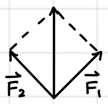
ch2: scalars & vectors

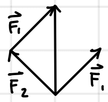
2-1: DEFINITIONS OF SCALARS AND VECTORS

- scalars require only # & unit
- vectors require a magnitude, unit, & direction
  - ↳ to describe them, need frame of reference (convention abt what's "at rest") & coordinate system
- polar notation is used to describe 2D vector in terms of magnitude & angle it makes w/ +ve x-axis
  - ↳ e.g. 30 m/s [25°]
- orthogonal coordinate system has perpendicular axes
  - ↳ e.g. xyz-coordinate system
  - ↳ can be described in terms of their components
    - $v_x = v\cos\theta$
    - $v_y = v\sin\theta$
    - $v = \sqrt{v_x^2 + v_y^2}$
    - $\tan\theta = \frac{v_y}{v_x}$
- scalar notation: rep of vector using components

2-2: VECTOR ADDITION

- geometric method:
  - ↳ parallelogram rule is tail-to-tail
  - ↳ triangle construction is head-to-tail





  - ↳ $\vec{F_R} = \vec{F_1} + \vec{F_2}$
- algebraic method:
  - ↳ $F_{Rx} = F_{1x} + F_{2x} + \ldots + F_{Nx}$
  - ↳ $F_{Ry} = F_{1y} + F_{2y} + \ldots + F_{Ny}$
  - ↳ $F_R = \sqrt{F_{Rx}^2 + F_{Ry}^2}$
  - ↳ $\tan\theta = \frac{F_{Ry}}{F_{Rx}}$
- vector subtraction: $\vec{F_1} - \vec{F_2} = \vec{F_1} + (-\vec{F_2})$

2-3: CARTESIAN VECTOR NOTATION

- unit vector has dimensionless magnitude of 1
- in Cartesian notation, any 3D vector can be rep as $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$
  - ↳ vector addition: $\vec{F_R} = (\sum_{a=1}^{N} F_{ax})\hat{i} + (\sum_{a=1}^{N} F_{ay})\hat{j} + (\sum_{a=1}^{N} F_{az})\hat{k}$
- to find unit vector in direction of given force: $\hat{u}_F = \frac{\vec{F}}{F} = \frac{F_x}{F}\hat{i} + \frac{F_y}{F}\hat{j} + \frac{F_z}{F}\hat{k}$
  - ↳ $\hat{u}_F$ is dimensionless vector in direction of $\vec{F}$
  - ↳ $\vec{F} = F\hat{u}_F$

- position vector $\vec{r}$ describes location of point in space relative to some fixed point (usually origin)
  - ↳ $\vec{r}_A = x_A\hat{i} + y_A\hat{j} + z_A\hat{k}$ for point $A(x_A, y_A, z_A)$
- displacement vector from A to B: $\vec{r}_{A\to B} = (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}$

horizontal latitude lines are parallels
- vertical longitude lines are meridans

- coordinate direction angles are angles that vector makes w/each coordinate axis
  - ↳ denoted as α, β, γ
  - ↳ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
  - ↳ for 2D vectors, α & β are complementary (i.e. $\alpha + \beta = 90°$)

2-4. DOT PRODUCT OF TWO VECTORS

- dot/scalar product used to find projection of vector onto any given axis
- $\vec{A} \cdot \vec{B} = AB\cos\theta$
  - ↳ θ is angle btwn vectors
- properties of dot product:
  - ↳ unit is units of vectors being multiplied
  - ↳ btwn 2 $\perp$ vectors is 0
  - ↳ greatest when vectors are parallel & smallest when antiparallel (b/c $\cos 180° = -1$)
  - ↳ btwn $\vec{A}$ & $\hat{u}_B$ rep projection of vector $\vec{A}$ onto direction of vector $\vec{B}$
  - ↳ commutative
  - ↳ distributive
    - $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
      $= AB\cos\theta_{A\text{-}B} + AC\cos\theta_{A\text{-}C}$
- Cartesian notation: $\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$

- translation of coordinate system, components (except for position vector) & magnitudes remain same
- rotation of coordinate system abt origin, components change but magnitudes remain same

2-5: CROSS PRODUCT OF VECTORS

- cross/vector product defines vector $\perp$ to 2 given vectors
- $\vec{A} \times \vec{B} = \vec{C}$
  - ↳ $C = AB\sin\theta$
  - ↳ direction using RHR: curl fingers from $\vec{A}$ to $\vec{B}$, thumb points in direction of $\vec{C}$
- properties of cross product:
  - ↳ unit is units of vectors being multiplied
  - ↳ magnitude is area of parallelogram w/ $\vec{A}$ & $\vec{B}$ as its side
  - ↳ btwn parallel & antiparallel vectors is 0 (b/c $\sin 0° = \sin 180° = 0$)
  - ↳ greatest when vectors are $\perp$
  - ↳ anticommutative
    - $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
  - ↳ can be multiplied by scalar
    - $a(\vec{A} \times \vec{B}) = (a\vec{A}) \times \vec{B} = \vec{A} \times (a\vec{B})$
    - $|a(\vec{A} \times \vec{B})| = aAB\sin\theta$
  - ↳ $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
- Cartesian notation: $\vec{A} \times \vec{B} = (x_A\hat{i} + y_A\hat{j} + z_A\hat{k}) \times (x_B\hat{i} + y_B\hat{j} + z_B\hat{k})$

$$= \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} \times \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$$

$$= \begin{bmatrix} y_Az_B - z_Ay_B \\ z_Ax_B - x_Az_B \\ x_Az_B - y_Ax_B \end{bmatrix}$$

ch 3: motion in 1D

NOTE: $\frac{m}{s} \times 3.6 = \frac{km}{h}$
$\frac{km}{h} \div 3.6 = \frac{m}{s}$

3-1: DISTANCE AND DISPLACEMENT

- displacement is vector connecting object's initial & final positions
  - ↳ $\vec{\Delta x} = x - x_0$ [$\Delta x = m$]
    - SI units are m
  - ↳ straight-line path btwn endpoints of trajectory (path in space followed by moving object)
- distance is path length covered by object
  - ↳ $d = \Sigma_i |\delta x_i|$

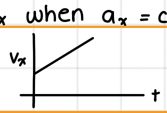
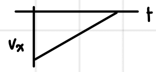
3-2: SPEED AND VELOCITY

- speed is scalar quantity describing how fast object is moving
- velocity is vector indicating both how fast & direction of motion
- average speed $v_{avg} = \frac{d}{\Delta t} = \frac{d}{t - t_0}$ [$v_{avg} = \frac{m}{s}$]
- average velocity: $v_{x,avg} = \frac{\Delta x}{\Delta t}$ [$\vec{v}_{avg} = \frac{m}{s}$]
  - ↳ slope of chord btwn 2 points on position vs time plot

- instantaneous velocity is derivative of position wrt/time
  - ↳ $v_x(t) = \lim_{\Delta t \to 0} v_{x,avg} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx(t)}{dt} = x'(t)$
  - ↳ instataneous speed is magnitude of instantaneous velocity

3-3: ACCELERATION

- average acceleration is $a_{x,avg} = \frac{\Delta v_x}{\Delta t} = \frac{v_x - v_{x0}}{t - t_0}$ [$a = \frac{m}{s^2}$]
  - ↳ slope of chord btwn 2 points on velocity vs time graph
  - ↳ can be non-zero w/ constant speed b/c direction can change
- instataneous acceleration is derivative of velocity wrt/time
  - ↳ $a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x(t)}{\Delta t} = \frac{dv_x(t)}{dt} = v_x'(t)$
  - ↳ second derivative of position wrt/time : $a_x = v_x'(t) = x''(t)$
-

