# chapter 1

## FUNCTIONS, VARIABLES, AND CONSTANTS

- in calculating fields, use functions to describe how magnitude of field changes w/distance
- variables used to describe pos wrt origin
- conventions are:
  - ↳ origin of coordinate system is O
  - ↳ arbitrary point on physical object is A
  - ↳ point in space where observations are made is P
  - ↳ variable r used to denote distance btwn 2 points
- physical dimensions are constants
  - ↳ e.g. R is radius of sphere

## VISUALIZATION

- use graphs to visualize how field amplitudes vary in space
  - ↳ plot independent variable on horizontal axis
  - ↳ plot dependent variable on vertical axis

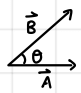
## VECTORS AND UNIT VECTORS

fields are vector elements (i.e. has magnitude & dir)

- ↳ vector $\vec{A}$ has magnitude ($|\vec{A}|$ or $A$) & unit vector ($\hat{a}$)
  - $\hat{a} = \frac{\vec{A}}{A}$
- position vector $\vec{r}$ defines distance & dir btwn 2 points
  - ↳ using variable b/c may want to move one point arbitrarily to diff points

## VECTOR PRODUCTS: DOT PRODUCT

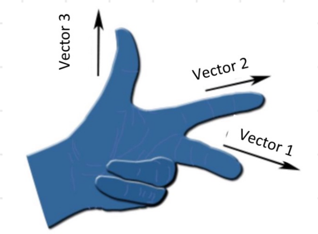
- when 2 vectors are multiplied, result is either scalar/vector
- dot/scalar product of 2 vectors $\vec{A}$ & $\vec{B}$ is defined geometrically as:




$\vec{A} \cdot \vec{B} = AB\cos\theta$

- ↳ commutative · $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- ↳ distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- magnitude of vector can be calculated by taking dot product w/itself
  - ↳ $\vec{A} \cdot \vec{A} = A^2$

## CO-ORDINATE SYSTEMS




- co-ordinate systems allow us to define point uniquely w/ 3 vectors
  - ↳ 3 vectors must be orthogonal to each other (i.e. mutually perpendicular)
  - ↳ when moving in one dir, don't move in other 2 dirs
- use left-hand thumb rule where middle finger, index finger, & thumb rep orthogonal axes

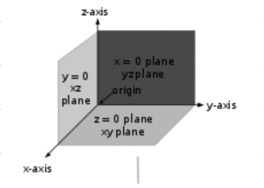
## CARTESIAN COORDINATES

- defined by 3 vectors w/dirs given by $\hat{x}$, $\hat{y}$, & $\hat{z}$

$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

- ↳ $A_x$ is distance moved in $\hat{x}$-dir & same for $A_y$ & $A_z$

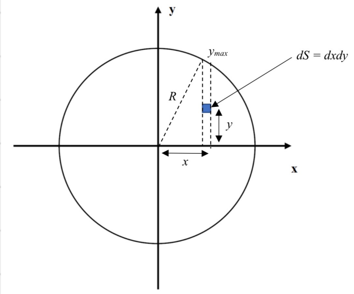
- physical object is defined by 3 planes




- to define arbitrary shapes, use differential elements (i.e. tiny step in one axis of coordinate system)
  - ↳ 3 differential elements in length : $dx$, $dy$, & $dz$
- differential length element in arbitrary dir is $\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$
- differential surface elements ($dS$) are $dx\,dy$, $dy\,dz$, & $dx\,dz$
  - ↳ for $\vec{dS}$, magnitude is area of element & dir is unit vector normal to surface
    - $\vec{dS}_z = dx\,dy\,\hat{z}$
    - $\vec{dS}_x = dy\,dz\,\hat{x}$
    - $\vec{dS}_y = dx\,dz\,\hat{y}$
  - ↳ for arbitrary surface shape, integrate differential elements over their individual planes
- differential volume element is $dv = dx\,dy\,dz$

## POLAR COORDINATES

- to find area of circle using differential elements & integration in Cartesian coordinates, use $dS = dx\,dy$




**NOTE**

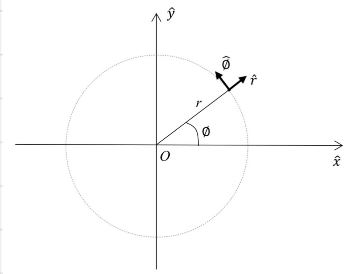
- in double integrals, solve integral whose variable depends on other (i.e. limits are dependent)

  - ↳ if we change $x$ from 0 to R, $y$ can only change from 0 to $\sqrt{R^2 - x^2}$
  - ↳ use double integrals to solve: $A = 4\int_{x=0}^{x=R}\int_{y=0}^{y=\sqrt{R^2-x^2}} dy\,dx$

$$= 4\int_{x=0}^{R}\sqrt{R^2-x^2}\,dx$$

(use sub $x = R\sin\theta$ to evaluate)

- polar coordinates use compass strategy to draw circle
  - ↳ put radial force on compass to pull it out by magnitude $r$
  - ↳ put perpendicular force at end of line to change angle $\varphi$ it makes w/ $x$-axis




**NOTE**

- unlike Cartesian, if origin is changed, a diff circle is created & we have to give coordinate axes diff names

**NOTE**

- to convert from polar to Cartesian coordinates:
  - ↳ $x = r\cos\varphi$
  - ↳ $y = r\sin\varphi$

  - ↳ 2 dir vectors :
    - $\hat{r}$ changes radius
    - $\hat{\varphi}$ changes angle $\varphi$ ccw (aka azimuthal vector)

2 differential elements are $dr$ in $\hat{r}$-dir & $r\,d\varphi$ in $\varphi$-dir

· to calculate circumference, $C = \int_{\varphi=0}^{2\pi} r\,d\varphi$

$= r\int_0^{2\pi} d\varphi$

$= r[\varphi]_0^{2\pi}$

$= 2\pi r$

· surface area element is multiplication of 2 orthogonal length elements so area of circle can be calculated as follows: $A = \int_{r=0}^{R}\int_{\varphi=0}^{2\pi} r\,d\varphi\,dr$

$= \int_{r=0}^{R} r\left(\int_{\varphi=0}^{2\pi} d\varphi\right) dr$

$= \int_0^R r[\varphi]_0^{2\pi}\,dr$

$= \int_0^R 2\pi r\,dr$

$= [\pi r^2]_0^R$

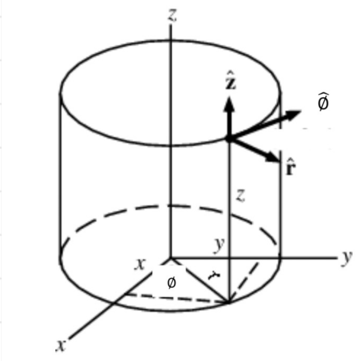
$= \pi R^2$

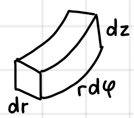
· summary table of diff elements of polar coordinates

| UNIT VECTORS | DIFFERENTIAL LENGTH ELEMENTS | DIFFERENTIAL AREA ELEMENT |
|---|---|---|
| ↳ $\hat{r}$ to change radius<br>↳ $\hat{\varphi}$ to change angle line makes w/ x-axis | ↳ dr (small step in radial dir)<br>↳ $r\,d\varphi$ (arc length when line is rotated by small angle $d\varphi$) | ↳ $dS = r\,dr\,d\varphi$ |

## CYLINDRICAL COORDINATES

· to convert polar coordinates to 3D, make a cylinder by adding 3rd differential length dz for height




↳ follow left-hand rule so order's important
- $\hat{r}$ is 1st vector (points radially outward)
- $\hat{\varphi}$ is 2nd vector (points along angle r makes w/ x-axis)
- $\hat{z}$ is 3rd vector (changes height)

· properties of cylindrical coordinates:

| VECTORS | DIFFERENTIAL LENGTH ELEMENTS | DIFFERENTIAL SA ELEMENTS | DIFFERENTIAL VOLUME ELEMENT |
|---|---|---|---|
| ↳ $\hat{r}$<br>↳ $\hat{\varphi}$<br>↳ $\hat{z}$ | ↳ dr (along $\hat{r}$-dir)<br>↳ $r\,d\varphi$ (along $\hat{\varphi}$-dir)<br>↳ dz (along $\hat{z}$-dir) | ↳ $r\,dr\,d\varphi$ (creates disk of cylinder, normal vector is $\hat{z}$)<br>↳ $r\,d\varphi\,dz$ (curved surface of cylinder, normal vector is $\hat{r}$)<br>↳ $dr\,dz$ (flat surface if cylinder is sliced through centre, normal vector is $\hat{\varphi}$) | ↳ $dv = r\,dr\,d\varphi\,dz$ |

· surface area of curved surface of cylinder w/ radius R & length L: $SA = \int_{z=0}^{L}\int_{\varphi=0}^{2\pi} R\,d\varphi\,dz$

$= \int_0^L R(2\pi)\,dz$

$= 2\pi R[z]_0^L$

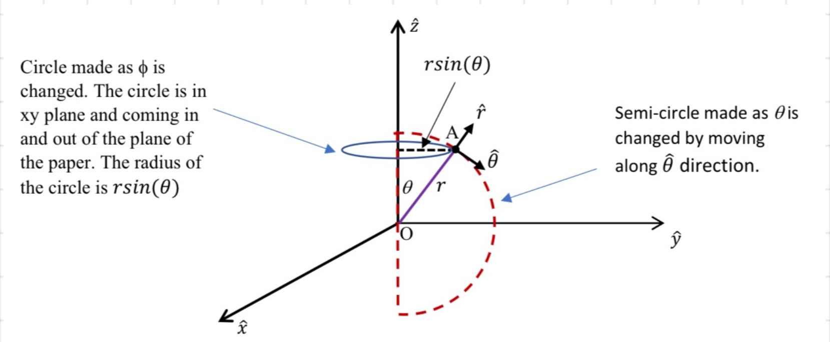
$= 2\pi RL$

- to calculate volume: $V = \int_{z=0}^{L} \int_{\varphi=0}^{2\pi} \int_{r=0}^{R} r\,dr\,d\varphi\,dz$

$$= \int_{z=0}^{L} \int_{\varphi=0}^{2\pi} \left[\tfrac{1}{2}r^2\right]_0^R d\varphi\,dz$$
$$= \int_{z=0}^{L} \int_{\varphi=0}^{2\pi} \tfrac{1}{2}R^2\,d\varphi\,dz$$
$$= \int_0^L \tfrac{1}{2}R^2(2\pi)\,dz$$
$$= \int_0^L \pi R^2\,dz$$
$$= \pi R^2 L$$

## SPHERICAL COORDINATES

- to make sphere, rotate disk made in xy-plane within polar coordinates & rotate it abt x-axis by $\pi$




**NOTE**
- in spherical coordinates, r is distance of point from origin
- in cylindrical coordinates, r is perpendicular distance of point from z-axis

↳ make z-axis principle axis of symmetry

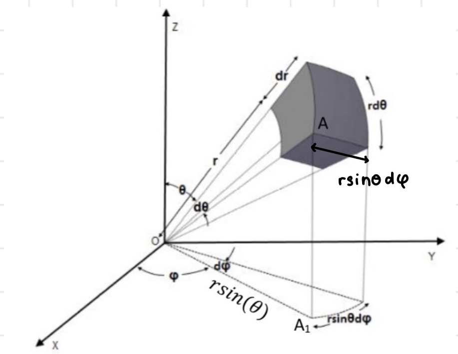
↳ use radial vector $\hat{r}$ to change line of radius r

↳ use polar vector $\hat{\theta}$ to change angle of line wrt z-axis

↳ use azimuthal vector $\hat{\varphi}$ to rotate line in xy-plane & create circle

↳ in summary:

1) vary r to create line of length R(radius)
2) vary $\theta$ to create vertical semi-circle
3) vary $\varphi$ to rotate semi-circle & create sphere




- project point A onto xy-plane as $A_1$ so $\Delta OAA_1$ is right-angled triangle
- $OA_1$ is parallel to line making circle in xy-plane & has length $r\sin\theta$
- $\varphi$ is angle $OA_1$ makes w/ x-axis

- using LHR, vectors in order are $\hat{r}$, $\hat{\theta}$, & $\hat{\varphi}$
- differential length elements are:
  - ↳ $dr$ in $\hat{r}$-dir
  - ↳ $r\,d\theta$ (arc length) in $\hat{\theta}$-dir
  - ↳ $r\sin\theta\,d\varphi$ (arc length) in $\hat{\varphi}$-dir
- main SA element of interest is one that creates spherical surface
  - ↳ radius doesn't change
- summary table of elements in spherical coordinate system:

| VECTORS | DIFFERENTIAL LENGTH ELEMENTS | LIMITS TO MAKE FULL SPHERE OF RADIUS R | SA ELEMENT ON SPHERICAL SURFACE | VOLUME ELEMENT |
|---|---|---|---|---|
| ↳ $\hat{r}$ (radial) | ↳ $dr$ in $\hat{r}$-dir | ↳ $r$: 0 to R | ↳ $d\vec{S} = r^2\sin\theta\,d\theta\,d\varphi\,\hat{r}$ | ↳ $dV = r^2 dr\sin\theta\,d\theta\,d\varphi$ |
| ↳ $\hat{\theta}$ (polar) | ↳ $r\,d\theta$ in $\hat{\theta}$-dir | ↳ $\theta$: 0 to $\pi$ | ∘ normal is $\hat{r}$ | |
| ↳ $\hat{\varphi}$ (azimuthal) | ↳ $r\sin\theta\,d\varphi$ in $\hat{\varphi}$-dir | ↳ $\varphi$: 0 to $2\pi$ | | |

- to calculate SA of sphere: $SA = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} R^2 \sin\theta \, d\theta \, d\varphi$

$$= R^2 \int_{\varphi=0}^{2\pi} [-\cos\theta]_0^{\pi} \, d\varphi$$
$$= R^2 \int_0^{2\pi} (1-(-1))\, d\varphi$$
$$= R^2 \int_0^{2\pi} 2 \, d\varphi$$
$$= 2R^2 [\varphi]_0^{2\pi}$$
$$= 4\pi R^2$$

- to calculate volume: $V = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^2 \, dr \sin\theta \, d\theta \, d\varphi$

$$= \int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{r=0}^{R} r^2 dr$$
$$= [\varphi]_0^{2\pi} [-\cos\theta]_0^{\pi} [\tfrac{1}{3}r^3]_0^R$$
$$= 2\pi(2)(\tfrac{1}{3}R^3)$$
$$= \frac{4}{3}\pi R^3$$

**TIP**

- if elements of double / triple integral are independent of each other, treat as multiplication of single integrals

## IMPORTANT INTEGRALS

- for integrals $\int_a^b \sin^2(\theta)\, d\theta$ & $\int_a^b \cos^2(\theta)\, d\theta$ , if $b-a = m\pi$ where $m \in \mathbb{Z}$ , then value of integrals is $\frac{m\pi}{2}$
- for $\int_a^b \sin(\theta)\cos(\theta)\, d\theta$ , if $b-a = m\pi$ where $m \in \mathbb{Z}$, then value of integral is 0

# chapter 2

## INTRODUCTION

- triboelectric effect: when 2 objects are rubbed against each other, small amounts of charges are transferred from one object to another
- triboelectric series gives tendency of material to acquire +ve charge
  - ↳ if higher in series, more likely to lose $e^-$ & become +ve
  - ↳ e.g. what happens when glass rod is rubbed w/ silk?

    SOLUTION

    Glass is higher in triboelectric series so it loses $e^-$ to become +ve & silk becomes -ve.


