$$\xi > \sqrt{\xi^2 - 1}.$$

Hence,

$$\begin{aligned} s_{\pm} &= -\xi\omega \pm \omega\sqrt{\xi^2 - 1} \\ &= \omega(-\xi \pm \sqrt{\xi^2 - 1}) \\ &< 0. \end{aligned}$$

Thus the poles of the transfer function have negative real parts and therefore any input that is bounded (i.e. has no poles with positive real part) cannot resonate to cause the system response to be unbounded.

For completion, we will find the system's impulse response. Using the coverup method and simplifying gives

$$A_{\pm} = \frac{\pm\omega}{2\sqrt{\xi^2 - 1}}$$

and hence the system's impulse response is

$$h(t) = A_+ e^{s_+ t} + A_- e^{s_- t}$$

which does decay as $t \to \infty$!

Relating this to A3Q1, this is the overdamped case of the harmonic oscillator.

Case 2: One real repeated pole: If $\xi^2 - 1 = 0$, or simply if $\xi = 1$, then there is a single repeated pole at

$$s_{\text{root}} = -\omega.$$

This root has a negative real part and hence the system is again well behaved.

For completion we compute the system's impulse response to examine the solution. In this case

$$H(s) = \frac{\omega^2}{(s - -\omega)}$$

$$H(s) = \frac{\omega^2}{(s + \omega)^2}.$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{\omega^2}{(s+\omega)^2}\right\}$$

$$= e^{-\omega t}\mathcal{L}^{-1}\left\{\frac{\omega^2}{s^2}\right\}$$

$$= \omega^2 e^{-\omega t} t$$

Hence the system's impulse response is simply

$$h(t) = \omega^2 t e^{-\omega t}$$

Relating this to A3Q1, this is the critically damped case of the harmonic oscillator.

(not on Im axis)

Case 3: Two complex conjugate poles with a non-zero real component: If $\xi^2 - 1 < 0$, or simply if $0 < \xi < 1$, then we have complex conjugate poles at

$$s_{\pm} = -\xi\omega \pm \omega j\sqrt{1-\xi^2}$$

where for ease in future computations, we pulled out the negative from inside the square root.

These roots have negative real parts and hence the system is again well behaved.

We now find the system's impulse response. We know that the solution will be decaying sine waves so we prepare the transfer function to use the exponential modulation rule and the sine transform.

$$\begin{aligned} H(s) &= \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2} \\ &= \frac{\omega^2}{(s+\xi\omega)^2 + \omega^2 - \xi^2\omega^2} && \leftarrow \text{complete the square or just note the roots} \\ &= \frac{\omega^2}{(s+\xi\omega)^2 + \omega^2(1-\xi^2)} \\ &= \frac{\omega^2}{(s+\xi\omega)^2 + (\omega\sqrt{(1-\xi^2})^2} \end{aligned}$$

from the LT table we have

$$h(t) = \frac{\omega}{\sqrt{1-\xi^2}} e^{-\xi\omega t} \sin((\omega\sqrt{1-\xi^2})t)$$

Relating this to A3Q1, this is the underdamped case of the harmonic oscillator.

(on Im axis)

Case 4: Two complex conjugate poles with a zero real component: If $\xi^2 - 1 = 0$, or simply if $\xi = 0$, then we have complex conjugate poles at

$$s_{\pm} = \pm\omega j$$

These roots have zero real parts and hence the system may not be well behaved is there is resonance!

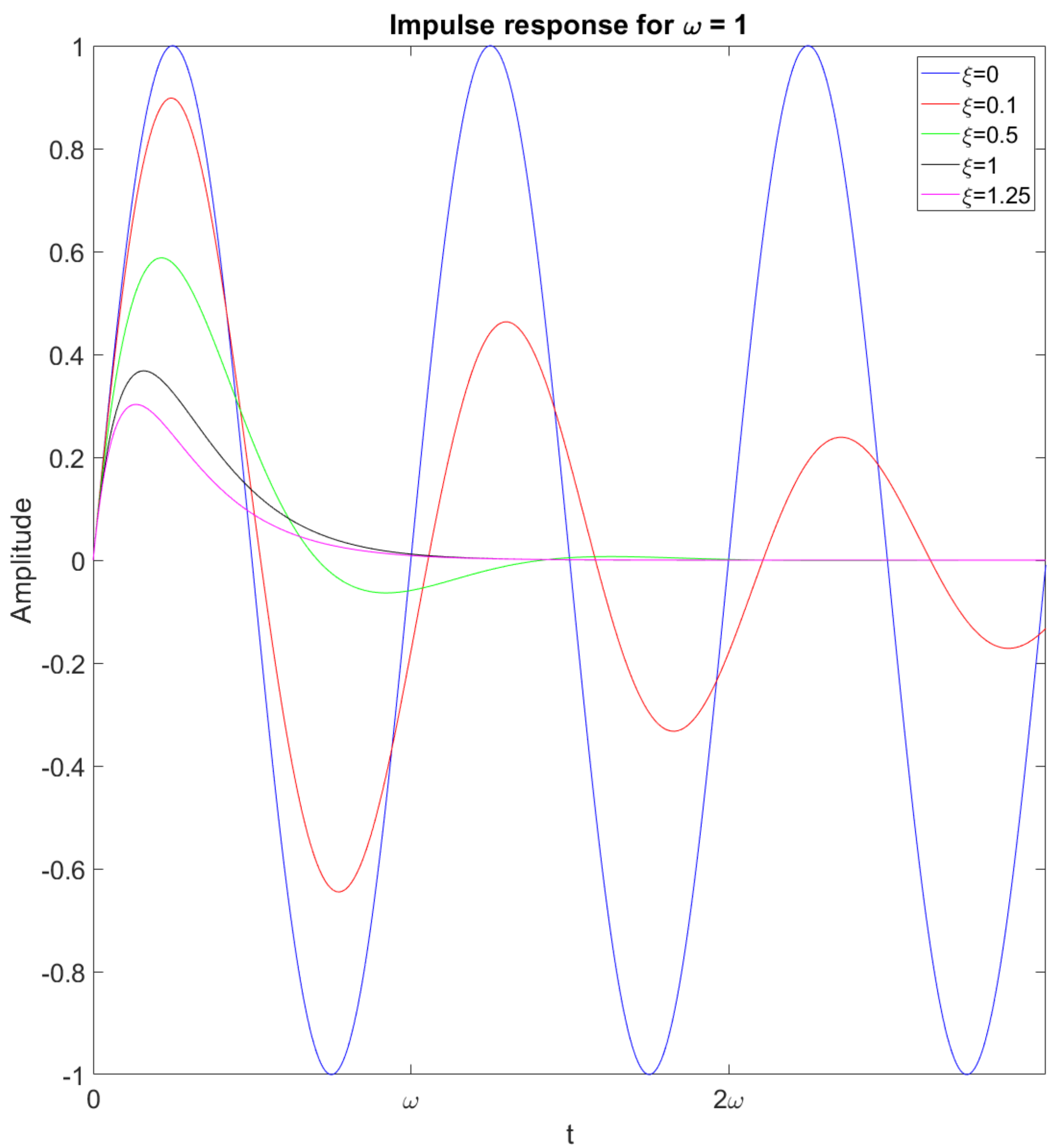
We now find the system's impulse response.

$$H(s) = \frac{\omega^2}{s^2 + \omega^2}$$

so from the LT table we have

$$h(t) = \omega \sin(\omega t)$$

Relating this to A3Q1, this is the undamped case of the harmonic oscillator. We can now plot the impulse response to all these systems. Here is a plot!!



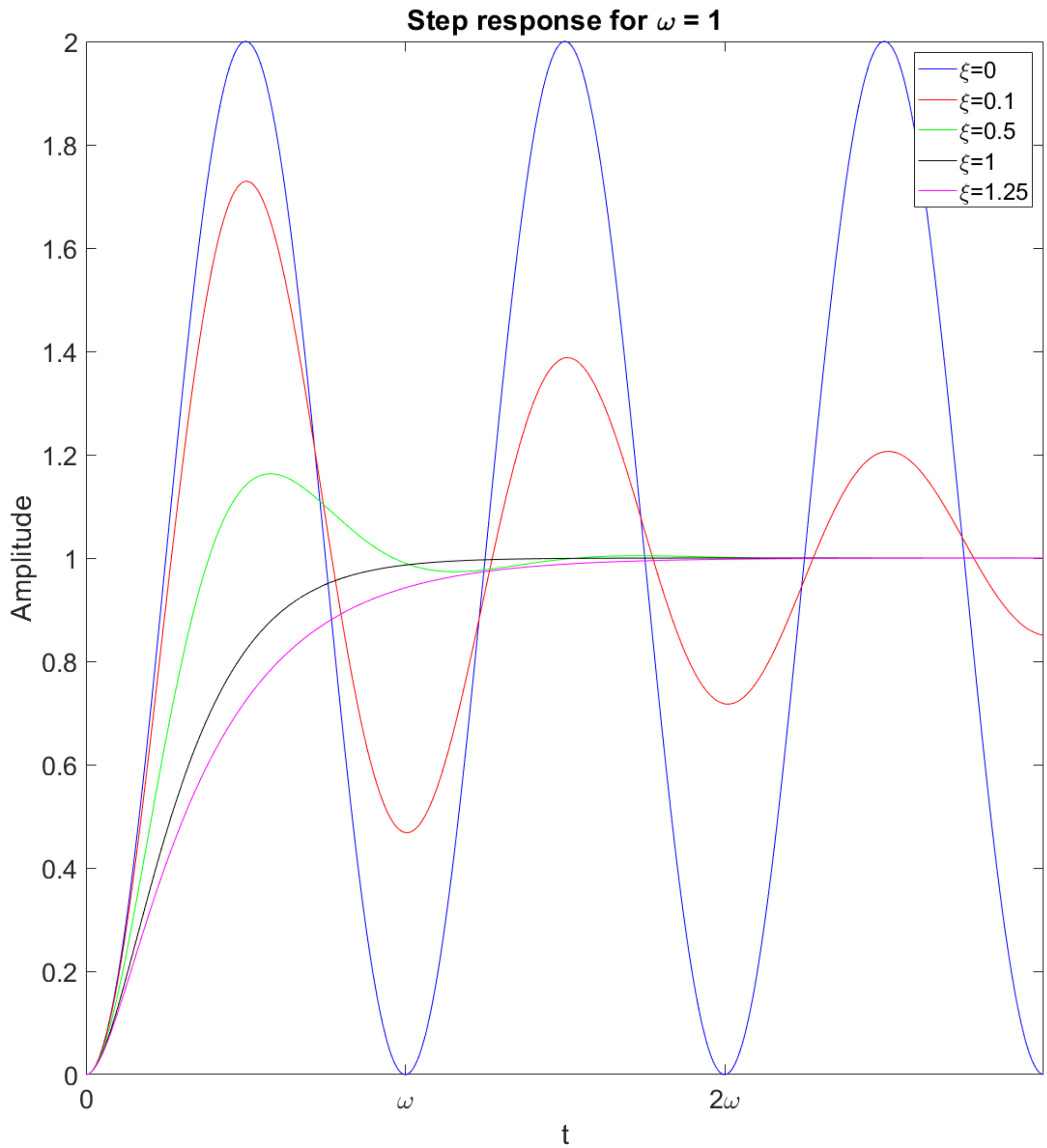
See the L16 .m file for a video!

We can also compute the step responses and plot them, we will not show the computations but the step response for $\xi \geq 0$ other than $\xi \neq 1$ is

$$1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi \omega t} \sin\left((\omega\sqrt{1-\xi^2})t + \theta\right)$$

phase: deals w/ cos terms

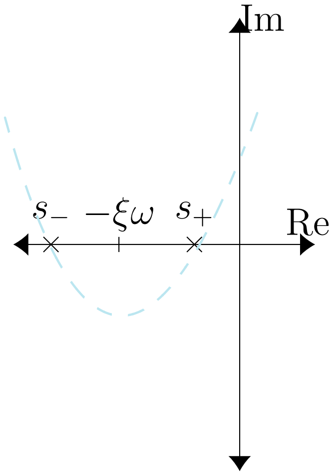
where $\theta = \cos^{-1}\xi$. At $\xi = 1$ we have a t scaled exponential. Here is a plot of some sample solutions



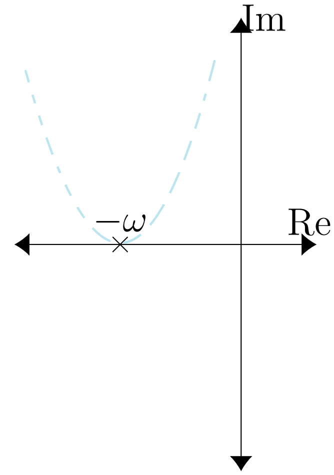
See the L16 .m file for a video!

Pole location summary (poles are marked with an $\times$):

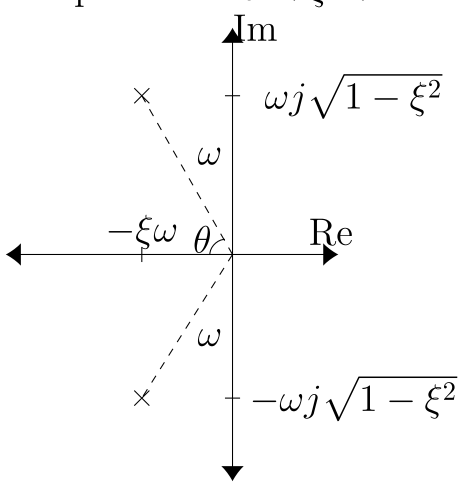
Overdamped case: $\xi > 1$



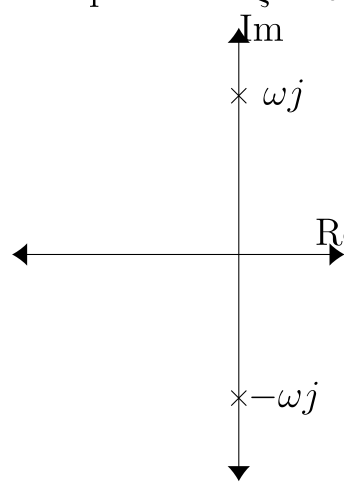
Critically damped case: $\xi = 1$



Underdamped case: $0 < \xi < 1$



Undamped case: $\xi = 0$ (blows up when f(t) resonates w/ poles)



In the underdamped cases:

If ω increases the poles move further from the origin and we have a faster response.

If ξ increases, θ decreases and we have more damping.

Matlab video!!! See Lecture 16 .m file for a nice video of the poles and the step response.

Example 1: Cruise control critical dampening

Recall that the transfer function for an integrally controlled car with mass m, drag coefficient b and integral controller constant k_i is

$$H_{RS}(s) = \frac{k_i/m}{s^2 + \frac{b}{m}\ s + \frac{k_i}{m}}. \quad = \quad \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}$$

Find k_i such that the curse control system is critically damped and hence adjust to the response $r(t) = v$ for $v \in \mathbb{R}_{\geq 0}$ as fast as possible without experiencing overshot.

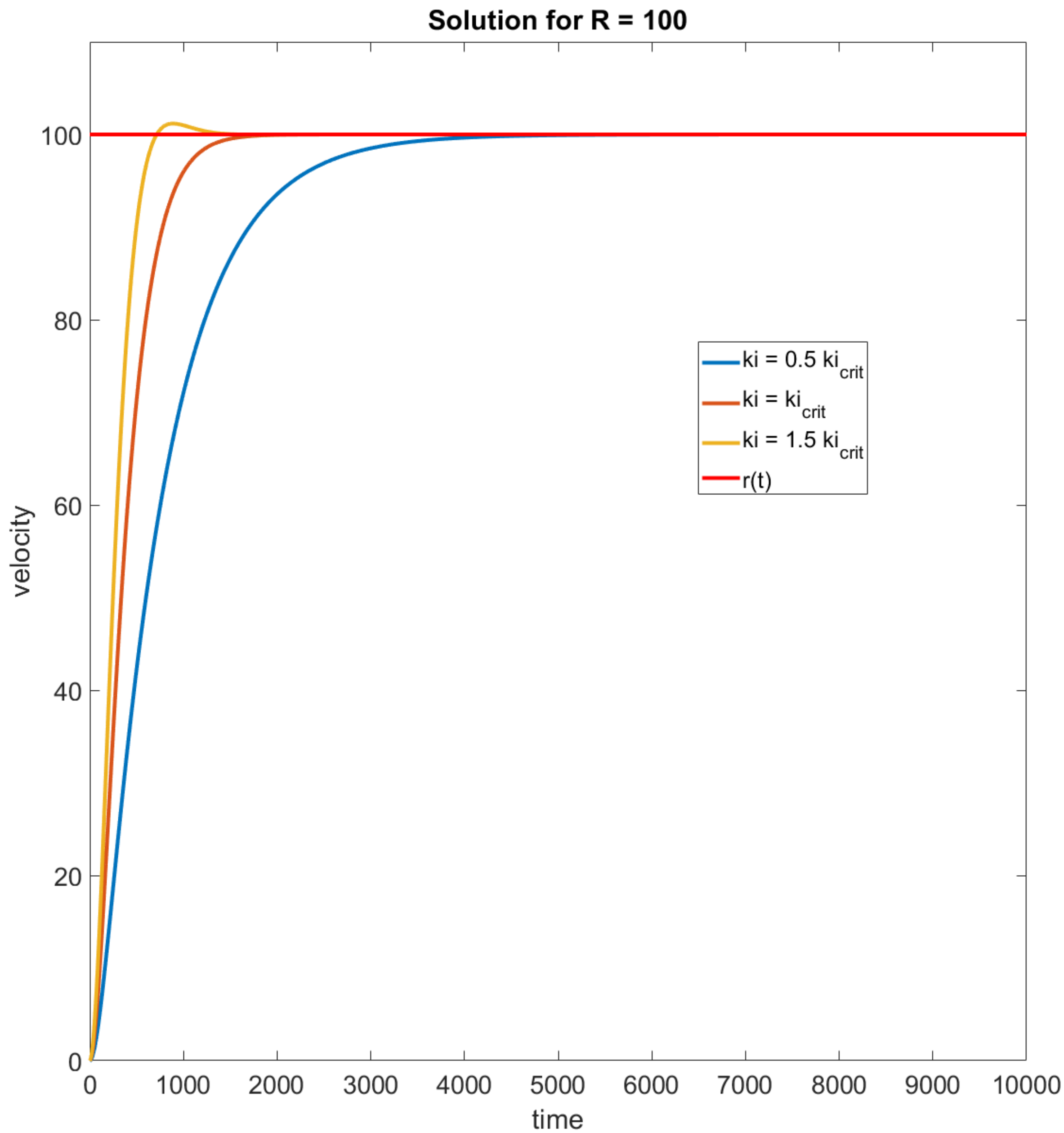
Solution: This is a second order system with $\omega = \sqrt{k_i/m}$ and

$$\begin{aligned} 2\xi\omega &= \frac{b}{m} \\ \xi &= \frac{b}{2m\omega} \\ &= \frac{b}{2\sqrt{mk_i}} \end{aligned}$$

To be critically damped we need $\xi = 1$ and thus k_i needs to satisfy

$$\begin{aligned} 1 &= \frac{b}{2\sqrt{mk_i}} \\ \sqrt{k_i} &= \frac{b}{2\sqrt{m}} \\ k_i &= \frac{b^2}{4m} \end{aligned}$$

Here is a plot of the behaviour of the controlled system's response for values of k_i around this critical value:

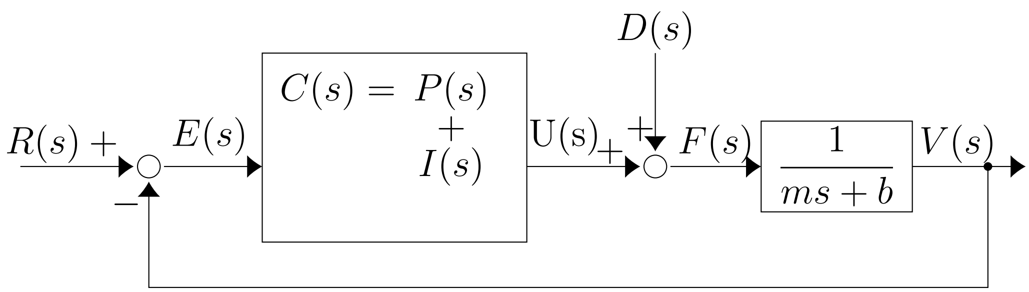


See the second L16 .m file if you want to explore this more!

PI controller:

For the cruise control problem, the P controller by itself gets us to a close velocity quickly but misses the exact velocity and the I controller by itself gets us to the exact velocity but is slow to do so.

Let's use them together!!



The transfer function for this system is

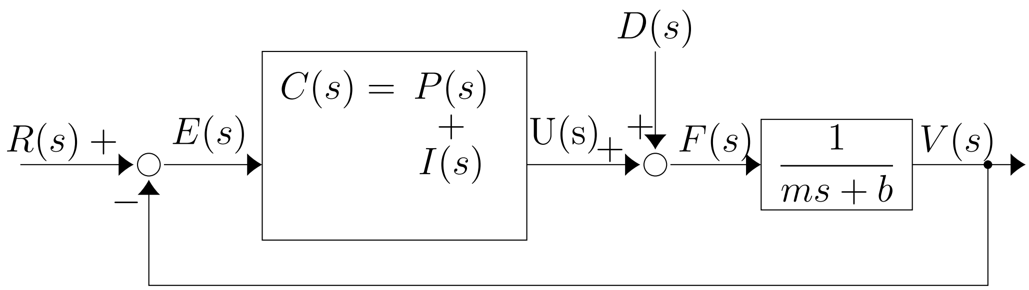
$$\begin{aligned}
H_{RV}(s) &= \frac{C(S)S(s)}{1 + C(s)S(s)} && \leftarrow \text{ Previous result} \\
&= \frac{\left(k_p + \frac{k_i}{s}\right)\frac{1}{ms+b}}{1 + \left(k_p + \frac{k_i}{s}\right)\frac{1}{ms+b}} \\
&= \frac{\frac{k_i}{m}\left(\frac{k_p}{k_i}s + 1\right)}{s^2 + \left(\frac{b+k_p}{m}\right)s + \frac{k_i}{m}} && \leftarrow \text{ Algebra}
\end{aligned}$$

Next lecture we will analyze this transfer function and how its zeros and poles change the system response and then look at more complex transfer functions.