Description of motion	v_x	a_x	v_x when a_x = const
Speeding up in +ve x-dir	+	+	
Slowing down in -ve x-dir	-	+	
Slowing down in +ve x-dir	+	-	
Speeding up in -ve x-dir	-	-	

- object is in free fall if gravity is only force acting on object
  - ↳ experiences gravitational acceleration $\vec{g}$
  - ↳ avg value of g (i.e. magnitude of $\vec{g}$) is 9.81 m/s²

3-4. MATHEMATICAL DESCRIPTION OF 1D MOTION WITH CONSTANT ACCELERATION

if object has constant acceleration, $a_{x,avg} = a_x$

- $\vec{v}_2 = \vec{v} + \vec{a}\Delta t$
- $\vec{v}_{avg} = \frac{\vec{v}_1 + \vec{v}_2}{2}$
- $x_2 = x_1 + \vec{v}_1 \Delta t + \frac{1}{2}\vec{a}(\Delta t)^2$
- $\vec{v}_2^{\,2} = \vec{v}_1^{\,2} + 2\vec{a}\Delta x$
- $x_2 - x_1 = \left(\frac{\vec{v}_1 + \vec{v}_2}{2}\right)\Delta t$

when $\vec{a}$ is constant

3-5: ANALYZING RELATIONSHIPS BETWEEN x(t), v(t), AND a(t) PLOTS

- $x(t)$ is integral of $v(t)$ which is integral of $a(t)$
- $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

3-6: FREE FALL

- free fall means object is only acted upon by force of gravity
- sign for acceleration in free fall motion depends on choice of coordination system

3-7: RELATIVE MOTION IN 1D

- a frame of reference consists of coordinate system that includes position of origin, direction of axes, & units of measurement
  - ↳ include spatial (e.g. x, y, z) & temporal (e.g. t) coordinates
- $\vec{v}_{1/2} = \vec{v}_{1g} - \vec{v}_{2g}$
  - ↳ velocity of object 1 relative to object 2
- relative positions & displacements:
  - ↳ $x_{1/2} = x_{1g} - x_{2g}$
  - ↳ $\Delta x_{1/2} = \vec{v}_{1/2}\Delta t = (\vec{v}_{1g} - \vec{v}_{2g})\Delta t = \Delta x_{1g} - \Delta x_{2g}$
- relative acceleration $\vec{a}_{1/2} = \frac{\Delta \vec{v}_{1/2}}{\Delta t} = \frac{\Delta \vec{v}_{1g}}{\Delta t} - \frac{\Delta \vec{v}_{2g}}{\Delta t} = \vec{a}_{1g} - \vec{a}_{2g}$
- $\vec{v}_{1/2} = -\vec{v}_{2/1}$
- acceleration due to gravity makes no contribution to relative quantities if acting on both objects

3-8: CALCULUS OF KINEMATICS

- $\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t)\,dt$
- $x(t) = x_0 + \int_0^t \vec{v}(t)\,dt = x_0 + \int_{t_0}^t (\vec{a}(t)t + \vec{v}_0)\,dt$

ch 4: motion in 2D & 3D

4-1: POSITION, VELOCITY, AND ACCELERATION

- position vector is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
- displacement is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

- average velocity is $\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$
  - ↳ vector is parallel to $\Delta\vec{r}$ vector
- instantaneous velocity is $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

- average acceleration is $\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$
- instantaneous acceleration is $\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$
- acceleration can result in either speed or direction of velocity changing

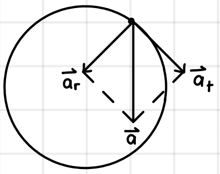
4-2: PROJECTILE MOTION

- for an object that experiences constant acceleration, the motion occurs in a plane
- in component form: $x_2\hat{i} + y_2\hat{j} = x_1\hat{i} + y_1\hat{j} + (\vec{v}_{1x}\hat{i} + \vec{v}_{1y}\hat{j})\Delta t - \frac{1}{2}g(\Delta t)^2\hat{j}$
  - ↳ $\vec{x}_2 = \vec{x}_1 + \vec{v}_{1x}\Delta t$
  - ↳ $\vec{y}_2 = \vec{y}_1 + \vec{v}_{1y}\Delta t - \frac{1}{2}g(\Delta t)^2$
  - ↳ $\vec{v}_{2x} = \vec{v}_{1x}$
    - velocity is constant in x-dir
  - ↳ $\vec{v}_{2y} = \vec{v}_{1y} - g\Delta t$

4-3: CIRCULAR MOTION

- object moving w/ uniform circular motion moves along circular path of fixed radius at constant speed
- at every point on circle, velocity of object is tangent to circle & ⊥ to position vector
- avg acceleration is $\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$
  - ↳ parallel to $\Delta\vec{v}$
- both $\Delta\vec{v}$ & $\vec{a}_{avg}$ are ⊥ to $\Delta\vec{r}$
- objects experience radial acceleration, which means continuous acceleration toward centre of circle
- $a_r = \frac{v^2}{r}$
  - ↳ period is $T = \frac{2\pi r}{v}$

- tangential acceleration, $\vec{a}_t$, is when speed changes as object travels in circle
- total acceleration is vector sum: $\vec{a} = \vec{a}_r + \vec{a}_t$
- $\vec{a}_t$ is always ⊥ to $\vec{a}_r$
  - ↳ total acceleration magnitude: $a = \sqrt{\vec{a}_t^2 + \vec{a}_r^2}$
  - ↳



4-4: RELATIVE MOTION IN 2D AND 3D

- relative motion compares results of 2 observers who move at a constant velocity wrt each other
- in notation used, 1st subscript is object being observed & 2nd subscript is ref frame in which it's

being observed in

- if there's a particle P being observed by 2 observers ; one's in ref frame A + other in ref frame B
  - ↳ $\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$
  - ↳ $\vec{v}_{AB} = -\vec{v}_{BA}$
- for relative acceleration :
  - ↳ $\frac{d\vec{v}_{PA}}{dt} = \frac{d\vec{v}_{PB}}{dt} + \frac{d\vec{v}_{BA}}{dt}$

    $\vec{a}_{PA} = \vec{a}_{PB}$ since $\frac{d\vec{v}_{BA}}{dt} = \vec{a}_{BA} = 0$

ch 5 : forces & motion

DYNAMICS AND FORCES

SI unit of force is newton (N)

↳ $1\,N = 1\,\frac{kg \cdot m}{s^2}$

- net force is resultant when all forces on an object are added as vectors
- FBD is core tool where object is rep by a point & all forces acting on it are drawn as vectors starting on that point

MASS AND FORCE OF GRAVITY

- attractive gravitational force is : $\vec{F}_g = m\vec{g}$
  - ↳ m is mass of object
  - ↳ $\vec{g}$ is acceleration due to gravity (-9.81 m/s² [↑])
- gravitational force is sometimes referred to weight

NEWTON'S LAWS OF MOTION

- Newton's 1st law: if no net force is exerted on an object, the object's velocity won't change
  - ↳ law of inertia
- Newton's 2nd law: acceleration of a mass is proportional to net applied force & inversely proportional to mass
  - ↳ $\vec{F}_{net} = m\vec{a}$
- inertia refers to object's resistance to being accelerated
  - ↳ quantitive measure is mass
    - requires large force to produce same acceleration
- inertial mass is defined by force required to produce a certain acceleration

Newton's 3rd law: if object 1 exerts force $\vec{F}_{12}$ on object 2, then object 2 will exert force $\vec{F}_{21}$ on object 1 that's equal in magnitude & opposite in direction