**NOTE**
- if objects are closer tgt in series, less $e^-$ being transferred

## CONSERVATION OF CHARGE

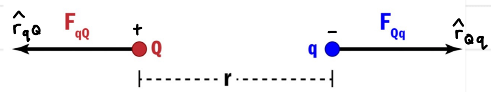
- total charge in isolated system remains the same
  - ↳ cannot create / destroy charge
  - ↳ can distribute / transfer to create particles & localized areas w/ varying charges
  - ↳ i.e. if charge Q is taken out of object st it becomes -Q, then +Q must go somewhere

## QUANTIZATION OF CHARGE

- atom: nucleus containing protons & neutrons w/ surrounding electron cloud
- units for magnitude of charge are Coulombs (C)
- e is called elementary charge
  - ↳ neutron: $q_n = 0$
  - ↳ electron: $q_e = -e = -1.602 \cdot 10^{-19}$ C
  - ↳ proton: $q_p = +e = +1.602 \cdot 10^{-19}$ C
- although $p^+$ & $e^-$ have same charge magnitudes, their masses are different
  - ↳ $m_p \simeq m_n \simeq 1.7 \cdot 10^{-27}$ kg
  - ↳ $m_e \simeq 9.1 \cdot 10^{-31}$ kg

## COULOMB'S LAW

- interaction btwn charges (i.e. forces btwn charges) obeys Coulomb's Law & defines electrostatic forces (i.e. electric force btwn stationary charges)




- shown above, 2 point charges are separated by distance
  - ↳ point charges have radii approaching 0
- Coulomb's law states:
  - ↳ force exerted by q on Q is $\vec{F}_{qQ} = \frac{kQq}{r^2}\hat{r}_{qQ}$

- $q$ & $Q$ are in Coulombs (C)
- $r$ is in meters (m)
- $F$ is in Newtons (N)

↳ force exerted by $Q$ on $q$ is $\vec{F}_{Qq} = \frac{kQq}{r^2}\hat{r}_{Qq} = -\vec{F}_{qQ}$

- $\hat{r}$ is position vector that gives dir of object acting on another object
- establish all position vectors $\hat{r}$ (always pointing twd object) first, then calculate force
  - ↳ actual dir of force depends on charges
    - if $\vec{F}$ is -ve, dir of force is in opp dir of $\hat{r}$
- Coulomb's constant : $k = \frac{1}{4\pi\epsilon_0} \simeq 8.99 \times 10^9 \ Nm^2/C^2$
  - ↳ $\epsilon_0 = 8.854 \times 10^{-12} \ m^{-3} kg^{-1} s^4 A^2$ is called permittivity of free space (universal constant)
    - free space means space that has no induced charges
    - $k$ will change if medium btwn charges change but $\epsilon_0$ won't

**TIP**

- dir of force is only opp of established $\hat{r}$-dir when one of the charges is -ve (i.e. they attract)

- +ve & -ve values we get for forces in mathematical rep of Coulomb's Law has physical meaning
  - ↳ +ve : repulsive force
  - ↳ -ve : attractive force

electrostatic forces are much larger than gravitational forces on Earth

- when there's more than 2 charges, to calculate net electrostatic force, use principle of superposition
  - ↳ add up individual vector forces on charge

## LIMITATIONS OF COULOMB'S LAW

- conditions when applying Coulomb's Law:
  - ↳ charged bodies must be very small compared to $r$
  - ↳ force must be inversely proportional to $r^2$
    - error of inverse-square law is $10^{-16}$
- Coulomb's Law has been experimentally verified w/ distance $r$ from $10^{-16}$ m to $10^8$ m
- difficult to use in dynamic situations

# chapter 3

## INTRODUCTION

- vector field : function that assigns vector value (magnitude & dir) to every point in space
- instead of calculating force btwn 2 charges w/ action at a distance perspective, say charge creates field around it & interaction happens through field

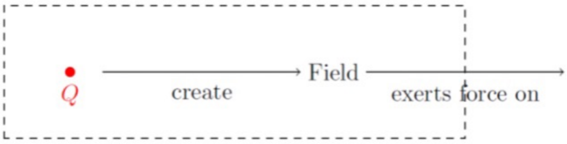



*Figure 3. Q creates an Electric field everywhere around it. Charge "q" interacts with this field at its position and experiences a force due to the interaction.*

## ELECTRIC FIELD DEFINITION

- looking at point charge Q in space, define E-field felt at point P, which is $\vec{r}$ away from Q, as force per +ve unit test charge
  - ↳ assume test charge is extremely small (negligible) so $\vec{E}(\vec{r}) = \lim_{q \to 0} \frac{\vec{F}_Q}{q}$
  - ↳ for point charge Q, E-field at point P which is $\vec{r}$ away from Q is given by $\vec{E}(\vec{r}) = \frac{F_Q}{q} = \frac{kQ}{r^2}\hat{r} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$ [N/C]
    - units are Newton / Coulomb
- electric force on another charge $Q_2$ placed at point P ($\vec{r}$ away from Q) is $\vec{F} = Q\vec{E}(\vec{r})$
- test charge can be +ve/-ve but E-fields are always defined wrt +ve test charge
  - ↳ i.e. E-field lines point in dir of force on +ve test charge

## ELECTRIC FIELD DUE TO MULTIPLE POINT CHARGES

- if there's more than 1 charge, E-field at point is superposition of E-field due to individual charges at that point
  - ↳ $\vec{E}_{tot} = \sum_{n=1}^{N} \vec{E}_n = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_N$
    - total of N charges

to calculate E-field for a charged region w/ continuous charge distributions, divide it into small charge elements dQ
  - ↳ each dQ produces E-field $d\vec{E} = \frac{k\,dQ}{r^2}\hat{r}$

- to determine dQ, need to work w/ charge densities ($\rho$)
  - ↳ 1D : $\rho_\ell$ is linear density so $dQ = \rho_\ell \, d\ell$
    - constant : $\rho_\ell = \frac{Q}{L}$
    - straight line : $dL = dx$
    - circular : $dL = r\,d\varphi$
  - ↳ 2D : $\rho_s$ is surface/area density so $dQ = \rho_s \, dS$
    - constant : $\rho_s = \frac{Q}{S}$
    - Cartesian : $dS = dx\,dy$
    - polar : $dS = r\,dr\,d\varphi$
    - cylindrical : $dS = r\,d\varphi\,dz$
    - spherical : $dS = r^2 \sin\theta \, d\theta \, d\varphi$
  - ↳ 3D : $\rho_v$ is volume density so $dQ = \rho_v \, dV$
    - constant : $\rho_v = \frac{Q}{V}$
    - Cartesian : $dV = dx\,dy\,dz$

- cylindrical: $dV = r\,dr\,d\varphi\,dz$
- spherical: $dV = r^2\,dr\,\sin\theta\,d\theta\,d\varphi$

· since there's infinite dQs, vector sum of all $d\vec{E}$s is $\vec{E} = \int d\vec{E} = \int \frac{kdQ}{r^2}\,\hat{r}$

## ELECTRIC FIELD VISUALIZATION

· to draw E-field lines, can use vectors or continuous field lines

**NOTE**

· E-field lines never intersect b/c the line is pointing in both dir at the same spot at the same time

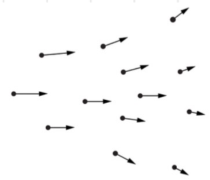


*Figure 6. A vector field may be represented by drawing a set of arrows whose magnitude and directions indicate the value of the vectors field at the points where the arrows are drawn.*

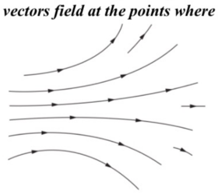


*Figure 7. A vector field may also be represented by lines which are tangent to the direction of the field vector at every point and whose density is proportional to the strength of the field.*

↳ w/ continuous field lines, field is stronger where lines are closer tgt

· to draw E-field lines, typically, 8 lines correspond to 1 charge of magnitude q

· dir of lines depend on polarity of charge Q

↳ if Q is +ve, lines point outward

↳ if Q is -ve, lines point inward

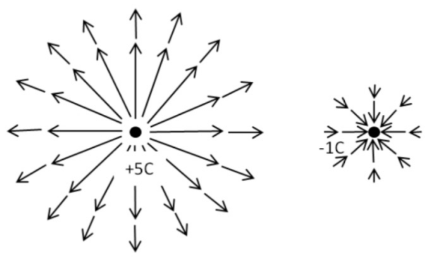


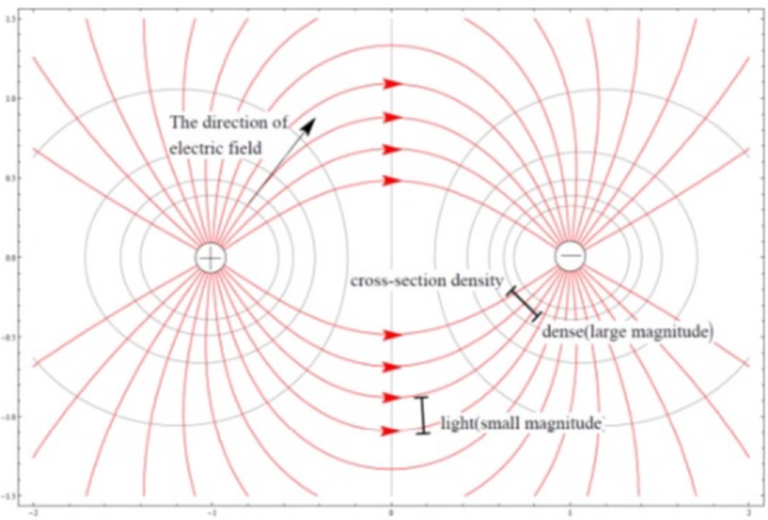

*Figure 8. Electric field distributions for +5 C and -1C charge.*

**NOTE**

· vector lengths rep field strength

· E-field = 0 when lines from charges all diverge away from each other

· electric dipole: 2 charges that are of equal value but opp polarity & separated by small distance

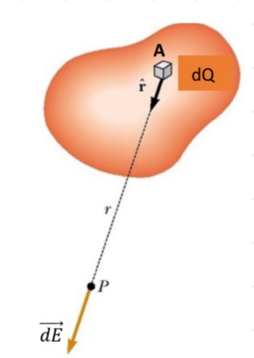


# chapter 4

## CHARGE DENSITY

- charge density: charge per unit length, surface area, or volume
  - ↳ can be treated like mass density except it can be -ve to rep -ve charge
- linear charge density: charge distributed over line that's infinitely thin (denoted as $\rho_\ell$)
  - ↳ $dQ = \rho_\ell \, d\ell$
    $Q = \int_L \rho_\ell \, d\ell$
  - ↳ if $\rho_\ell = \rho_0$ (i.e. constant), then $Q = \rho_0 L$
    - $\rho_0 = \frac{Q}{L}$
- surface charge density: charge distributed over a surface (denoted as $\rho_s$)
  - ↳ $dQ = \rho_s \, dS$
    $Q = \int_S \rho_s \, dS$
  - ↳ if $\rho_s = \rho_0$ (i.e. constant), then $Q = \rho_0 S$
    - $\rho_0 = \frac{Q}{S}$
- volume charge density: charge distributed over a volume (denoted as $\rho_v$)
  - ↳ $dQ = \rho_v \, dv$
    $Q = \int_v \rho_v \, dv$
  - ↳ if $\rho_v = \rho_0$ (i.e. constant), then $Q = \rho_0 v$
    - $\rho_0 = \frac{Q}{v}$

## CALCULATION OF ELECTRIC FIELD FROM DISTRIBUTED CHARGES




↳ using volume charge distribution as example, same applies for linear & surface
- only need to change dQ element

- from above diagram:
  - ↳ $dQ = \rho_v \, dv$
  - ↳ r is distance & $\hat{r}$ is unit vector from A to P
    - $\hat{r}$ is not same in previous coordinate systems where their radial vectors will have diff names (e.g. $r_1$)
  - ↳ $d\vec{E} = \frac{k\,dQ}{r^2}\hat{r}$
    $= \frac{k\rho_v\,dv}{r^2}\hat{r}$
  - ↳ using principle of superposition: $\vec{E} = \int d\vec{E} = \int \frac{k\,dQ}{r^2}\hat{r}$

## ALGORITHMIC STEPS IN CALCULATING ELECTRIC FIELD

0) calculate charge density from charge distribution
   ↳ in some problems, may be given
1) choose coordinate system & place origin appropriately
2) go to arbitrary point A on charge distribution ; rep A & differential element at A in terms of coordinate system (i.e. find dQ)

3) write $d\vec{E}$ in terms of $\rho$ & $\hat{r}$

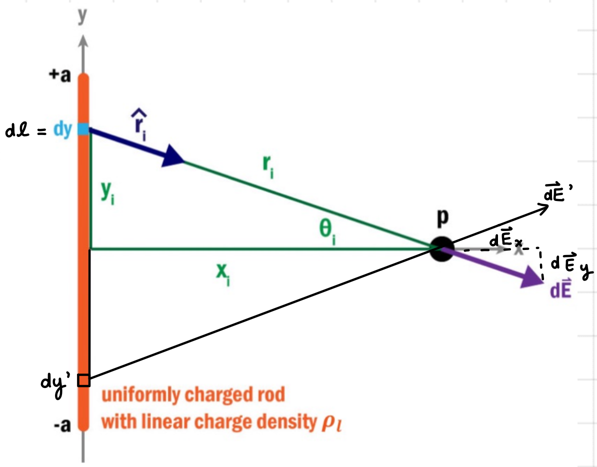
↳ consider its components ($dE_x$, $dE_y$, $dE_z$)

4) look for symmetry & see if any of the components cancel out

5) integrate remaining $d\vec{E}$ to find $\vec{E}$

↳ rep r & other terms that vary w/ pos in terms of parameters of point A's pos

· e.g. An electric field exists near a straight, thin rod of length $2a$, uniformly charged at $\rho_0$. Assume that the rod is placed along the $y$-axis, centered at the origin. Find the electric field at any point on the $x$-axis.




SOLUTION

0) $\rho_\ell = \rho_0$

1) Use Cartesian coordinates.

2) Length element $d\ell$ holds charge $dQ = \rho_\ell \, d\ell$

↳ since $d\ell = dy$, then $dQ = \rho_0 \, dy$

3) $d\vec{E} = \frac{k\,dQ}{r_i^2}\hat{r}_i$

↳ $r_i = \sqrt{x_i^2 + y_i^2}$

↳ $\hat{r}_i = \frac{\langle x_i, -y_i \rangle}{\sqrt{x_i^2+y_i^2}}$

Rewrite $d\vec{E}$ as $d\vec{E} = \frac{k\,dQ}{x_i^2+y_i^2} \cdot \frac{\langle x_i, -y_i\rangle}{\sqrt{x_i^2+y_i^2}}$

4) $dE_y$ components cancel out due to symmetry so only $dE_x$ remains

5) $\vec{E} = \int d\vec{E} = \int \frac{k\,dQ}{r_i^2}\hat{r}_i$

$= \int_{-a}^{a} \frac{k\rho_0\,dy}{x^2+y^2} \cdot \frac{\langle x, -y\rangle}{\sqrt{x^2+y^2}}$

Bounds:

$y : -a \rightarrow +a$

Since $y$-component cancels out, only need $x$-component

$\vec{E}_x = k\rho_0 \int_{-a}^{a} \frac{dy}{(x^2+y^2)} \cdot \frac{x}{\sqrt{x^2+y^2}}$

$= k\rho_0 \int_{-a}^{a} \frac{x\,dy}{(x^2+y^2)^{3/2}}$ ⟶ Let $\tan\theta = \frac{y}{x}$

$= k\rho_0 \int_{\theta_1}^{\theta_2} \frac{x^2\sec^2\theta}{x^3\sec^3\theta} \, d\theta$

$= \frac{k\rho_0}{x}\int_{\theta_1}^{\theta_2} \frac{1}{\sec\theta}\, d\theta$

$= \frac{k\rho_0}{x}\int_{\theta_1}^{\theta_2} \cos\theta \, d\theta$

$= \frac{k\rho_0}{x}\left[\sin\theta\right]_{\theta_1}^{\theta_2}$

$= \frac{k\rho_0}{x}\left(\frac{a}{\sqrt{x^2+a^2}} - \left(\frac{-a}{\sqrt{x^2+a^2}}\right)\right)$

$= \frac{2k\rho_0}{x\sqrt{x^2+a^2}}\hat{x}$

$\sec\theta = \frac{r_i}{x}$

$y = x\tan\theta$ $\qquad$ $\sec\theta = \frac{\sqrt{x^2+y^2}}{x}$

$dy = x\sec^2\theta \, d\theta$ $\qquad$ $x\sec\theta = \sqrt{x^2+y^2}$

$x^3\sec^3\theta = (x^2+y^2)^{3/2}$

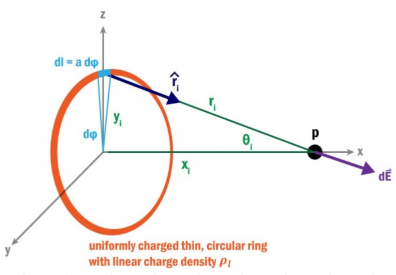
⟶ Changed bounds:

$\sin\theta_1 = \frac{-a}{\sqrt{x^2+a^2}}$ $\qquad$ $\sin\theta_2 = \frac{a}{\sqrt{x^2+a^2}}$

$\theta_1 = \sin^{-1}\left(\frac{-a}{\sqrt{x^2+a^2}}\right)$ $\qquad$ $\theta_2 = \sin^{-1}\left(\frac{a}{\sqrt{x^2+a^2}}\right)$

Thus, $\vec{E} = \frac{2k\rho_0}{x\sqrt{x^2+a^2}}\hat{x}$

· e.g. A uniformly charged circular ring with linear charge density $\rho_l$ and radius $a$ is in the $yz$-plane, centered at the origin. Find the electric field at any point $p$ on its $x$-axis.



SOLUTION

0) $\rho_\ell$ is constant

1) polar coordinates

2) $dQ = \rho_\ell \, d\ell$

$= \rho_\ell \, a \, d\varphi$

3) $\vec{dE} = \frac{kdQ}{r_i^2} \hat{r}_i$

↳ $r_i = \sqrt{x^2+a^2}$

• $r_i$ is a constant

↳ $\hat{r}_i = \frac{\langle x, y, z \rangle}{\sqrt{x^2+a^2}}$

4) $dE_y$ + $dE_z$ components cancel out due to symmetry

5) $\vec{E} = \int \vec{dE} = \int \vec{dE}_x \, \hat{x}$

$= \int \frac{kdQ}{r_i^2} \hat{x}$

$= \int_0^{2\pi} \frac{k\rho_\ell a d\varphi}{x^2+a^2} \cdot \frac{x}{\sqrt{x^2+a^2}}$

$= \int_0^{2\pi} \frac{xk\rho_\ell a}{(x^2+a^2)^{3/2}} d\varphi$

$= \frac{xk\rho_\ell a}{(x^2+a^2)^{3/2}} [\varphi]_0^{2\pi}$

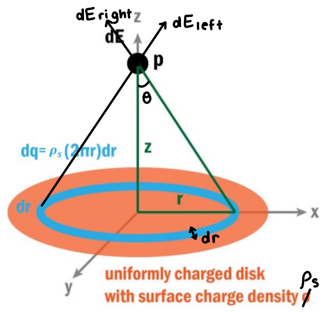
$= \frac{2\pi x k\rho_\ell a}{(x^2+a^2)^{3/2}} \hat{x}$

Bounds:

$\varphi : 0 \rightarrow 2\pi$

Thus, $\vec{E} = \frac{2\pi x k \rho_\ell a}{(x^2+a^2)^{3/2}} \hat{x}$.

· e.g. A uniformly charged circular disk has a radius of $r$ and a surface charge density of $\rho_s$ $\cancel{\sigma}$. Find the electric field along the axis of the disk.




SOLUTION

Define $dS = 2\pi r \, dr$, which is ring on circle   $dQ = 2\pi r dr \rho_s$

Due to symmetry, $\hat{x}$ + $\hat{y}$ - components cancel out.