MATH 213 - Lecture 17: PI controllers, zeros, extra poles and stability

Lecture goals: Analyze a PI controller for a first order system, know the effects of the zeros and ("extra") poles of a transfer function.

Recall that the system diagram for a PI controlled car is



and the transfer function for this system is

$$H_{RV}(s) = \frac{\frac{k_i}{m}\left(\frac{k_p}{k_i}s + 1\right)}{s^2 + \left(\frac{b+k_p}{m}\right)s + \frac{k_i}{m}}.$$

Example 1
Analyze the result of using a PI controller to attempt to control the velocity of a car.

We will first simplify the functional form of $H_{RV}(s)$ to something more familiar by noting that it is of the form

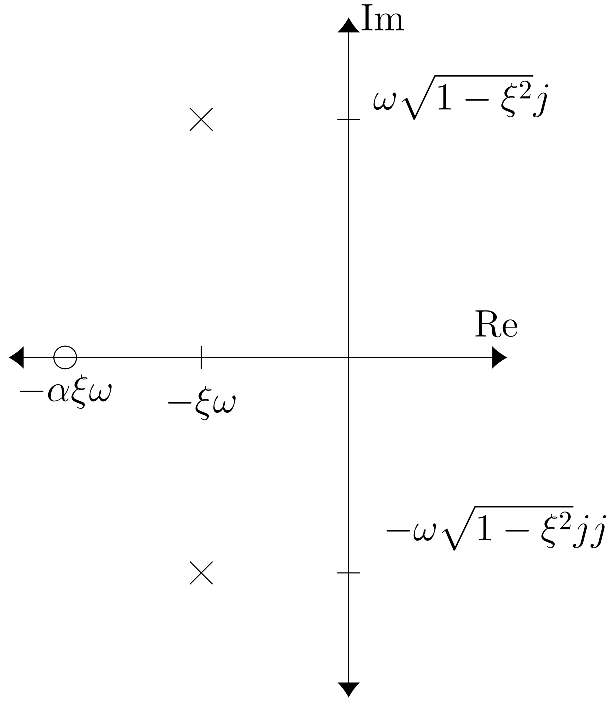
$$H_z(s) = \frac{\omega^2\left(\frac{s}{\alpha\xi\omega} + 1\right)}{s^2 + 2\xi\omega s + \omega^2}$$

where we note that this is almost of the form of the standard second order system but it has a zero.

To make the connection between $H_{RV}(s)$ and $H_z(s)$ note that

$$\omega = \sqrt{\frac{k_i}{m}}, \qquad \xi = \frac{(b+k_p)\sqrt{\frac{k_i}{m}}}{2k_i}, \qquad \text{and} \qquad \alpha = \frac{2k_i m}{k_p(b+k_p)}$$

Since $k_i, k_p, m, b > 0$, the poles and zeros can be plotted in the complex plane as



For the cruise control system we mostly care about the step response, so we will analyze the response to the step function.

$\mathcal{L}\{1\} = \frac{1}{s}$ is Laplace transform of step fcn

$$
\begin{aligned}
S(u(t)) &= H_z(s) \cdot \frac{1}{s} \\
&= \left(\frac{\omega^2 \frac{s}{\alpha\xi\omega}}{s^2 + 2\xi\omega s + \omega^2} + \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2} \right) \frac{1}{s} \\
&= \frac{1}{\alpha\xi\omega} \cdot \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2} + \frac{\omega^2}{(s^2 + 2\xi\omega s + \omega^2)s}
\end{aligned}
$$

This can be decomposed into different responses of the standard second order system

$$
S(u(t)) = \frac{1}{\alpha\xi\omega} \cdot \underbrace{\frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}}_{\text{Std. 2nd order impulse res.}} + \underbrace{\frac{\omega^2}{(s^2 + 2\xi\omega s + \omega^2)s}}_{\text{Std. 2nd order step response}} .
$$

We know what the standard second order impulse and step responses look like as a function of ω and ξ (L16), so we just need to explore how α changes the linear combination of these responses. Since α controls where the zero is, we are studying the effect of the zero on the response.

We will look at the extreme cases:

- If $\alpha \to \infty$ then $S(u(t)) \to \underbrace{\dfrac{\omega^2}{(s^2 + 2\xi\omega s + \omega^2)s}}_{\text{Std. 2nd order step response}}$. (1st term gets really small)

  In practice, if $\alpha\xi\omega \gtrsim 10$ then the effects of the impulse response term can ignored in many practical applications.

- If $\alpha \to 0$ then $\frac{1}{\alpha\xi\omega} \to \infty$. Thus

$$S(u(t)) \to \text{“}\infty\text{”} \cdot \underbrace{\frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}}_{\text{Std. 2nd order impulse res.}} .$$

  and we see a lot of overshooting/instability.

In terms of the cruise control example:

- If k_p increases for a fixed value of k_i then
  - $\omega = \sqrt{\frac{k_i}{m}}$ is unaffected.
  - $\xi = \frac{(b+k_p)\sqrt{\frac{k_i}{m}}}{2k_i}$ increases. This leads to a faster system response when $\xi > 1$ and a slower system response with $0 < \xi < 1$. It also leads to more oscillations when $\xi < 1$
  - $\alpha = \frac{2k_i m}{k_p(b+k_p)}$ decreases. This causes the impact of the standard second order impulse response part of the response to increase. i.e. we will see a larger “bump” in the initial velocity.

- If k_i increases for a fixed value of k_p then
  - $\omega = \sqrt{\frac{k_i}{m}}$ increases. This in isolation leads to a faster oscillating response.
  - $\xi = \frac{(b+k_p)\sqrt{\frac{k_i}{m}}}{2k_i}$ decreases. This has the opposite effect of k_p.
  - $\alpha = \frac{2k_i m}{k_p(b+k_p)}$ increases. This has the opposite effect of k_p.

Generally, for this application we care about getting to the steady state velocity quickly and do not want to overshoot. Hence we want no oscillations and we want the impact of the impulse term to not cause overshooting.

Combining the effects above, for **this** system we use k_i to adjust the speed of the response and k_p to control the dampening of the oscillations. The exact values of

k_i and k_p needed will depend on b and m but generally we want $\xi > 1$ to avoid the oscillations.

See the Lecture17_PI_controller.m for some examples of the controlled velocity.

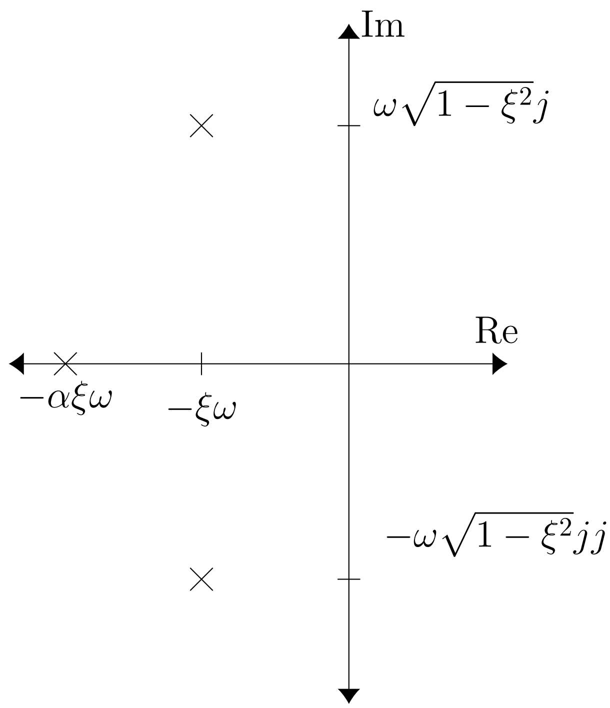
Adding a pole:

Example 2
Examine the impulse response of the system with transfer function

$$H(s) = \frac{\omega^2}{\left(\frac{s}{\alpha\xi\omega} + 1\right)(s^2 + 2\xi\omega s + \omega^2)}$$

for $\omega, \xi, \alpha > 0$ in terms of the standard first and second order systems.

Since $k_i, k_p, m, b > 0$, the poles and zeros can be plotted in the complex plane as



Based on the locations of the poles/functional form of $H(s)$ we note that we can decompose $H(s)$ as a linear combination of first and second order systems

$$H(s) = \underbrace{\frac{A}{\frac{s}{\alpha\xi\omega} + 1}}_{\text{Std. 1st order system}} + \underbrace{\frac{Bs + C}{s^2 + 2\xi\omega s + \omega^2}}_{\text{The system we just looked at}}$$

We need to know the relative values of A, B, C (and the decay rates of the system responses) to know what the impulse response looks like.

Evaluating the PF decomposition components gives

$$A = \frac{1}{1 - 2\alpha\xi^2 + \alpha^2\xi^2}$$
$$B = \frac{-\alpha\xi\omega}{1 - 2\alpha\xi^2 + \alpha^2\xi^2}$$
$$C = \frac{\alpha(\alpha - 2)\xi^2\omega^2}{1 - 2\alpha\xi^2 + \alpha^2\xi^2}.$$

α changes the relative weights of the impacts of the first order system and the second order system. What happens as α changes?

We again look at the extreme values:

- If $\alpha \to 0$ then $A \to 1$ and $B, C \to 0$.

  In this case, the impulse response looks like that of a standard first order system given that its pole is closer to the imaginary axis than the complex poles.

  The latter happens when $-\alpha\xi\omega > -\xi\omega$. In practice this happens for $\alpha \lesssim 0.1$ or so.

- If $\alpha \to \infty$ then $A, B \to 0$ and $C \to \omega^2$.

  In this case, the impulse response looks like that of a standard second order system given that the complex poles are closer to the imaginary axis than the real pole.

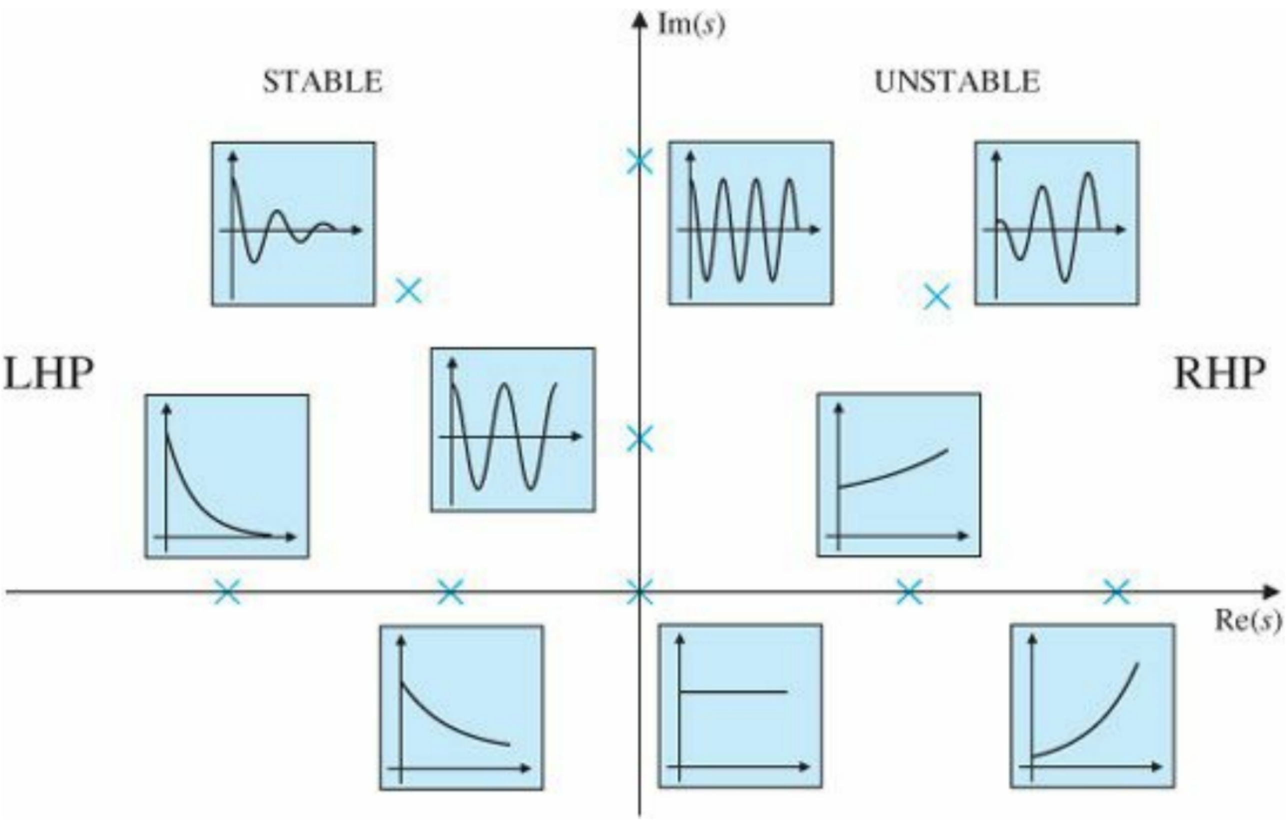
  Assuming the roots are complex valued (i.e $0 < \xi < 1$), the latter happens when $-\alpha\xi\omega < -\xi\omega$. In practice this happens for $\alpha \gtrsim 10$ or so. If the roots are not complex then the second order system decomposed into two first order systems but the result still holds.

See Lecture17_extra_poles.m for pictures showing this.

Definition 1
When a real pole or a complex-conjugate pair of poles are an order of magnitude closer to the imaginary axis than <u>all</u> other poles then we say that they are **dominant**.

Linear Stability:

Recall the general behaviours of the impulse response caused by a pole at some point $a + bj$ for $a, b \in \mathbb{R}$.



From Franklin, Powell and Emami-Naeini,
"Feedback Control of Dynamic Systems" 6th ed.

The impulse response of <u>**ALL**</u> Linear Time Invariant systems is a linear combination of the types of functions shown in the above plot.

Further since $\{\delta(t - \tau) | \tau \in \mathbb{R}\}$ is a basis for the set of all functions, the response of <u>**ANY**</u> LTI to <u>**ANY**</u> function $f(t)$, can be written as a convolution of $f(t)$ with a linear combination of the types of functions shown above.

Definition 2: System Stability
LTI, S, is **stable** if $S(\delta(t))$ decays to 0.

A LTI, S, is **unstable** if $S(\delta(t))$ is unbounded.

A LTI, S is **marginally stable** if $S(\delta(t))$ is bounded but does not decay to 0.

The type of stability mentioned above is often called linear stability.

Definition 3: Transfer function stability
A transfer function is stable if the system it is a transfer function for is stable.
A transfer function is unstable if the system it is a transfer function for is unstable.
A transfer function is marginally stable if the system it is a transfer function for is marginally stable.

Theorem 1: Stability
A transfer function is stable if all poles have a negative real part.

A transfer function is unstable if there is a pole with a positive real part OR there is a second order pole that has a real part of 0.

A transfer function is marginally stable if there are no poles with positive real parts OR second order poles with a real part of 0 and in addition there is at least order 1 pole that has a real part of 0.

Sketch of proof:

If all poles have negative real parts then the impulse response is a sum of exponentials (potentially with oscillations) that decay to 0 and hence the impulse response decays.

If there is any pole with a positive real part, then that component of the impulse response grows. Hence the impulse response will not be bounded.

If the first condition is met, then the system is not unstable and hence the impulse response does not grow without bound. If the second condition is met then there is either a sinusoidal component to the impulse response or a constant component. In either case, these terms will keep the system response from decaying to 0.

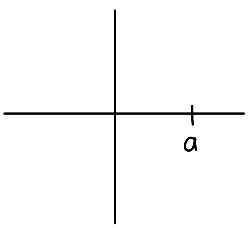
Definition 4: Bounded-input, bounded-output (BIBO) stable
A LTI, S, is **bounded-input, bounded-output (BIBO) stable** if $S(f)$ is bounded for all bounded functions f.

Theorem 2
A LTI system with a rational transfer function is BIBO stable if and only if its transfer function is both stable and proper.

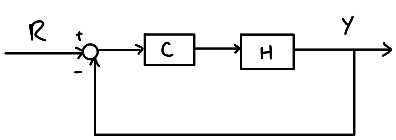
Example 3
Build a controller that turns the system with a transfer function

$$H(s) = \frac{1}{s-a} \quad , \quad a > 0$$

into a **stable** system.



Pole is +ve so $h(t)$ is unbounded.



$$T(s) = \frac{C(s)H(s)}{1 + C(s)H(s)} \longrightarrow C(s)H(s) = \frac{s-a}{(s+b)^2}\,\frac{1}{s-a}$$

gets rid of +ve pole

square to make proper rational (not necessary)

$$= \left(\frac{s-a}{(s+b)^2}\right)\left(\frac{1}{s-a}\right)\frac{1}{1 + \frac{s-a}{(s+b)^2}\frac{1}{s-a}}$$

$$= \frac{1}{(s+b)^2}\left(\frac{1}{1 + \frac{1}{(s+b)^2}}\right)$$

$$= \frac{1}{(s+b)^2}\left(\frac{(s+b)^2}{(s+b)^2 + 1}\right)$$

$$= \frac{1}{(s+b)^2 + 1}$$

If $H(s) = \frac{1}{s-(a+\varepsilon)}$, $\varepsilon > 0$ ← error

$$C(s) = \frac{s-a}{(s+b)^2}$$

$$T(s) = \frac{s-a}{(s-a-\varepsilon)(s+z)^2+s-a}$$

pole at $s = a+\varepsilon$

NASA tried this method for a Jupiter rocket (1957) and Atlas 4a rocket (1957) and others... don't do this!

I repeat **NEVER** cancel unstable poles in the above way.

Go to the tutorial if you want to see how to do this properly.

MATH 213 - Lecture 18: Bode plots

Lecture goals: Understand what the frequency response is and be able to generate and interpret Bode plots.

In lectures 13-17 we learned how to:

- compute the response of a LTI given an input function $f(t)$ by either evaluating $\mathscr{L}^{-1}\{H(s)\} * f(t)$ or computing $\mathscr{L}^{-1}\{H(s)F(s)\}$,
- analyze the general structure of the unit impulse and unit step responses by looking at the poles (and zeros) of the transfer function
- apply simple control systems (P, I, and PI) to control a system to have the behaviour we want it to have (i.e. cruse control problem).
- that the transfer function is the Laplace transform of the system's impulse response.
- that complex exponentials are the eigenfunctions of LTIs

These methods work well for many simple/academic problems but this method requires us either to find all of the poles (or potentially the dominant ones) or to evaluate a convolution integral.

Sometimes it is not possible to find all the poles/roots of the transfer function!

Sometimes we want to work in the time domain rather than the frequency domain.

In these cases, we need to use a different approach that has its own pros and cons.

Deriving the frequency response:

Recall Theorem 4 from lecture 13:

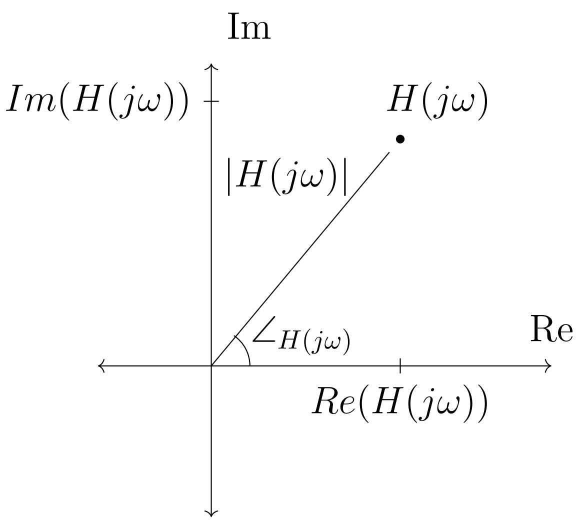
> **Theorem 1: LTI response to an exponential**
> If $S : f \to y$ is a LTI with transfer function $H(s)$ then for any $s \in \mathbb{C}$
> $$e^{st} \xrightarrow{S} H(s)e^{st}.$$

Now if $s = j\omega$ then the above gives $e^{j\omega t} \xrightarrow{S} H(j\omega)e^{j\omega t}$.

$H(j\omega)$ is a complex number so we can "recall" from MATH 115 that we can write it in polar form:

$$H(j\omega) = |H(j\omega)|e^{j\angle_{H(j\omega)}}$$

"Recall" from MATH 115 that this decomposition can be viewed geometrically:



Now we can write the system's response to $e^{j\omega t}$ as

$$\begin{aligned} H(j\omega)e^{j\omega t} &= \underbrace{|H(j\omega)|e^{j\angle_{H(j\omega)}}}_{H(s)} e^{j\omega t} \\ &= |H(j\omega)|e^{(\omega + \angle_{H(j\omega)})jt} \end{aligned}$$

Because of the above, $H(j\omega)$ is called the **frequency response**.

Explicitly, $H(j\omega)$ is the factor we need to scale the input signal $e^{j\omega t}$ by in order to find the system's response of $e^{j\omega t}$.

Theorem 2
If S is a LTI with transfer function $H(s)$, then

$$\sin(\omega t) \xrightarrow{S} |H(j\omega)| \sin(\omega t + \angle_{H(j\omega)})$$

Sketch of proof: Recall that

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}.$$

Use the above identity along with the previous results and then do some algebra.

Observations: The above says that the system response of an LTI to a sin wave of frequency ω:

- has an amplitude scaled by $|H(j\omega)|$
- has the same frequency
- Has a phase shifted by $\angle_{H(j\omega)}$

In the real world signals have a clear starting time (i.e. real world signals are one sided) so the above can't be used for many applications. Lucky for us:

Theorem 3
If S is a stable LTI with transfer function $H(s)$, then as $t \to \infty$

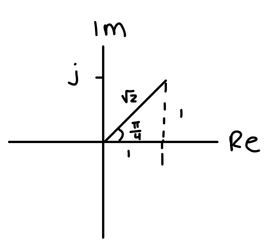
$$\sin(\omega t)u(t) \xrightarrow{S} |H(j\omega)| \sin(\omega t + \angle_{H(j\omega)}).$$

We will skip this proof.