- ↳ $\vec{F}_{21} = -\vec{F}_{12}$
- ↳ true whether or not motion is involved

APPLYING NEWTON'S LAWS

- normal force is ⊥ to a surface
  - ↳ $\vec{F}_N$ or $\vec{N}$
- tension force is being exerted rope/wire is being used to pull an object
  - ↳ $\vec{F}_T$ or $\vec{T}$
- to solve Newton's 2nd law problems:
  - ↳ identify object & all of forces that act on object
  - ↳ draw an FBD
  - ↳ decide on orientation of the axes
    - acceleration direction is usually along one of the axes
  - ↳ define +ve direction
  - ↳ divide forces into components
  - ↳ write $\Sigma\vec{F} = m\vec{a}$ for each axis direction
  - ↳ simultaneously solve equations
    - clearly state & interpret directions

- if objects are joined by ideal ropes, accelerations must have same magnitude
  - ↳ if ideal & massless, tension will also have same magnitude
- when pulleys have multiple complex rope connections, can't all parts have same acceleration magnitude
- when multiple objects are connected, simplify by drawing FBD for composite total object

FRICTION

- static friction is friction that acts btwn objects not moving relative to each other
- max force of static friction btwn 2 surfaces is directly proportional to normal force btwn 2 surfaces: $f_s^{max} = \mu_s N$
  - ↳ μ_s is coeff of static friction (unitless b/c it's ratio)
  - ↳ $0 \leq \vec{f}_s \leq \vec{f}_s^{max}$
- $\vec{f}_s$ is in direction to oppose tendency to move

- force of friction decreases when object starts moving
- kinetic friction is friction that acts on a moving object
- magnitude is directly proportional to magnitude of normal force btwn object & surface it's moving on: $f_k = \mu_k N$
  - ↳ μ_k is coeff of kinetic friction (usually less than μ_s)

SPRING FORCES AND HOOKE'S LAW

- equilibrium position is where spring is unstretched & exerts no force on mass
- as long as spring isn't extended beyond its elastic limit, F_s is directly proportional to $\vec{\Delta d}$ of free end of spring
- ideal springs follow Hooke's Law: $\vec{F}_s = -k\vec{\Delta x}$
  - ↳ k is spring constant (units are N/m = kg/s²)
    - ∘ k always has +ve value
  - ↳ $\vec{\Delta x}$ is extension/compression of spring from equilibrium state
  - ↳ minus sign indicates that force acts in opposite direction of $\vec{\Delta x}$
- system of 2 parallel springs has spring constant $k = k_1 + k_2$
- when 2 springs are connected in series, total $\vec{\Delta x}$ is equal to elongation of each spring
  - ↳ $$\begin{aligned} \vec{\Delta x}_T &= \vec{\Delta x}_1 + \vec{\Delta x}_2 \\ &= \frac{\vec{F}}{k_1} + \frac{\vec{F}}{k_2} \\ &= \vec{F}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \end{aligned}$$
  - ↳ $k = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1}$

FUNDAMENTAL AND NON-FUNDAMENTAL FORCES

- fundamental forces result from fundamental interactions in nature
  - ↳ act at a distance & don't require contact
- 4 fundamental interactions:
  - ↳ gravitational affects everything that has mass
  - ↳ electromagnetic affects electrically charged objects
  - ↳ weak affects subatomic particles called quarks & leptons (electrons, muons, tau particles, & their neutrinos)
  - ↳ strong has 2 sub-branches
    - ∘ fundamental interaction btwn quarks & gluons
    - ∘ residual interaction btwn hadrons, such as protons & neutrons

- non-fundamental forces depend on 1+ fundamental forces
  - ↳ e.g. friction, elastic, tension, & normal forces

↳ most involve contact / near contact

UNIFORM CIRCULAR MOTION

- an object moving w/uniform speed v along circular trajectory of radius r has radially inward centripetal acceleration
  - ↳ $a_r = \frac{v^2}{r}$
  - ↳ inward twd centre of circle
  - ↳ $v = \frac{2\pi r}{T}$
- centripetal force is sum of forces pointing in radial direction

REFERENCE FRAMES AND FICTITIOUS FORCES

- in an inertial reference frame, object will only accelerate if there's net force
- fictituous forces are only present in non-inertial reference frame
  - ↳ these frames are accelerated reference frames
- acceleration in non-inertial ref frames can be linear or rotational
  - ↳ in non-inertial rotating ref frame, there will be fictituous force pushing object radially outward
    - called centrifugal force
    - opposite in direction to centripetal force
- fictituous forces can always be eliminated by choosing inertial ref frame

MOMENTUM AND NEWTON'S SECOND LAW

- linear momentum ($\vec{p}$) is defined as : $\vec{p} = m\vec{v}$
  Newton's 2nd law can be written in terms of time derivative of linear momentum
  - ↳ $\vec{F}_{net} = \frac{d\vec{p}}{dt}$
- impulse can be calculated as $\vec{J} = \int d\vec{p} = \int \vec{F}_{net}\, dt$
  - ↳ impulse is vector change in linear momentum
  - ↳ $\vec{J} = \vec{F}_{avg}\Delta t = \Delta\vec{p}$

ch 6: work & energy

WORK DONE BY CONSTANT FORCE IN 1D

- preliminary definition of work: $W_F = \pm Fd$
  - ↳ i.e. work done by force on rigid body on frictionless surface is product of constant force F acting parallel to direction of motion & distance d it moves
  - ↳ +ve if force parallel to $\vec{\Delta d}$ is in same dir & -ve if they're opposite dir
  - ↳ if d = 0, no work is done even if force is applied
- work is a scalar quantity
  - ↳ unit is Joule
    - 1 J = 1 N·m

WORK DONE BY CONSTANT FORCE IN 2D AND 3D

- $W_F = F_{\parallel} d = (F\cos\theta)d = F(d\cos\theta)$
  - ↳ work done by constant force that makes angle θ in dir of displacement
- formal definition of work by constant force: $W = \vec{F} \cdot \vec{\Delta d}$

  sign convention for 2D/3D motion:
  - ↳ $0° \le \theta < 90°$ or $270° < \theta < 360°$: work is +ve
  - ↳ $\theta = 90°$ or $\theta = 270°$: work is 0
  - ↳ $90° < \theta < 270°$: work is -ve
- $W_{net} = \vec{F}_{net} \cdot d$

WORK DONE BY VARIABLE FORCES

- work done by force in displacing object from x_1 to x_2 is area under graph of F(x) btwn x_1 & x_2
- $W_F = \int_{r_1}^{r_2} \vec{F}(\vec{r}) \cdot d\vec{r}$

  $= \int_{r_1}^{r_2} F_x(\vec{r})dx + \int_{r_1}^{r_2} F_y(\vec{r})dy + \int_{r_1}^{r_2} F_z(\vec{r})dz$

- force exerted by spring: $\vec{F} = -k\vec{x}$
  - ↳ dir of force is opposite to dir of displacement
  - ↳ type is called restoring force
- work done by spring: $W_{by\text{-}spring} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2$
  - ↳ if $x_1 = 0$ (unstretched position) & $x_2 = x$ (final position), then:
    - $W_{by\text{-}spring} = -\frac{1}{2}kx^2$
    - $W_{on\text{-}spring} = \frac{1}{2}kx^2$

KINETIC ENERGY: WORK-ENERGY THEOREM

- kinetic energy is $K = \frac{1}{2}mv^2$
- work done by force to object's initial & final speeds: $W_F = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

  $= K_2 - K_1$

  $= \Delta K$

- work-energy theorem: work done on an object is equal to change in object's kinetic energy
  - ↳ when +ve work is done, speed increases
  - ↳ when -ve work is done, speed decreases
- $W_{net} = K_f - K_i$
- work done by force of gravity is mgh

CONSERVATIVE FORCES AND POTENTIAL ENERGY

- a force $\vec{F}$ is conservative when net work done by force in moving object over any closed path is 0
  - ↳ $\oint \vec{F}(\vec{r}) \cdot d\vec{r} = 0$
    - $\oint$ means taken over closed path
  - ↳ closed path starts & ends at same point
  - ↳ work done by conservative force in moving object btwn 2 points is independent of path
  - ↳ e.g. gravity & spring are conservative but friction is non-conservative