$d\vec{E}_z = \frac{kdQ}{(\sqrt{r^2+z^2})^2} \cdot \frac{z}{\sqrt{r^2+z^2}} \hat{z}$

$= \frac{k 2\pi r dr \rho_s z}{(r^2+z^2)^{3/2}} \hat{z}$

$\vec{E} = \vec{E}_z = \int_0^r \frac{2\pi k r \rho_s z}{(r^2+z^2)^{3/2}} dr \, \hat{z}$

$= 2\pi k \rho_s z \int_0^r \frac{r dr}{(r^2+z^2)^{3/2}} dr \, \hat{z}$ $\longrightarrow$ Let $\tan\theta = \frac{r}{z}$   $\sec\theta = \frac{\sqrt{r^2+z^2}}{z}$

$= 2\pi k \rho_s z \int_{\theta_1}^{\theta_2} \frac{z\tan\theta}{z^3 \sec^3\theta} (z \sec^2\theta \, d\theta) \hat{z}$   $r = z\tan\theta$   $z\sec\theta = \sqrt{r^2+z^2}$

$= 2\pi k \rho_s z \int_{\theta_1}^{\theta_2} \frac{z^2 \tan\theta \sec^2\theta}{z^3 \sec^3\theta} \hat{z}$   $dr = z\sec^2\theta \, d\theta$   $z^3\sec^3\theta = (r^2+z^2)^{3/2}$

$= 2\pi k \rho_s z \int_{\theta_1}^{\theta_2} \frac{\tan\theta}{z\sec\theta} \hat{z}$

$= 2\pi k \rho_s z \int_{\theta_1}^{\theta_2} \sin\theta \, \hat{z}$

$= 2\pi k \rho_s z [-\cos\theta]_{\theta_1}^{\theta_2} \hat{z}$ $\longrightarrow$ $r_1 = 0 \longrightarrow \theta_1 = 0$   $r_2 = r \longrightarrow \cos\theta_2 = \frac{z}{\sqrt{r^2+z^2}}$

$= 2\pi k \rho_s z \left(-\frac{z}{\sqrt{r^2+z^2}} - (-1)\right) \hat{z}$   $\cos\theta_1 = \cos 0$   $\theta_2 = \cos^{-1}\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

$= 2\pi k \rho_s z \left(1 - \frac{z}{\sqrt{r^2+z^2}}\right) \hat{z}$   $= 1$

Thus, $\vec{E} = 2\pi k \rho_s z \left(1 - \frac{z}{\sqrt{r^2+z^2}}\right) \hat{z}$.

# chapter 5

## CONCEPT OF FLUX

- electric flux : measure of how many electric field lines pass through given area
  - ↳ denoted as $\Phi_E$
- to quantify electric flux, define surface differential $dS$ w/ dir vector $\hat{n}$
  - ↳ $\hat{n}$ is unit vector normal to surface
- electric flux of surface is dot product of field $\vec{E}$ & surface normal vector $\vec{n}$
  - ↳ $\Phi_E = \vec{E} \cdot \vec{n} = |\vec{E}||\vec{n}|\cos\theta$
    - electric flux is scalar quantity
    - value of 0 when $\vec{E}$ & $\vec{n}$ are perpendicular (i.e. surface is parallel to E-field)
- electric flux can apply to both open & closed surface
  - ↳ can work independently from Gauss's Law
- when finding flux through closed surface w/ enclosed net charge, $\Phi_E = \oint_{surface} \vec{E} \cdot d\vec{S}$
  - ↳ $d\hat{S} = \hat{n}$
  - ↳ not complete definition of electric flux & we use $\Phi_E = \oint_{surface} \vec{D} \cdot d\vec{S} = \oint_{surface} \epsilon_0 \vec{E} \cdot d\vec{S}$
    - $\vec{D}$ is electric flux density / displacement & in free space, $\vec{D} = \epsilon_0 \vec{E}$
    - $\epsilon_0$ is constant & can be taken out of integral

## ELECTRIC FLUX OVER A CLOSED SURFACE

- flux is calculated as $d\Phi_E = \vec{E} \cdot d\vec{S}$

$$= |\vec{E}||d\vec{S}|\cos\theta$$

  - ↳ if $\vec{E} \parallel d\vec{S}$, then $\cos\theta = \cos 0 = 1$
  - ↳ if $\vec{E} \perp d\vec{S}$, then $\cos\theta = \cos\frac{\pi}{2} = 0$
    - electric flux is 0
- to develop Gauss's Law, consider point charge +Q & put imaginary spherical surface around it
  - ↳ analyze E-field at every point on surface
  - ↳ $d\vec{S} = R^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$ w/ a normal vector pointing radially outward
  - ↳ $\vec{E} = \frac{kQ}{R^2} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{R^2}\right)$
  - ↳ $d\Phi_E = \vec{E} \cdot d\vec{S} = |\vec{E}||d\vec{S}|\cos\theta$
    - $\theta = 0$ so $\cos 0 = 1$
  - ↳ $\Phi_E = \oint \vec{E} \; d\vec{S}$

$$= |\vec{E}| \oint d\vec{S}$$

$$= |\vec{E}| A$$

$$= \frac{Q}{4\pi\epsilon_0 R^2}(4\pi R^2)$$

$$\Phi_E = \frac{Q}{\epsilon_0}$$

    - flux is only dependent on enclosed charge Q

> **NOTE**
> - for a closed surface, $d\vec{S}$ always points normal to surface away from enclosed volume

- when E-field enters closed surface, electric flux is -ve & when it leaves, flux is +ve
- if charge is outside closed surface, electric flux is 0
  - ↳ whatever flux flows in will flow out
- total flux through closed surface is only equal to total charge enclosed within volume, no matter how charge is distributed, shape of charge, & shape of closed surface
  - ↳ i.e. can use electric flux & Gauss's Law on any closed surface
  - ↳ total flux through non-spherical surface is same as that through sphere

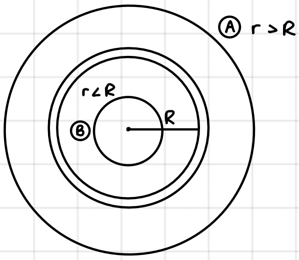
# GAUSS'S LAW

- Gauss's Law states that total electric flux through closed surface is equal to net electric charge inside surface, divided by $\epsilon_0$
  ↳ $\Phi_E = \oint \vec{E} \cdot d\vec{S} = \frac{Q_{enclosed}}{\epsilon_0}$
    - $d\vec{S}$ element rep mathematical surface we draw in space to calculate flux cutting through it & has no relation to how charge is distributed
  ↳ Law is only applicable on closed surfaces
- always true & based on $\frac{1}{r^2}$ nature of force law
- if $\varphi_E > 0$, net +ve charge enclosed within surface (i.e. more field flows out than in)
- if $\varphi_E < 0$, net -ve charge enclosed within surface (i.e. more field flows in than out)

# CALCULATING ELECTRIC FIELD WITH GAUSS'S LAW

- Gauss's Law on its own can't calculate E-field
- to use Gauss's Law for E-field calculations, must create surface that exhibits Gaussian symmetry
- Gaussian symmetry: guarantee that $d\vec{S}$ must have same $\theta$ w/ $\vec{E}$ at every point on surface
  ↳ i.e. $\vec{E} \parallel d\vec{S}$ or $\vec{E} \perp d\vec{S}$
- $\vec{E}$ can be decomposed into normal & tangential dir to surface (i.e. $\vec{E} = E_n\hat{n} + E_t\hat{t}$)
  ↳ Gauss Law becomes $\oint \vec{E} \cdot d\vec{S} = \oint E_n dS = \frac{Q_{encl}}{\epsilon_0}$
  ↳ if there's surface where $E_n$ is constant ($\cos 0 = 1$) or zero ($\cos\frac{\pi}{2} = 0$), $E_n$ can be taken out of integral
    - $E_n \oint dS = \frac{Q_{encl}}{\epsilon_0}$
      $\vec{E} = E_n = \frac{Q_{encl}}{\epsilon_0 \oint dS}$
- process of using Gauss's Law:
  1) draw diagram to find E-field dir (symmetry)
  2) draw Gaussian surface
     - field should have constant magnitude around surface
     - field is $\perp$ to surface & in parts that it's not, it's $\parallel$
     - draw surface around point we want to calculate E-field at ; although not necessary to enclose all charge, surface should be symmetrical to all charge (both inside & out)
  3) evaluate flux over surface using Gauss's Law
  4) find field by evaluating $\Phi_E$ & $Q_{encl}$
- 3 cases where Gaussian symmetry exists:
  ↳ spherical surface around charge distributed isotropically symmetrically in sphere
    - isotropic means charge density looks same in any dir (no variation in $\theta$ or $\varphi$)
  ↳ cylindrical surface around charge distributed geometrically symmetrically in infinite cylinder
    - infinite line is cylinder w/ radius 0
  ↳ pill box surface around charge distributed symmetrically in infinite plane
- e.g.
  A charge $Q$ is spread uniformly throughout a sphere of radius $R$. Find the electric field at all points (a) outside and (b) inside the sphere.

SOLUTION




## NOTE

symmetrical spherical distribution outside sphere looks like point charge if total charge is sitting at centre of charge

There's spherical symmetry so Gaussian sphere can be constructed $\vec{dS}$ is always parallel to $\vec{E}$.

a) $Q_A = Q$

$$\oint_{sphere} \vec{E} \cdot \vec{dS} = \frac{Q_A}{\epsilon_0}$$

$$\oint |E||dS| \cos\theta = \frac{Q_A}{\epsilon_0} \longrightarrow \theta = 0 \text{ everywhere}$$

$$E \oint dS = \frac{Q_A}{\epsilon_0}$$

$$E = \frac{Q_A}{\epsilon_0 (4\pi r^2)}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

b) $Q_B = Q\left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\right)$ $\longrightarrow$ $\rho_v = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3}$ $\quad$ $Q_B = \rho_v V_B = \frac{Q}{\frac{4}{3}\pi R^3}\left(\frac{4}{3}\pi r^3\right) = Q\left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\right)$

$$= Q\left(\frac{r^3}{R^3}\right)$$

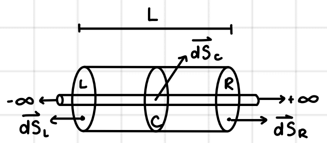
$$\oint \vec{E} \cdot \vec{dS} = \frac{Q_B}{\epsilon_0}$$

$$E = \frac{Q_B}{\epsilon_0 4\pi r^2} = \frac{Qr^3}{R^3}\left(\frac{1}{4\pi\epsilon_0 r^2}\right) = \frac{Qr}{4\pi\epsilon_0 R^3}$$

$$\vec{E} = \frac{Qr}{4\pi\epsilon_0 R^3} \hat{r}$$

· e.g. Find the electric field of an infinite line of charge that carries charge density $\rho_0$ C/m

SOLUTION




A Gaussian cylinder around line charge is constructed.

$$\Phi_{tot} = \Phi_L + \Phi_R + \Phi_C \qquad Q_{encl} = \rho_0 L$$

$$= \Phi_C$$

$$\oint \vec{E} \cdot \vec{dS} = \frac{Q_{encl}}{\epsilon_0}$$

$$E = \frac{\rho_0 L}{\epsilon_0 (2\pi r L)} = \frac{\rho_0}{2\pi r \epsilon_0}$$

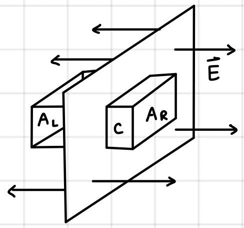
$$\vec{E} = \frac{\rho_0}{2\pi r \epsilon_0} \hat{r}$$

Curved: $\vec{E} \parallel \vec{dS}$ so $\theta = 0$ everywhere

Caps: $\vec{E} \perp \vec{dS}$ so $\theta = 1$ everywhere

· e.g. An infinite sheet of charge carries uniform surface charge density ~~$\phi$~~ $\rho_s$ C/m$^2$. Find the electric field.

SOLUTION




A Gaussian pillbox shape is constructed.

$$\Phi_{tot} = \oint \vec{E}_L \cdot \vec{dS}_{AL} + \oint \vec{E}_R \cdot \vec{dS}_{AR} + \oint \vec{E}_C \cdot \vec{dS}_C$$

$$\vec{E}_L \parallel \vec{dS}_{AL} \qquad \vec{E}_R \parallel \vec{dS}_{AR} \qquad \vec{E}_C \perp \vec{dS}_C$$

$$\Phi_{tot} = \oint \vec{E}_L \cdot \vec{dS}_{AL} + \oint \vec{E}_R \cdot \vec{dS}_{AR}$$

$$\Phi_{tot} = 2E \int dS_A$$

$$2E \int dS_A = \frac{Q_{encl}}{\epsilon_0}$$

$$E = \frac{\rho_s A}{2A\epsilon_0} = \frac{\rho_s}{2\epsilon}$$

$$\vec{E} = \frac{\rho_s}{2\epsilon} \hat{r}$$

**TIP**

· use cylinder as pillbox shape b/c pill box can have caps of any shape

· for a point charge, E-field decreases as $\frac{1}{r^2}$

· for infinite line, E-field decreases as $\frac{1}{r}$

· for infinite plane, E-field doesn't decrease

## OTHER CASES WHERE GAUSS'S LAW CAN BE USED TO CALCULATE ELECTRIC FIELDS

· case 1: problem consists of individual symmetrical parts

↳ apply Gauss's Law to calculate individual E-fields + use vector addition to get final answer

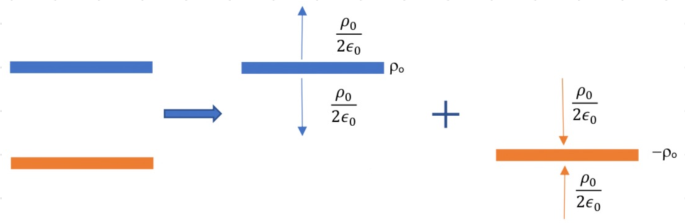
· case 2: if we know region where E-field is 0 or value of field in another region, place Gaussian surface go into that region

↳ since $\vec{E} = 0$, then $\Phi = 0$

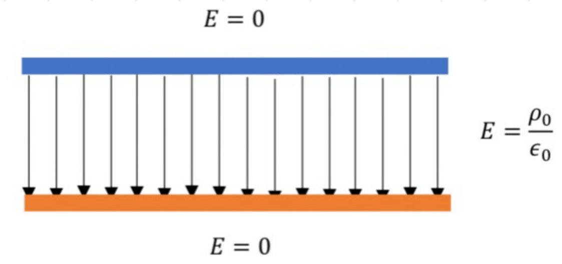
e.g. Ex. We have two very large (can be considered infinite) charge sheets separated by a distance "d". The top sheet has uniform surface charge density, $\rho_o$, while the bottom sheet has uniform surface charge density given by $-\rho_o$ as shown in the figure below. Cross-section is shown and the sheets come in and out of the paper. Calculate Electric field everywhere in space.


SOLUTION

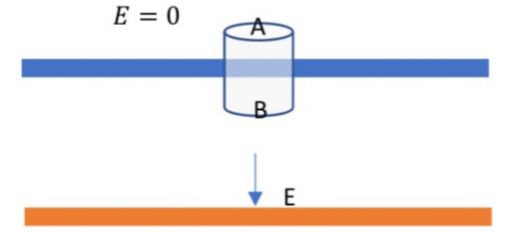
Case 1: use superposition to separate 2 charges into 2 separate sheets




↳ since E-field doesn't vary in magnitude as we go distance away from sheet, add fields tgt to get:




Case 2: since we have 2 large planes w/ equal but opp charges, outside E-field is 0




↳ $\Phi_A = 0$ & $\Phi_{curved} = 0$

↳ flux only exists on surface B & we can apply Gauss's Law

↳ $\oint \vec{E} \cdot d\vec{s} = \frac{Q_{encl}}{\epsilon_0}$

$$ES = \frac{\rho_o S}{\epsilon_o}$$

$$E = \frac{\rho_o}{\epsilon_o}$$

## SUMMARY

· $\Phi = \int \vec{E} \cdot d\vec{s}$ is general flux equation for any surface (open/closed)

· Gauss's Law (closed surfaces only): $\Phi_E = \oint \vec{E} \cdot d\vec{s} = \frac{Q_{encl}}{\epsilon_o}$

↳ flux for Q: $\Phi_E = \frac{Q_{encl}}{\epsilon_o}$

↳ flux for E: $\Phi_E = \oint \vec{E} \cdot d\vec{s}$

$$= \oint \vec{E}_1 \cdot d\vec{s}_1 + \oint \vec{E}_2 \cdot d\vec{s}_2 + \dots$$

- for however many faces on closed object

↳ electric field : $\oint \vec{E} \cdot \vec{dS} = \frac{Q_{encl}}{\epsilon_0}$

$$E \oint dS = \frac{Q_{encl}}{\epsilon_0}$$

- requires Gaussian symmetry to factor out E

# chapter 6

## INTRODUCTION TO ELECTROSTATIC POTENTIAL

work : measure of energy transfer when there's movement from external force

electric potential energy : capacity for doing work (movement from one area to another)

- electric potential : difference in PE per unit charge btwn 2 locations in E-field
  - ↳ aka voltage
  - ↳ scalar field that gives picture of how electric PE is changing in space

- work is done when force acts over certain distance
  - ↳ if we do +ve work on system, we lose energy & system gains
  - ↳ $W = \vec{F} \cdot \vec{d} = Fd\cos\theta$
    - ∘ $\theta$ is angle btwn vectors
- potential difference is $\Delta U_{AB} = U_B - U_A = -W_{AB}$
  - ↳ when +ve work is done, define potential function as reduction in potential
  - ↳ 2 ways to look at it:
    - ∘ $W_{AB}$ is work done by system (+ve)
    - ∘ $W_{I, A\to B} = -W_{AB}$ is work done by us (-ve)
- electric force is conservative force
  - ↳ U is equal to work we must do against $\vec{F}_E$ to move object
  - ↳ since $\vec{F}_E = q\vec{E}$, $\Delta U_{AB} = -W_{AB}$

$$= -q\vec{E} \cdot \Delta\vec{r}$$
$$= -qE\,\Delta r\cos\theta$$

**NOTE**
- +ve work done by external force (us) = system gains potential energy

## MATHEMATICAL DEFINITION

- consider charge Q in space; in order to move test charge q from ∞ to point P in presence of Q, we must do work
  - ↳ E-field by Q puts force $\vec{F}_E = q\vec{E}$ on q (in opp dir to force we're putting on charge)
- work we do to move charge from ∞ to point P is $W_I = \int_{\text{point}\,\infty}^{P} \vec{F}_I \cdot d\vec{\ell}$

$$W_I = -q\int_{\infty}^{P} \vec{E}\, d\vec{\ell}$$

  - ↳ $\vec{F}_I = -\vec{F}_E = -q\vec{E}$
  - ↳ $d\vec{\ell}$ is differential step in path twd P
  - ↳ work done by us is -ve so we're adding energy to system (i.e. $\Delta U > 0$)