Example 1
Suppose $H(s) = \frac{1}{RCs+1}$ for $RC = 0.01$. Find the magnitude and phase shift for the system response to $0.5\sin(100t)$

$$H(j \cdot 100) = \frac{1}{0.01(j \cdot 100)+1} = \frac{1}{1+j} = \frac{1}{\sqrt{2}e^{\frac{\pi}{4}j}} = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}j}$$

$1+j$



Output:

$$\underbrace{\frac{0.5}{\sqrt{2}}}_{\text{magnitude}} \sin(100t \underbrace{- \tfrac{\pi}{4}}_{\text{phase shift}})$$

Looking ahead: Fourier series/transforms will allow us to decompose functions as a sum/integral of sin and cos waves.

Hence if we know what the LTI does so all complex exponentials, then we can decompose a signal into a sum/integral of complex exponentials, scale and phase shift them, and then add the results together to see the response to the original signal.

We want a nice way to display the scaling factors and phase shifts so we introduce Bode plots.

Bode plots:

- Bode plots are a graphical representation of the frequency response.
- We need two plots: one for $|H(j\omega)|$ in "decibels" (dB) vs $\log_{10}(\omega)$ and one for $\angle H(i\omega)$ vs $\log_{10}(\omega)$
- The aforementioned plots are called the "magnitude" and "phase" curves respectively.

Using these conventions has two benefits:

- it allows curves to be approximated by piecewise lines (we will show this via examples).
- allows plots for complex transfer functions to be build by adding plots from simpler transfer functions.

To see the basic idea behind this approach suppose $H(j\omega) = H_1(j\omega)H_2(j\omega)$ and that we have the Bode plots for H_1 and H_2.

In this case

$$\begin{aligned} H(j\omega) &= |H_1(j\omega)|e^{j\angle_{H_1(j\omega)}}|H_2(j\omega)|e^{j\angle_{H_2(j\omega)}} \\ &= |H_1(j\omega)||H_2(j\omega)|e^{j(\angle_{H_1(j\omega)}+\angle_{H_2(j\omega)})} \end{aligned}$$

The angles are additive so we can simply add the phase curves of H_1 and H_2 to get the phase curve for H.

The magnitudes are not additive but... "recall" that $\log(ab) = \log(a) + \log(b)$ so if we use decibels for the magnitude, then the magnitude curves become additive.

> **Definition 1: Decibels**
> $|H(j\omega)|$ in decibels is $20\log_{10}(|H(j\omega)|)$.

Using decibels for the magnitude: if $H(j\omega) = H_1(j\omega)H_2(j\omega)$ then we have

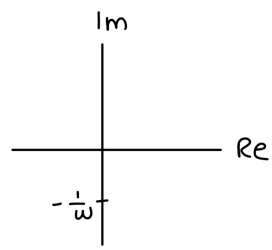
$$20\log_{10}(|H(j\omega)|) = 20\log_{10}(|H_1(j\omega)|) + 20\log_{10}(|H_2(j\omega)|).$$

So if we use decibels, then we can just add the magnitude curves to find the magnitude curve of $H(j\omega)$.

Bode plot examples:

> **Example 2**
> Find the Bode plot for the system with transfer function $H(s) = \frac{1}{s}$ and find the system response to $\sin(\omega t)$.

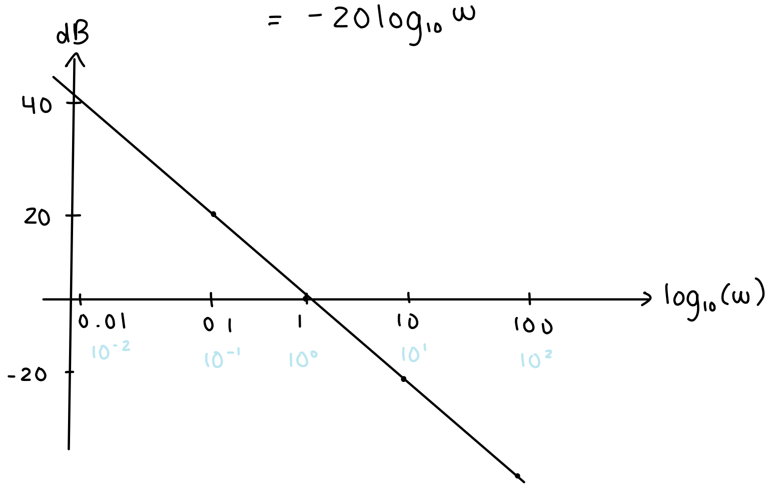
$$H(j\omega) = \frac{1}{j\omega}$$
$$= -\frac{1}{\omega}j$$



$$|H(j\omega)| = \left|\frac{1}{j\omega}\right|$$
$$= \frac{|1|}{|j\omega|}$$
$$= \frac{1}{\omega}$$

In decibels:

$$|H(j\omega)|_{dB} = 20\log_{10}\frac{1}{\omega}$$
$$= 20(\log_{10}1 - \log_{10}\omega)$$
$$= -20\log_{10}\omega$$

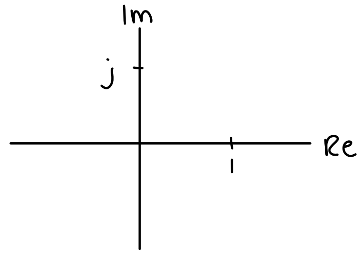


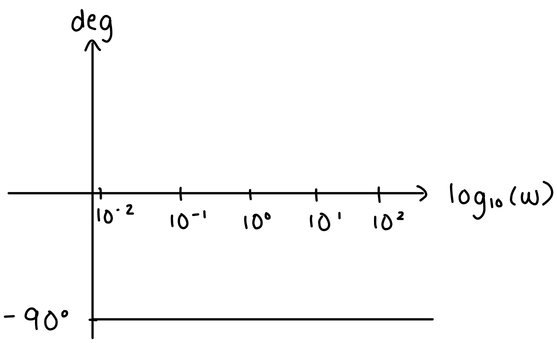
$$\angle H(j\omega) = \angle \frac{1}{j\omega}$$

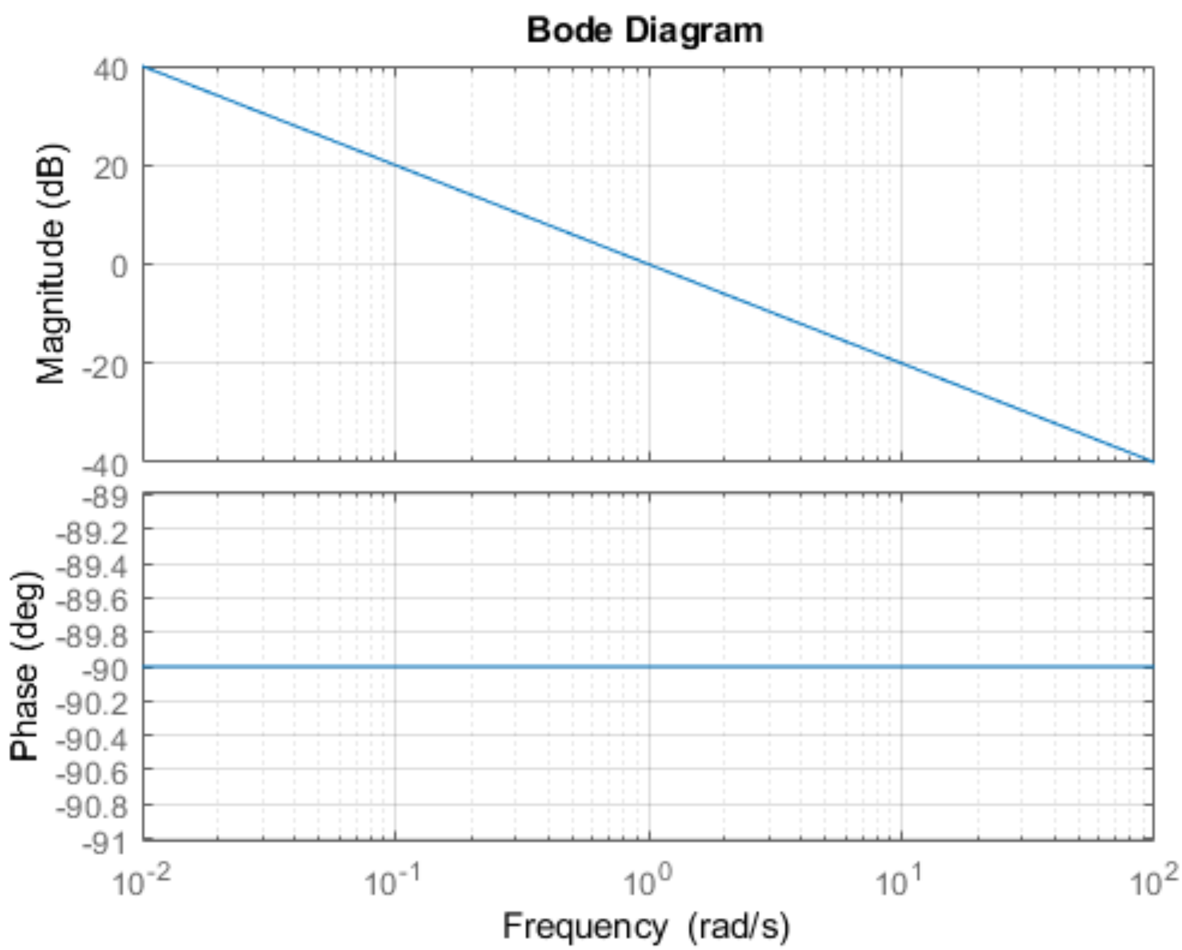
$$= \angle(1) - \angle(j\omega)$$

$$= 0 - 90^\circ$$

$$= -90^\circ$$







Example 3

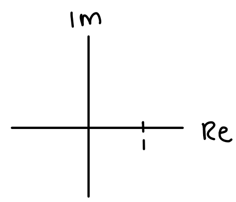
Find the relation between the Bode plots of $H(s)$ and $G(s) = \frac{1}{H(s)}$. Use this result to find the Bode plot of s given the plot from the previous example.

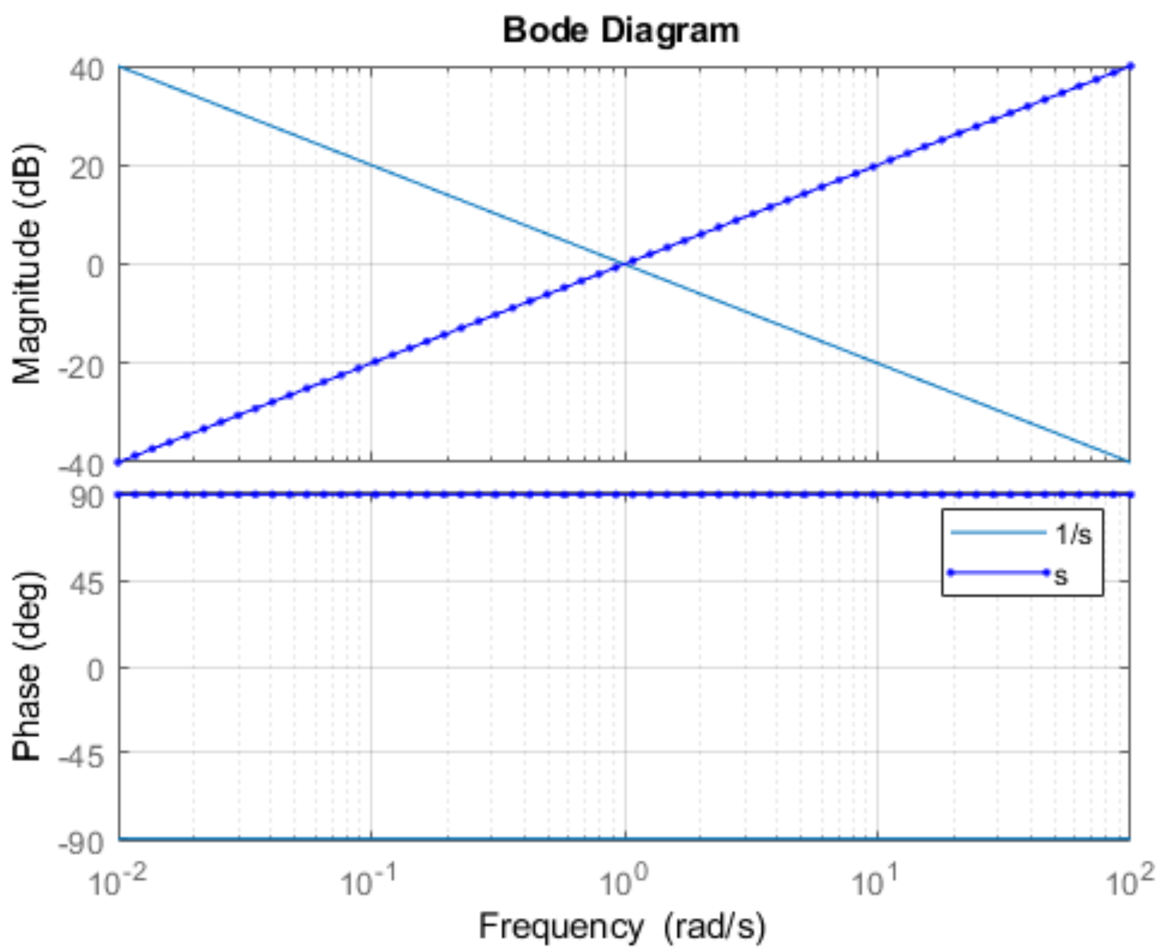
$$G(j\omega) = \frac{1}{H(j\omega)}$$

$$|G(j\omega)| = \frac{1}{|H(j\omega)|}$$

$$\begin{aligned} |G(j\omega)|_{dB} &= 20\log_{10}|G(j\omega)| \\ &= 20\log_{10}\left(\frac{1}{|H(j\omega)|}\right) \\ &= -20\log_{10}(|H(j\omega)|) \end{aligned}$$

$$\begin{aligned} \angle(G(j\omega)) &= \angle\left(\frac{1}{H(j\omega)}\right) \\ &= \angle(1) - \angle(H(j\omega)) \\ &= -\angle(H(j\omega)) \end{aligned}$$





Bode Diagram
Magnitude (dB)
40
20
0
-20
-40
90
45
0
-45
-90
Phase (deg)
1/s
s
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Example 4

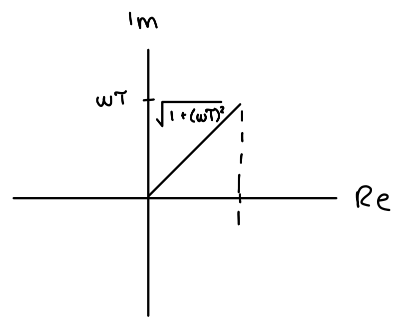
Find the Bode plot for the standard first order system:

$$H(s) = \frac{\kappa}{s\tau + 1}, \kappa, \tau > 0.$$

$$|H(j\omega)| = \left| \frac{K}{j\omega\tau + 1} \right|$$

$$= \frac{K}{|1 + \omega\tau j|}$$

$$= \frac{K}{\sqrt{1 + (\omega\tau)^2}}$$



If $\omega^2\tau^2 \ll 1$, then $\omega \ll \frac{1}{\tau}$

$\hookrightarrow |H(j\omega)| \simeq K$

$|H(j\omega)|_{dB} \simeq 20\log_{10} K$

If $\omega^2\tau^2 \gg 1$, $\omega \gg \frac{1}{\tau}$

$\hookrightarrow |H(j\omega)| \simeq \frac{K}{\omega\tau}$

$$|H(j\omega)|_{dB} \simeq 20\log_{10}\left(\frac{K}{\omega\tau}\right)$$

$$\simeq 20\log_{10}(K) - 20\log_{10}(\omega) - 20\log_{10}(\tau)$$

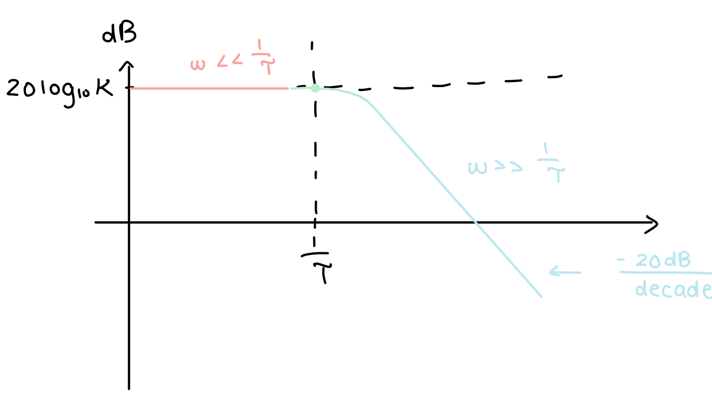
$$\simeq 20\log_{10}(K) + 20\log_{10}\left(\frac{1}{\tau}\right) - 20\log_{10}(\omega)$$

If $\omega = \frac{1}{\tau}$

$\hookrightarrow |H(j\omega)| = \frac{K}{\sqrt{2}}$

$|H(j\omega)|_{dB} = 20\log_{10} K - 20\log_{10}\sqrt{2}$

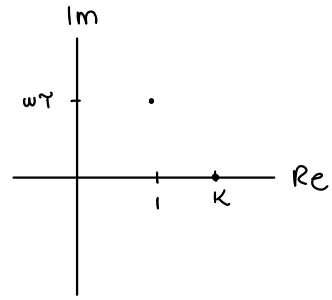
$\simeq 20\log_{10} K - 3$



$$\angle H(j\omega) = \angle\left(\frac{K}{1+\omega\tau j}\right)$$

$$= \angle K - \angle(1+\omega\tau j)$$

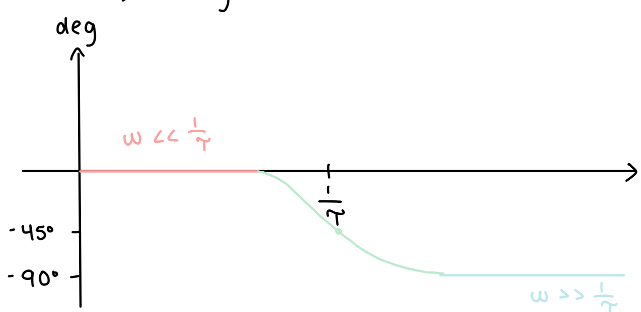
$$= -\angle(1+\omega\tau j)$$

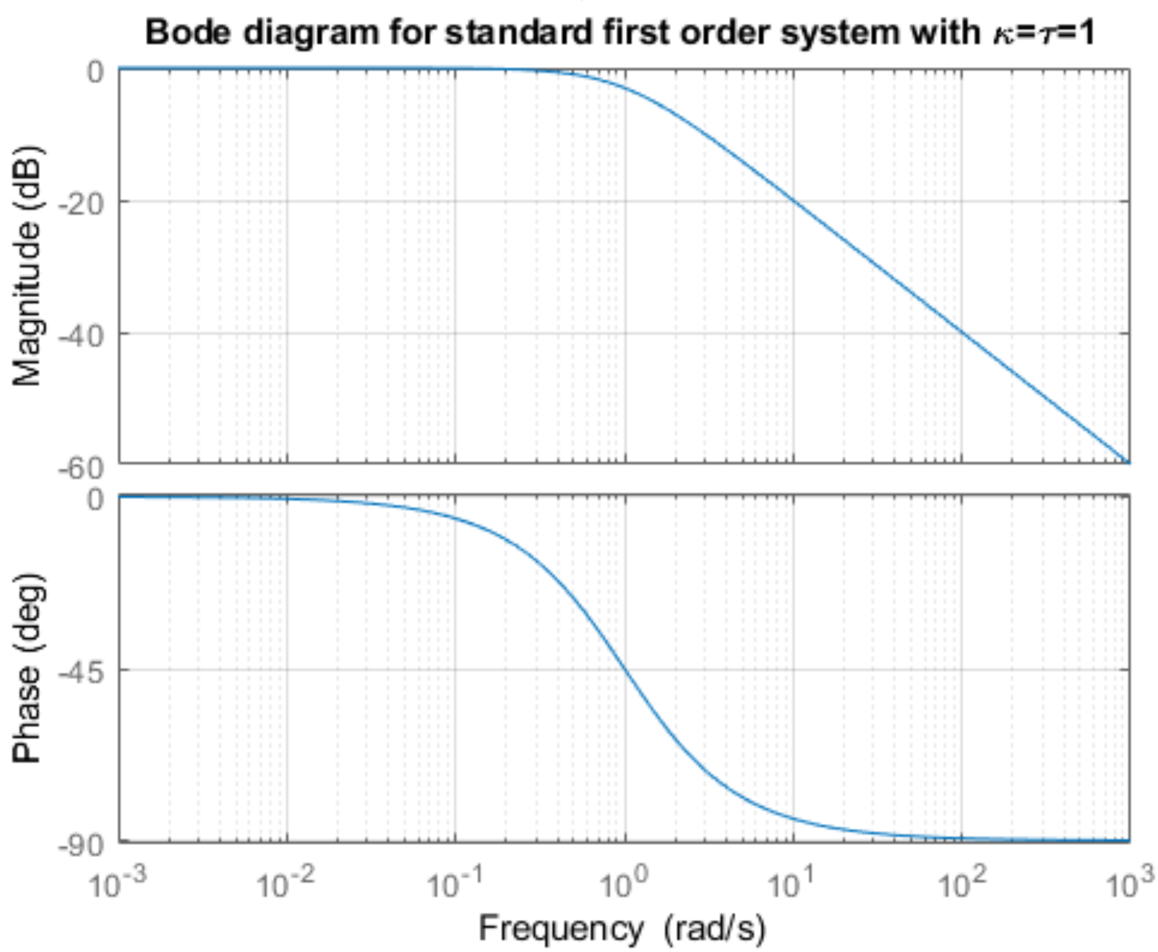


If $\omega << \frac{1}{\tau}$, $\angle H(j\omega) \simeq 0$

If $\omega >> \frac{1}{\tau}$, $\angle H(j\omega) \simeq -90°$

If $\omega = \tau$, $\angle H(j\omega) = -45°$





Note that the Bode plot for a system with no poles and a zero is simply this plot flipped.

Example 5

Find the Bode plot for the standard second order system:

$$H(s) = \frac{w^2}{s^2 + 2\xi ws + w^2}, \quad w > 0$$

for cases where $\xi = 0$ and then where $0 < \xi < 1$.