- potential energy is capacity of object to do work
  - ↳ $\Delta U = -\int_a^b \vec{F}(\vec{r}) \cdot d\vec{r}$
    - work done by conservative force in moving object from a to b (w/no change in its kinetic energy) is equal to change in PE
- gravitational potential energy (GPE) is $\Delta U_g = mg\Delta h$
  - ↳ GPE at certain height h wrt ground level is $U_g = mgh$
  - ↳ assumed that g is constant
- elastic potential energy (EPE) is $\Delta U_s = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$
  - ↳ if start w/ unstretched spring, $U_s = \frac{1}{2}kx_2^2$
  - ↳ valid as long as elastic limit of spring isn't exceeded

CONSERVATION OF MECHANICAL ENERGY

- total work done on object classified into 2 categories: work done by conservative forces (W_c) & by non-conservative forces (W_{nc})
  - ↳ $\Delta K = W_{net} = W_{nc} + W_c$

    $\Delta K = -\Delta U + W_{nc}$

    $\Delta K + \Delta U = W_{nc}$
- mechanical energy is sum of changes in KE & PE of system
  - ↳ rep by E / E_m
- in absence of non-conservative forces, ME is zero
  - ↳ $\Delta K + \Delta U = 0$
- law of conservation of mechanical energy: in absence of non-conservative forces, sum of KE & PE of an object / system remains constant
  - ↳ $K_1 + U_1 = K_2 + U_2 = \text{constant}$

FORCE FROM POTENTIAL ENERGY

- potential energy related to constant conservative force, $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, that displaces object by distance $\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$ is $\Delta U = -\vec{F} \cdot \Delta\vec{r}$
- in 1D, conservative force is -ve of derivative of PE wrt/ displacement
  - ↳ $F_x = -\frac{dU}{dx}$
  - ↳ e.g. elastic restoring force: $F_x = -\frac{dU}{dx}$

    $= -\frac{d}{dx}\left(\frac{1}{2}kx^2\right)$

    $= -kx$

- in 3D, when taking derivative of PE wrt/ x, keep y & z constant
  - ↳ i.e. partial derivative & denoted as $\frac{\partial}{\partial x}$
- $F_x = -\frac{\partial U}{\partial x}$, $F_y = -\frac{\partial U}{\partial y}$, $F_z = -\frac{\partial U}{\partial z}$
  - ↳ define vector differential operator called gradient / del: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$
  - ↳ in compact notation: $\vec{F}(x,y,z) = -\vec{\nabla}U(x,y,z)$
  - ↳ e.g. particle in region has PE given by $U = (3x^2 - 2xy + zx)$ & coeff have units J/m^2; find expression for force acting on particle & determine magnitude of force at A(0.5m, -1.0m, 0.5m)

SOLUTION

$$F_x = -\frac{\partial U}{\partial x} = -\frac{\partial}{\partial x}(3x^2 - 2xy + zx)$$
$$= -(6x - 2y + z)$$
$$= -6x + 2y - z$$
$$F_y = -\frac{\partial U}{\partial y} = -\frac{\partial}{\partial y}(3x^2 - 2xy + zx)$$
$$= -(-2x)$$
$$= 2x$$
$$F_z = -\frac{\partial U}{\partial z} = -\frac{\partial}{\partial z}(3x^2 - 2xy + zx)$$
$$= -(x)$$
$$= -x$$

Subbing in A:

$$\vec{F} = (-6x + 2y - z)\hat{i} + (2x)\hat{j} + (-x)\hat{k}$$
$$= -5.5\hat{i} - 1\hat{j} - 0.5\hat{k}$$
$$F = \sqrt{(-5.5)^2 + (-1)^2 + (-0.5)^2}$$
$$\simeq 5.612$$

$\therefore$ The magnitude of force at A is 5.6N.

POWER

- power is rate at which work is done
  - ↳ $P_{avg} = \frac{\Delta W}{\Delta t}$
- to calculate instantaneous power: $P = \frac{dW}{dt}$
$$P = \frac{d}{dt}(F dx)$$
$$P = F\frac{dx}{dt}$$
$$P = Fv$$
  - ↳ for forces acting in more than 1D: $P = \vec{F} \cdot \vec{v}$
- SI unit for power is watt(W)
  - ↳ $1\ W = 1\ J/s$
  - ↳ another common unit is horsepower
    - $1\ hp = 746\ W$

ch 7: linear momentum, collisions, & systems of particles

LINEAR MOMENTUM

- linear momentum is a conserved quantity
  - ↳ $\vec{p} = m\vec{v}$
  - ↳ units are kg·m/s
- can write kinetic energy in terms of momentum
  - ↳ $K = \frac{1}{2}mv^2$ and $\vec{v} = \frac{\vec{p}}{m}$
    $v^2 = \frac{p^2}{m^2}$
  - ↳ $K = \frac{p^2}{2m}$ or $p = \sqrt{2mK}$
- momentum is vector quantity that depends on velocity while kinetic energy is scalar quantity that depends on speed
- there's only one type of linear momentum + it can't be converted like energy does

RATE OF CHANGE OF LINEAR MOMENTUM AND NEWTON'S LAWS

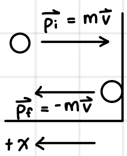
- Newton's second law written in terms of momentum: $\vec{F}_{net} = \frac{d\vec{p}}{dt}$
  - ↳ $\Sigma \vec{F}_{avg} = \frac{\Delta \vec{p}}{\Delta t}$
    - average net force is equal to finite change in momentum over time interval

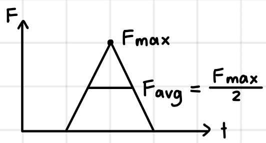
IMPULSE

- impulse is change in momentum
  - ↳ $\vec{I} = \Delta\vec{p} = \int_{t_1}^{t_2} \vec{F}_{net}\, dt = \Sigma \vec{F}_{avg}\, \Delta t$
- linear momentum remains constant when $\Sigma\vec{F} = 0$
- e.g. golf ball of mass 46.0 g strikes wall horizontally at speed 130 km/h + bounces back w/same speed in time of 0.900 ms; find impulse to ball by wall, avg force experienced by ball, plot graph assuming force changes linearly w/time, + estimate max value of force of impact

NOTE: when using linear apx graph for force of impact, $F_{max} = \frac{F_{avg}}{2}$

SOLUTION





$130\,\frac{km}{h} \times \frac{1000\,m}{3600\,s} = 36.111\ m/s$

$\vec{p}_i = (0.046)(-36.111)$
$= -1.661\ kg\cdot m/s\ [\rightarrow]$

$\vec{p}_f = (0.046)(36.111)$
$= 1.661\ kg\cdot m/s\ [\rightarrow]$

$\vec{I} = \vec{p}_f - \vec{p}_i$
$= 1.661 - (-1.661)$
$= 3.322\ kg\cdot m/s\ [\rightarrow]$

$\Sigma\vec{F}_{avg} = \frac{\Delta\vec{p}}{\Delta t} = \frac{3.322}{0.0009}$
$= 3691.111\ N\ [\rightarrow]$

$\vec{F}_{max} = 2(\Sigma\vec{F}_{avg}) = 2(3691.111)$
$= 7382.222\ N\ [\rightarrow]$