- electric potential at point P is work we do per +ve test charge under limit of test charge → 0
  - ↳ $V_P = \lim_{q\to 0} \frac{W_I}{q}$

$$V_P = -\int_{\infty}^{P} \vec{E} \cdot d\vec{\ell}$$

  - ↳ unit is Volts (V)
- EP is always defined assuming +ve test charge
- in general, ref is ∞ so EP at ∞ for charge distribution is 0
- -ve in equation is b/c EP is defined wrt to work we do, which is opp of work done by E-field
  - ↳ must be part of equation

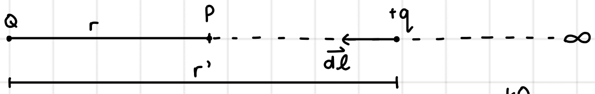
- electrostatic fields are conservative fields so it doesn't matter what path is taken to reach

point P, potential will be same
- ↳ $d\vec{l}$ only depends on change in dr (radial displacement)
- ↳ choose most favourable path, which is usually along E-field line

- potential diff for point A w/ itself is always 0 b/c displacement is 0
  - ↳ $\Delta V_{A \to A} = -\oint \vec{E} \cdot d\vec{l} = 0$
    - i.e. KVL

## ELECTRIC POTENTIAL DUE TO POINT CHARGE

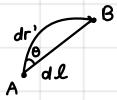
- to calculate potential at point P in system due to test charge Q, take straight path from ∞ to P along E-field line




- ↳ E-field at r' away from Q is $\vec{E} = \frac{kQ}{(r')^2}\hat{r}$
- ↳ $dV = -\vec{E} \cdot d\vec{l}$

$= -E\,dl\cos\theta \longrightarrow \cos\theta = \frac{dr'}{dl}$

$= -E\,dr' \qquad dl\cos\theta = dr'$




- ↳ $V_P = \int dV$

$$= -\int_{\infty}^{r} E\,dr'$$
$$= -\int_{\infty}^{r} \frac{kQ}{(r')^2}\,dr'$$
$$= -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0 (r')^2}\,dr'$$
$$= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r'}\right]_{\infty}^{r}$$
$$= -\frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)$$
$$V_P = \frac{Q}{4\pi\epsilon_0 r}$$

- if Q > 0, then $V_P > 0$ & charges repel
  - ↳ to bring q to P, we'll have to do +ve work so $V_P > 0$
- if Q < 0, then $V_P < 0$ & charges attract
  - ↳ to bring q to P, we'll have to do -ve work so $V_P < 0$

**NOTE**
- ∞ is where electric potential is considered 0

## ELECTRIC POTENTIAL DUE TO DISTRIBUTED CHARGES

- 2 options to calculate electric potential
  - ↳ calculate E-field as function of r & do integral $-\int_{\infty}^{r} E\,dr$
  - ↳ use superposition & consider charge distribution as summation of charges
    - differential potential at P due to point charge is $dV_P = \frac{dQ}{4\pi\epsilon_0 r} = \frac{\rho_v dv}{4\pi\epsilon_0 r}$
    - potential at point P is $V_P = \int \frac{\rho_v dv}{4\pi\epsilon_0 r}$
- superposition method is similar to E-field integrals except it's a scalar so we don't need to worry abt $\hat{r}$-dir & look for symmetries
- to decide on what method to use to calculate electric potential:

Is there Gaussian symmetry in charge distribution?

Yes → Calculate Electric field using Gauss Law, Calculate potential with integral $V_P = -\int_{\infty}^{r} Edr$. Electric field is going to be quite symmetrical and this integral will be straightforward.

No ↓

Can we break the distribution into sum of individual problems which are individually symmetric?

Yes → Calculate Electric field using Gauss Law, Calculate potential with integral $V_P = -\int_{\infty}^{r} Edr$. Electric field is going to be quite symmetrical and this integral will be straightforward.

No ↓

Calculate Electric field as $\vec{E} = \int \frac{kdQ}{r^2}\hat{r}$ and $V_P = \int \frac{kdQ}{r}$

These are separate integrals. Please do not try to calculate V from integral of E on the contour.

**NOTE**
- $dV = \frac{kdQ}{r}$
- $\Delta V_{A \to B} = -\int_a^b E\,dr$

- when calculating potential through E-field calcs first, always go outside in
  - ↳ if outside alr calculated first, start from boundary instead of $\infty$
- e.g. **Example 3.** Charge Q is uniformly distributed over the volume of a sphere of radius R. Calculate the Electric potential as a function of r from r = 0 to ∞.

SOLUTION

Region bounded by $R \le r \le \infty$:

We apply Gauss's Law by enclosing the sphere in a Gaussian sphere.

$$\oint \vec{E}_2 \cdot d\vec{S} = \frac{Q_{encl}}{\epsilon_0}$$

$$\vec{E}_2 = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$V(r) = -\int_\infty^{P_2} E\,dr$$

$$= -\int_\infty^r \frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_\infty^r$$

$$= \frac{Q}{4\pi\epsilon_0 r}$$

Region bounded by $0 \le r \le R$:

$$\oint \vec{E}_1 \cdot d\vec{S} = \frac{Q_{encl}}{\epsilon_0} \longrightarrow \quad \rho_v = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3} \qquad Q_{encl} = \rho_v V = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\frac{4}{3}\pi r^3\right) = \frac{Qr^3}{R^3}$$

$$E_1 \oint dS = \frac{Qr^3}{\epsilon_0 R^3}$$

$$E_1 = \frac{Qr^3}{4\pi\epsilon_0 r^2 R}$$

$$\vec{E}_1 = \frac{Qr}{4\pi\epsilon_0 R}\hat{r}$$

$$V(r<R) = -\int_\infty^{P_1} E\,dr$$

$$= -\int_\infty^R E_2\,dr + \left(-\int_R^r E_1\,dr\right)$$

$$= \frac{Q}{4\pi\epsilon_0 R} - \int_R^r \frac{Qr}{4\pi\epsilon_0 R}\,dr$$

$$= \frac{Q}{4\pi\epsilon_0 R} + \frac{Q}{4\pi\epsilon_0 R}\int_r^R r\,dr$$

$$= \frac{Q}{4\pi\epsilon_0 R} + \frac{Q}{4\pi\epsilon_0 R}\left[\frac{r^2}{2}\right]_r^R$$

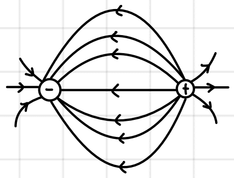
$$= \frac{Q}{4\pi\epsilon_0 R}\left(1 + \frac{1}{2}(R^2 - r^2)\right)$$

## ELECTRIC POTENTIAL AND ELECTROSTATIC POTENTIAL ENERGY

- one charge alone in space has no potential energy b/c it has no force acting on it
  - ↳ if bring charge $Q_2$ into system from $\infty$, work we have done is stored as potential energy in system
- change in PE $\Delta U_e$ when bringing new charge q to P is given as $\Delta U_e = qV_P$
  - ↳ $V_P$ doesn't relate to total energy stored in system
- if there's 2 charges, $Q_1$ & $Q_2$, in system, then total energy is: $U_e = 0 + \Delta U_e = Q_2 V_P$
  - ↳ initial energy is 0 b/c w/ $Q_1$, energy is 0
  - ↳ electric potential changes due to 2 charges & at one point, be sum of potential individually due to $Q_1$ & $Q_2$
- electric potential is ability of system to gain/lose energy if we ever brought new charge into system
- if we want to assemble N charges, do pair wise addition of PE stored in each unique pair of charges to get total PE of system
  - ↳ $U = k\sum_{i=1}^{N}\sum_{j>i}^{N}\frac{Q_i Q_j}{r_{ij}}$

## ELECTRIC POTENTIAL DIFFERENCE BETWEEN POINTS

- electric potential difference is $\Delta V = \frac{\Delta U}{q}$
  - ↳ for non-uniform fields, $\Delta V_{A\to B} = -\int_A^B \vec{E}\cdot d\vec{\ell}$
  - ↳ if we alr know potential at points A & B, $\Delta V_{A\to B} = V_B - V_A$
- potential at centre of dipole = 0
  - ↳ $V = \frac{kQ}{r}$ & if we add individual potentials, they cancel out



## VISUALIZING ELECTRIC POTENTIAL

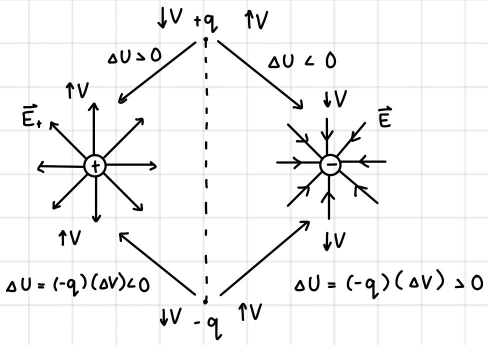
- $V = \frac{kQ}{r}$ , $\Delta V = \frac{\Delta U}{q}$

  $\Delta U = q\Delta V$

  ↳ $\Delta U$ : change in potential energy

  ↳ $V$ : potential

  - defined by location (test charge is always +ve)




> **NOTE**
> - $V$ & $\Delta V$ are based purely on location
> - $\Delta U$ is based on whether charge is +ve/-ve

## ELECTRIC FIELD FROM ELECTRIC POTENTIAL

- using Cartesian coordinates, $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$ & $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

  ↳ $dV = -(E_x dx + E_y dy + E_z dz)$

  E-field given as $\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right) = -\nabla V$
- $\vec{E} = 0$ when $V$ is constant (not necessarily 0)

## EQUIPOTENTIAL SURFACES

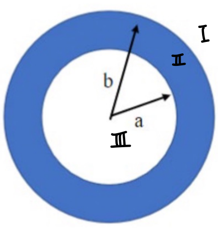
- equipotential lines are analogous to contour lines on a map

  ↳ tell us where E-field has same value

  ↳ i.e. take uniform steps in value of electric potential
- if lines are closer, field is stronger & potential is higher
- E-field is always $\perp$ to equipotential surfaces

## EXAMPLES

Charge is uniformly distributed in a volume of spherical shell of inner radius $a$ and outer radius $b$ with volume charge density $\rho_0$ as shown in the figure. Calculate the electric field and electric potential for all values of $r$. Accurately plot the amplitude of the field and potential with $r$.




SOLUTION

Apply Gauss's Law to find $E_I, E_{II}, E_{III}$ b/c there's spherical symmetry so $\vec{E} \parallel d\vec{s}$ at every point.

1) $V_I = -\int_\infty^r E_I\,dr$ or $V_I = \frac{kQ}{r}$ (2 methods)

2) $V_{II} = -\int_\infty^r E\,dr = -\int_\infty^b E_I\,dr - \int_b^r E_{II}\,dr$

3) $V_{III} = -\int_{\infty}^{r} E dr = -\int_{\infty}^{b} E_I dr - \int_{b}^{a} E_{II} dr - \int_{a}^{r} E_{III} dr$

Region I : $r > b$

$E_I (4\pi r^2) = \frac{Q_{encl}}{\epsilon_0} \longrightarrow Q = \int_a^b \rho_0 dV$

$E_I (4\pi r^2) = \frac{\rho_0}{\epsilon_0}(\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3) \qquad = \rho_0 (\frac{4}{3}\pi b^3 - \frac{4}{3}\pi a^3)$

$E_I (4\pi r^2) = \frac{\rho_0}{\epsilon_0}(\frac{4}{3}\pi)(b^3 - a^3)$

$\vec{E}_I = \frac{\rho_0 (b^3 - a^3)}{3\epsilon_0} (\frac{1}{r^2}) \hat{r}$

Shortcut:

$|E_I| = $ "$kQ$" $(\frac{1}{r^2})$ & $V_I = $ "$kQ$" $(\frac{1}{r})$ $\longrightarrow$ shortcut by observation

$V_I = \frac{\rho_0 (b^3 - a^3)}{3\epsilon_0} (\frac{1}{r})$

↳ $|E|$ must be proportional to $\frac{1}{r^2}$

Long:

$V_I = -\int_{\infty}^{r} E_I dr$

$= -\int_{\infty}^{r} \frac{\rho_0 (b^3 - a^3)}{3\epsilon_0} \frac{1}{r^2} dr$

$= \frac{-\rho_0 (b^3 - a^3)}{3\epsilon_0} \lim_{t \to \infty} \int_t^r r^{-2} dr$

$= \frac{-\rho_0 (b^3 - a^3)}{3\epsilon_0} \lim_{t \to \infty} [-\frac{1}{r}]_t^r$

$= \frac{\rho_0 (b^3 - a^3)}{3\epsilon_0} (\frac{1}{r})$

Region II : $a < r < b$

$E_{II} (4\pi r^2) = \frac{Q_{encl}}{\epsilon_0}$

$E_{II} (4\pi r^2) = \frac{\rho_0}{\epsilon_0}(\frac{4\pi}{3}(r^3 - a^3))$

$E_{II} = \frac{\rho_0 (r^3 - a^3)}{3\epsilon_0} (\frac{1}{r^2}) \hat{r}$

$E_{II} = \frac{\rho_0}{3\epsilon_0} (r - \frac{a^3}{r^2}) \hat{r}$

$V_{II} = -\int_{\infty}^{r} E dr$

$= -\int_{\infty}^{b} E_I dr - \int_{b}^{r} E_{II} dr$

$= V_I (r = b) - \int_b^r E_{II} dr$

$= \frac{\rho_0 (b^3 - a^3)}{3\epsilon_0 b} - \int_b^r \frac{\rho_0}{3\epsilon_0} (r - \frac{a^3}{r^2}) dr$

$= \frac{\rho_0 (b^3 - a^3)}{3\epsilon_0 b} - \frac{\rho_0}{3\epsilon_0} [\frac{r^2}{2} + \frac{a^3}{r}]_b^r$

$V_{II} = \frac{\rho_0}{3\epsilon_0} (\frac{b^3 - a^3}{b}) + \frac{\rho_0}{6\epsilon_0} (b^2 - r^2) + \frac{\rho_0 a^3}{3\epsilon_0} (\frac{1}{b} - \frac{1}{r})$

Region III : $r < a$

$E_{III} (4\pi r^2) = \frac{Q_{encl}}{\epsilon_0} \longrightarrow Q = 0$

$\vec{E}_{III} = 0 \hat{r}$

$V_{III} = -\int_{\infty}^{b} E_I dr - \int_{b}^{a} E_{II} dr - \int_{a}^{r} E_{III} dr$

$= V_{II} (r = a)$

$V_{III} = \frac{\rho_0}{3\epsilon_0} (\frac{b^3 - a^3}{b}) + \frac{\rho_0}{6\epsilon_0} (b^2 - a^2) + \frac{\rho_0 a^3}{3\epsilon_0} (\frac{1}{b} - \frac{1}{a})$

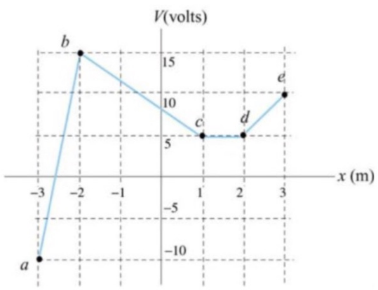
Electric potential varies along the x-axis and its plot is shown below. The potential does not vary in the y or z direction. Of the regions shown, determine the intervals where the electric field, $E_x$, has the:

A. Greatest amplitude. What is the amplitude and direction of the E-field in the region?

B. Least amplitude. What is the amplitude and direction of E-field in the region?

C. Plot the value of $E_x$ ($E$ pointing in the $+x$ axis is positive)

D. Can you suggest a charge distribution which can create this electric field?




SOLUTION

$\vec{E} = -\nabla V = -\frac{dV}{dx} \hat{x}$

A) greater slope so greater $|\vec{E}_x|$

↳ btwn a & b

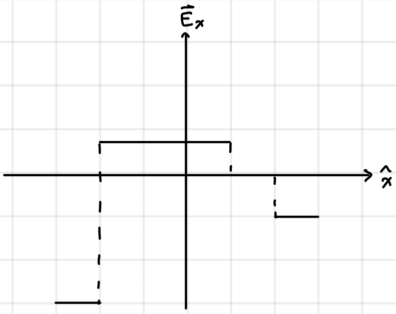
B) smaller slope so smaller $|\vec{E}_x|$

↳ btwn c & d

c) $\vec{E}_x = -\nabla V_x$

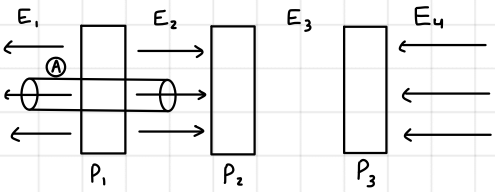
$= -\frac{25}{1} + (\frac{10}{3}) + 0 + (-\frac{5}{1})$