ϖ : omega (ω)

w : letter w

$$H(j\varpi) = \frac{w^2}{(j\varpi)^2 + 2\xi w\varpi j + w^2}$$

$$= \frac{w^2}{-\varpi^2 + 2\xi w\varpi j + w^2}$$

$$= \frac{1}{1 - (\frac{\varpi}{w})^2 + 2\xi \frac{\varpi}{w} j}$$

$$|H(j\varpi)| = \frac{1}{\sqrt{(1-(\frac{\varpi}{w})^2)^2 + (2\xi \frac{\varpi}{w})^2}}$$

If $\varpi \ll w$, then $\frac{\varpi}{w} \ll 1$ so $\frac{\varpi}{w} \simeq 0$:

$$|H(j\varpi)| \simeq \frac{1}{\sqrt{1}} = 1$$

$$|H(j\varpi)|_{dB} = 20\log_{10}(1)$$
$$= 0$$

If $\varpi \gg w$, then $\frac{\varpi}{w} \gg 1$ (essentially ∞):

$$|H(j\varpi)| = \frac{1}{\sqrt{1 - 2(\frac{\varpi}{w})^2 + \underbrace{(\frac{\varpi}{w})^4}_{\text{blows up fastest b/c } x^4 \gg x^2 \text{ for } x \gg 1} + 4\xi(\frac{\varpi}{w})^2}}$$

$$\simeq \frac{1}{\sqrt{(\frac{\varpi}{w})^4}}$$

$$= \frac{1}{(\frac{\varpi}{w})^2}$$

$$= \frac{w^2}{\varpi^2}$$

$$|H(j\varpi)|_{dB} = 20(\log_{10} w^2 - \log_{10} \varpi^2)$$

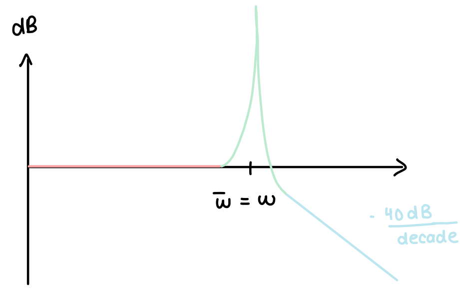
$$= 40\log_{10} w \underbrace{- 40\log_{10} \varpi}_{\text{dominates b/c } \varpi \gg w}$$

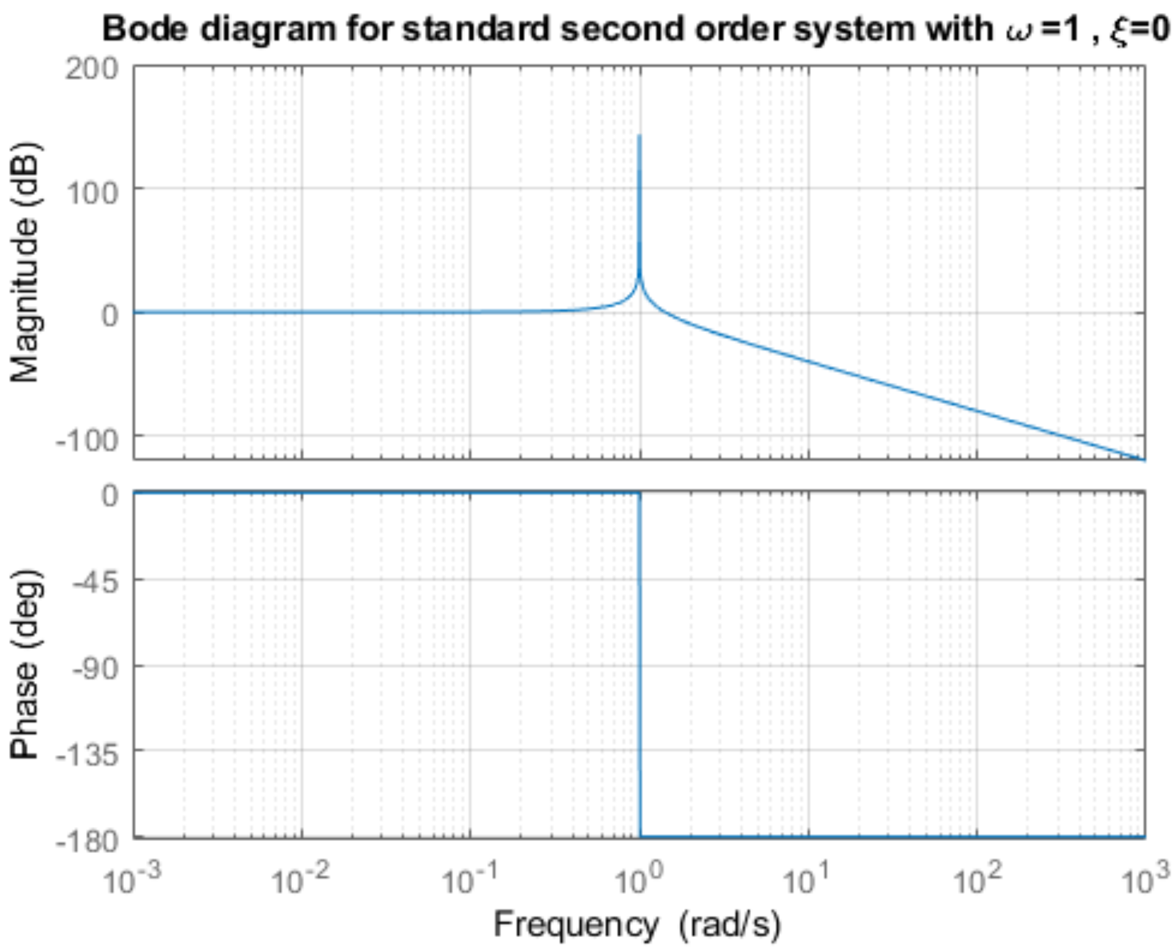
If $\bar{\omega} = \omega$:

$$|H(j\bar{\omega})| = \frac{1}{\sqrt{0^2 + (2\varepsilon)^2}}$$

$$= \frac{1}{2\varepsilon}$$

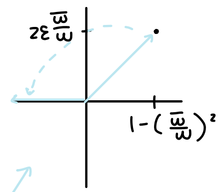
For $\varepsilon = 0$:





We will use this as out idealized plot for second order systems (that cannot be decomposed into two standard first order systems). Note that the Bode plot for a system with no poles and zeros that are complex-conjugate pairs is simply this plot flipped.

$$\angle (H(j\bar{\omega})) = \angle \left(\frac{1}{(1-(\frac{\bar{\omega}}{\omega})^2) + 2\varepsilon \frac{\bar{\omega}}{\omega} j} \right)$$



If $\bar{\omega} \ll \omega$, then $\frac{\bar{\omega}}{\omega} \simeq 0$

$$\angle (H(j\bar{\omega})) = 0° - 0°$$
$$= 0°$$

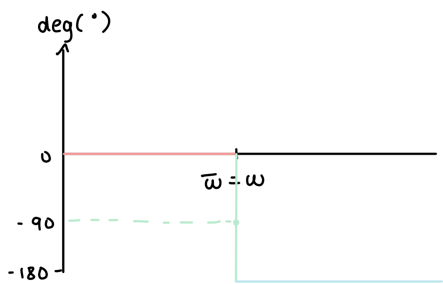
If $\bar{\omega} \gg \omega$, then $\frac{\bar{\omega}}{\omega} \simeq \infty$

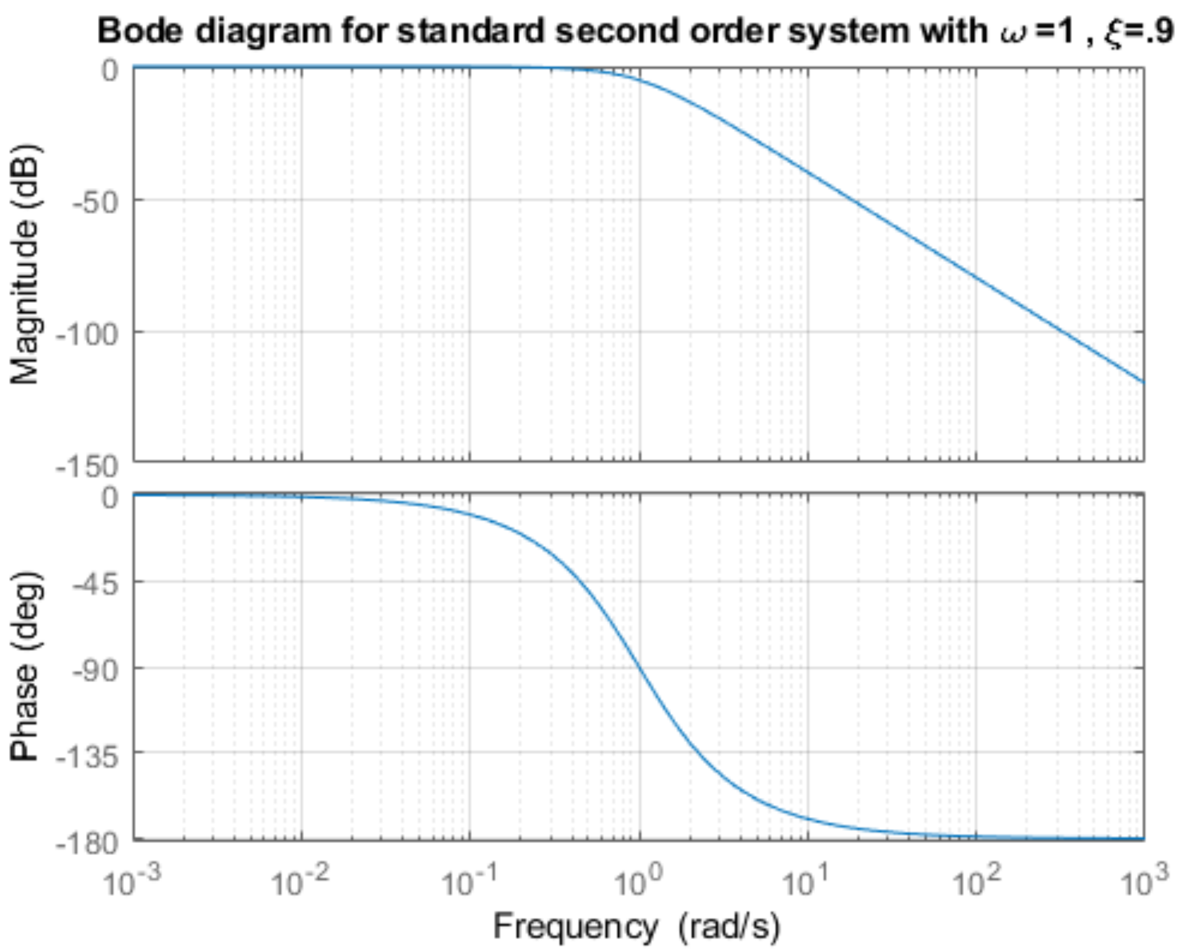
$$\angle (H(j\bar{\omega})) = 0° - 180°$$
$$= -180°$$

$x^2 \gg x$

If $\bar{\omega} = \omega$, then $\frac{\bar{\omega}}{\omega} = 1$

$$\angle (H(j\bar{\omega})) = 0° - 90°$$
$$= -90°$$





If given a transfer function $H(s) = H_1(s) \cdot H_2(s) \cdots H_k(s)$ then the Bode plot for $H(j\omega)$ is found by

- Finding the magnitude and phase curves for each $H_i(j\omega)$.
- Adding the magnitude and phase curves.

MATH 213 - Lecture 19: Bode plots 2

Lecture goals: Refine our ability to quickly draw bode plots, approximate the transfer function from bode plots, and determine stability of the system/closed loops using Bode plots.

Summary of main results from Lecture 18:

- The "starting value" of the magnitude curve is $|H(0)|_{dB}$ and the "starting value" of the phase curve is $\angle H(0)$.

  This will be not exist (i.e. the curve is unbounded) if there is a pole with a real value of 0.

- If the transfer function has an order one **stable** pole at a point then the magnitude curve will experience an extra decrease of -20 dB/Decade and the phase curve will experience a drop of $-90°$ (see L18 Ex 4).

  These phase adjustment will be rather "slow" taking an order of magnitude (before and after the pole) to adjust.

- If the transfer function has complex conjugate poles that are stable then the magnitude curve will experience an extra decrease of -40 dB/Decade and the phase curve will experience a gain of $-180°$. The location of this adjustment is given by ω where ω is the term in the standard second order system (see L18 Ex 5).

  The speed of adjustment is related to ξ:

  - $\xi \approx 0$ causes fast adjustments and overshooting in the amplitude plot for nearby ωs.
  - $\xi \approx 1$ causes slow adjustments and undershooting in the amplitude plot for nearby ωs.

- If we have zeros of order 1 or of order 2, at some point then the magnitude and phase curves experience the opposite effects as listed in the above two points.

What if we have an unstable pole?

In this case Bode plots do not make sense to apply as they assume there is a steady state solution but... for completion

Example 1
Find the Bode plot for the "standard" unstable linear system

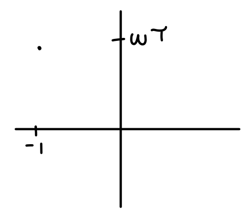
$$\frac{\kappa}{s\tau - 1}, \kappa, \tau > 0$$

$$|H(j\omega)| = \frac{\kappa}{|j\omega\tau - 1|}$$

$$= \frac{\kappa}{\sqrt{1+(\omega t)^2}}$$ ← same as std 1st order system

$$\angle H(j\omega) = \angle(\kappa) - \angle(j\omega\tau - 1)$$

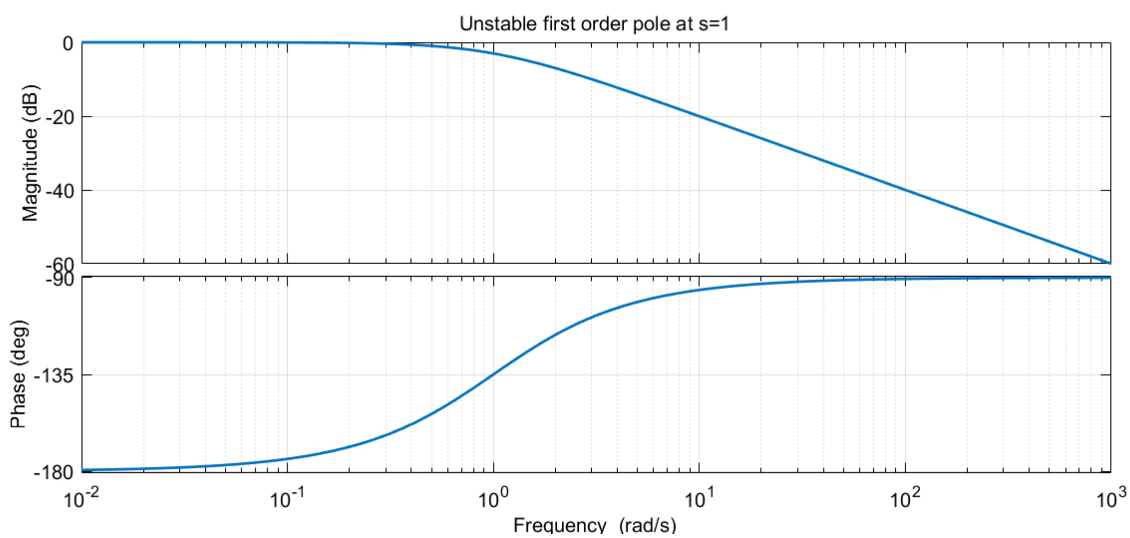
$$= 0 - \angle(j\omega\tau - 1)$$



If $\omega \ll \frac{1}{\tau}$, $\angle H(j\omega) \simeq -180°$
$\omega\tau \simeq 0$

If $\omega = \frac{1}{\tau}$, $\angle H(j\omega) = -135°$
$\omega\tau = 1$

If $\omega \gg \frac{1}{\tau}$, $\angle H(j\omega) \simeq -90°$
$\omega\tau \simeq \infty$



Similar result for second order systems but they are a bit more complex to analyze and often there are better ways to test stability so we skip the analysis.
In general if you see a decrease in the slope of the asymptotic amplitude curve and an increase in the phase, then you can conclude that the system is unstable (and hence a Bode plot should have never been made). You can not conclude anything about stability of the system modelled by the transfer function used to generate the plot though.

To quickly generate plots, we can use the results derived from lectures 18 and 19 by either adding the curves or using the relation of the poles/zeros and the curves to sketch asymptotic lines. The latter is often nicer.

Example 2

Sketch the Bode plot for

$$T(s) = \left(\frac{0.01}{s+0.01}\right)\left(\frac{s+100}{s+10}\right)$$

$$|T(0)| = \left(\frac{0.01}{0.01}\right)\left(\frac{100}{10}\right)$$

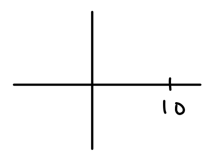
$$= 10$$

$$|T(0)|_{dB} = 20\log_{10} 10$$

$$= 20$$

$$\angle(T(0)) = \angle(10)$$

$$= 0^\circ$$



Poles / zeros:

$$\omega = \underset{10^{-2}}{\overset{P}{0.01}},\ \overset{P}{10},\ \underset{10^{2}}{\overset{Z}{100}}$$

$\underset{\text{pole}}{10^{-2}} \leq \omega \leq 10$:

$$\frac{d}{d\omega}(\text{amp}) = -20$$

$$\text{phase} = 0^\circ - 90^\circ$$

$\underset{\text{pole}}{10} \leq \omega \leq 10^2$:

$$\frac{d}{d\omega}(\text{amp}) = \underset{\text{old}}{-20} \underset{\text{new}}{- 20}$$

$$= -40$$

$$\text{phase} = -90^\circ - 90^\circ$$

$$= -180^\circ$$

$10^2 \leq \omega$:

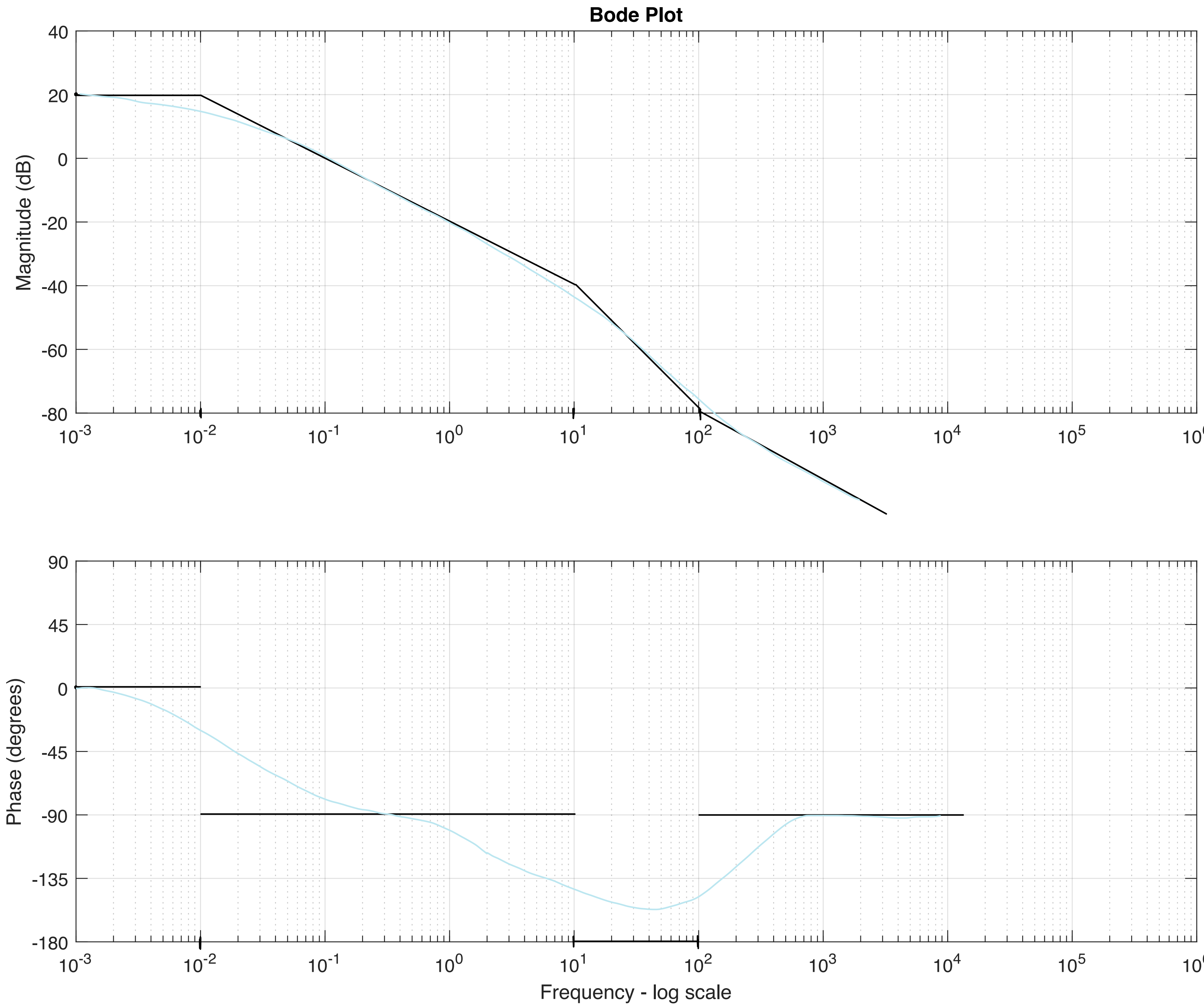
$$\frac{d}{d\omega}(\text{amp}) = -40 + 20$$

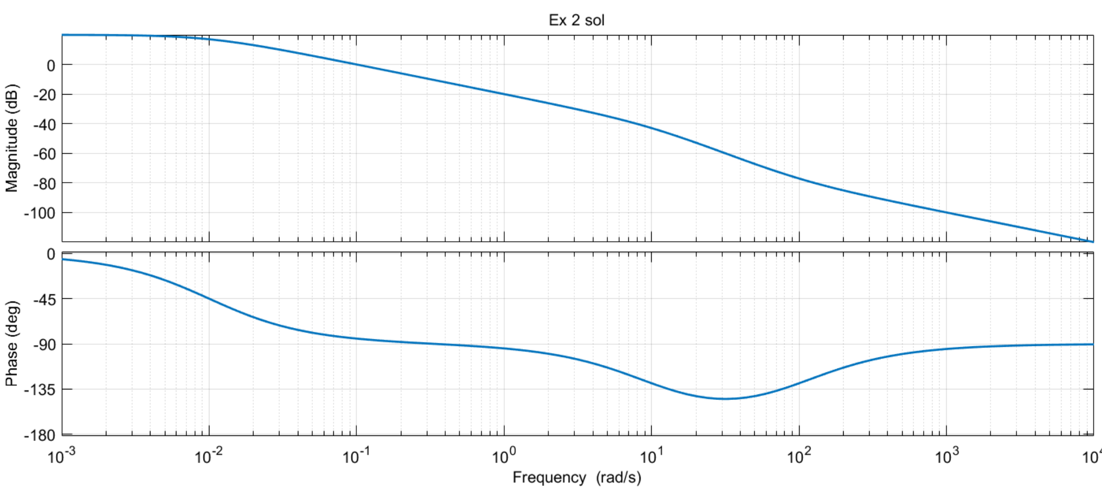
$$= -20$$

$$\text{phase} = -180^\circ + 90^\circ$$

$$= -90^\circ$$

NOTE
(for stable only)
Single pole w/order 1:
↳ dec amp slope by 20
↳ dec phase by 90°
Single zero:
↳ inc amp slope by 20
↳ inc phase by 90°


Bode Plot
Magnitude (dB)
40
20
0
-20
-40
-60
-80
Phase (degrees)
90
45
0
-45
-90
-135
-180
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
Frequency - log scale



Ex 2 sol
Magnitude (dB)
0
-20
-40
-60
-80
-100
Phase (deg)
0
-45
-90
-135
-180
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency (rad/s)

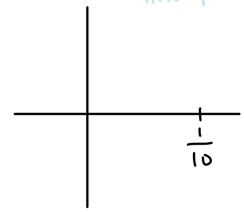
Example 3
Sketch the Bode plot for

$$T(s) = \left(\frac{s+1}{s+1.01}\right)\left(\frac{s+10}{(s+3)^2+100}\right) \longrightarrow s^2+6s+109$$

$s^2 + 2\varepsilon\omega s + \omega^2$

determines speed

only consider to find pole



$$|T(0)| = \frac{1}{1.01} \quad \frac{10}{9+100}$$

$$\simeq 1 \quad \frac{1}{10}$$

$$\angle(T(0)) = 0°$$

$$|T(0)|_{dB} = 20\log_{10} 10^{-1}$$

$$= -20$$

Poles / zeros : $\omega = 1,\ 1.01,\ 10,\ \sqrt{109} \simeq 10$

zero + pole basically cancel

$0 \lesssim \omega \lesssim 10^1$.