NOTE: important to choose coordinate system when finding centre of mass

SYSTEMS OF PARTICLES AND CENTRE OF MASS

- point mass is object w/ non-zero mass but zero size
- centre of mass is point corresponding to avg position of mass in system
  - ↳ mass of system/object usually treated as point mass, located at centre of mass
- position of centre of mass of system of n particles: $x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{M_T}$
  - ↳ M_T is total mass of all particles
- for systems in more than 1D, use $\vec{r}_i$ for each particle: $\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n}{M_T}$
  - ↳ in component form:
    - $x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{M_T}$
    - $y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{M_T}$
    - $z_{cm} = \frac{\sum_{i=1}^{n} m_i z_i}{M_T}$
- centre of mass of continuous distribution of mass: $\vec{r}_{cm} = \frac{\int_V \vec{r}\, dm}{\int_V dm}$
  - ↳ integration limit V means integral taken over volume of object

SYSTEMS OF PARTICLES AND CONSERVATION OF MOMENTUM

- $\vec{p}_{cm} = \vec{p}_1 + \vec{p}_2 + \dots + \vec{p}_n$
  $\frac{d\vec{p}_{cm}}{dt} = \Sigma \vec{F}_{net}$
  - ↳ if $\Sigma \vec{F}_{net} = 0$, acceleration of centre of mass of system is zero

in system of particles, internal forces btwn particles don't contribute to net force
  - ↳ $\frac{d\vec{p}_{cm}}{dt} = \vec{F}_{ext}$
  - ↳ if $\vec{F}_{ext} = 0$, then $\frac{d\vec{p}_{cm}}{dt} = 0$ so $\vec{p}_i = \vec{p}_f$

conservation of momentum states that if mechanical system experiences zero net external force, momentum is conserved
  - ↳ $\vec{p}_i = \vec{p}_f$
  - ↳ if $\vec{F}_{ext} = 0$, centre of mass of system doesn't move

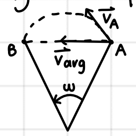
COLLISIONS

- inelastic collision is when colliding objects lose KE
  - ↳ in completely inelastic collision, 2 objects stick together after colliding
    - after collision, $\vec{p}_f = (m_1 + m_2)\vec{v}$ meaning they can be treated as one object
  - ↳ although KE is lost, momentum is conserved
- during elastic collision, both total linear momentum + total KE of system are conserved
  - ↳ $\vec{p}_i = m\vec{v} + M\vec{u}$ and $\vec{p}_f = m\vec{v}' + M\vec{u}'$
    - $\vec{v}$ + $\vec{u}$ rep the 2 masses' velocities
  - ↳ $\vec{u}' = \frac{2m}{m+M}\vec{v} + \frac{M-m}{m+M}\vec{u}$
  - ↳ $\vec{v}' = \frac{m-M}{m+M}\vec{v} + \frac{2M}{m+M}\vec{u}$

ch 8: rotational kinematics & dynamics

ANGULAR VARIABLES

- for particle moving along circumference of circle w/radius r, length of arc is $\Delta s = r\Delta\theta$
  - ↳ $\Delta\theta$ is angular displacement in radians
- differential displacement along circumference is $ds = r d\theta$
  - ↳ tangential speed: $\frac{ds}{dt} = r\frac{d\theta}{dt}$
  - ↳



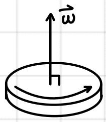
- angular velocity is IROC of angular position wrt time
  - ↳ angular speed equation: $\omega = \frac{d\theta}{dt}$
  - ↳ SI units are rad/s

tangential speed written as function of time: $v(t) = \omega(t) r$

- take derivative of tangential speed to get tangential acceleration: $\frac{d^2s}{dt^2} = r\frac{d^2\theta}{dt^2}$
  - ↳ angular acceleration: $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
- tangential acceleration as function of time: $a_t(t) = \alpha(t) r$
-

	ω = constant	$\omega \neq$ constant
a_t	0	$\alpha r = \frac{d\omega}{dt} r$
a_r	$\frac{v^2}{r} = \omega^2 r$	$\frac{v^2}{r} = \omega^2 r$

- use RHR to find dir of angular velocity: curl fingers in dir of motion of rotating object & thumb will point in correct dir
  - ↳ angular velocity vector is ⊥ to plane of rotation
  - ↳



  - ↳ if angular speed is increasing, angular acceleration is same dir; if it's decreasing, angular acceleration is opp dir

NOTE: to find # of revolutions, $\Delta\theta\left(\frac{1 \text{ rev}}{2\pi}\right)$

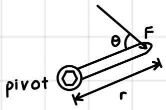
KINEMATIC EQUATIONS FOR ROTATION

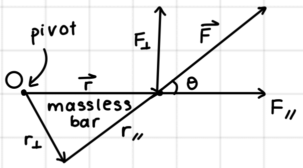
- rotational kinematics equations when angular acceleration is constant are same as kinematics equations developed for constant linear acceleration
-

Angular Quantity	Symbol / Expression
angular position	θ
angular velocity	ω
angular acceleration	α
kinematic equations for constant α	· $\omega_2 = \omega_1 + \alpha\Delta t$ · $\omega_{avg} = \frac{\omega_1 + \omega_2}{2}$ · $\Delta\theta = \omega_1\Delta t + \frac{1}{2}\alpha(\Delta t)^2$ · $\omega_2^2 = \omega_1^2 + 2\alpha\Delta\theta$ · $\Delta\theta = \left(\frac{\omega_1 + \omega_2}{2}\right)\Delta t$

TORQUE

- net torque causes rotation & angular acceleration
- axis of rotation runs through the pivot (centre of rotation)
- torque is measure of effectiveness of force in causing rotations
  - ↳ depends on 3 things.
    - strength of applied force
    - angle θ btwn applied force & arm to point of application of force
    - distance r btwn pivot & point where force is applied
  - ↳ e.g.





  - ↳ torque of force $\vec{F}$ abt pivot O is $\tau_0 = rF\sin\theta$
    - SI units are N·m
  - ↳ moment arm $r_\perp$ (aka torque/lever arm) is $\perp$ distance btwn point of application of force & pivot
    - $\tau = (r\sin\theta)F = r_\perp F$
- torque is cross product btwn 2 vectors: $\vec{\tau}_0 = \vec{r} \times \vec{F}$
- use RHR to find dir of torque
  - ↳ move all vectors to common origin
  - ↳ line up thumb w/axis of rotation
  - ↳ $\vec{r}$ vector is sitting in plane of palm of hand & curl fingers twd $\vec{F}$ so then, thumb is aligned w/ $\vec{\tau}$
- can also use 3-finger RHR to find dir of torque
  - ↳ esp useful when taking cross product of 2 orthogonal vectors
  - ↳ line up thumb w/ $\vec{r}$ & index finger w/ $\vec{F}_\perp$ so middle finger points in dir of $\vec{\tau}$
    - keep all fingers orthogonal

MOMENT OF INERTIA OF A POINT MASS

$\vec{\tau}_{net} = mr^2\vec{\alpha}$
- ↳ rotational equivalent of $\vec{F} = m\vec{a}$
- ↳ resistance to acceleration depends on both mass & distance from rotation axis
- rotational inertia/moment of inertia is $I_0 = mr^2$
  - ↳ SI units are kg·m²
  - ↳ only for point mass
  - ↳ $\vec{\tau}_{net} = I\vec{\alpha}$