$= -25 + \frac{10}{3} + 0 - 5$




D) Should have 3 infinite sheets to apply Gauss's Law

↳ to find charge density, need Gauss's Law since we alr know $\vec{E}$




A) $E_1 A + E_2 A = \frac{\rho_1 A}{\epsilon_0}$

$\rho_1 = \epsilon_0 (E_1 + E_2)$

$= \epsilon_0 (25 + \frac{10}{3})$

$= \frac{85}{3} \epsilon_0$

B) $-E_2 A + 0 = \frac{\rho_2 A}{\epsilon_0}$

$\rho_2 = -\frac{10}{3} \epsilon_0$

C) $-E_4 A = \frac{\rho_3 A}{\epsilon_0}$

$\rho_3 = 5\epsilon_0$

# chapter 7

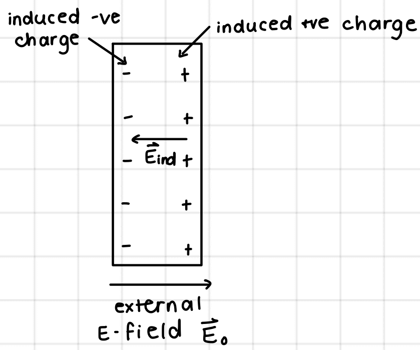
## CONDUCTORS

- conductors: materials that have large # of $e^-$ that are free to move when influenced by E-field
- ideal conductors have following properties:
  - ↳ free $e^-$ in conduction band which are very weakly bound to atoms
    - when external E-field is applied, $e^-$ are stripped off of atom & free to move abt
  - ↳ no resistance to motion of $e^-$
  - ↳ infinite free $e^-$ available so conductor has unlimited supply of free charges
  - ↳ $e^-$ will only move within conductor
- conductors are neutral b/c there's $p^+$ (which can't move) & $e^-$ (which can move) so in absence of external E-field, -ve charges are attracted to +ve charges but attraction is weak
  - ↳ if external E-field is applied, $e^-$ move very easily
- insulators: materials where $e^-$ aren't free to move around
  - ↳ dielectric: type of insulator which has some free $e^-$ & get polarized in presence of E-field

## CONDUCTORS INSIDE EXTERNAL E-FIELDS

- external means E-field is created by charges not on conductor

1) E-field inside conductor is always 0




- ↳ originally, material is neutral but $\vec{E}_0$ causes force of $-e\vec{E}$ on $e^-$ in conductor so they all move to left until they hit surface of conductor
  - creates net +ve charge on right
- ↳ creates induced charge that produces E-field of its own ($\vec{E}_{ind}$)
- ↳ $\vec{E}_{ind}$ is opp in dir but can usually assume $|\vec{E}_{ind}| = |\vec{E}_0|$ so E-fields cancel out within conductor (i.e. $\vec{E}_0 + \vec{E}_{ind} = \vec{0}$)
- ↳ in electrostatic equilibrium, there can't be any E-field in conductor
  - induced charges on surface

2) charge density is 0 inside conductor

- ↳ if $\vec{E} = \vec{0}$ inside conductor, then through Gauss's Law: $\oint \cancel{\vec{E}}^{\,0} \, d\vec{S} = \frac{Q_{encl}}{\epsilon_0}$

  $Q_{encl} = 0$
- ↳ $\rho = 0$ so that $Q_{encl} = 0$ (i.e. net charge density is 0)
- ↳ i.e. there's no induced charge in volume of conductor

3) any net charge resides on surface

- ↳ since net charge density is 0, if there's net charge, then only possible location is on surface
- ↳ i.e. charges will be induced only on surface of conductor

4) conductor is equipotential

- ↳ since $\vec{E} = \vec{0}$ & $\vec{E} = -\nabla V$, then $V$ must be constant
  - as such, $\Delta V_{A \to B} = -\int_A^B \vec{E} \cdot d\vec{l} = 0$
- ↳ all points on conductor (whether on surface or inside volume) are equipotential

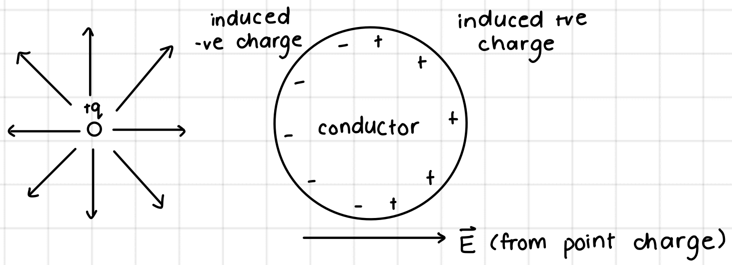
• i.e. all have same value of potential diff

5) any external E-field lines must be ⊥ to conductor surface
   ↳ no tangential component b/c E-fields are always ⊥ to equipotential lines
   ↳ conductors in external E-fields will bend fields so they hit conductor perpendicularly
      • achieved by induced surface charges

6) Faraday's Cage · E-field inside cavity due to external E-field will be 0
   ↳


   ↳ charge will redistribute on outer surface of conductor
   ↳ E-field inside cavity can't exist due to external E-fields so there's no charge on inside cavity surface
   ↳ Faraday's Cage states that by putting smth inside conductor box (whatever shape as long as it's closed), we shield it from any external E-field

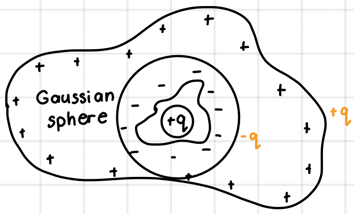
· place charge +q outside spherical conductor




   ↳ will attract each other
   ↳ +q will pull ⊖ toward it & "repel" ⊕ to other side

## CHARGE INSIDE CAVITY

· +q charge goes inside hollow conductor
   ↳ fixed charge, meaning that it's fixed in specific place inside cavity
   ↳




   ↳ $\vec{E}=0$ inside conductor so inside Gaussian sphere, charges must cancel
      • $\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\epsilon_0}$

$$Q_{encl} = 0$$

$$(+q) + (\text{ind}) = 0$$

$$\text{ind} = -q$$

   ↳ charge redistributes itself so -q gathers on inside surface & +q is on outside surface
      • overall, conductor is still neutral

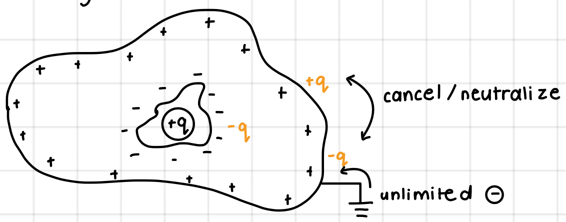
· there will be E-field outside conductor b/c total charge enclosed by Gaussian surface is +q, which results in non-zero $\Phi$ & E-field

· conductor w/ hollow cavity shields inside from external E-field but allows E-field from charge inside to go outside

# GROUNDING THE CONDUCTOR

- grounding means connecting conductor w/Earth
  - ↳ Earth has infinite amount of +ve & -ve free charges
  - ↳ $V_{Earth} = 0$

when we ground conductor, connect it to Earth & its unlimited supply of +ve & -ve charges

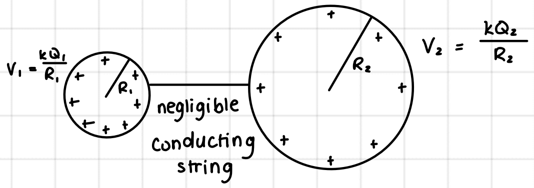



↳ no E-field outside conductor so we've also shielded inside charge

- enclosed charge outside conductor is 0 & its electric potential will also be 0V (same as Earth)

# CHARGES RELEASED INTO CONDUCTOR

- if touching conductor w/ charged rod, charge would go to surface b/c it can't be distributed into volume of conductor
- consider 2 spherical conductors that are far enough away from each other st induced charges from the other is negligible; one of them has excess charge Q on it




↳ connecting the 2 conductors gives you 1 conductor w/same electric potential
  - $Q = Q_1 + Q_2$

↳ since V is the same after electrostatic equilibrium is achieved.
  - $\frac{kQ_1}{R_1} = \frac{kQ_2}{R_2}$

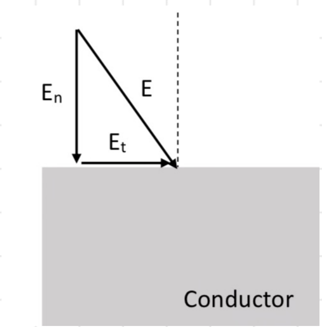
↳ solving gives $Q_1 = \frac{R_1}{R_1+R_2} Q$ & $Q_2 = \frac{R_2}{R_1+R_2} Q$
  - since $R_2 > R_1$, larger sphere has greater charge

↳ solving for charge density gives $\rho_1 = \frac{Q_1}{4\pi R_1^2}$ & $\rho_2 = \frac{Q_2}{4\pi R_2^2}$
  - ratio is $\frac{\rho_1}{\rho_2} = \frac{R_2}{R_1}$
  - charge density of smaller sphere is larger than that of larger sphere's

- smaller radius of curvature = larger charge density on surface

# BOUNDARY CONDITIONS

- define how E-fields & induced charges will be related at boundary of 2 diff materials (i.e. mediums)
- assume E-field incident on conductor from another medium has 2 components: tangential & normal

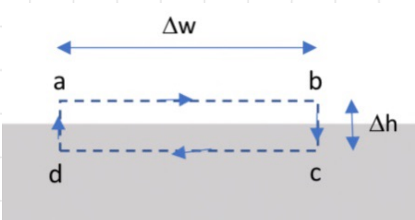



for tangential boundary conditions, draw closed loop contour abcda across boundary

↳ width $\Delta w = ab = cd$

↳ height $\Delta h = bc = ad$

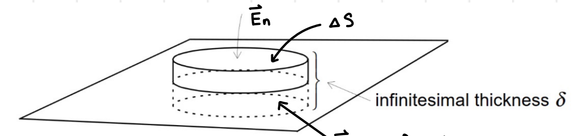
- make $\Delta h \to 0$ to approach boundary

↳ 


- $\oint_a^a \vec{E}\cdot d\vec{l} = E_t \Delta w = 0$ b/c we're taking a closed path

↳ there can be no tangential component at boundary of conductor, even in other medium

for normal boundary conditions, construct Gaussian pillbox

↳ 


- $\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\epsilon_0}$

  $-E_n \Delta S = \frac{\rho_s \Delta S}{\epsilon_0}$

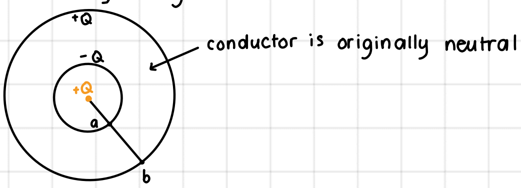
  $E_n = -\frac{\rho_s}{\epsilon_0}$ ($\rho_s$ is charge induced on surface of conductor)

↳ if $E_n$ is incident (i.e. going into) on conductor, $\rho_s < 0$

↳ if $E_n$ is coming out, $\rho_s > 0$

## EXTRA TIPS

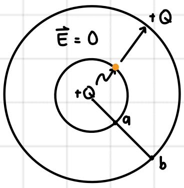
- stationary charge:




↳ $\rho_a = \frac{-Q}{4\pi a^2}$ (induced)

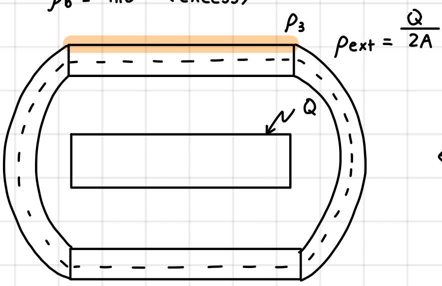
↳ $\rho_b = \frac{Q}{4\pi b^2}$ (induced)

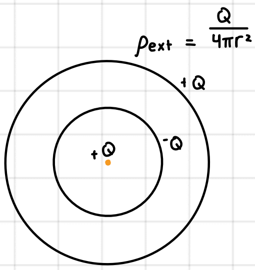
Q touching conductor:




↳ $\rho_a = 0$

↳ $\rho_b = \frac{Q}{4\pi b^2}$ (excess)



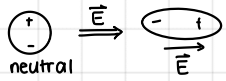




↳ connecting 2 plates w/ conducting wire is equivalent to making single conducting surface that surrounds the enclosed charge

# chapter 8

## INTRODUCTION

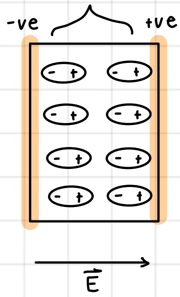
- in an insulator, $e^-$ are tightly bound to atom & can't freely move
- when we apply external E-field to insulator, $e^-$ in atom still feel force & displaced to one side in nucleus
- dielectric : insulating material that can be polarized when external $\vec{E}$ is applied

↳ 


  - bound $e^-$ are displaced by very small amount b/c there will be charge separation

## POLARIZATION VECTOR

- visual rep of polarized dielectric & creation of dipoles when external E-field is applied:




- dipole is characterized by dipole moment $\vec{p} = q\vec{d}$
  - ↳ q is charge value
  - ↳ d is distance btwn equal & opp charges $-q$ & $+q$
    - vector is always from $-q$ to $+q$
- use polarization vector $\vec{P}$ to describe polarization that occurs & how induced dipoles affect material macroscopically
  - ↳ $\vec{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=0}^{n\Delta v} \vec{p}_i$
    - n is # of atoms per unit volume
    - construct small volume $\Delta v$ around point & add all dipole moments vectorially within $\Delta v$
  - ↳ $\vec{P}$ is volume density of electric dipole moment
- at edge of polarized dielectric, there's induced surface charge
  - ↳ $\rho_s = \vec{P} \cdot \hat{n}$
    - $\hat{n}$ is normal unit vector pointing out of dielectric
- inside, there's induced volume charge density
  - ↳ $\rho_v = -\nabla \cdot \vec{P}$
  - ↳ given by divergence of $\vec{P}$
- divergence of vector field A at a point is net outward flux of A per unit volume as volume abt point tends to 0
  - ↳ $\text{div}\,\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}$
  - ↳ in Cartesian coordinates, $\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

## ELECTRIC FLUX DENSITY AND DIELECTRIC CONSTANT

- from equation $\oint_S \vec{E} \cdot d\vec{s} = \frac{Q_{encl}}{\epsilon_0}$, charge is source of E-field (sink if -ve) & flux flows from charge

↳ i.e. E-field diverges out of/into charge

- in point form, Gauss Law is $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
  - ↳ $\rho$ includes both free charges & induced charges due to E-fields created by free charges
  - ↳ in dielectrics, induced charges are related to $\vec{P}$:

$$\nabla \cdot \vec{E} = \frac{\rho_{free} + \rho_v}{\epsilon_0}$$
$$\nabla \cdot \vec{E} = \frac{\rho_{free} - \nabla \cdot \vec{P}}{\epsilon_0}$$
$$\rho_{free} = \nabla \cdot \epsilon_0 \vec{E} + \nabla \cdot \vec{P}$$
$$\rho_{free} = \nabla(\epsilon_0 \vec{E} + \vec{P})$$

- electric flux displacement vector $\vec{D}$ describes how much E-field is affected by $\vec{P}$
  - ↳ $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$
  - ↳ units of $\vec{D}$ are $C/m^2$
  - ↳ new Gauss Law in point form is $\nabla \cdot \vec{D} = \frac{\rho_{free}}{\epsilon_0}$
    - unlike in conductors, don't worry abt induced charges in calculations
- rewrite Gauss Law: $\oint \vec{D} \cdot \vec{dS} = Q_f$
  - ↳ $Q_f$ is free charge
  - ↳ 1st Maxwell equation
- when dielectric properties of medium are linear & isotropic (i.e. doesn't change w/ dir), $\vec{P} \propto \vec{E}$
  - ↳ proportionality constant is $\vec{P} = \epsilon_0 \chi_e \vec{E}$
    - $\chi_e$ is dimensionless quantity called electric susceptibility
    - medium is linear if $\chi_e$ is independent of $\vec{E}$ & homogeneous if $\chi_e$ is independent of space coordinates
  - ↳ $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$
$$\vec{D} = \epsilon_0 (1 + \chi_e) \vec{E}$$

    - relative permittivity/dielectric constant is $\epsilon_r = 1 + \chi_e$
  - ↳ $\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$
    - $\epsilon = \epsilon_0 \epsilon_r$ is absolute permittivity
    - $\epsilon_0$ is free space permittivity
    - $\epsilon_r$ is relative permittivity

- first, calculate $\vec{D}$ using Gauss Law (for symmetric problems) & only worry abt free charges (i.e. charges placed in system) while not considering induced charges
- calculate $\vec{E}$ by dividing $\vec{D}$ w/ $\epsilon$
- calculate P once we have $\vec{D}$ & $\vec{E}$
  - ↳ $\vec{P} = \vec{D} - \epsilon_0 \vec{E}$

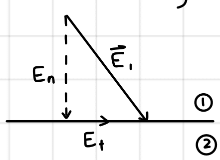
$$= \epsilon_0 \epsilon_r \vec{E} - \epsilon_0 \vec{E}$$
$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}$$
$$= \epsilon_0 (\epsilon_r - 1)\vec{E}$$

**NOTE**

- $\epsilon$ is local relation specific to point so only consider material property at point we're calculating $\vec{E}$ at

## BOUNDARY CONDITIONS

- rules for boundary conditions for dielectrics:
  - ↳




  - ↳ tangential boundary condition: $E_{1t} = E_{2t}$
  - ↳ normal boundary condition: $D_{1n} = D_{2n}$

- $\vec{D}$ is continuous at interface since there's no free charge (i.e. $\rho_{free} = 0$)
- this means $\epsilon_2 E_{2n} = \epsilon_1 E_{1n}$ & we see if $\epsilon_2 > \epsilon_1$, then $E_{1n} > E_{2n}$
- if $\rho_{free} \neq 0$, then $D_{2n} - D_{1n} = \rho_{free}$

# chapter 9

## CAPACITANCE

- potential on conductor is proportional to charge on conductor
  - ↳ $Q = CV$
    - C is proportionality constant
- capacitance: electric charge that must be added to conducting body to increase its electric potential by 1V
  - ↳ i.e. ratio of charge to voltage
  - ↳ $C = \frac{Q}{V}$ [F]
    - SI units are 1 Farad (F) $= \frac{C}{V}$
  - ↳ wherever we have conductor w/ charge on it, capacitance exists
- e.g. calculate capacitance of spherical conductor w/ radius R & charge Q deposited on it

SOLUTION

$V = \frac{kQ}{R}$ (for spherical conductor)