Amp = -20

phase = 0°

$10^1 \lesssim \omega$:

$\frac{d}{d\omega}(-20) = 0$ $s+10$ $(s+3)^2+100$ is order 2 pole

$$\frac{d}{d\omega}(\text{Amp}) = 0 + 20 - 40$$

$$= -20$$

$$\text{phase} = 0° + 90° - 180°$$

$$= -90°$$

$$s^2 + 6s + 109 = s^2 + 2\varepsilon\omega s + \omega^2$$

$$\omega = \sqrt{109}$$

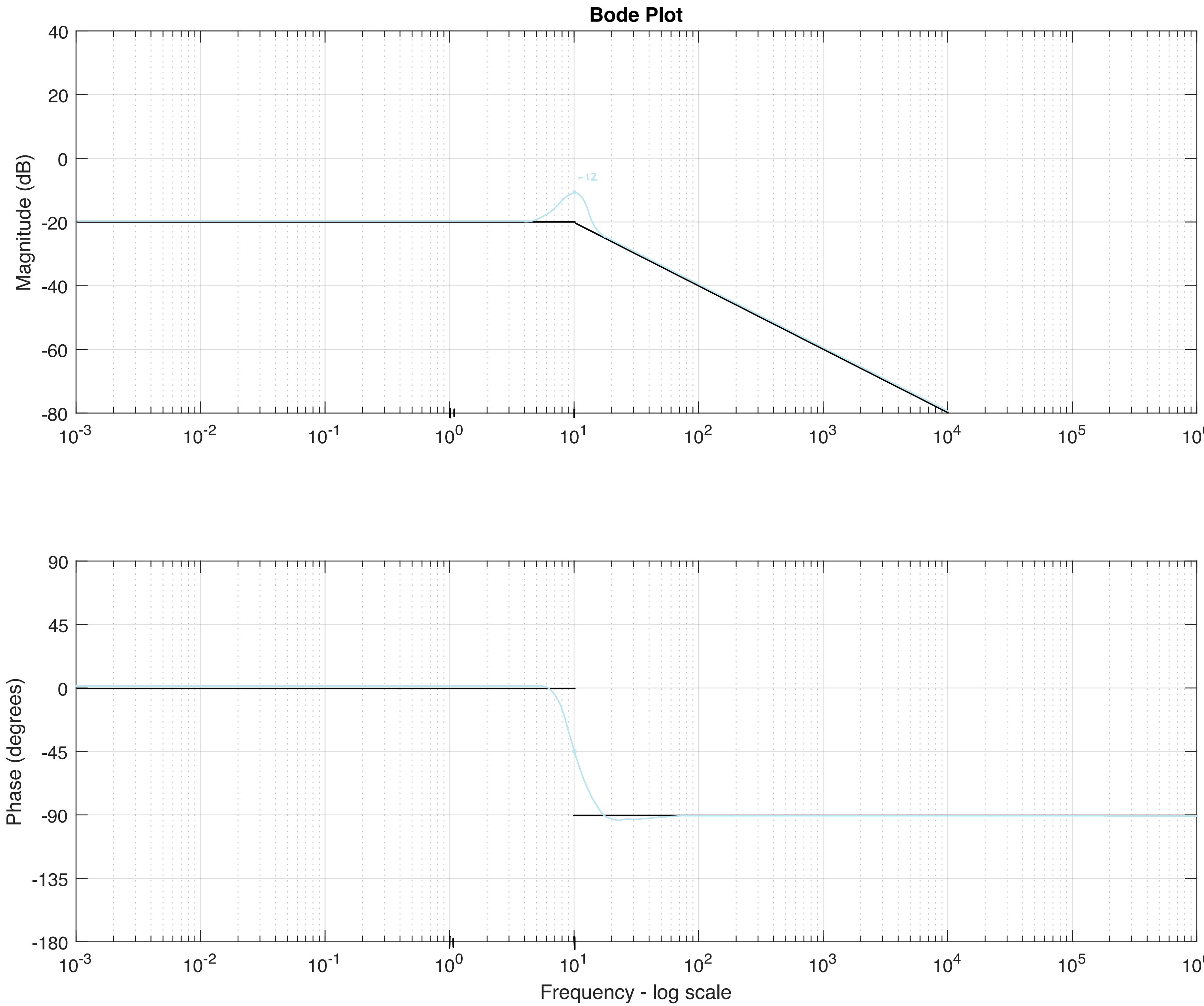
$$\varepsilon = \frac{6}{2\omega}$$

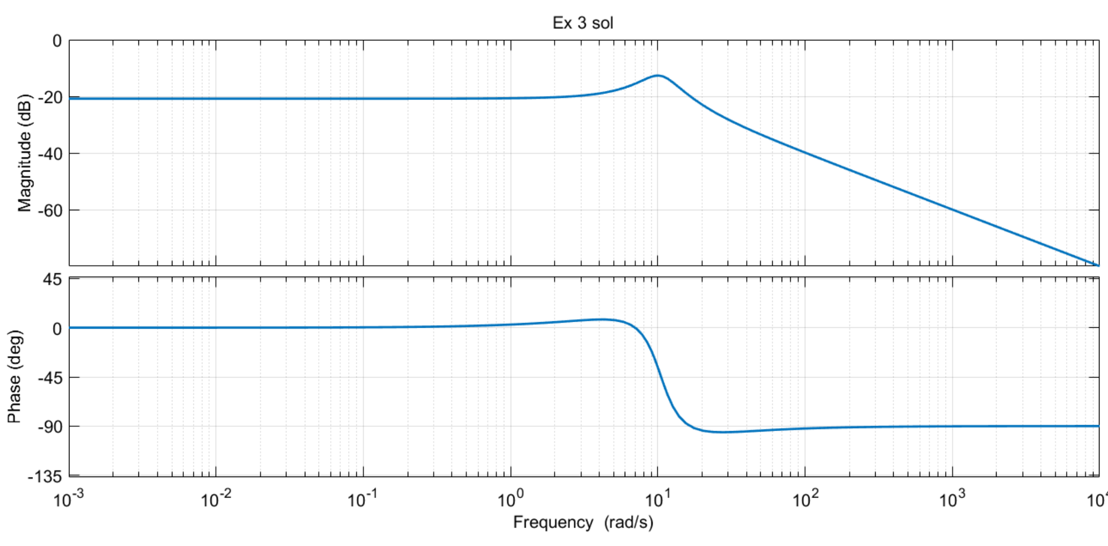
$$= \frac{6}{2\sqrt{109}} \ll 1$$

relatively quick drop but not instant

$$|T(10)| \simeq 0.2$$

$$|T(10)|_{dB} \simeq -12$$


Bode Plot
Magnitude (dB)
40
20
0
-20
-40
-60
-80
-12
Phase (degrees)
90
45
0
-45
-90
-135
-180
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
Frequency - log scale


Ex 3 sol
Magnitude (dB)
0
-20
-40
-60
45
0
-45
-90
-135
Phase (deg)
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency (rad/s)

Example 4
Sketch the Bode plot for

$$T(s) = \left(\frac{s-1}{(s+1)^2}\right)\left(\frac{10^5(s+0.01)}{(s+10)^2+1000}\right)$$

$|T(0)| \simeq 1 \cdot \frac{10^5 \cdot 10^{-2}}{10^3}$ ← ignore $10^2 = 100$ b/c $10^2 + 10^3 \simeq 10^3$

$= 1$

$|T(0)|_{dB} = 0$

$\angle(T(0)) = \angle(-1)$

$= -180°$

$\underset{Z}{\omega = 10^{-2}}, \underset{P}{1}, \underset{P}{\sqrt{1000+100}} \simeq 3 \cdot 10^1$

$0 \leq \omega \lessapprox 10^{-2}$:
Amp $= 0$
phase $= -180°$

$10^{-2} \lessapprox \omega \lessapprox 10^0$: $(s+0.01)$

$\frac{d}{d\omega}(\text{Amp}) = 0 + 20$
$= 20$

phase $= -180° + 90°$
$= -90°$

$10^0 \lessapprox \omega \lessapprox 3 \cdot 10^1$. $(s+1)^2$, $(s-1)$

$\frac{d}{d\omega}(\text{Amp}) = 20 - 2(20) + 20$
$= 0$

phase $= -90° - 2 \cdot 90° \underbrace{-90°}_{\text{s-1 is unstable zero}}$
$= -360°$

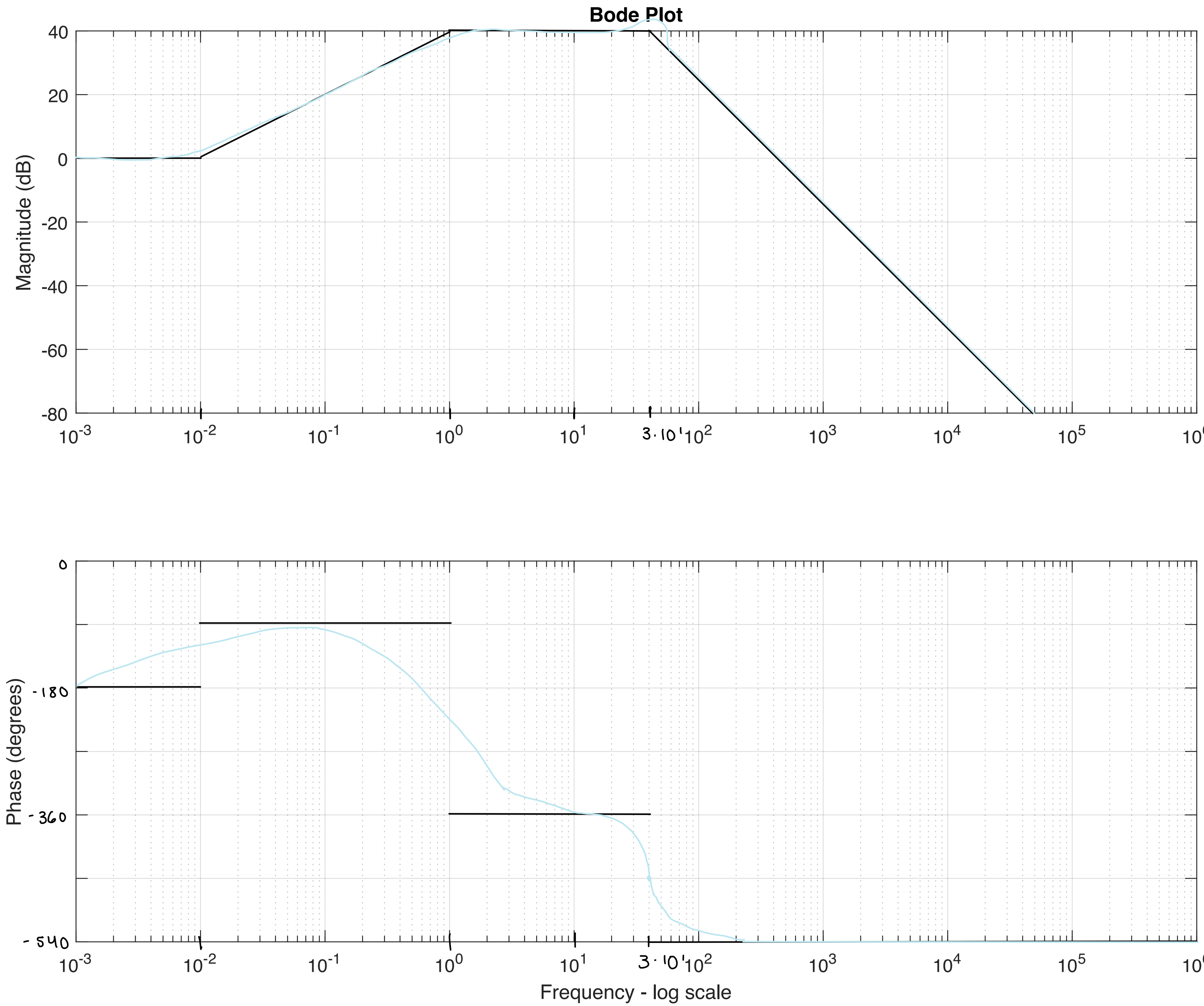
NOTE
Unstable zero (i.e. Re < 0) dec phase by 90° instead of inc

$3 \cdot 10^1 \lessapprox \omega$: $(s+10)^2 + 1000$
$\frac{d}{d\omega}(\text{Amp}) = 0 - 40$
$= -40$
phase $= -360° - 180° = -540°$

$(s+10)^2 + 1000 = s^2 + 20s + 1000 + 100$

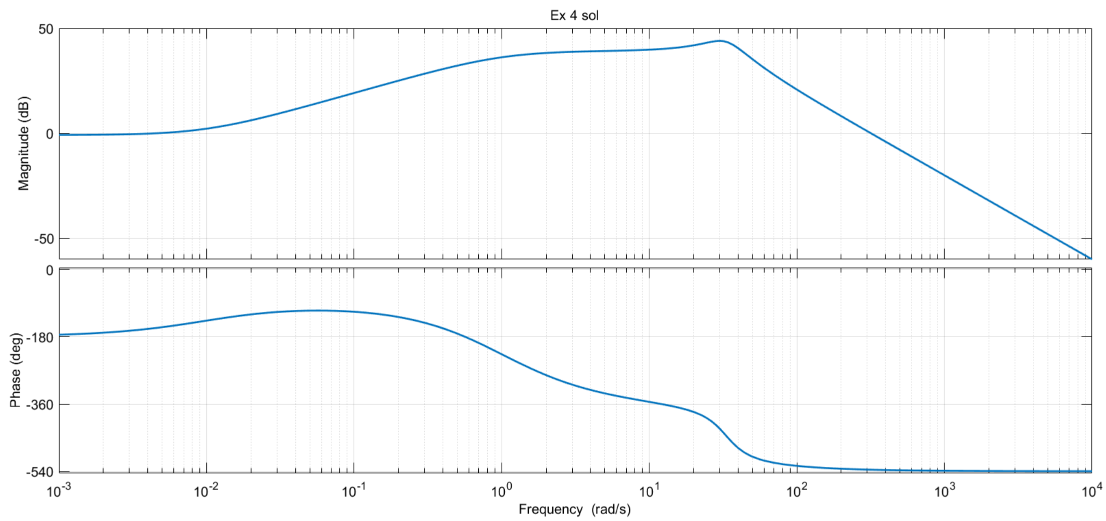
$\omega = 3 \cdot 10^1$

$\varepsilon = \frac{20}{2 \cdot 3 \cdot 10} \simeq 0.3$

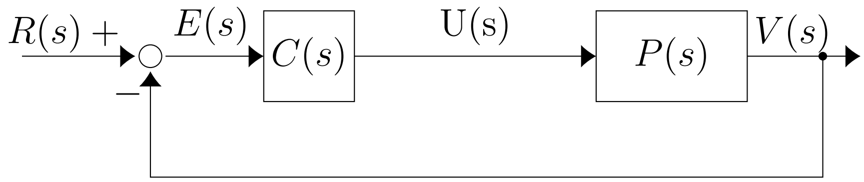


NOTE

Unstable poles (dec net amp change + inc phase) make system unstable. Unstable zeroes don't, but can cause more oscillations.



Recall that for a closed loop system with system diagram given by:



The transfer function for this system is

$$\frac{P(s)C(s)}{1+P(s)C(s)}$$

Hence the closed system will be stable when both $C(s)$ and $P(s)$ are stable and when

$$1+P(s)C(s) \neq 0.$$

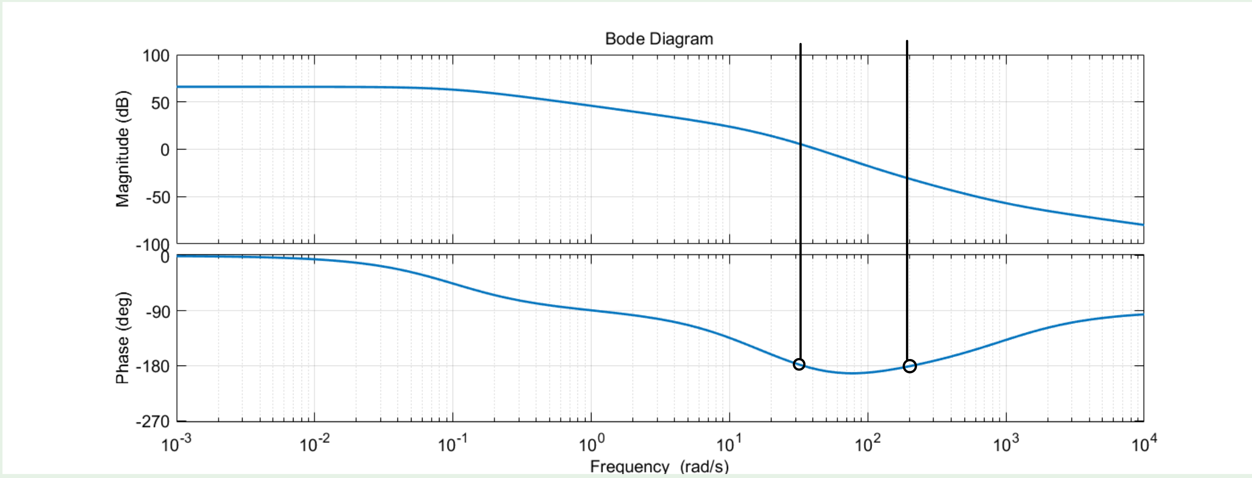
Hence if $C(s)$ and $P(s)$ are both stable, then the closed look control system is **unstable** when

$$\begin{aligned} P(s)C(s) &= -1 \\ &= 1 \cdot e^{i\pi}. \end{aligned}$$

This happens when the amplitude curve is $|1|_{dB} \approx 0$ while the phase is either $180°$ or $-180°$ (or some other equivalent angle).

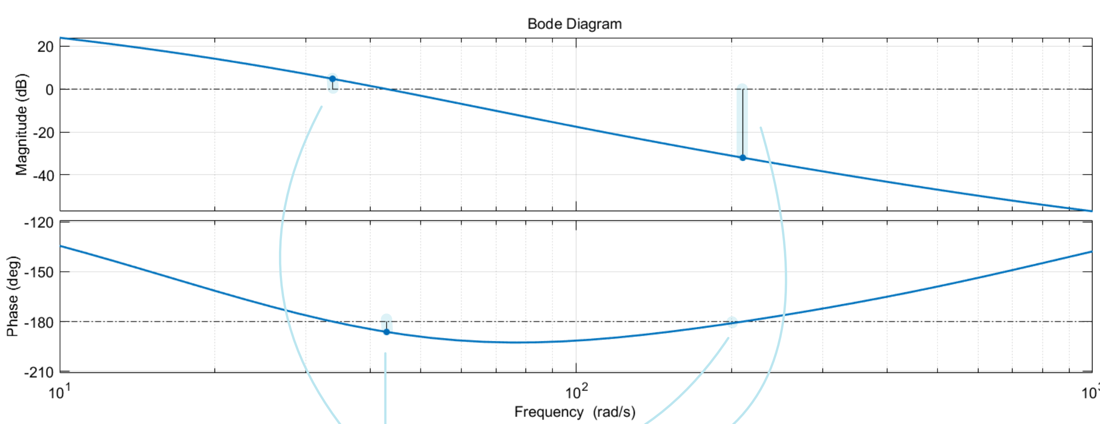
In general to talk about stability we will not look for these exact conditions but will instead look at the points where either the amplitude or the phase is critical and then look at how far we are from stability. We generally require some threshold to be "far enough" from being unstable.