SYSTEMS OF PARTICLES AND RIGID BODIES

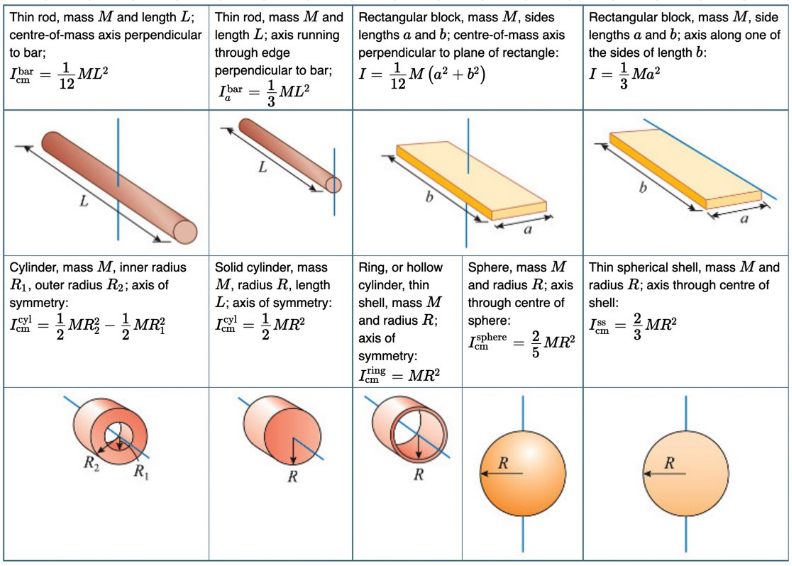
- moment of inertia of 2D system of point masses abt an axis a $\perp$ to plane of particles is equal to sum of individual moments of inertia: $I_a = \sum_{i=1}^{N} m_i r_i^2$
  - ↳ N is # of particles
  - ↳ can be used as long as system is held tgt as rigid body (i.e. non-deformable)
- when object has continuous mass distribution, $I = \int_{object} r^2\,dm$
  - ↳ object is uniform/homogeneous when its properties (e.g. density) don't change over its volume

↳ differential mass is $dm = \rho dV$
- ρ is mass density

· moment of inertia of thin ring: $I_{cm} = MR^2$
· moment of inertia of solid disk: $I_{cm} = \frac{MR^2}{2}$

for composite object, add all moments of inertia for individual parts abt their common axis of rotation

· parallel-axis theorem: $I_a^{object} = I_{cm}^{object} + Md^2$
↳ moment of inertia of an object abt arbitrary axis is equal to moment of inertia of that object abt a parallel axis running through its centre of mass, plus product of total mass & square of distance btwn $axis_{cm}$ & $axis_a$



ROTATIONAL KINETIC ENERGY AND WORK

rotational kinetic energy for point mass: $K_{rot} = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}I_o\omega^2$
↳ rotational translational speed $v = \omega r$

· for a constant torque: $W_T = \tau_{net}\Delta\theta = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$
↳ for varying torque: $W_T = \int \tau_{net}\, d\theta = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$

ANGULAR MOMENTUM

· angular momentum of point mass abt an axis through point O is cross product of position vector of object wrt pivot/axis & linear momentum of object: $\vec{L}_o = \vec{r} \times \vec{p}$
↳ magnitude is $L_o = rp\sin\theta$
$= rmv\sin\theta$

any particle that has linear momentum also has angular momentum abt specified axis/pivot
↳ $\vec{L}_o = 0$ when $\vec{r}$ & $\vec{p}$ are collinear (or either is 0)
↳ particle can have angular momentum wrt axis even when it doesn't rotate around it

· only ⊥ components of linear momentum & position of particle relative to axis/pivot matter

· use RHR or 3-finger used for torque to determine dir of angular momentum
for rotations in 2D: $\vec{L}_o = I_o\vec{\omega}$
↳ angular momentum & angular velocity are in same dir
↳ can apply for any rigid rotating body

· $\vec{\tau}_{net} = \vec{r} \times \vec{F}_{net} = \frac{d\vec{L}}{dt}$
↳ $\vec{\tau}_{net}^{avg} = \frac{\Delta\vec{L}_{net}}{\Delta t}$

- change in linear momentum is angular impulse $\vec{\Delta L}$
  - ↳ $\vec{\Delta L} = \int_{interval} \vec{\tau}_{net}\, dt = \vec{\tau}_{net} \Delta t$
- conservation of angular momentum states that in absence of net external torque, angular momentum won't change

ch 9: rolling motion

ROLLING AND SLIPPING

- rolling is rotation abt an axis that's translating
  - ↳ e.g. rolling pin to flatten dough
- spinning is when rotational speed of surface of object is too high in comparison to translational speed of axis abt which object rotates
  - ↳ e.g. truck trying to emerge from muddy hole & wheels spinning fast while truck moves slowly/doesn't move
- skidding is when translational speed of rolling object is too high compared to rotational speed of its surface
  - ↳ e.g. wheels lock when depress brake pedal so wheels stop rotating while car still moves
- slipping refers to either slipping/skidding
  - ↳ e.g. presence of kinetic friction btwn tire & road
- rolling w/o slipping means object neither spins nor skids as it rolls
  - ↳ i.e. ideal rolling, perfect rolling
  - ↳ e.g. relative speed btwn part of wheel in contact w/road & pavement surface is 0

RELATIONSHIPS BETWEEN ROTATION AND TRANSLATION FOR A ROLLING OBJECT

- consider wheel of radius r rotating in clockwise dir abt fixed axis through centre of symmetry; choose right to be +ve dir so $\vec{v}_{top} = \omega r$, $\vec{v}_{bottom} = -\omega r$, & $\vec{v}_{cm} = 0$
  - ↳ $v_{cm} = v_{bottom} + \omega r$
  - ↳ $v_{cm} = v_{top} - \omega r$
  - ↳ $v_{top} = v_{bottom} + 2\omega r$
- same relationship btwn acceleration of cm & tangential component of acceleration at top & bottom of wheel
  - ↳ $a_{cm} = a_{bottom} + \alpha r$
  - ↳ $a_{cm} = a_{top} - \alpha r$
  - ↳ $a_{top} = a_{bottom} + 2\alpha r$
- above relationships apply regardless of whether rotation axis is stationary/moving

ROLLING MOTION: 2 PERSPECTIVES

- for object that rolls w/o slipping, velocity of part of object in contact w/surface is 0: $v_{bottom} = 0$
  - ↳ $v_{cm} = \omega r$
    - $v_{top} = 2\omega r$
  - ↳ $a_{cm} = \alpha r$
    - $a_{top} = 2\alpha r$, which gives tangential component of acceleration
    - $a_{bottom} = 0$
- rolling is translation of cm combined w/rotation of object abt an axis through cm
  - ↳ for translational dynamics of cm, $\Sigma \vec{F} = m\vec{a}_{cm}$
  - ↳ for rotational dynamics abt cm, $\Sigma \vec{\tau}_{cm} = I_{cm}\vec{\alpha}$

as object rolls, successive points on its circumference will touch surface & each of them will be stationary wrt/surface

- rolling can also be considered as continuous series of rotations abt a contact point, called momentary pivot
  - ↳ label point b as momentary centre of rotation
  - ↳ since b is at rest, sum up torques abt pivot point & apply rotational equivalent of Newton's 2^{nd} law:

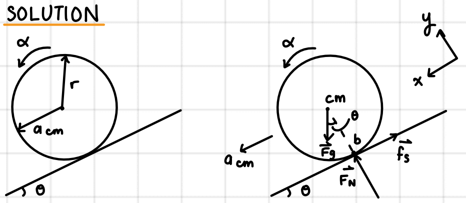
$\boxed{\sum \vec{\tau}_b = I_b \vec{\alpha}}$

- must use I_b rather than I_{cm}

NEWTON'S 2ND LAW AND ROLLING

- e.g. solid uniform disk starts from rest & rolls w/o slipping down ramp at angle θ ; derive expressions for acceleration of cm & force of friction btwn surface of ramp & disk

SOLUTION



Approach A: translation of cm + rotation of object abt cm

↳ translation: $\sum F_x = F_g \sin\theta - f_s$

$$m a_{cmx} = mg\sin\theta - f_s$$

$$f_s = mg\sin\theta - m a_{cmx} \qquad 1)$$

↳ rotation: $\sum \tau_{cm} = r f_s$

$$I_{cm}\alpha = r f_s \qquad 2)$$

- $\vec{F}_N$ & $\vec{F}_g$ exert no torque on cm b/c their lines of action pass through cm ($r = 0$)