$C = \frac{Q}{\frac{kQ}{R}}$

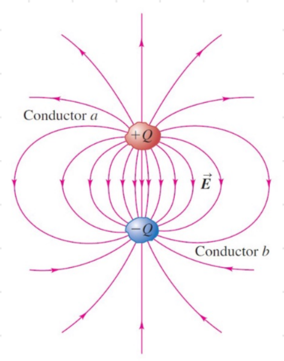
$= Q\left(\frac{R}{kQ}\right)$

$= \frac{R}{k}$

$C = 4\pi\epsilon_0 R$

## CAPACITOR

- capacitor is formed when 2 conductors are separated by free space / dielectric medium
  - ↳ conductors are arbitrary shapes & we connect battery / DC voltage source to conductors
    - +Q charge transferred to one conductor via +ve terminal of source
    - -Q charge transferred to other conductor via -ve terminal of source
  - ↳



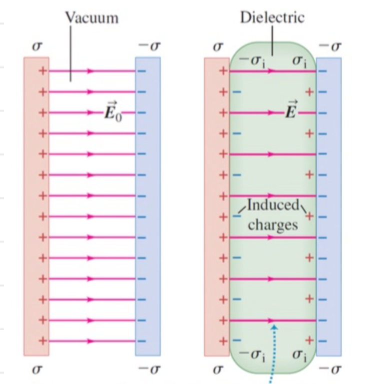

    - charging a capacitor means transferring e⁻ from one conductor to another
- potential diff is created across 2 conductors ($\Delta V_{2\to1}$) & capacitance is $C = \frac{Q}{\Delta V_{2\to1}}$
  - ↳ $\Delta V_{2\to1} = -\int \vec{E} \cdot d\vec{l}$
- capacitance is physical property of capacitor & only depends on geometry of conductors, distance btwn them, & permittivity of dielectric medium in btwn
  - ↳ capacitor still has capacitance even when there's no voltage applied to it & no free charge exists on its conductors
- if DC source is connected to 2 conductor system & charge is deposited on them, +Q & -Q attract & create force btwn conductors
  - ↳ thus, energy is stored in capacitor
  - ↳ if source is removed, charges remain & energy's still stored

↳ can put capacitor in circuit & release stored energy

## CALCULATION OF CAPACITANCE

- steps to calculate capacitance:
  1) choose coord system
  2) assume +Q/-Q charge on each conductor
  3) calculate $\vec{E}$ w/ Gauss Law or other methods
  4) find $\Delta V_{2\to1}$ by evaluating integral $\Delta V_{2\to1} = -\int \vec{E} \cdot d\vec{\ell}$
     - calculate potential diff from -Q conductor to +Q conductor so that $\Delta V_{2\to1} > 0$ & $C > 0$
     - all capacitors being analyzed are +ve
  5) find C using $C = \frac{Q}{\Delta V_{2\to1}}$
     - Q values should cancel
- dielectric materials (contain molecular dipoles but no free charge) can be added to capacitor to reduce potential diff btwn plates

↳




For a given charge density $\sigma$, the induced charges on the dielectric's surfaces reduce the electric field between the plates.

↳ molecular dipoles align w/ -ve end twd +ve plate

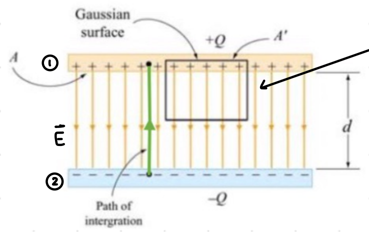
↳ dipoles' E-field superposes w/ original $\vec{E}$ (i.e. goes in opp dir) & reduces net $\vec{E}$

↳ charge Q stays same but reduced $\vec{E}$ results in lower $\Delta V_{2\to1}$ & larger C

- e.g. A parallel plate capacitor consists of two parallel conducting plates of cross-sectional area A and separated by a distance d. The space between the plates is filled with a dielectric of constant permittivity, $\epsilon$. Determine the capacitance.

SOLUTION

If A >> d, then assume charge is almost uniformly distributed & draw diagram:




Gaussian pillbox

To find $\vec{E}$:

$$\oint \vec{D} \cdot d\vec{S} = Q_{free}$$
$$D \oint dS = \rho_o A$$
$$D = \frac{\rho_o A}{A}$$
$$D = \rho_o$$
$$\epsilon E = \frac{Q}{A}$$
$$E = \frac{Q}{A\epsilon}$$

To find $\Delta V_{2\to1}$:

$$\Delta V_{2\to1} = -\int_2^1 \vec{E} \cdot d\vec{\ell}$$
$$= -\int_2^1 |\vec{E}||d\vec{\ell}|\cos\theta \longrightarrow \text{since } \theta = \pi,\ \cos\theta = -1$$
$$= E\int_2^1 d\ell$$
$$= Ed$$
$$= \frac{Qd}{A\epsilon}$$

To find C:

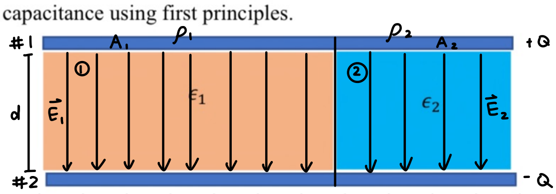
$$C = \frac{Q}{\Delta V_{2\to1}}$$

$$= \frac{Q}{\frac{Qd}{A\epsilon}}$$

$$= \frac{A\epsilon}{d}$$

↳ capacitance of parallel plate capacitor is $C = \frac{\epsilon_0 \epsilon_r A}{d}$

· e.g. Consider a parallel plate capacitor where we have a dielectric with dielectric permittivity $\epsilon_1$ covering cross-sectional area $A_1$ and another dielectric with dielectric permittivity $\epsilon_2$ covering a cross-sectional area $A_2$ as shown in the figure below. The separation between the plates is d. Calculate the capacitance using first principles.




SOLUTION

To find E:

$\rho_1 = \frac{Q_1}{A_1}$ , $\rho_2 = \frac{Q_2}{A_2}$

↳ charge is conserved. $\rho_1 A_1 + \rho_2 A_2 = Q$

$$\rho_2 = \frac{Q - \rho_1 A_1}{A_2} \quad 1)$$

There's 2 $\vec{E}$s to consider but they share a boundary

↳ $E_{1t} = E_{2t}$

$E_1 = E_2$ ⟵ all tangential on boundary

$\frac{\rho_1}{\epsilon_1} = \frac{\rho_2}{\epsilon_2}$ ⟵ values for E-fields are obtained by same method in prev example

$\rho_1 = \frac{\epsilon_1 \rho_2}{\epsilon_2}$ ⟵ sub in eq 1

$\rho_1 = \frac{\epsilon_1 (Q - \rho_1 A_1)}{\epsilon_2 A_2}$

$\rho_1 \epsilon_2 A_2 = Q\epsilon_1 - \rho_1 \epsilon_1 A_1$

$Q\epsilon_1 = \rho_1 (\epsilon_2 A_2 + \epsilon_1 A_1)$

$\rho_1 = \frac{Q\epsilon_1}{\epsilon_1 A_1 + \epsilon_2 A_2}$ ⟶ using same method $\rho_2 = \frac{Q\epsilon_2}{\epsilon_1 A_1 + \epsilon_2 A_2}$

↳ $E = E_1 = E_2$

$= \frac{\rho_1}{\epsilon_1}$

$E = \frac{Q}{\epsilon_1 A_1 + \epsilon_2 A_2}$

To find $\Delta V_{2\to1}$:

$\Delta V_{2\to1} = -\int_2^1 \vec{E} \cdot d\vec{l}$

$= Ed$

$= \frac{Qd}{\epsilon_1 A_1 + \epsilon_2 A_2}$

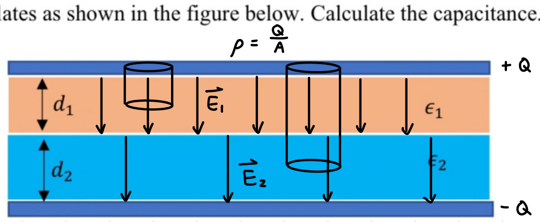
To find C:

$C = \frac{Q}{\Delta V}$

$= \frac{\epsilon_1 A_1 + \epsilon_2 A_2}{d}$

↳ for capacitors in parallel, $C = C_1 + C_2 + \ldots = \Sigma C_i$

· e.g. Consider a parallel plate capacitor where we have a dielectric with dielectric permittivity $\epsilon_1$ with thickness $d_1$ and another dielectric with dielectric permittivity $\epsilon_2$ and thickness $d_2$ sandwiched between the plates as shown in the figure below. Calculate the capacitance.




SOLUTION

To find E:

Apply Gauss Law & let A rep area of pillbox cap.

$\oint \vec{D} \cdot d\vec{S} = Q_{free}$

$DA = \rho A$

$D = \rho$

$$E_1 = \frac{D}{\epsilon_1} = \frac{Q}{A\epsilon_1} \qquad E_2 = \frac{D}{\epsilon_2} = \frac{Q}{A\epsilon_2}$$

To find $\Delta V_{2\to1}$:

$$\Delta V_{2\to1} = -\int_2^1 \vec{E}\cdot d\vec{l}$$
$$= E_1 d_1 + E_2 d_2$$
$$= \frac{Qd_1}{A\epsilon_1} + \frac{Qd_2}{A\epsilon_2}$$

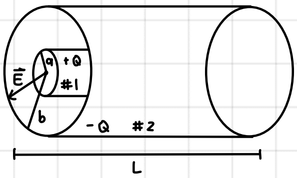
To find C:

$$C = \frac{Q}{\Delta V}$$
$$C = \frac{1}{\frac{d_1}{A\epsilon_1} + \frac{d_2}{A\epsilon_2}}$$
$$\frac{1}{C} = \frac{d_1}{A\epsilon_1} + \frac{d_2}{A\epsilon_2}$$

↳ for capacitors in series, $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots = \sum \frac{1}{C_i}$

- e.g. A coaxial cable consists of an inner-conductor of radius $a$ surrounded by another conductor of radius $b$. The space between the conductors is filled with a dielectric of permittivity, $\epsilon$. Calculate the capacitance per unit length of the cable.

SOLUTION




To find $\vec{E}$:

Construct Gaussian cylinder w/ radius $a < r < b$.

$$\oint \vec{D}\cdot d\vec{s} = Q_f$$
$$D(2\pi r L) = Q$$
$$D = \frac{Q}{2\pi r L}$$
$$E = \frac{D}{\epsilon} = \frac{Q}{2\pi\epsilon r L}$$

To find $\Delta V_{2\to1}$:

$$\Delta V_{2\to1} = -\int_2^1 \vec{E}\cdot d\vec{l}$$
$$= -\int_b^a \frac{Q}{2\pi\epsilon r L}\,dr$$
$$= -\frac{Q}{2\pi\epsilon L}\int_b^a \frac{1}{r}\,dr$$
$$= \frac{Q}{2\pi\epsilon L}\int_a^b \frac{1}{r}\,dr$$
$$= \frac{Q}{2\pi\epsilon L}(\ln b - \ln a)$$
$$= \frac{Q}{2\pi\epsilon L}\ln\left(\frac{b}{a}\right)$$

To find C.

$$C = \frac{Q}{\Delta V} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}$$
$$\frac{C}{L} = \frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)}$$ ⟵ capacitance per unit length

## ENERGY STORED IN A CAPACITOR

- capacitors are used to store electrical energy in terms of E-fields
  - ↳ amount of energy = work done to charge it
  - ↳ PE of uncharged capacitor is 0 so $W = U$
- to move $dQ$ from -ve to +ve plate of capacitor, we must do work ($dW$)
  - ↳ potential diff V btwn plates
  - ↳ $V = \frac{dW}{dQ}$

    $dW = V\,dQ$
  - ↳ capacitance is given by $C = \frac{dQ}{dV}$

$$dQ = C\,dV$$

- ↳ combining 2 equations tgt: $dW = V\,dQ$

$$dW = VC\,dV$$

- ↳ work done to charge capacitor : $W = \int dW$

$$= \int C V\,dV$$

$$= C\int V\,dV$$

$$W = \frac{1}{2}CV^2$$

- we know $W = U$ & also $Q = CV$ so potential energy stored in capacitor is $U_e = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}$
  - ↳ use $U_e = \frac{1}{2}CV^2$ when potential diff across conductors remain same
  - ↳ use $U_e = \frac{Q^2}{2C}$ when stored charge remains same

## ENERGY DENSITY OF ELECTRIC FIELD

- energy in capacitor is stored in E-field itself
- to calculate energy stored in parallel plate capacitor, use $C = \frac{\epsilon A}{d}$ & $V = Ed$
  - ↳ $U = \frac{1}{2}CV^2$

$$= \frac{1}{2}\left(\frac{\epsilon A}{d}\right)(Ed)^2$$

$$= \frac{1}{2}\epsilon E^2 Ad$$

  - ↳ Ad is volume of capacitor & since E-field is same everywhere, energy density is

$$u_e = \frac{U}{Ad}$$

$$u_e = \frac{1}{2}\epsilon E^2$$

    - in vacuum, $u_e = \frac{1}{2}\epsilon_0 E^2$
- generally, energy density is written as $u_e = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\epsilon\vec{E}\cdot\vec{E} = \frac{1}{2}\epsilon E^2$

# chapter 10

## INTRODUCTION

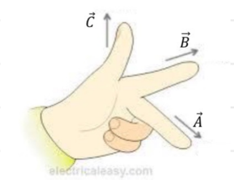
- magnetism involves a moving electric charge
- magnetic field is interaction of magnetic forces
  - ↳ aka magnetic flux density $\vec{B}$
    - units are webers / m² ( $Wb/m^2$ )
  - ↳ current / moving charges creates $\vec{B}$ around it
  - ↳ $\vec{B}$ only interacts w/ moving charges & creates force on moving charge
- force on moving charge q is $\vec{F}_m = q\vec{u} \times \vec{B}$
  - ↳ q is moving in magnetic field $\vec{B}$ w/ velocity $\vec{u}$

  when both E-field & magnetic field in space, total force is $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$
  - ↳ aka Lorentz's force equation
- comparison table of electrostatics vs magnetostatics:

| Electrostatics | Magnetostatics |
|---|---|
| ↳ stationary charges | ↳ uniform moving charges<br>• creates steady magnetic field |
| ↳ $\vec{E}$ = E-field | ↳ $\vec{B}$ = magnetic field ( $\frac{Wb}{m^2}$ ) |
| ↳ $\Phi_E$ = electric flux | ↳ $\Phi_B$ = magnetic flux (Wb) |
| ↳ charge at rest : $\vec{F}_e = q\vec{E}$ | ↳ charge moving at constant velocity $\vec{u}$ : $\vec{F}_m = q\vec{u} \times \vec{B}$ |

## CROSS PRODUCT

- cross product is given by $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin\theta$
  - ↳ θ is angle btwn 2 vectors
  - ↳ calculate magnitude & get dir w/ Fleming's LHR :




  - all fingers are extended w/ 90° to each other
  - middle is 1st vector, index is 2nd vector, & thumb gives dir of cross product
- to decide if cross product is +ve / -ve & what dir in coordinate system, use order prev defined
  - ↳ Cartesian: $\hat{x} \rightarrow \hat{y} \rightarrow \hat{z}$
    - result is next vector following arrows
    - +ve : follow arrow
    - -ve : go against arrow
    - e.g $\hat{x} \times \hat{y} = \hat{z}$ , $\hat{y} \times \hat{z} = \hat{x}$ , $\hat{z} \times \hat{x} = \hat{y}$
    - e.g. $\hat{y} \times \hat{x} = -\hat{z}$ , $\hat{z} \times \hat{y} = -\hat{x}$ , $\hat{x} \times \hat{z} = -\hat{y}$
  - ↳ cylindrical : $\hat{r} \rightarrow \hat{\varphi} \rightarrow \hat{z}$
  - ↳ spherical : $\hat{r} \rightarrow \hat{\theta} \rightarrow \hat{\varphi}$

## GAUSS LAW FOR MAGNETISM

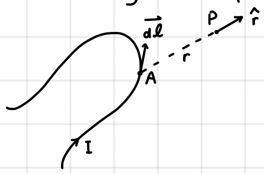
- magnetic field lines always go from N→S
- unlike charge, can't isolate a single pole

↳ i.e. every magnet has N & S pole ; there's no such thing as monopole

- magnetic field lines close upon themselves & make closed loop
- flux on closed surface is 0 : $\oint_S \vec{B} \cdot d\vec{S} = 0$
  - ↳ i.e. any field lines that leave will enter again somewhere else
  - ↳ $2^{nd}$ Maxwell equation

## BIOT-SAVART LAW DUE TO CURRENT

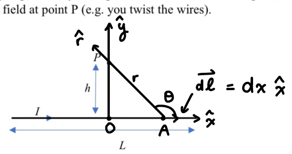
- to find $\vec{B}$ through first principles, consider diagram below:




- Biot-Savart Law gives $d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$
  - ↳ $\mu_0$ is permeability of free space: $\mu_0 = 4\pi \cdot 10^{-7}$ Tm/A
    - T is Tesla
  - ↳ $\vec{B} = \int_{circuit} \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$
- $d\vec{B}$ is perpendicular to plane containing $d\vec{l}$ & $\hat{r}$
  - ↳ to rep w/ picture:
    - pointing out of screen is ⊙
    - pointing into screen is ⊗
  - ↳ for many problems, dir of dB remains same even when we go to diff parts of circuit
- to evaluate $\vec{B}$:
  1) choose coord system based on shape of circuit
  2) define point A arbitrarily on circuit & define its coords w/chosen coord system
     - define $dl$ in terms of variables in coord system
  3) evaluate $d\vec{l} \times \hat{r}$, including its amp & dir
  4) look for symmetry to see if $\vec{B}$ only exists in 1 dir
     - if so, decompose $d\vec{B}$ in that dir
  5) integrate

## BIOT-SAVART LAW FOR MOVING CHARGES

- since $I = \frac{dq}{dt}$, we can write
$$dB = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$
$$= \frac{\mu_0}{4\pi} \frac{dq}{dt} \frac{d\vec{l} \times \hat{r}}{r^2}$$
$$= \frac{\mu_0}{4\pi} dq \frac{d\vec{l}/dt \times \hat{r}}{r^2}$$
$$d\vec{B} = \frac{\mu_0}{4\pi} dq \frac{\vec{v} \times \hat{r}}{r^2}$$
  - ↳ $\vec{v}$ is velocity of moving charge
- e.g. A thin, straight wire is carrying current I as shown in the figure below. Calculate the magnetic field at point P on an axis going half-way through the wire. We are assuming that the leads to the ends of the wire cancel the field at point P (e.g. you twist the wires).