Example 5
Determine if the closed system with Bode diagram given by



is stable.

Formally stable, but close to unstable.



In practice,
these would be unstable
b/c of how close it is

MATH 213 - Lecture 20: Signals Intro

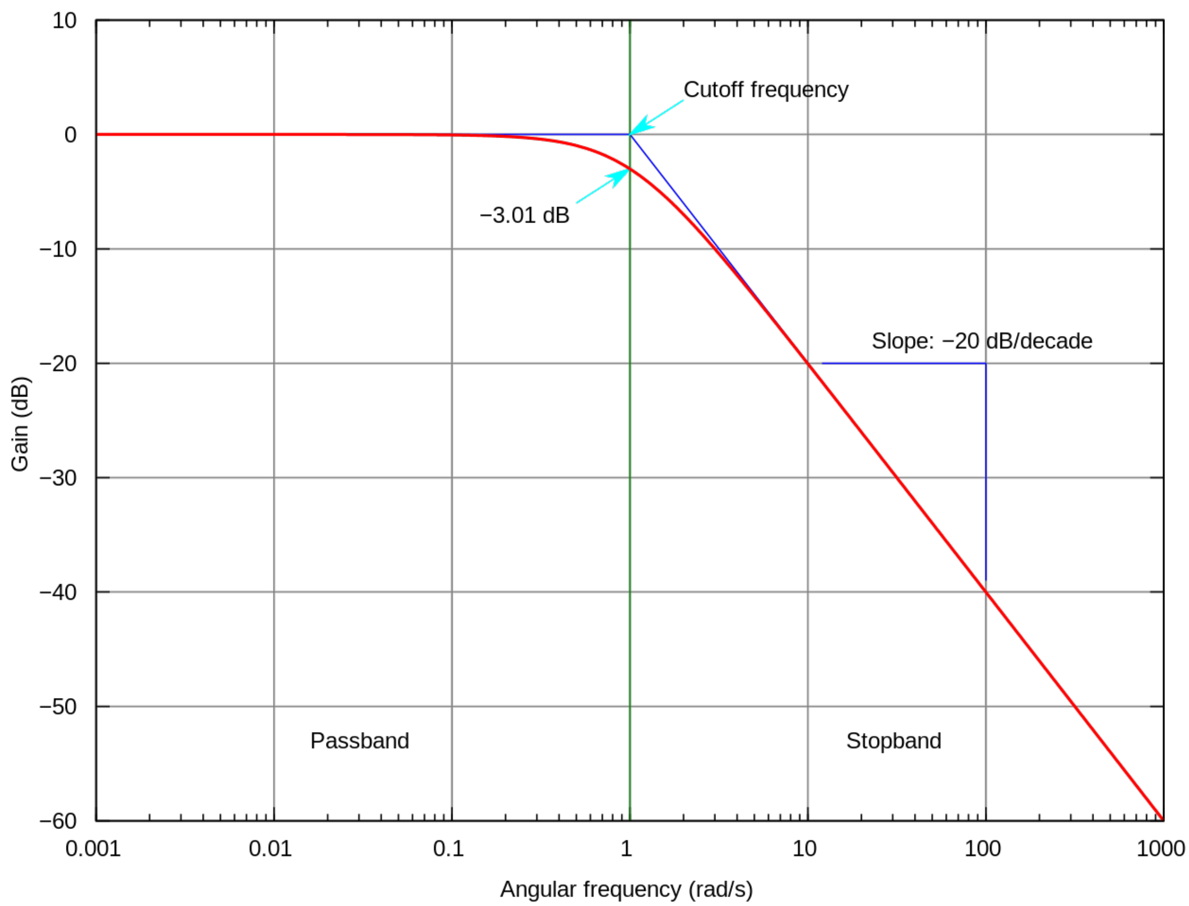
Lecture goals: know the big picture of Fourier series/transforms.

Bode plots allows us to quickly see how a system responds to input signals of the form $\sin(\omega t)$.

This can be used to quickly tell what type of filtering/amplifying the LTI does to the sin wave.

For example a low pass filter is a LTI that removes the “high” frequency waves. Explicitly, they reduce the amplitude of all waves with a frequency larger than some cutoff w.

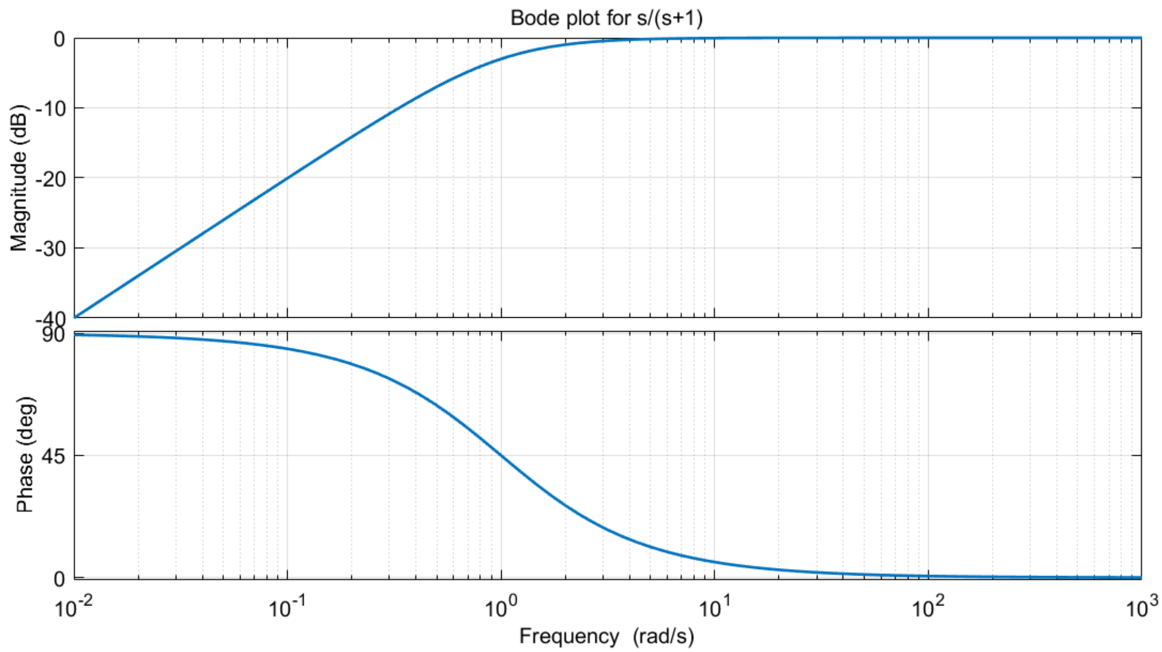
All stable systems with a single pole and no zeros are low pass filters.
Here is the hopefully familiar amplitude plot (stolen from wiki because I liked their annotations):



As another example for a high pass filters we need the amplitude curve to go to 0dB at some cutoff point. The standard high pass filter transfer function is

$$T(s) = \frac{as}{s + \omega_0}$$

for $a, \omega_0 > 0$ and its Bode plot is

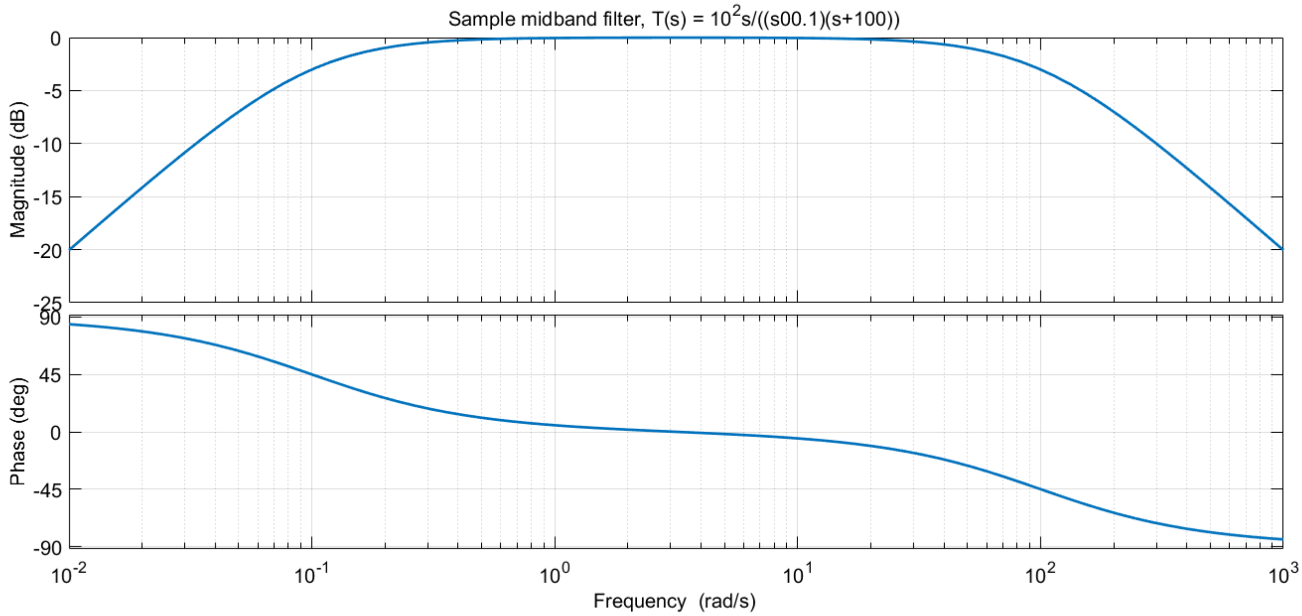


If you wanted stronger drops in the frequency, you could simply add more zeros near 0 and poles to counter them!!!

If you wanted a medium band filter, then you can add another pole later:

$$T(s) = \frac{as}{(s + \omega_0)(s + \omega_1)}$$

for $a, \omega_0, \omega_1 > 0$. Here is a sample Bode plot



This is all fair and good but it only works if we are working with sine waves.

Most functions are not sine waves..... but what if they (i.e. functions we care about) can be written as a linear combination of sine waves?

Fourier series:

Fourier series are obviously useful when working with systems, but they are also useful when working with signals on their own.

We will first work with a classic signal example that was a major problem in the days of the telegraph,

in the early days of landline phones,

in the early days of cell phones and the early Internet,

is the reason why we developed 5G,

and will continue to be a problem for the foreseeable future.

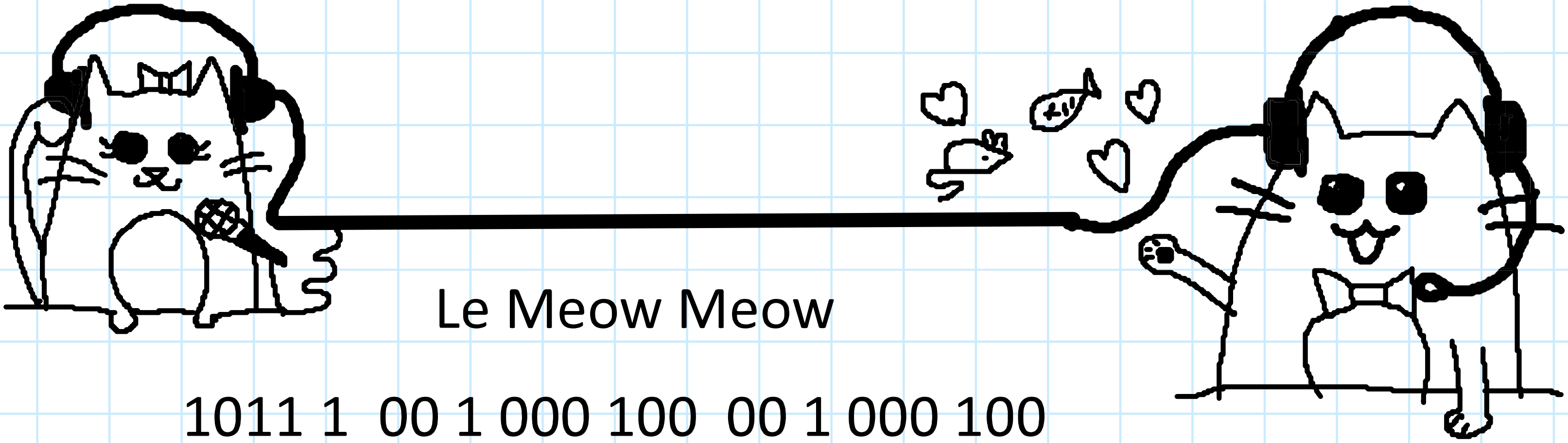
Lecture 23

March 3, 2022 9:20 PM

Fourier series!!!!

Suppose you want to send a message down a wire to a cat who lives in a different city.

You can do this via say some binary code!!



Le Meow Meow

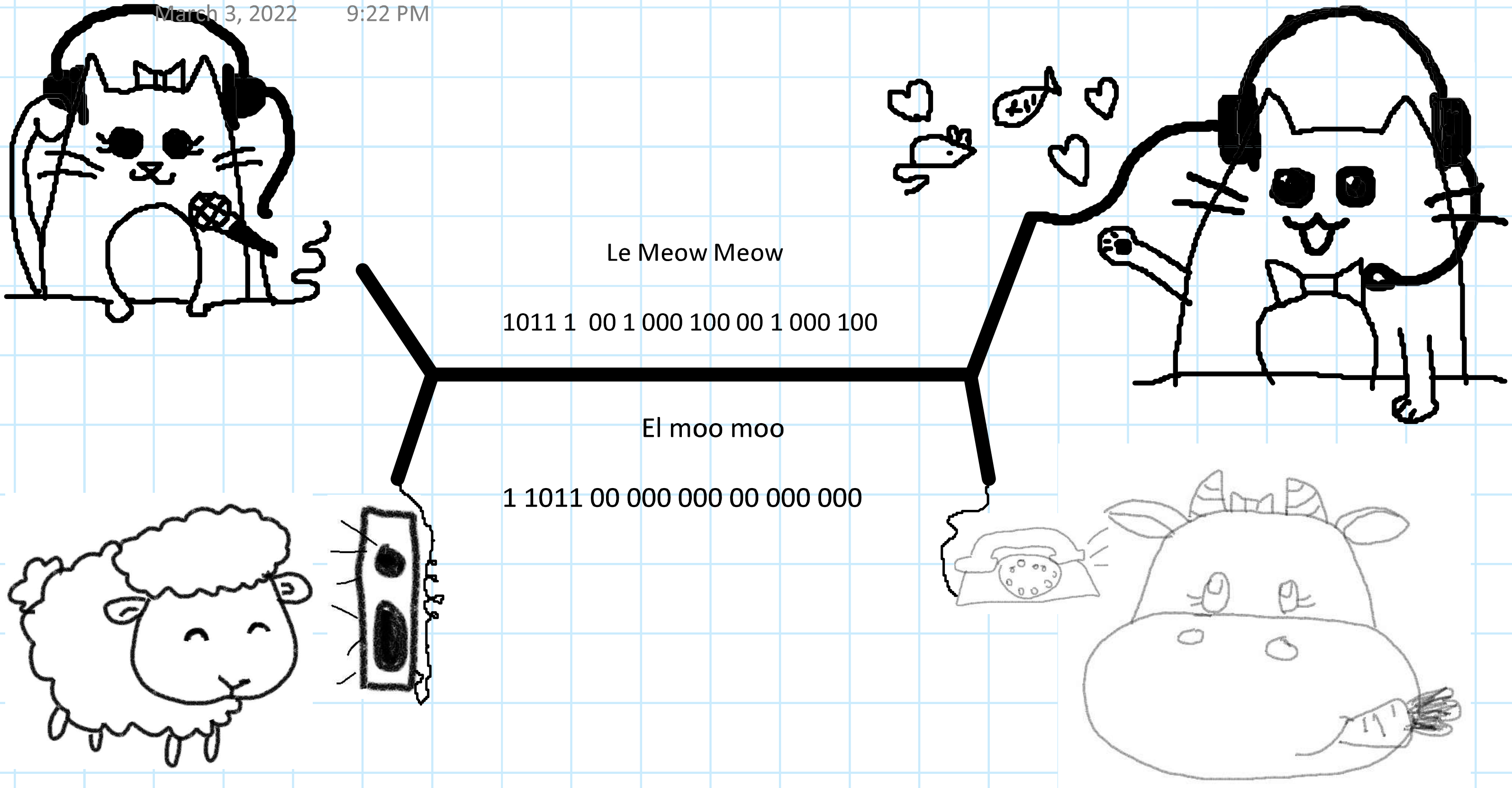
1011 1 00 1 000 100 00 1 000 100

Your friend then finds out about your wire and also wants to send a message at the same time.

You say sure because what can possibly go wrong...

P2

March 3, 2022 9:22 PM



What comes out the other end???

11111101000100001000100

Or

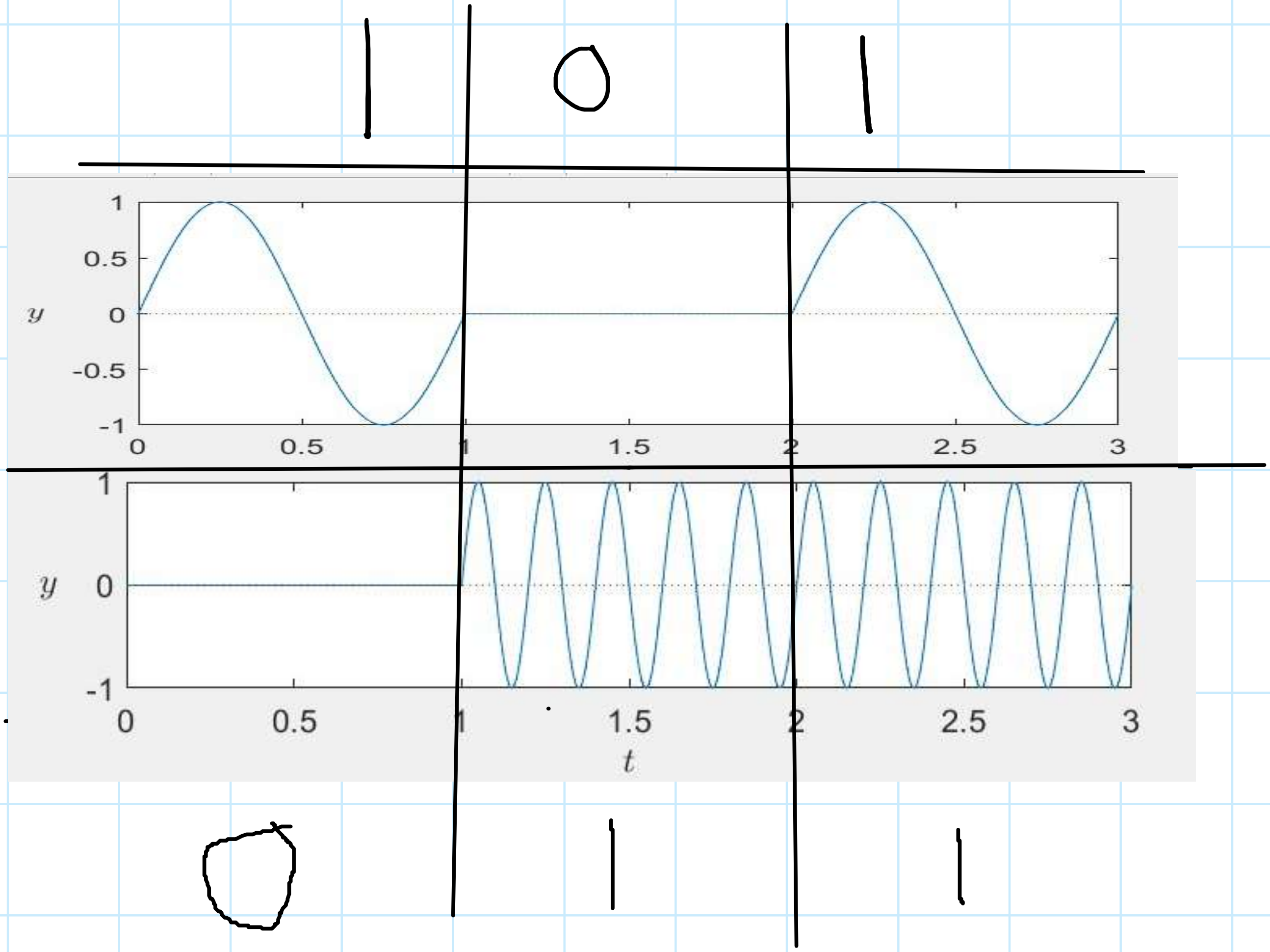
I S T W K O W W in our code……

How can we fix this?

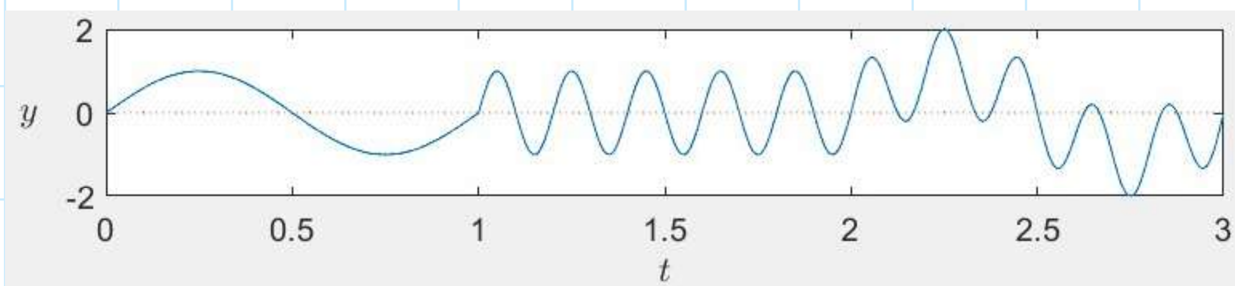
P3

Friday, March 4, 2022 1:35 PM

We can use different frequency waves!!!



When we add these together we get



If the recipients know the frequency to use then they can decode their message!!

"Recall" Taylor's theorem from MATH 119:

Theorem 1: Taylor's Theorem
Let $k \geq 1$ be an integer and let f be a real valued function that is differentiable at least k times at some point $a \in \mathbb{R}$. Then there exists a real valued function h_k such that

$$f(x) = \left(\underbrace{\sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!}(x-a)^i}_{\text{Taylor Polynomial}} \right) + h_k(x)(x-a)^{k+1}$$

and $\lim_{x \to a} h_k(x) = 0$.