↳ $a_{cm} = \alpha r$

$$\alpha = \frac{a_{cm}}{r} \qquad 3)$$

Sub 1) & 3) into 2):

$$I_{cm}\,\alpha = r f_s$$

$$I_{cm}\left(\frac{a_{cm}}{r}\right) = r(mg\sin\theta - m a_{cm})$$

$$I_{cm} a_{cm} = r^2 mg\sin\theta - r^2 m a_{cm}$$

$$a_{cm}(I_{cm} + r^2 m) = r^2 mg\sin\theta$$

$$a_{cm} = \frac{r^2 mg\sin\theta}{I_{cm} + r^2 m} \longrightarrow I_{cm} = \frac{mr^2}{2}$$

$$= \frac{r^2 mg\sin\theta}{\frac{mr^2}{2} + r^2 m}$$

$$= r^2 mg\sin\theta\left(\frac{2}{r^2 m + 2r^2 m}\right)$$

$$= r^2 mg\sin\theta\left(\frac{2}{3r^2 m}\right)$$

$$= \frac{2g\sin\theta}{3}$$

To calculate f_s, sub a_{cm} into 1):

$$f_s = mg\sin\theta - m a_{cmx}$$

$$= mg\sin\theta - m\,\frac{2g\sin\theta}{3}$$

$$= \frac{3mg\sin\theta - 2mg\sin\theta}{3}$$

$$= \frac{mg\sin\theta}{3}$$

As such, $a_{cm} = \frac{2g\sin\theta}{3}$ & $f_s = \frac{mg\sin\theta}{3}$.

NOTE: advantage of approach B is that it eliminates f_s b/c moment arm points straight away from b & is 0

Approach B: successive rotations abt momentary pivot

$$\sum \tau_b = r F_g \sin\theta$$

$$I_b \alpha = rmg\sin\theta$$

↳ $\vec{F}_N$, $\vec{f}_s$, & $\vec{F}_g\cos\theta$ exert no torque on b b/c their lines of action pass through b ($r = 0$)

$$\alpha = \frac{rmg\sin\theta}{I_b}$$

$$a = \alpha r$$

$$= \frac{r^2 mg\sin\theta}{I_b} \longrightarrow I_b = I_{cm} + md^2$$

$$= \frac{r^2 mg\sin\theta}{\frac{3mr^2}{2}} \qquad = \frac{mr^2}{2} + mr^2$$

$$= \frac{2g\sin\theta}{3} \qquad = \frac{3mr^2}{2}$$

MECHANICAL ENERGY AND ROLLING

- using rotation/translation approach:

- $K_{trans} = \frac{1}{2} m v_{cm}^2$
- $K_{rot} = \frac{1}{2} I_{cm} \omega^2$
- total kinetic energy is sum of K_{trans} & K_{rot} : $K = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$

· using momentary-pivot approach:
- $K = \frac{1}{2} I_b \omega^2$

ROLLING WITHOUT FRICTION

· for car wheel rolling down incline, frictional force can be 0 if engine delivers enough torque (τ_e) to accelerate it as required

· free rolling on smooth horizontal surface means there's no frictional force

· if object w/ non-zero angular velocity is thrown on frictionless surface, initial condition of $v_{cm} = \omega r$ maintained indefinitely
- since this is condition for rolling w/o slipping, will roll indefinitely

ROLLING FRICTION

· rolling friction / resistance can be caused by adhesion btwn surface & object or deformation

· rolling friction opposes dir of motion, acting to slow down rolling object
- can be apx quantified using coeff of rolling friction μ_r
- $f_r = \mu_r N$

ch 10: equilibrium

CONDITIONS FOR EQUILIBRIUM

- for point mass to be in equilibrium, net force acting on it must be 0 so acceleration is also 0
  - ↳ if it's stationary, it's in static equilibrium
  - ↳ if it's moving at a constant velocity, it's in dynamic equilibrium
- when object is on verge of tipping, normal force acts at pivot point b/c it'll be only point of contact btwn surface & object
- in order for extended object to be in equilibrium:
  - ↳ vector sum of external forces on object must be 0
    - $\Sigma \vec{F} = 0$
    - i.e. translational equilibrium
    - linear momentum is constant: $\frac{d\vec{p}}{dt} = 0$
  - ↳ vector sum of all external torques acting on object abt any pivot/axis must be 0
    - $\Sigma \vec{\tau} = 0$
    - i.e. rotational equilibrium
    - angular momentum abt any axis is constant: $\frac{d\vec{L}}{dt} = 0$
- for extended object to be in static equilibrium, speed of cm & rotational speed must both be 0
  - ↳ if not, object's in dynamic equilibrium
- classify states of equilibrium by considering what will happen to object if it undergoes small displacements from its equilibrium position
  - ↳ stable equilibrium: object tends to return to its equilibrium position
    - e.g. ball resting at bottom of bowl
  - ↳ unstable equilibrium: small displacement will cause object to leave its equilibrium position
    - e.g. ball resting on top of bowl
  - ↳ neutral equilibrium: if object is displaced, it'll remain in new position
    - e.g ball on horizontal surface

CENTRE OF GRAVITY

- centre of gravity is single point on object where we treat force of gravity to be acting on
- can find object's centre of gravity by suspending it & allowing it to come to static, stable equilibrium
  - ↳ centre of gravity must be directly below point of suspension
- for objects in uniform gravitational field, centre of gravity coincides w/ cm
  - ↳ can treat as if total mass of object is located at centre of mass
  - ↳ torque exerted on continuous mass distribution in presence of uniform gravitational field is $r_{cm} \times (M\vec{g})$

APPLYING CONDITIONS FOR EQUILIBRIUM

- guidelines when approaching equilibrium problems:
  - ↳ look for points on object where single unknown force produces torque, then sum torques abt that point to find unknown force
  - ↳ remember torque equilibrium condition valid for any point on object & can be used multiple times in any solution
  - ↳ monitor if there's a single unknown force/force component in any dir b/c summing forces in that dir will immediately give value for unknown

NOTE: incorrect choice for dir of unknown force doesn't affect magnitude of force

ch 13: oscillations

NOTE: can also use sine function since $\cos(\theta - \frac{\pi}{2}) = \sin\theta$

PERIODIC MOTION

- periodic / harmonic motion is motion that repeats after finite amount of time
  - ↳ period of motion is amount of time after which motion repeats
    - denoted by T

when object repeats motion periodically, both displacement & velocity of object will be same at start & end of any interval of T seconds
- ↳ $x(t + nT) = x(t)$, $n \in \mathbb{Z}$
- ↳ $v(t + nT) = v(t)$, $n \in \mathbb{Z}$

- oscillation is one complete back-and-forth motion
- frequency f is # of oscillations completed in 1s
  - ↳ $f = \frac{1}{T}$
  - ↳ SI units are hertz ($1\ Hz = 1\ s^{-1}$)

SIMPLE HARMONIC MOTION

- when displacement of oscillating object from equilibrium position varies w/time as cosine (or sine) function, periodic motion called simple harmonic motion
  - ↳ object undergoing simple harmonic motion is simple harmonic oscillator
- $x(t) = A\cos(\omega t + \phi)$
  - ↳ A is amplitude
    - +ve # & describes max displacement of oscillator from its equilibrium
    - constant & doesn't change w/time
    - SI units are metres (m)
  - ↳ ω is angular frequency
    - $\omega = 2\pi f = \frac{2\pi}{T}$
    - measured in rad/s but radians are dimensionless so simply use s^{-1}
  - ↳ ϕ is phase constant
    - angle in rad determined by position & velocity of motion at given time (usually $t=0$)