SOLUTION

To evaluate $d\vec{l} \times \hat{r}$:

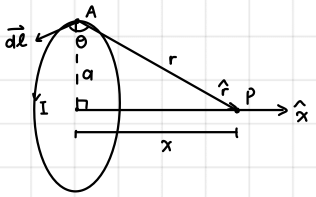
- ↳ using LHR, $d\vec{B}$ points outward (i.e. $\hat{z}$-dir) & doesn't change as we move A to diff parts of line
  - don't to look for symmetry
- ↳ for magnitude: $|d\vec{l} \times \hat{r}| = |d\vec{l}||\hat{r}| \sin\theta$
$$= dx(1)\sin(\pi - \theta)$$

$$= \frac{h}{r}\,dx$$

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{I\vec{d\ell}\times\hat{r}}{r^2}$$
$$= \frac{\mu_0}{4\pi}\frac{I}{r^2}\left(\frac{h}{r}\,dx\right)\hat{z}$$
$$= \frac{\mu_0 I h}{4\pi r^3}\,dx\ \hat{z}$$
$$\vec{B} = \int_{-L/2}^{L/2}\frac{\mu_0 I h}{4\pi r^3}\,dx\ \hat{z}$$
$$= \frac{\mu_0 I h}{4\pi}\int_{-L/2}^{L/2}\frac{1}{r^3}\,dx\ \hat{z} \longrightarrow r = \sqrt{x^2+h^2}$$
$$= \frac{\mu_0 I h}{4\pi}\int_{-L/2}^{L/2}\frac{1}{(x^2+h^2)^{3/2}}\,dx\ \hat{z}$$
$$= \frac{\mu_0 I h}{4\pi}\left[\frac{x}{h^2\sqrt{x^2+h^2}}\right]_{-L/2}^{L/2}\hat{z}$$
$$= \frac{\mu_0 I h}{4\pi}\left(\frac{L}{h^2\sqrt{\frac{L^2}{4}+h^2}}\right)$$
$$= \frac{\mu_0 I L}{4\pi h\sqrt{\frac{L^2}{4}+h^2}}\hat{z}$$

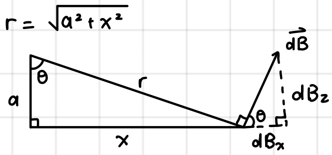
- e.g find $\vec{B}$ on axis of circular loop w/ current $I$ & radius $a$

SOLUTION




Note that $\theta = \frac{\pi}{2}$ for any point A.

$r = \sqrt{a^2+x^2}$



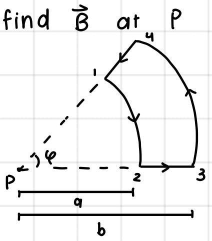

$$dB_x = \vec{dB}\cos\theta$$
$$\cos\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2+x^2}}$$

$$|\vec{d\ell}\times\hat{r}| = |\vec{d\ell}||\hat{r}|\sin\theta$$
$$= d\ell$$

Due to symmetry, only $dB_x$ remains.

$$dB_x = \vec{dB}\cos\theta$$
$$= \frac{\mu_0}{4\pi}\frac{I\vec{d\ell}\times\hat{r}}{r^2}\cos\theta\ \hat{x}$$
$$= \frac{\mu_0 I}{4\pi}\frac{d\ell}{r^2}\left(\frac{a}{r}\right)\hat{x}$$
$$= \frac{\mu_0 I a}{4\pi}\frac{1}{(a^2+x^2)^{3/2}}\,d\ell\ \hat{x}$$
$$B_x = \int\frac{\mu_0 I a}{4\pi}\left(\frac{1}{(a^2+x^2)^{3/2}}\right)d\ell\ \hat{x}$$
$$= \frac{\mu_0 I a}{4\pi(a^2+x^2)^{3/2}}\int d\ell\ \hat{x}$$
$$= \frac{\mu_0 I a\, 2\pi a}{4\pi(a^2+x^2)^{3/2}}\hat{x}$$
$$= \frac{\mu_0 I a^2}{2(a^2+x^2)^{3/2}}\hat{x}$$

- e.g. find $\vec{B}$ at P




Let ⊙ be $\hat{z}$.

SOLUTION

2 - 3 :

$$|\vec{d\ell}\times\hat{r}| = |\vec{d\ell}||\hat{r}|\overbrace{\sin\theta}^{\sin 180^\circ}$$
$$= 0$$

$\vec{dB}$ due to $2\to3 = 0$

4 - 1 :

$\vec{dB}$ due to $4\to1 = 0$

1 - 2 :

$$|\vec{d\ell}\times\hat{r}| = |\vec{d\ell}||\hat{r}|\sin\theta$$
$$= d\ell\,(1)\sin 90^\circ$$
$$= d\ell$$
$$\vec{dB} = \frac{\mu_0}{4\pi}I\frac{d\ell}{a^2}\ \otimes$$
$$= \frac{\mu_0}{4\pi}I\frac{d\ell}{a^2}(-\hat{z}) \longleftarrow$$ vector is constant in integral b/c $\vec{dB}$ dir doesn't change
$$\vec{B}_{1\to2} = \int\frac{\mu_0}{4\pi}I\frac{d\ell}{a^2}(-\hat{z})$$
$$= \frac{\mu_0 I}{4\pi a^2}\int d\ell\,(-\hat{z})$$
$$= \frac{\mu_0 I a\varphi}{4\pi a^2}(-\hat{z})$$
$$\vec{B}_{1\to2} = \frac{\mu_0 I\varphi}{4\pi a}(-\hat{z})$$

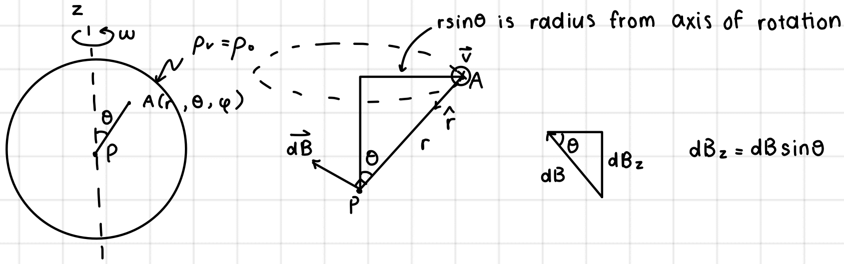
3 - 4: use ans from 1 - 2

$$\vec{B}_{3\to4} = \frac{\mu_0 I \varphi}{4\pi b}(\hat{z})$$

$$\vec{B} = \vec{B}_{1\to2} + \vec{B}_{3\to4}$$

$$\vec{B} = \frac{\mu_0 I \varphi}{4\pi}\left(-\frac{1}{a} + \frac{1}{b}\right)(\hat{z})$$

· e.g. find $\vec{B}$ at centre of uniformly charged sphere rotating w/ angular freq $\omega$




$dB_z = dB\sin\theta$

SOLUTION

$$\vec{dB} = \frac{\mu_0}{4\pi} dq \frac{\vec{v}\times\hat{r}}{r^2}$$

To find dq:

$dq = \rho_v dv$

$dq = \rho_0 r^2 dr \sin\theta d\theta d\varphi$

$|\vec{v}\times\hat{r}| = |\vec{v}||\hat{r}|\sin 90°$

$= v$

$= (r\sin\theta)\omega$

$$dB = \frac{\mu_0}{4\pi}(\rho_0 r^2 \sin\theta dr d\theta d\varphi)\left(\frac{r\sin\theta\omega}{r^2}\right)$$

$$= \frac{\mu_0\rho_0\omega}{4\pi} r\sin^2\theta dr d\theta d\varphi$$

Due to symmetry, only $dB_z$ remains.

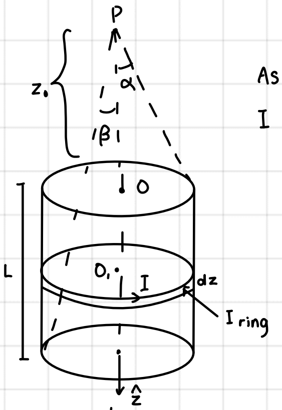
$$\vec{dB} = dB_z = \frac{\mu_0\rho_0\omega}{4\pi} r\sin^3\theta dr d\theta d\varphi \ \hat{z}$$

$$\vec{B} = \frac{\mu_0\rho_0\omega}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi}\sin^3\theta d\theta \int_0^R r \ \hat{z}$$

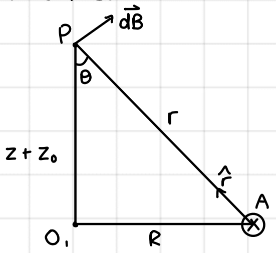
$$= \frac{\mu_0\rho_0\omega}{4\pi}(2\pi)\left(\frac{4}{3}\right)(R) \ \hat{z}$$

$$\vec{B} = \frac{2\mu_0\rho_0\omega R}{3} \ \hat{z}$$

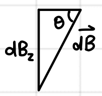
· find $\vec{B}$ at P due to solenoid of length L w/ N turns over it




Assume wire is very thin & tightly packed. Consider loop to be circle. I is current carried in each loop.




$dl = R d\varphi$




$dB_z = dB\sin\theta$

$\sin\theta = \frac{R}{r}$

$r = \frac{R}{\sin\theta}$

$\tan\theta = \frac{R}{z+z_0}$

$z + z_0 = \frac{R}{\tan\theta}$

$\frac{d}{d\theta}(z+z_0) = \frac{d}{d\theta}\left(\frac{R}{\tan\theta}\right)$

$\frac{dz}{d\theta} = -R\csc^2\theta$

$dz = -\frac{R}{\sin^2\theta} d\theta$

SOLUTION

# of turns/unit : $n = \frac{N}{L}$ ⟵ length over which loops are wound

For 1 ring:

↳ # of turns in dz = ndz

↳ each turn is creating its own field

↳ field due to ndz turns is $(dB)(ndz) \propto I\,ndz$

↳ effectively, current in ring is $ndzI$

$$\vec{dB} = \frac{\mu_0}{4\pi}(nIdz)\frac{\vec{dl}\times\hat{r}}{r^2}$$

$|\vec{dl}\times\hat{r}|$

$= |\vec{dl}||\hat{r}|\sin 90°$

$= dl$

Due to symmetry, only $dB_z$ remains.

$$\vec{dB} = dB_z = \frac{\mu_0}{4\pi}(nIdz)\left(\frac{dl}{r^2}\right)\sin\theta \ \hat{z}$$

$$dB_{z\,(ring)} = \frac{\mu_0}{4\pi} nIdz \frac{\sin\theta}{r^2} \int d\ell \qquad \longleftarrow \text{ nIdz, } \sin\theta, \text{ \& } r \text{ don't change in ring}$$

$$= \frac{\mu_0}{4\pi} nIdz \frac{\sin\theta}{r^2} \, 2\pi R$$

$$= \frac{\mu_0 nIR}{2} \left( \frac{\sin\theta \, dz}{r^2} \right)$$

$$\vec{B} = \int dB_{z\,(ring)}$$

$$= \frac{\mu_0 nIR}{2} \int \frac{\sin\theta dz}{r^2}$$

$$= \frac{\mu_0 nIR}{2} \int \frac{\sin\theta}{(R^2/\sin^2\theta)} \left( - \frac{R}{\sin^2\theta} \right) d\theta$$

$$= \frac{\mu_0 nIR}{2} \int \frac{\sin^3\theta}{R^2} \left( - \frac{R}{\sin^2\theta} \right) d\theta$$

$$= \frac{\mu_0 nIR}{2} \int - \frac{\sin\theta}{R} \, d\theta$$

$$= \frac{\mu_0 nI}{2} \int_\alpha^\beta - \sin\theta \, d\theta$$

$$\vec{B} = \frac{\mu_0 nI}{2} (\cos\beta - \cos\alpha) \quad (-\hat{z})$$

# chapter 11

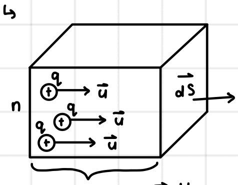
## INTRODUCTION

Ampere's Law can be used to calculate magnetic field for circuits w/unique symmetry

- analogous to Gauss Law, it relates $\vec{B}$ to encircled current within closed Amperian loop
- Ampere's Law for this chapter only works for DC currents & apx correct when current changes at low freq

## CURRENT DENSITY $\vec{J}$

- current is rate at which charges are crossing some surface S
  - ↳ i.e. $I = \frac{\Delta Q}{\Delta t}$
- assume solid rectangular volume of $(\vec{u}\,dt)(\vec{dS})$ w/ $\vec{dS}$ // to current flow
  - ↳




distance = $\vec{u}\,dt$

↳ $dI = \frac{dQ(t)}{dt}$

$= \frac{nq\,(dS\,\vec{u}\,dt)}{dt}$

$dI = nq\,dS\,\vec{u}$

- current density vector $\vec{J}$ is in dir of motion of +ve charge
  - ↳ $\vec{J} = nq\vec{u}$
    - n is # of charges per unit volume
    - $\vec{u}$ is uniform velocity at which the charges move at
  - ↳ units are $A/m^2$
  - ↳ volume charge density is $\rho_v = nq$ so $\vec{J} = \rho_v \vec{u}$
- current is defined as $dI = nq\,\vec{u}\,dS$

$= \vec{J} \cdot \vec{dS}$

$I = \int_S \vec{J} \cdot \vec{dS}$

  - ↳ integral is done over complete surface
- velocity $\vec{u}$ depends on conductor properties & applied E-field
  - ↳ $\vec{J}$ is proportional to $\vec{E}$ : $\vec{J} = \sigma\vec{E}$
    - $\sigma$ is conductivity of material measured in Siemens/m (S/m)

**NOTE**
after defining $\vec{J}$, can see current thru wire remains same even if we cut thru it at diff angle (i.e. dS changes)
↳ angle change in dot product accounts for it

## AMPERE'S LAW

- Ampere's Law is $\oint_C \vec{B}\cdot\vec{dl} = \mu_0 I_{encl} = \mu_0 \int_S \vec{J}\cdot\vec{dS}$
  - ↳ loop C is Amperian loop & it's boundary of region
  - ↳ surface S is enclosed by loop & it's open surface
- use right-hand thumb rule to determine current dir: curl fingers in dir of $\vec{dl}$ (i.e. integration path) & thumb points in dir of $\vec{dS}$
  - ↳ current flowing in dir of $\vec{dS}$ is +ve & in opp dir is -ve
  - ↳ if $\vec{dl}$ is ccw, $\vec{dS}$ points up
  - ↳ if $\vec{dl}$ is cw, $\vec{dS}$ points down

## CALCULATION OF MAGNETIC FIELD WITH AMPERE'S LAW

- like Gauss Law, use symmetric charge distribution w/ Ampere's Law to solve questions:

1) identify symmetry by sketching magnetic field lines

2) draw Amperian loop to evaluate $\oint \vec{B} \cdot d\vec{l}$

- guarantee $\vec{B}$ is parallel / perpendicular to $d\vec{l}$ in all areas so $|\vec{B}|$ can be factored out
- $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

$$B \oint d\vec{l} = \mu_0 I_{encl}$$

$$B \oint d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{S}$$

3) rearrange & solve for $|\vec{B}|$: $|\vec{B}| = \frac{\mu_0 I_{encl}}{\oint dl} = \frac{\mu_0 \int \vec{J} \cdot d\vec{S}}{\oint dl}$

- $\oint d\vec{l}$ is circumference of loop

4) determine dir of $\vec{B}$ by using RH thumb rule

- place thumb in dir of current so fingers curl around dir of $\vec{B}$

· Ampere's Law can work for 4 geometries:

↳ infinite cylinders & wires w/ isotropic current densities

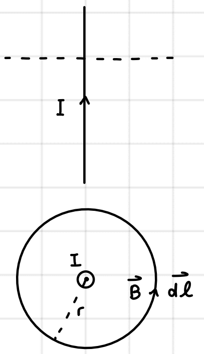
↳ infinite sheets carrying uniform current density

↳ infinitely long solenoids

↳ toroids

· e.g. infinitely long current-carrying wire; calculate $\vec{B}$ at distance r away

SOLUTION




Take cross-section ⊥ to current. Using LHR for $d\vec{l} \times \hat{r}$, $\vec{B}$ is circling around wire so can form Amperian loop around.

Ampere's Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

$$\oint B\, dl = \mu_0 I$$

$$B = \frac{\mu_0 I}{\oint dl}$$

$$= \frac{\mu_0 I}{2\pi r}$$

$\vec{B}$ is in $\hat{\varphi}$-dir since it's going ccw.

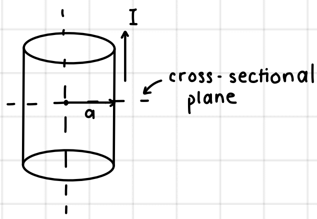
Thus, $\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\varphi}$.

**NOTE**

· can't use Ampere's Law for finite length line

· e.g. wire of radius a is carrying current I w/ uniform charge density, calculate B-field for diff values of radial distance, r

SOLUTION

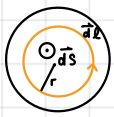



We will use circular loop to calculate $\vec{B}$.

Current density:

$J = \frac{I}{\pi a^2}$ ⟵ area of wire

For $r < a$:




Since $d\vec{S}$ is pointing out of page, $d\vec{l}$ goes ccw.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{S}$$

$$\oint B\, dl = \mu_0 \int J\, dS$$

$$B \oint dl = \mu_0 J \int dS$$

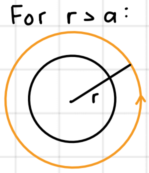
$$B = \frac{\mu_0 J (\pi r^2)}{2\pi r}$$

$$= \frac{\mu_0 J r}{2}$$

$$= \frac{\mu_0 I r}{2\pi a^2}$$

For $r < a$, $\vec{B} = \frac{\mu_0 I r}{2\pi a^2} \hat{\varphi}$.