For infinitely differential functions f we can write

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Examples: $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, or $\sin(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}$.

Here we used the basis $\{x^n | n \in \{0, 1, 2, \ldots\}\}$ but what if...

somehow... $\{\sin(nx) | n \in \{0, 1, 2, 3, \ldots\}\}$ could be used as a basis for some set of functions?

This would synchronize well with LTIs and it is what we will study for the rest of this course.

MATH 115 orthogonal basis "review" part 2 (see L3 for the version we used for Laplace transforms):

Suppose that there is a orthogonal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$.

This means that $\vec{v}_i \cdot \vec{v}_j = a_i \delta[i-j]$ where $a_i \neq 0$.

If we wanted to write $\vec{x} \in \mathbb{R}^n$ in this basis we would need to solve the system of equations

$$\vec{x} = x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n.$$

We can solve this using orthogonality (as in L3). Explicitly to find a_i we take the dot product with $\vec{v}_i$:

$$\begin{aligned} \vec{v}_i \cdot \vec{x} &= \vec{v}_i \cdot (x_1\vec{v}_1 + \ldots + x_n\vec{v}_n) \\ &= x_1(\vec{v}_i \cdot \vec{v}_1) + \ldots + x_n(\vec{v}_i \cdot \vec{v}_n) && \text{Dot product is bilinear} \\ &= x_i(\vec{v}_i \cdot \vec{v}_i) && \text{Dot products other that } \vec{v}_i \cdot \vec{v}_i \text{ vanish} \\ &= x_i a_i && \vec{v}_i \cdot \vec{v}_i = a_i \end{aligned}$$

Thus $x_i = \dfrac{\vec{v}_i \cdot \vec{x}}{a_i}$.

To make this method work for $\{\sin(nx)|n \in \{0, 1, 2, 3, \ldots\}\}$ we **MUST**

- Find a "dot product", called an inner product, for functions such that "$\sin(nx) \cdot \sin(mx) = a_n\delta[n-m]$" and compute the a_ns.
- Define (the different types of) convergence for series of **functions** i.e. $f(x) = \sum_{i=0}^{\infty} a_i \sin(nx)$.

To fft_fun.m

To voice analyzing phone thing!!

In subsequent lectures we will:

- Generalize the dot product to functions
- Learn how to use this new dot product to compute a few different versions of Fourier series (sin, cos, half sin, half cos and complex exponential),
- Talk about convergence and what the Fourier series for $f(x)$ converges to (it is not always $f(x)$!!) and
- Talk about the Fourier transform (which is just the 2-sided Laplace transform when $s = j\omega$).

MATH 213 - Lecture 21: L^2 space, inner product on L^2, and computing Fourier coefficients

Lecture goals: know what the standard inner product in L^2 is and know how to use it to compute Fourier coefficients of $\tau-$periodic functions. Also get you to fill out the SCP survey.

In general properly defining a dot product for functions is a huge issue so we will only work with a special class of functions called "Lebesgue square integrable functions" or L^2 functions which makes things nice:

Definition 1: L^2 functions
A complex valued function f is in the class $L^2([a, b]])$ if

$$\int_a^b |f(x)|^2 dx$$

exists and is finite.

f is in the class L^2 if

$$\int_{-\infty}^{\infty} |f(x)|^2 dx$$

exists and is finite.

L^2 and $L^2([a, b])$ form vector spaces so ideas from MATH 115 can be used (with proper adjustments). Now if f is a member of $L^2([-\tau/2, \tau/2])$ for some fixed τ then our goal is to write

$$f(t) \text{``}=\text{''} \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau} jt} \qquad \text{for} \qquad t \in [-\tau/2, \tau/2].$$

To do this we need to somehow solve for the c_ns....

By comparison with how we solved

$$\vec{b} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

for an orthogonal basis $\{v_1, \ldots, v_n\}$
We need a....


Come closer...
I need...
I need...
A DOT PRODUCT
imgflip.com

Inner product for $L^2([-\tau, \tau])$

Recall that if $\vec{x}, \vec{y} \in \mathbb{C}^n$ then

$$\vec{x} \cdot \vec{y} = x_1\overline{y_1} + x_2\overline{y_2} + \ldots + x_n\overline{y_n}.$$

Now if f and g are complex valued functions, then the "dot product", which will be called an inner product, should follow a similar definition.

The summation becomes integration!!

Definition 2: Standard Inner product on $L^2([a, b])$
If f and g are complex valued functions in $L^2([a, b])$ then the **standard inner product** is

$$\langle f, g\rangle = \frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt.$$

Theorem 1: Existence of Inner Product
If $f, g \in L^2([a, b])$ then $\langle f, g\rangle$ exists and is finite.

We skip the proof since it needs some real analysis...

Theorem 2
The set of complex exponentials $\left\{e^{\frac{2\pi n}{\tau}jt} | n \in \{0, \pm 1, \pm 2, \ldots\}\right\}$ is an orthonormal basis for a subspace of $L^2([-\tau/2, \tau/2])$.

Partial proof: We will not prove that the collection is linearly independent but will prove that they are orthonormal.

$$\left\langle e^{\frac{2\pi n}{\tau}jt}, e^{\frac{2\pi m}{\tau}jt} \right\rangle = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{\frac{2\pi n}{\tau}jt} \cdot \overline{e^{\frac{2\pi m}{\tau}jt}} \, dt$$

$$= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{\frac{2\pi n}{\tau}jt} \cdot e^{-\frac{2\pi m}{\tau}jt} \, dt$$

$$= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{\frac{2\pi(n-m)}{\tau}jt} \, dt$$

$$= \begin{cases} 1 & , n = m \\ 0 & , \text{else} \end{cases}$$

Now that we have an orthonormal basis for a subset of $L^2([-\tau/2, \tau/2])$, we can project any function in $L^2(-\tau/2, \tau])$ into our basis $\{e^{\frac{2\pi n}{\tau}jt} | n \in \{0, \pm 1, \pm 2, \ldots\}\}$ by using our inner product.

Note that since we are doing a projection and the basis may not be (is not...) a basis for $L^2([-\tau/2, \tau/2])$, the result of projecting into this basis may not be equal to the original function in the traditional sense.

Definition 3: Fourier Series - Complex Form
If $f \in L^2([-\tau/2, \tau/2])$ then the **Fourier series in complex form** of $f(t)$ is $\sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau}jt}$ where the c_n are found by projecting f into the basis of complex exponentials.

Theorem 3: Fourier Coefficients for Series in Complex Form
If $f \in L^2([-\tau/2, \tau/2])$ then the Fourier coefficients c_n of $f(t)$ are

$$c_n = \left\langle f(t), e^{\frac{2\pi n}{\tau}jt} \right\rangle.$$

If f is real valued than $c_n = \overline{c_{-n}}$.

Proof:

We know $f(t)$ "$=$" $\sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau}jt}$.

$n, m \in \mathbb{Z}$

$$\left\langle f(t), e^{\frac{2\pi m}{\tau}jt} \right\rangle = \left\langle \sum_n c_n e^{\frac{2\pi n}{\tau}jt}, e^{\frac{2\pi m}{\tau}jt} \right\rangle$$

$$= \sum_n c_n \left\langle e^{\frac{2\pi n}{\tau}jt}, e^{\frac{2\pi m}{\tau}jt} \right\rangle$$ only true for nice fcns

$$= \sum_n c_n \begin{cases} 1 & , n = m \\ 0 & , \text{else} \end{cases}$$

$$= c_m$$ $0 + 0 + 0 \ldots + c_m + \ldots 0 + \ldots$

Compute the things!!

Example 1

Compute the Fourier series of $f : [-0.5, 0.5] \to \mathbb{R}$ defined by

$$f(t) = \sin(2\pi t)$$

If possible simplify the complex exponentials to real valued terms.

$$f(t) \text{ "="} \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau} jt}$$

$$\tau = 0.5 - (-0.5)$$
$$= 1$$

Option 1:

$$c_n = \frac{1}{1} \int_{-0.5}^{0.5} \sin(2\pi t) \cdot e^{-\frac{2\pi n}{1} jt} \, d\tau$$

Option 2:

$$\sin 2\pi t = \frac{e^{2\pi jt} - e^{-2\pi jt}}{2j}$$

$$c_n = \begin{cases} \frac{1}{2j} & , n = 1 \\ -\frac{1}{2j} & , n = -1 \\ 0 & , \text{else} \end{cases}$$

NOTE

$$c_n = \overline{c_{-n}}$$

Real val: $\sin(2\pi t)$

Example 2

Compute the Fourier series of $f : [-\tau/2, \tau/2] \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} -1 & t \in [-\tau/2, 0) \\ 1 & t \in [0, \quad \tau/2] \end{cases}$$

If possible simplify the complex exponentials to real valued terms.
Plot $f(t)$ along with several terms of the Fourier series.

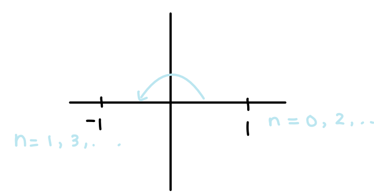
$$c_n = \langle f(t), e^{\frac{2\pi n}{\tau} jt} \rangle$$

$$= \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)\, e^{\frac{-2\pi n}{\tau} jt}\, dt$$

$$= \frac{1}{\tau} \left(\int_{-\frac{\tau}{2}}^{0} -e^{\frac{-2\pi n}{\tau} jt}\, dt + \int_{0}^{\frac{\tau}{2}} e^{-\frac{2\pi n}{\tau} jt}\, dt \right)$$

Case 1: $n \neq 0$

$$e^{n\pi j} = (-1)^n$$



$$c_n = \frac{1}{\tau} \left(\frac{\tau}{2\pi n j} e^{\frac{-2\pi n}{\tau} jt} \Big|_{-\frac{\tau}{2}}^{0} + \frac{-\tau}{2\pi n j} e^{-\frac{2\pi n}{\tau} jt} \Big|_{0}^{\frac{\tau}{2}} \right)$$

$$= \frac{1}{\tau} \left(\frac{\tau}{2\pi n j} \left(1 - e^{n\pi j}\right) + \frac{-\tau}{2\pi n j} \left(e^{-n\pi j} - 1\right) \right)$$

$$= \frac{1}{\tau} \left(\frac{\tau}{2\pi n j} \left(1 - (-1)^n\right) - \frac{\tau}{2\pi n j} \left((-1)^n - 1\right) \right)$$

$$= \frac{1}{2\pi n j} \left(2 - 2(-1)^n \right)$$

$$= \frac{1}{\pi n j} \left(1 + (-1)^{n+1} \right)$$

Case 2: $n = 0$

$$c_0 = \frac{1}{\tau} \left(\int_{-\frac{\tau}{2}}^{0} -e^{\frac{2\pi n}{\tau} jt}\, dt + \int_{0}^{\frac{\tau}{2}} e^{-\frac{2\pi n}{\tau} jt}\, dt \right)$$

$$= \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{0} -1\, dt + \int_{0}^{\frac{\tau}{2}} 1\, dt \;)$$

$$= 0$$

$$c_n = \begin{cases} 0 & , n = 0 \\ \frac{1}{\pi n j} \left(1 + (-1)^{n+1}\right) & , n \neq 0 \end{cases}$$

$$f(t) \simeq \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{2\pi n}{\tau} jt}$$

Real version:

$$c_{-n} e^{-\frac{2\pi n}{\tau} jt} + c_n e^{\frac{2\pi n}{\tau} jt}$$

$$= -\frac{1}{\pi n j}\left(1-(-1)^n\right) e^{-\frac{2\pi n}{\tau} jt} + \frac{1}{\pi n j}\left(1-(-1)^n\right) e^{\frac{2\pi n}{\tau} jt}$$

$$= \frac{1}{\pi n j}\left(1-(-1)^n\right)\left(e^{\frac{2\pi n}{\tau} jt} - e^{-\frac{2\pi n}{\tau} jt}\right)$$

$$= \frac{2\left(1-(-1)^n\right)}{\pi n}\left(\frac{e^{\frac{2\pi n}{\tau} jt} - e^{-\frac{2\pi n}{\tau} jt}}{2j}\right) \qquad \sin t = \frac{e^{jt} - e^{-jt}}{2j}$$

$$= \frac{2\left(1-(-1)^n\right)}{\pi n}\left(\sin\left(\frac{2\pi n}{\tau} t\right)\right)$$

$$= \begin{cases} \frac{4}{\pi n}\left(\sin\left(\frac{2\pi n}{\tau} t\right)\right) & , \text{ n is odd} \\ 0 & , \text{ n is even} \end{cases}$$

$$f(t) \simeq \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(\frac{2\pi n}{\tau} t\right)$$

Example 3

$\frac{T}{2}$ so $T = 2\pi$

Compute the Fourier series of $f : [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(t) = \frac{1}{2}(\pi - |t|)$$

If possible simplify the complex exponentials to real valued terms.
Plot $f(t)$ along with several terms of the Fourier series.

$$c_n = \langle f(t), e^{\frac{2\pi n}{T} jt} \rangle$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\pi - |t|)\, e^{-njt}\, dt$$

$$= \frac{1}{4\pi} \left(\int_{-\pi}^{\pi} \pi e^{-njt}\, dt - \left(\int_{-\pi}^{0} -t e^{-njt}\, dt + \int_{0}^{\pi} t e^{-njt}\, dt \right) \right)$$

$$= \frac{1}{4\pi} \left(\pi \left(\frac{e^{-nj\pi}}{-nj} - \frac{e^{nj\pi}}{-nj} \right) - \left(\frac{-1 + e^{j\pi n}(1 - j\pi n)}{n^2} + \frac{-1 + e^{-j\pi n}(1 + j\pi n)}{n^2} \right) \right)$$

since $n \in \mathbb{Z}$, $e^{nj\pi} = e^{-nj\pi} = (-1)^n$

$$= \frac{1}{4\pi} \left(\pi \underbrace{\cancel{\left(\frac{(-1)^n - (-1)^n}{-nj} \right)}}_{0} - \left(\frac{-2 + (-1)^n (1 - \cancel{j\pi n}) + (-1)^n (1 + \cancel{j\pi n})}{n^2} \right) \right)$$

$$= \frac{1 + (-1)^{n+1}}{2\pi n^2}$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\pi - |t|)\, e^{-0jt}\, dt$$

$$= \frac{\pi}{4}$$

$$f(t) \simeq \frac{\pi}{4} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1 + (-1)^{n+1}}{2\pi n^2} e^{njt}$$

Real ver:

$$f(t) \simeq \frac{\pi}{4} + \sum_{n=0}^{\infty} \frac{2}{\pi (2n+1)^2} \cos((2n+1)t)$$

MATH 213 - Lecture 22: Real Fourier series and convergence

Lecture goals: Know how to find the real Fourier series quickly and know the convergence properties of Fourier series. Get you to fill out the SCP survey.

τ periodic functions:

Definition 1: τ periodic functions
A f function defined on $\mathbb{R}$ is τ **periodic** if for all $t \in \mathbb{R}$

$$f(t) = f(t + \tau).$$

Generally, we pick the smallest value of τ such that the above holds.

Theorem 3 from L21 holds for τ periodic functions and we would integrate over one period.

If I ask you to find the Fourier series without telling you a domain for f and f is τ periodic then you first find out the τ period of f and do the computation over a period of f.

Fourier sin/cos series:

For a real valued function sometimes we found the Fourier series simplified to a sum of sine waves and sometimes it resulted in a sum of cos waves. We now elaborate on this:

Definition 2: Even and odd functions
A function f is **even** if

$$f(-t) = f(t). \quad \forall t \in \mathbb{R}$$

A function f is **odd** if

$$f(-t) = -f(t) \quad \forall t \in \mathbb{R}$$

Example 1
$\sin(x)$ is an odd function. $\cos(x)$ is an even function.

Theorem 1
For a real valued, τ periodic function $f \in L^2([-\tau/2, \tau/2])$:

- If f is an even then the Fourier series can be simplified to a sum of cos waves.

- If f is an odd then the Fourier series can be simplified to a sum of sin waves.

The sums above are called **Fourier cosine series** and **Fourier sine series** respectively.

For completion we will write Theorem 3 from L21 for sin and cos series.

Theorem 2: Fourier Sine and Cosine Coefficients
If f is a real valued function that is in $L^2([-\tau/2, \tau/2])$ then

- If f is even then the Fourier cosine series for f is

$$\sum_{n=0}^{\infty} c_n \cos\left(\frac{2\pi n}{\tau} t\right)$$

where

$$c_n = \begin{cases} \langle f(t), 1 \rangle & n = 0 \\ 2\left\langle f(t), \cos\left(\frac{2\pi n}{\tau} t\right)\right\rangle & n > 0 \end{cases}$$

- If f is odd then the Fourier sine series for f is

$$\sum_{n=1}^{\infty} s_n \sin\left(\frac{2\pi n}{\tau} t\right)$$

where

$$s_n = 2\left\langle f(t), \sin\left(\frac{2\pi n}{\tau} t\right)\right\rangle$$

Sketch of proof:

- Start with the coefficients in Theorem 3,
- write the complex exponential in terms of sin/cos waves,

- simplify the integrals by using the properties that
  - the integral of an odd function over $[-a, a]$ for any $a \in \mathbb{R}$ is 0,
  - that $g(-x) + g(x) = 0$ for an odd function g and
  - that $g(-x) + g(x) = 2g(x)$ for an even function g.

I strongly suggest not blindly memorizing this but to instead memorize Theorem 3 and then understand how it sill simplify if there are no sin or cos terms.

This is similar to the computations seen in the last lecture

Theorem 3
If f is real then we can decompose it into even and odd functions as follows:

$$f_{even}(t) = \frac{f(t) + f(-t)}{2} \qquad \text{and} \qquad f_{odd}(t) = \frac{f(t) - f(-t)}{2}.$$

$f(t) = f_{even}(t) + f_{odd}(t)$.

Sketch of proof: Show that f_{even} is even, that f_{odd} is odd and that when summed we get f.

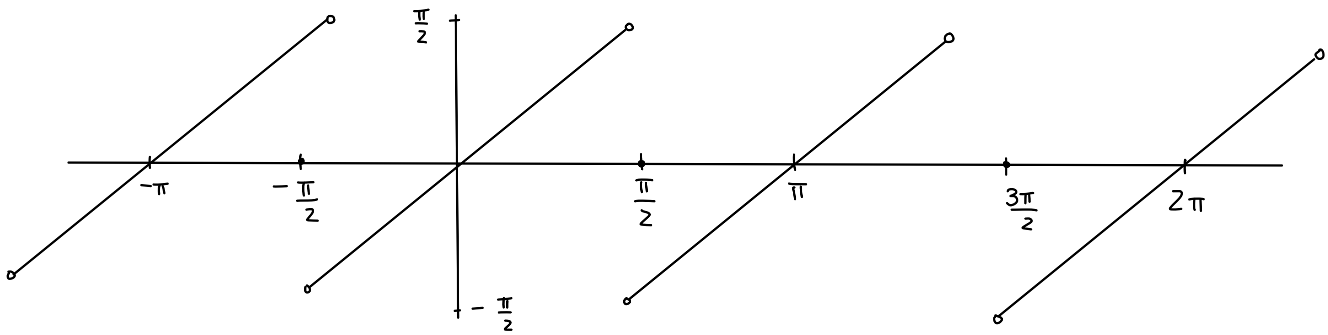
Theorem 4
Every real valued function in $L^2([-\tau/2, \tau/2])$ admits a real valued Fourier series with some sin and/or cos terms. i.e. the complex form will always simplify to sin and cos terms.

Sketch of proof: Since f is real, it can be decomposed into an even and odd part. The even part gives us a cos series and the off part gives us a sin series.

Example 2

Find the Fourier series for the π periodic version of

$$f(x) = x$$



$$f(-x) = -x = -f(x)$$

odd so sin fcn

$$f(x) \simeq \sum_{n=1}^{\infty} s_n \sin\left(\frac{2\pi n}{\tau} x\right) \qquad \tau = \pi$$

$$s_n = 2\langle x, \sin(2nx)\rangle$$ ← don't need to compute s_0 b/c $f(x)$ is odd

$$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin(2nx)\,dx$$

$$= \frac{2}{\pi}\left(\frac{\overset{0}{\cancel{\sin(2nx)}} - 2nx\cos(2nx)}{4n^2}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$\sin(\pm\pi n) = 0$

$\cos(\pm\pi n) = (-1)^n$

$$= \frac{-2(2)n}{\pi(4n^2)}\left(\frac{\pi}{2}\underbrace{\cancel{\cos(\pi n)}}_{(-1)^n} - \left(-\frac{\pi}{2}\right)\underbrace{\cancel{\cos(-\pi n)}}_{(-1)^n}\right)$$

$$= -\frac{1}{\pi n}\left(\frac{\pi}{2}\right)\left(2(-1)^n\right)$$

$$= \frac{(-1)^{n+1}}{n}$$

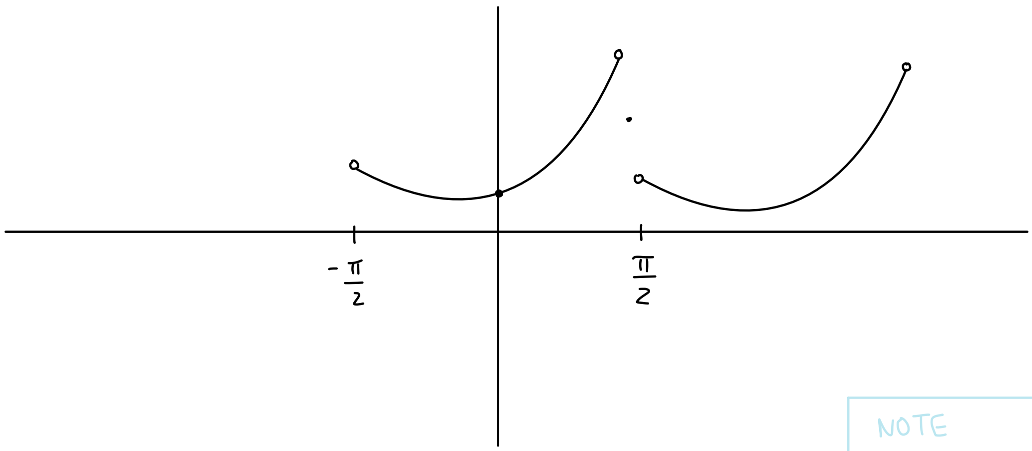
$$f(x) \simeq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2nx)$$

$[-\frac{\pi}{2}, \frac{\pi}{2}]$

Example 3

Find the Fourier series for the π periodic version of

$$f(x) = x^2 + x + 1$$



NOTE

Fourier series of any constant # is just that #

$$f(x) = \underbrace{x^2 + 1}_{\text{even}} + \underbrace{x}_{\text{odd}}$$

$$x^2 \simeq c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{2\pi n}{\pi} x\right)$$

$$c_0 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x^2\,dx$$

$$\vdots$$

$$= \frac{\pi^2}{12}$$

$$c_n = 2\langle x^2, \cos 2nx\rangle$$

$$\vdots$$

$$= \frac{(-1)^n}{n^2}$$

$$f(x) \simeq 1 + \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(2nx) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(2nx)$$

Types of Convergence:

In MATH 119 you dealt with the convergence of series of real numbers. Things like

$$\sum_{i=1}^{\infty} 0.9^i.$$

In these cases there is a natural way to define convergence (though this is not actually unique).