- velocity of simple harmonic oscillator is $v(t) = -\omega A\sin(\omega t + \phi)$
  - ↳ $v_{max} = \omega A$
  - ↳ in displacement vs time graph, dir of velocity is given by slope of graph at that time

$x(t)$ & $v(t)$ are $\frac{\pi}{2}$ rad out of phase
- ↳ $v(t) = \omega A\cos(\omega t + \phi + \frac{\pi}{2})$
- ↳ phase of $v(t)$ - phase of $x(t)$ = $(\omega t + \phi + \frac{\pi}{2}) - (\omega t + \phi) = \frac{\pi}{2}$
- ↳ when $x = \pm A$, then $v = 0$; when $x = 0$, then $v = \pm\omega A$

- acceleration of simple harmonic oscillator is $a(t) = -\omega^2 A\cos(\omega t + \phi)$
  - ↳ $a_{max} = \omega^2 A$
  - ↳ $a(t) = -\omega^2 x(t)$
    - for motion to be simple harmonic motion, acceleration of object must be proportional to its displacement from equilibrium position & opp in dir at all times

$x(t)$ & $a(t)$ are π rad out of phase
- ↳ $a(t) = \omega A\cos(\omega t + \phi + \pi)$
- ↳ phase of $a(t)$ - phase of $x(t)$ = $(\omega t + \phi + \pi) - (\omega t + \phi) = \pi$

- as object executes simple harmonic motion, acceleration changes w/time so there must be time-dependent force acting on object
  - ↳ $F(t) = ma(t)$
    $= -m\omega^2 x(t)$
    - since $m\omega^2$ is constant, then $F(t) \propto -x(t)$ (known as Hooke's law)
- simple harmonic motion is motion by object that's subjected to restoring force
  - ↳ restoring force causes object to return to equilibrium position
- to determine whether object will undergo simple harmonic motion under influence of external force:
  1) displace object from equilibrium position by small displacement x
  2) calculate net force F on object at displaced position
  3) determine whether F can be written as $F = -\text{constant} \cdot x$
     - if so, $\text{constant} = m\omega^2$
       $\omega = \sqrt{\frac{\text{constant}}{m}}$

UNIFORM CIRCULAR MOTION AND SHM

Uniform Circular Motion	Simple Harmonic Motion
Radius: R	Amplitude: A
Angular speed: ω	Angular frequency: ω
1 complete revolution	1 complete oscillation
Time taken for 1 complete revolution: $\frac{2\pi}{\omega}$	Time period for 1 complete oscillation: $\frac{2\pi}{\omega}$
Starting angle: ϕ	Phase constant: ϕ

MASS-SPRING SYSTEMS

- for a horizontal mass-spring system, gravity doesn't play any role in motion
  - ↳ $\omega = \sqrt{\frac{k}{m}}$
  - ↳ $T = 2\pi\sqrt{\frac{m}{k}}$ & $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$
    - frequency of oscillation is directly proportional to stiffness of spring (k) & inversely proportional to mass
    - light mass attached to stiff spring has large frequency & small period
- frequency of oscillation is independent of amplitude of motion

- for mass suspended vertically from spring of unstretched length L_0, at equilibrium, spring stretches by length ΔL st elastic restoring force on mass balances force of gravity
  - ↳ $\Delta L = \frac{mg}{k}$
- vertical mass-spring system also undergoes SHM
  - ↳ can use same equations for angular frequency of oscillation as horizontal system

displaced harmonic oscillator is when origin of coordinate system isn't located at equilibrium

ENERGY CONSERVATION IN SHM

in absence of frictional forces, total energy of oscillating horizontal mass-spring system is sum of K & U_s

- when mass is at max compression $x = A$, $E = K + U_s$
  $= 0 + \frac{1}{2}kA^2$
  $= \frac{1}{2}kA^2$
- energy at some time t is $E = \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t)$
- $\frac{1}{2}kA^2 = \frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t)$
  - ↳ velocity for given position x is $v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$
  - ↳ position for given velocity v is $x = \pm\sqrt{A^2 - \frac{m}{k}v^2}$
- mass moves w/max speed at equilibrium b/c $U_s = 0$

↳ $E = \frac{1}{2}mv_m^2$

$\frac{1}{2}kA^2 = \frac{1}{2}mv_m^2$

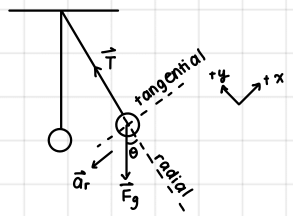
$v_m = \sqrt{\frac{k}{m}}\, A = \omega A$

- $U_s = \frac{1}{2}kx^2(t) = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$
- $K = \frac{1}{2}mv^2(t) = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi)$

  $= \frac{1}{2}kA^2\sin^2(\omega t + \phi)$
- total energy of mass-spring oscillator is proportional to square of its amplitude

SIMPLE PENDULUM

- simple pendulum consists of point mass attached to string of negligible mass that doesn't stretch
  - ↳ when displaced from vertical equilibrium, swings back & forth w/gravity as restoring force



radial axis: $ma_r = T - F_g\cos\theta$

$\frac{mv^2}{L} = T - mg\cos\theta$

$T = \frac{mv^2}{L} + mg\cos\theta$

- tangential axis: $ma_t = -F_g\sin\theta$

  $a_t = -g\sin\theta$

  $a_t = -g\sin\left(\frac{s}{L}\right)$

  - ↳ s is length of arc travelled so $s = L\theta$
  - ↳ in general, pendulum motion isn't SHM since a_t isn't proportional to s
    - use small-angle apx (reasonable for ~0.2 rad), $a_t = -\left(\frac{g}{L}\right)s$
- for small angular displacements, motion of pendulum is SHM
  - ↳ $\omega = \sqrt{\frac{g}{L}}$
  - ↳ $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$
  - ↳ $f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$

- at max angular displacements ($\pm\theta_m$), pendulum is momentarily stationary & its energy is all GPE
  - ↳ $E = mgL(1-\cos\theta_m)$
- at equilibrium, all energy is kinetic so $\frac{1}{2}mv_{max}^2 = mgL(1-\cos\theta_m)$
  - ↳ max speed is independent of mass
- $E = K + U_g$

  $mgL(1-\cos\theta_m) = \frac{1}{2}mv^2 + mgL(1-\cos\theta)$

  $v^2 = 2gL(\cos\theta - \cos\theta_m)$

  - ↳ v is 0 at turning points ($\theta = \pm\theta_m$) & max at $\theta = 0$
- relationship btwn period & length of pendulum string is $T \propto \sqrt{L}$

PHYSICAL PENDULUM

- physical pendulum is extended object displaced from equilibrium & oscillates abt axis
- magnitude of torque is $\tau = -(Mg)(L_{cm})\sin\theta$
  - ↳ $L_{cm} = L + R$ is distance btwn pivot point & cm of rod-sphere pendulum
  - ↳ $I_p\alpha = \tau$

    $\alpha = -\left(\frac{MgL_{cm}}{I_p}\right)\sin\theta$

    - by parallel-axis theorem, $I_p = I_{cm} + M(L_{cm})^2$

- for small angular displacements, use apx $\sin\theta = \theta$ so $\alpha = -\left(\frac{MgL_{cm}}{I_p}\right)\theta$
  - ↳ sphere undergoes SHM
  - ↳ $\omega = \sqrt{\frac{MgL_{cm}}{I_p}}$
  - ↳ $T = 2\pi\sqrt{\frac{I_p}{MgL_{cm}}}$
- angular displacement of oscillating sphere is $\theta(t) = \theta_m \cos(\omega t + \phi)$
- physical & simple pendulum have same time period when $L = \frac{I_p}{ML_{cm}}$