With sums of functions like this

$$\sum_{i=0}^{\infty} f_i(x),$$

there are many different ways to define convergence:

Definition 3: Some Types of Convergence
If $f_1, f_2, \ldots, f_n \ldots$ is a sequence of L^2 functions defined on $[a, b]$ then we say that

- the sequence **converges in the $L^2([a, b])$ norm**, or **converges in the mean** or **converges almost everywhere**, to f if

$$\lim_{n\to\infty} \sqrt{\int_a^b |f_n(x) - f(x)|^2 dx} = 0$$

  tldr; the "average error" goes to 0.

- the sequence **pointwise converges** to f, if for any $x \in [a, b]$

$$\lim_{n\to\infty} (f_n(x) - f(x)) = 0$$

  tldr; the error at each point goes to 0.

- The sequence **uniformly converges** to f if

$$\lim_{n\to\infty} \max_{[a,b]} |f_n(x) - f(x)| = 0$$

  If the maximum does not exist then we replace it with the smallest upper bound (called the sup).

  tldr; the maximum error converges to 0.

Note that each of the things we take the limit of are just numbers!
Metanote, the first bullet point can be called by all those names only when we talk about functions on a finite interval (that is the case for this course).

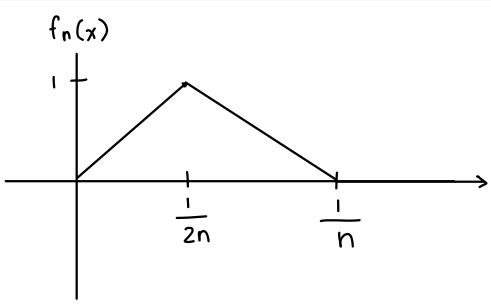
Example 4

Determine if the sequence of functions defined by

$$f_n(x) = \begin{cases} 2nx & 0 \leq x \leq \frac{1}{2n} \\ 1 - 2nx & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & else \end{cases}$$

converges in the L^2 norm, pointwise, or uniformly.

If it converges, then what is the limiting function(s) in each case?



Pointwise: need to show $\forall x \in \mathbb{R}$, $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$

↳ case 1: $x \leq 0$

$f_n(x) = 0$

↳ case 2: $x > 0$

- case 2a: $x > 1$

  $f_n(x) = 0$

- case 2b: $0 < x < 1$

  If $x > \frac{1}{n}$, then $f(x) = 0$. So, we need $\frac{1}{x} < n$. Then, if $n \geq \lfloor \frac{1}{x} \rfloor + 1$, $f_n(x) = 0$.

converges

L^2 norm:

$\sqrt{\int_{-\infty}^{\infty} |f_n(x) - 0|^2 dx}$

$= \frac{A}{\sqrt{n}}$, $A \in \mathbb{R}$

$\lim_{n\to\infty} \frac{A}{\sqrt{n}} = 0$

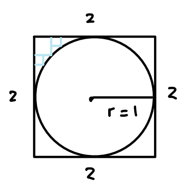
converges

Uniform:

$\max(f_n(x)) = 1$

$\lim_{n\to\infty} (1-0) = 1 \neq 0$

doesn't converge



circumference = 2π

↳ converges pointwise, but not uniformly

Convergence of Fourier series:

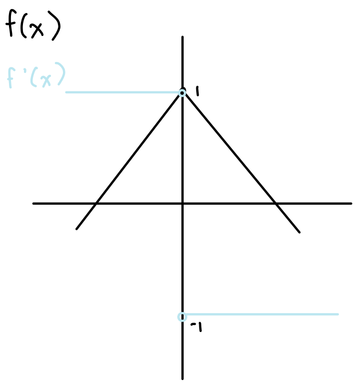
Before talking about the convergence of Fourier series we need to introduce some new definitions

Definition 4: Piecewise C^1
A function f is **Piecewise C^1 (PWC1)** on the interval $[a, b]$ if there is a finite partition $a = t_0 < t_1 < \ldots < t_k = b$ such that:

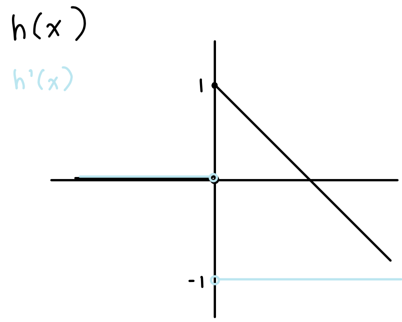
- f' exists on each interval (t_i, t_{i+1}),
- f' is continuous on each interval (t_i, t_{i+1}),
- f and f' are bounded on each interval (t_i, t_{i+1}).

Example 5
Determine if the following are continuous, piecewise continuous, differentiable, and/or piecewise C^1.

- $f(x) = 1 - |x|$
- $g(x) = 1 - x^{2/3}$
- $h(x) = u(x)(1 - |x|)$
- $\ell(x) = u(x)(1 - x^{2/3})$

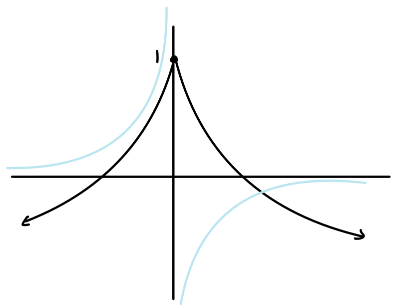


cont? ✓
PWC? ✓
dif ? ✗
PWC1? ✓

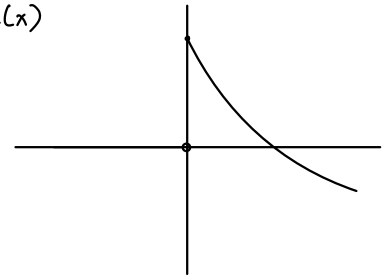


cont? ✗
PWC? ✓
dif? ✗
PWC1? ✓

$g(x)$
$g'(x)$



$\ell(x)$



cont? ✓
PWC? ✓
dif? X
PWC1? X

$g'(x) = -\frac{2}{3x^{1/3}}$, $x \neq 0$

(not bounded / blows up at $x = 0$)

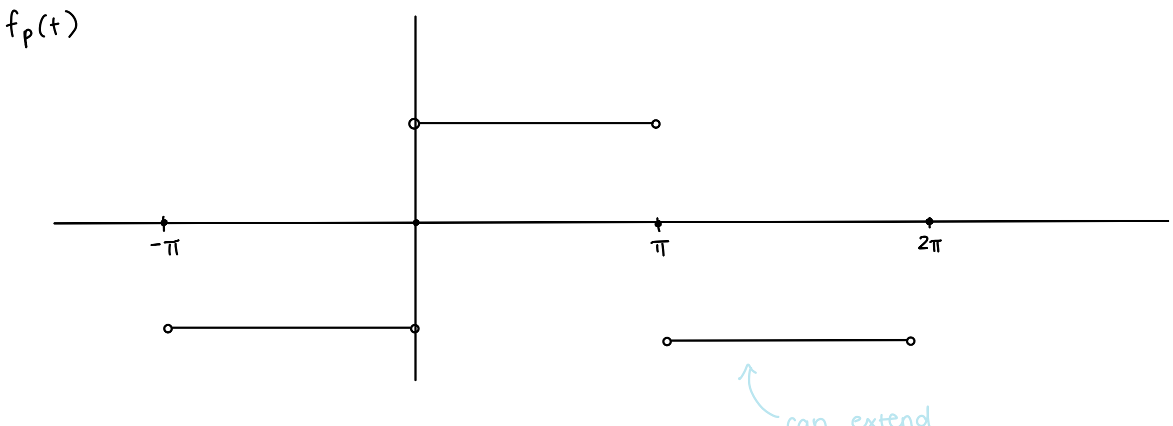
cont? X
PWC? ✓
dif? X
PWC1? X

Definition 5: Periodic Extension
The **periodic extension** of a function f defined on $[a, b]$ is the $b - a$ periodic function f_p such that

- $f_p(t) = f(t)$ for $t \in (a, b)$ where $f(t)$ is continuous.
- $f_p(t) = \frac{f(t^-)+f(t^+)}{2}$ for $t \in (a, b)$ where $f(t)$ is not continuous.
- $f_p(a) = \frac{f(a)+f(b)}{2} = f_p(b)$

Example 6
Draw the periodic extension of

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 \leq t < \pi \end{cases}$$



$$f_p(-\pi) = \frac{f(-\pi^-) + f(-\pi^+)}{2} = \frac{1 + -1}{2} = 0$$

$$f_p(0) = \frac{f(0^-) + f(0^+)}{2} = \frac{-1+1}{2} = 0$$

$$f_p(\pi) = \frac{f(\pi^-) + f(\pi^+)}{2} = \frac{1 + -1}{2} = 0$$

Theorem 5: Convergence of Fourier series
Let f_p be the periodic extension of a function $f \in L^2([-\tau/2, \tau/2])$.

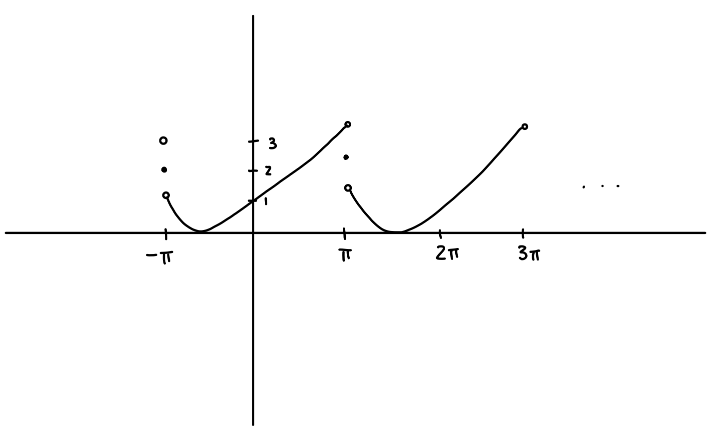
- The Fourier series of f converges in the L^2 norm (also in the mean and almost everywhere) to f (and also f_p) on any finite subinterval of $[-\tau/2, \tau/2]$.
- If f_p is piecewise C^1 then the Fourier series of f converges pointwise to f_p for all $x \in \mathbb{R}$
- If f_p is piecewise C^1 and continuous then the Fourier series of f converges uniformly f_p on any finite interval of $\mathbb{R}$.

The proof is beyond the scope of this course.

Example 7
Draw the function that the Fourier series of f defined on $[-\pi, \pi]$

$$f(x) = \frac{x^2}{\pi^2} + \frac{x}{\pi} + 1$$

would converge to. Explain the type of convergence.



NOTE
Draw at least 2 periods.

conv in L^2? ✓

PWC1? ✓

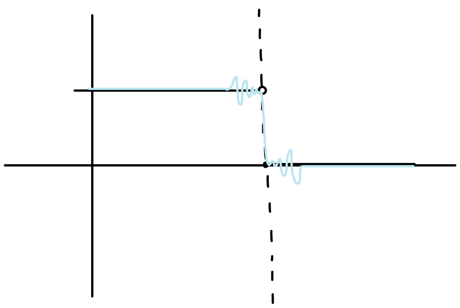
cont? ✗

converges pointwise
doesn't converge uniformly

Definition 6: Gibbs Phenomenon

For a $L^2([a,b])$ function f with periodic extension f_p, if f_p is not continuous at some point t_0 then truncated Fourier series of f will have growing oscillations near the point t_0. This is called **Gibbs Phenomenon**.

These oscillations do not appear in the infinite sum.



MATH 213 - Lecture 23: Differentiating and integrating Fourier series, Persivalle and Dirichlet theorems and Fourier transform

Lecture goals: Know when you can term-wise integrate and differentiate Fourier series, what the Persivalle and Dirichlet theorems state and how to use Persivalle's theorem. Get you to fill out the SCP survey.

Integrating and differentiating Fourier series:

For general series of piecewise convergent functions

$$f(x) = \sum_{n=-\infty}^{\infty} f_n(x)$$

it is **NOT** true that

$$f'(x) = \sum_{n=-\infty}^{\infty} f'_n(x)$$

or that

$$\int f(x)dx = \sum_{n=-\infty}^{\infty} \int f_n(x)dx.$$

So when can we do this for Fourier series?

Theorem 1: Term-by-term integration of Fourier series
The Fourier series of a PWC1 $\tau-$periodic L^2 function $f(t)$ can be term-by-term integrated to give a convergent series that **pointwise converges** (and sometimes uniformly) over any finite interval.

TLDR: If PWC1, integral has uniform convergence. Else, pointwise convergence.

Explicitly if f_p is in L^2 and

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi njt}{\tau}}$$

then

$$\int_a^t f_p(t)dt = \sum_{n=-\infty}^{\infty} c_n \int_a^t e^{\frac{2\pi njt}{\tau}} dt$$

and the convergence is at least pointwise.

Note 1: It is because of this theorem, that we can guarantee the existence of the Fourier coefficients derived in L21.
Note 2: If the Fourier series has a non-trivial constant a_0 term then when integrated we obtain $a_0 t$ which means that the RHS might not be a Fourier series.

Theorem 2: Term-by-term differentiation of Fourier series
Let f

- be a PWC1 $\tau-$periodic function,
- be a continuous function and
- satisfy $f(-\tau/2) = f(\tau/2)$.

If the above conditions are met then the series for f can be term-by-term differentiated to give a pointwise convergent series that **pointwise converges** to $f_p'(t)$ on for all t such that $f_p''(t)$ exists

Explicitly if f satisfied the above and

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi njt}{\tau}}$$

then

$$f_p'(t)dt = \sum_{n=-\infty}^{\infty} c_n \frac{d}{dt} e^{\frac{2\pi njt}{\tau}}$$

and the convergence is pointwise for all points where $f_p''(t)$ exists.

Example 1

In Lecture 22 we showed that Fourier series for the π periodic version of x^2 is

$$\frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(2nx)$$

and the Fourier series for the π periodic version of x is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2nx)$$

Use these facts to find the Fourier series for the π periodic version of x^3 and then x^4.

$$x^3 = \sum c_n e^{\frac{2\pi n}{T} jt}$$

$$x^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(2\pi x)$$

$$\int_0^x x^2 dx = \int_0^x \frac{\pi^2}{12} dx + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_0^x \cos(2nx) dx$$

$$\frac{x^3}{3} = \frac{\pi^2}{12} x + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{\sin(2nx)}{2n}$$

$$x^3 = \frac{\pi^2}{4} x + \sum_{n=1}^{\infty} \frac{3(-1)^n \sin(2nx)}{2n^3}$$

$$x^3 = \frac{\pi^2}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2nx) + \frac{3(-1)^n \sin(2nx)}{2n^3}$$

$$x^3 = \sum_{n=1}^{\infty} \underbrace{\left(\frac{\pi^2 (-1)^{n+1}}{4n} + \frac{3(-1)^n}{2n^3} \right)}_{c_n} \sin(2nx)$$

$$\int_0^x x^3 dx = \sum_{n=1}^{\infty} c_n \int_0^x \sin(2nx) dx$$

$$\frac{x^4}{4} = \sum_{n=1}^{\infty} c_n \left(-\frac{\cos(2nx)}{2n} - \frac{-1}{2n}\right)$$

$$\frac{x^4}{4} = \sum_{n=1}^{\infty} \frac{-c_n}{2n} \cos(2nx) + \sum_{n=1}^{\infty} \frac{c_n}{2n}$$

$$x^4 = \sum_{n=1}^{\infty} \frac{-4c_n}{2n} \cos(2nx) + \sum_{n=1}^{\infty} \frac{4c_n}{2n}$$

$$x^4 \simeq \sum_{n=1}^{\infty} c'_n \cos(2nx) + c_0$$

$$c_0 = \sum_{n=1}^{\infty} \frac{2c_n}{n}$$

$$= \langle x^4, 1 \rangle$$

$$= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^4 dx$$

$$= \frac{\pi^4}{80}$$

$$x^4 = \sum_{n=1}^{\infty} \left(\frac{\pi^2 (-1)^n}{2n^2} + \frac{12(-1)^{n+1}}{4n^4} \right) \cos(2nx) + \frac{\pi^4}{80}$$

Recall that (although I did not previously name it in L22 Theorem 5):

Theorem 3: Dirichlet's theorem
If f is PWC1 and τ periodic then the Fourier series of $f(t)$ converges pointwise to $f_p(t)$.

If we define the error of a truncated Fourier series to be

$$e_N(t) = f_p(t) - \sum_{n=-N}^{N} c_n e^{\frac{2\pi njt}{\tau}}$$

Then the above theorem tells us that for PWC1 τ periodic functions f,

$$\lim_{N\to\infty} \langle e_N, e_N \rangle = 0$$

i.e. the average error goes to zero in the limit. In other words:

Theorem 4: Parseval's theorem
If f is $L^2[-\tau/2, \tau/2]$ and τ periodic function then

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

The term $\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |f(t)|^2 dt$ is the average energy of the wave and hence

- this gives us a way to compute the energy of a wave from the Fourier coefficients (i.e. using fft),
- to determine how many terms we need to include in a truncation of the Fourier series to approximate the original signal to some given power,
- or to derive many useful formulas for computing sums...

Before giving an example, here is the real valued version of Parseval's theorem:

Theorem 5: Parseval's theorem
If f is a real valued PWC1 and τ periodic function then

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |f(t)|^2 dt = c_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} \left(c_n^2 + s_n^2\right)$$

where c_n and s_n are the Fourier sine and cosine coefficients.

Example 2
Recall from L21 that on $t \in [-\tau/2, \tau/2]$ and for $\tau > 0$,

$$\begin{cases} -1 & t \in (-\tau/2, 0) \\ 0 & else \\ 1 & t \in (0, -\tau/2) \end{cases} = \frac{4}{\pi} \sum_{n=1,2,3,\ldots} \frac{1}{n} \sin\left(\frac{2\pi n}{\tau} t\right)$$

pointwise. Use the above with Perceval's theorem to find an infinite series that evaluates to π.

$$\frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 1 \, dt = 0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(0^2 + \left(\frac{4}{\pi n}\right)^2\right)$$

$$\frac{1}{\tau}\left(\frac{\tau}{2} - \left(-\frac{\tau}{2}\right)\right) = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{4}{\pi n}\right)^2$$

$$1 = \frac{16}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\pi^2 = 8 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\pi = 2\sqrt{2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

Fourier transform (tldr;):

Recall that the two sided Laplace transform was defined by

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

given that the integral converged.

When working with (causal) DEs, we used the one sided version of the above.

In our work with Fourier series, we required that the functions be τ periodic. If this is not the case, then our methods do not work!!

What do we do with these cases? Recall

$$f_\tau(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n t j}{\tau}}, \qquad c_n = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f_\tau(t)e^{-\frac{2\pi n t j}{\tau}}dt$$

If $\tau \to \infty$ then n is not allowed to be continuous so c_n becomes a function of $\mathbb{R}$ and the discrete sum becomes an integral. Further we would need to do some regularization to deal with the averaging in c_n. Doing this defining ω to be the limiting values of $\frac{2\pi n}{\tau}$ (and some complex analysis to deal with the regularization) gives:

$$f_\infty(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\omega t j}d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f_\infty(t)e^{-\omega t j}dt.$$

$F(\omega)$ is called the Fourier transform of f_∞ and we write $F(\omega) = \mathcal{F}\{f_\infty\}$ explicitly:

Definition 1: Fourier Transform

If $f \in L^1$ then the Fourier transform of f is

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-\omega t j}dt$$

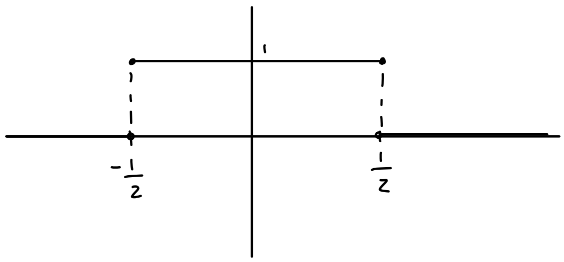
and the inverse Fourier transform of $F(\omega)$ is

$$f(t) = \mathcal{F}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\omega t j}d\omega.$$

Note that the Fourier transform/inverse transform is just the two-sided Laplace transform in the case that $s = \omega j$.

Example 3
Compute the Fourier transform of

$$f(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & else \end{cases}$$



$$F(\omega) = \mathfrak{F}\{f(t)\}$$

$$= \int_{-\infty}^{\infty} f(t) e^{-\omega t j} \, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\omega t j} \, dt$$

$$= \left(-\frac{1}{\omega j} e^{-\omega t j}\right)\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{e^{-\frac{\omega j}{2}} - e^{\frac{\omega j}{2}}}{-\omega j}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$= \frac{2}{\omega} \frac{e^{\frac{\omega j}{2}} - e^{-\frac{\omega j}{2}}}{2j}$$

$$= \frac{2}{\omega} \sin\left(\frac{\omega}{2}\right)$$

$$= \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

$$F(\omega) = \begin{cases} 1 & , \omega = 0 \\ \dfrac{\sin(\frac{\omega}{2})}{\frac{\omega}{2}} & , \omega \neq 0 \end{cases} \longleftarrow \operatorname{sinc}\left(\frac{\omega}{2}\right)$$