For $r > a$:




$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

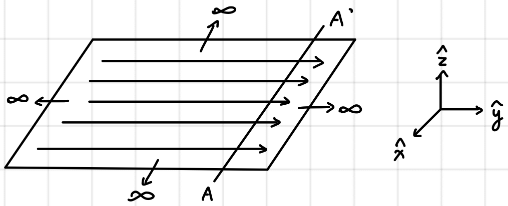
$$B \oint dl = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

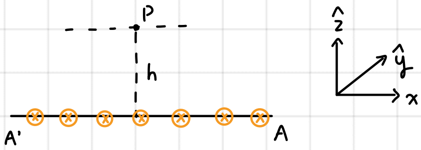
For $r > a$, $\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\varphi}$.

· e.g. very large & thin sheet in xy-plane carries uniform current density $J_0 \hat{y}$ ; calculate $\vec{B}$ at height h above sheet

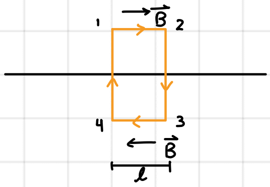
SOLUTION




Look twd A'A cross-section :




Make into closed loop :




Ampere's Law :

$$\oint \vec{B}\cdot \vec{d\ell} = \mu_0 I_{encl}$$

$$\int_1^2 B\,d\ell + \underbrace{\int_2^3 \vec{B}\cdot\vec{d\ell}}_{\vec{B}\perp\vec{d\ell}} + \int_3^4 \vec{B}\cdot\vec{d\ell} + \underbrace{\int_4^1 \vec{B}\cdot\vec{d\ell}}_{\vec{B}\perp\vec{d\ell}} = \mu_0 I_{encl}$$

same as $\int_1^2 B\,d\ell$ since both $\vec{B}$ & $\vec{d\ell}$ change dir

$$B\ell + B\ell = \mu_0 \int \vec{J}\cdot dS$$

$$2B\ell = \mu_0 \int J_0\, dx$$

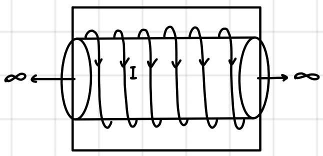
$$2B\ell = \mu_0 J_0 \ell$$

$$B = \frac{\mu_0 J_0}{2}$$

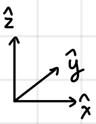
By inspection, $\vec{B} = \frac{\mu_0 J_0}{2}\hat{x}$ for $h>0$ & $\vec{B} = \frac{\mu_0 J_0}{2}(-\hat{x})$ for $h<0$ Note $\vec{B}$ stays constant & doesn't depend on h.

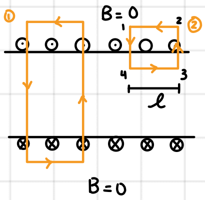
· e.g. very long solenoid has n turns per unit length & carries current I ; calculate $\vec{B}$ in diff regions

SOLUTION




Cut solenoid lengthwise.




Enclosing w/ loop 1.

$$\oint \vec{B}\ \vec{d\ell} = \mu_0 I_{encl} \longleftarrow \text{current cancels out}$$

$$\oint B\,d\ell = 0$$

$$2B\oint d\ell = 0$$

$$B = 0$$

Loop 2 :

↳ # of wires is $n\ell$ so $I_{encl} = n\ell I$

$$\oint \vec{B}\ \vec{d\ell} = \mu_0 I_{encl}$$

$$\underbrace{\int_1^2 B\,d\ell}_{0} + \underbrace{\int_2^3 \vec{B}\cdot\vec{d\ell}}_{\vec{B}\perp\vec{d\ell}} + \int_3^4 \vec{B}\cdot\vec{d\ell} + \underbrace{\int_4^1 \vec{B}\cdot\vec{d\ell}}_{\vec{B}\perp d\ell} = \mu_0 n\ell I$$

$$B\int_3^4 d\ell = \mu_0 n\ell I$$

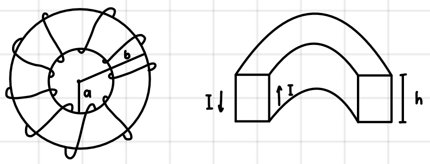
$$B = \mu_0 n I$$

If there's N loops over long length L, then $n = \frac{N}{L}$ so $\vec{B} = \mu_0 \frac{N}{L} I \,(\hat{x})$.

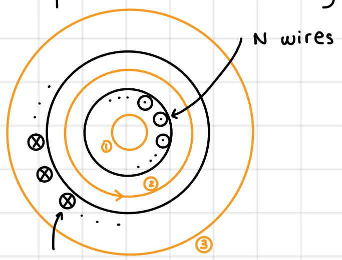
· e.g. magnetic field inside toroid

SOLUTION

↳ toroid is donut w/ wire wrapped around N times




Cut plane ⊥ to current by slicing it in middle (think of cutting bagel).




Region 1:

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

$B(2\pi r) = \mu_0 (0)$ ⟵ no current enclosed

$B = 0$

$\therefore \vec{B} = \vec{0}$

Region 2:

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

$B(2\pi r) = \mu_0 (NI)$

$B = \frac{\mu_0 NI}{2\pi r}$

$\therefore \vec{B} = \frac{\mu_0 NI}{2\pi r} \hat{\varphi}$

Region 3:

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$

↓

N wires carrying current up & down

$B \oint dl = \mu_0 (0)$

$B = 0$

$\therefore \vec{B} = \vec{0}$

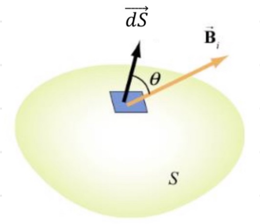
# chapter 12

## INTRODUCTION

- electromagnetic induction: if magnetic field varied w/ time, it creates E-field around it
- in one of Faraday's experiments, when magnet is moved around conducting coil, induced voltage is observed on ends of coil
  - ↳ when magnet's stationary, there's no induced voltage
- time-varying magnetic fields also create E-fields in free space or non-conducting materials
  - ↳ induced voltage/current isn't created

## MAGNETIC FLUX

- consider magnetic field $\vec{B}$ passing thru open surface S:




  - ↳ magnetic flux thru $\vec{dS}$ (which has normal vector $\hat{n}$ to surface) is given by $d\Phi_m = \vec{B}\ \vec{dS} = B\,dS\cos\theta$
- total magnetic flux thru open surface S is $\Phi_m = \int_S \vec{B}\cdot\vec{dS} = \int_S B\,dS\cos\theta$
  - ↳ units are Webers (1 Weber = 1 Tm)
  - ↳ don't confuse w/ Maxwell's 2nd eq, $\oint_S \vec{B}\cdot\vec{dS} = 0$
    - although flux is 0 over closed surface, it can exist over open one

## FARADAY'S LAW

- Faraday's Law states that if magnetic flux enclosed within loop changes w/ time, E-field is created which loops around changing flux
  - ↳ strength of E-field created is proportional to ROC of enclosed magnetic flux
  - ↳ $\oint_C \vec{E}\cdot\vec{dl} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\left(\int_S \vec{B}\cdot\vec{dS}\right)$
    - -ve sign comes from Lenz's Law which states that E-field is created in such a way that opposes ROC of magnetic flux
- comparing to Ampere's Law $\oint_C \vec{B}\cdot\vec{dl} = \mu_0\int_S \vec{J}\cdot\vec{dS}$, Faraday's Law has same form except for time derivative
  - ↳ time varying magnetic flux creates E-field that curls around magnetic field
- Lenz's Law guarantees that total energy is conserved b/c if magnetic flux is time-varying, magnetic energy is being converted to electric energy
  - ↳ magnetic fields are created by moving charges so they're equivalent to kinetic energy
  - ↳ E-fields create potential around them so electric energy is equivalent to potential energy
  - ↳ can compare Faraday's Law to saying time varying KE gets converted to PE

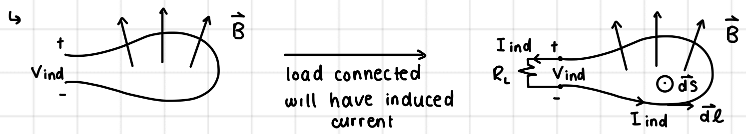
- for electrostatic fields, $\oint_C \vec{E}\cdot\vec{dl} = 0$ but in Faraday's Law, $\oint_C \vec{E}\cdot\vec{dl} = -\frac{d\Phi_m}{dt}$
  - ↳ if magnetic flux is time varying, walking on 2 diff loops across same point could give diff enclosed flux
    - results in diff potential diff across same point (i.e path matters)
- Faraday's Law is non-conservative

# ELECTROMAGNETIC INDUCTION

- consider space where magnetic flux is changing w/ time & place thin conducting wire there
  - ↳ E-field that circles magnetic field is created & applies force on e⁻ in conductor
  - ↳ if conducting loop is closed, current will start flowing
  - ↳ if there's break in loop, charges will collect around break
    - one end will be +ve & other will be -ve
    - voltage diff is created btwn 2 ends
- induced electromotive force / voltage across 2 points in conductor is

$$V_{ind} = \mathcal{E} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{s}\right)$$

↳



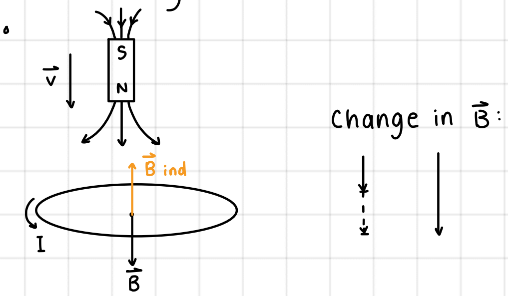

- can choose $d\vec{l}$ to be ccw & $d\vec{s}$ pointing up
  - ↳ if answer is +ve, $I_{ind}$ flows in dir of chosen $d\vec{l}$
  - ↳ if answer is −ve, $I_{ind}$ flows in opp dir of chosen $d\vec{l}$

# LENZ'S LAW

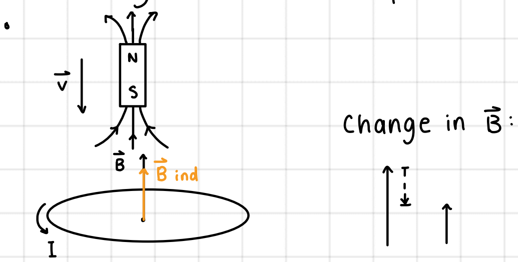
Lenz's Law states that E-field is created in such a way that opposes ROC of magnetic flux

- ↳ if $\vec{B}$ is inc w/ time, induced magnetic field will be in dir opp to $\vec{B}$
- ↳ if $\vec{B}$ is dec w/ time, induced magnetic field will be in same dir as $\vec{B}$

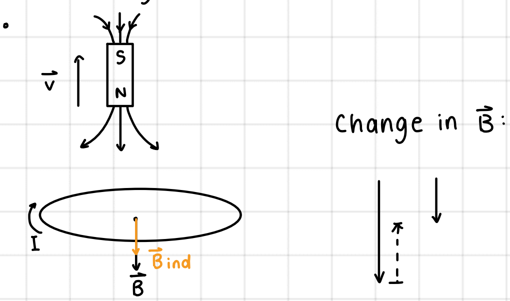
- 4 possible scenarios:
  - ↳ motion of magnet causes inc down flux



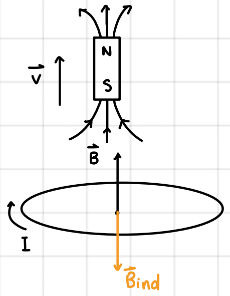

  - ↳ motion of magnet causes dec upward flux




  - ↳ motion of magnet causes dec down flux




  - ↳ motion of magnet causes inc up flux




Change in $\vec{B}$:

> **NOTE**
> · for motional EMF, draw circuit at some time $t$ & calc flux at that time
> ↳ $\Phi_m$ will be function of $t$

## TYPES OF INDUCED EMF

· flux can change due to 2 reasons:
- ↳ magnetic field changes w/ time so induced EMF is called **transformer EMF**
  - e.g. wire carrying low freq AC current sits next to loop
- ↳ magnetic flux changes w/ time due to motion of loop so induced EMF is called motional EMF
  - flux may be changing b/c loop is expanding / contracting so area changes or loop is rotating in magnetic field so angle btwn $\vec{B}$ & $d\vec{S}$ changes

· motional EMF can be better conceptualized w/ Lorentz's force eq
- ↳ force that moving charge sees in magnetic field is $\vec{F}_m = q(\vec{u} \times \vec{B})$
- ↳ $\vec{E} = \frac{\vec{F}_m}{q} = \vec{u} \times \vec{B}$
- ↳ induced field is $V_{ind} = \oint_c \vec{E} \cdot d\vec{l} = \oint (\vec{u} \times \vec{B}) \cdot d\vec{l}$

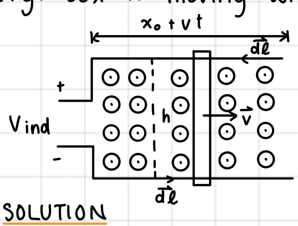
## FLUX LINKAGE

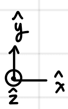
· flux linkage is how magnetic flux links to induced current / field (symbol is $\Lambda$)
- ↳ $V_{ind} = -\frac{d\Lambda}{dt}$
- ↳ i.e. how the "batteries" created link tgt

consider circuit w/ N loops in region w/ changing magnetic flux
- ↳ case 1: loops go in same dir so total induced voltage is N times $V_{ind}$ of single loop
  - i.e. batteries are added up
  - $\Lambda = N\Phi_m$
  - $V_{ind} = -\frac{d\Lambda}{dt} = -N\frac{d\Phi_m}{dt}$
- ↳ case 2: loops change dir every turn
  - polarities of batteries change every turn & cancel each other out
  - if N is even, $\Lambda = 0$ & $V_{ind} = 0$ even though flux exists in every region
  - if N is odd, $\Lambda = \Phi_m$

## EXAMPLES

· e.g. box is moving w/ $\vec{v}$ on conducting rails in region w/ $\vec{B} = B_0\hat{z}$; calc induced voltage




SOLUTION

Assume at $t=0$, bar is at $x_0$ so at time $t$, bar is at $x_0 + vt$.

Assume $d\vec{l}$ is ccw & $d\vec{S}$ points up.

$\Phi_m = \int_S \vec{B} \cdot d\vec{S}$

$= B\int dS$

$= B_0 h(x_0 + vt)$

Since there's only 1 loop, $\Lambda = \Phi_m$

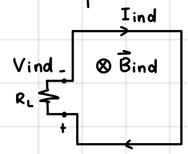
$$\Lambda = B_0 h (x_0 + vt)$$

$$V_{ind} = -\frac{d\Lambda}{dt}$$

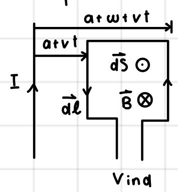
$$\boxed{V_{ind} = -B_0 h v}$$

Since $V_{ind}$ is -ve, its acc polarity is opp to our assumption so $I_{ind}$ is cw

↳ $\vec{B}_{ind}$ points into page

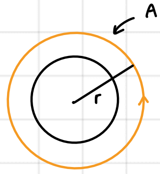



e.g. rectangular loop is moving away $\vec{v}$ from long wire carrying current I ; calc induced EMF on loop




SOLUTION

Assume $\vec{dl}$ is ccw & $\vec{dS}$ points up.




$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{encl}$$

$$B \oint dl = \mu_0 I$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\varphi}$$

At plane of loop, B is pointing into page.

$$\vec{dS} = h dr \odot$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \otimes$$

$$\Phi_m = \int_S \vec{B} \cdot \vec{dS}$$

$$= \int_S - B dS$$

$$= -\int_{a+vt}^{a+w+vt} \frac{\mu_0 I}{2\pi r} (h dr)$$

$$= -\frac{\mu_0 I h}{2\pi} \int_{a+vt}^{a+w+vt} \frac{1}{r} dr$$

$$= -\frac{\mu_0 I h}{2\pi} (\ln(a+w+vt) + \ln(a+vt))$$

$$\Lambda = \Phi_m$$

$$V_{ind} = -\frac{d\Lambda}{dt}$$

$$= \frac{d}{dt}\left(\frac{\mu_0 I h}{2\pi} (\ln(a+w+vt) + \ln(a+vt))\right)$$

$$= \frac{\mu_0 I h}{2\pi}\left(\frac{v}{a+w+vt} - \frac{v}{a+vt}\right)$$

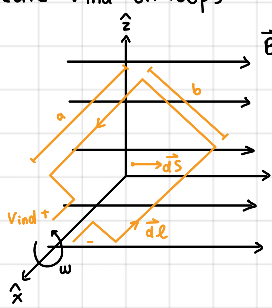
$$= \frac{\mu_0 I h}{2\pi}\left(\frac{va + v^2 - va - vw - v^2}{(a+w+vt)(a+vt)}\right)$$

$$\boxed{V_{ind} = \frac{-\mu_0 I h v w}{2\pi (a+w+vt)(a+vt)}}$$

Since $V_{ind}$ is -ve, $I_{ind}$ is cw & $\vec{B}_{ind}$ should be pointing into page.

e.g. generator: N loops are placed in uniform magnetic field & they're rotating w/ angular velocity ω ; calc $V_{ind}$ on loops

$$\vec{B} = B_0 \hat{y}$$

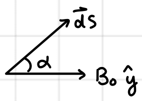



SOLUTION

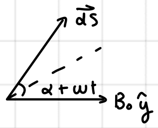
As time changes, $\vec{dS}$ moves w/ loop

↳ $\vec{dS}$ moves in ccw circle

t = 0:




t = t:




$$d\Phi_m = \vec{B} \cdot \vec{dS}$$

$$= B_0 \, dS \cos(\alpha + \omega t)$$

$$\Phi_m = B_0 \cos(\alpha + \omega t) \int_S dS$$ ⟵ both $B_0$ & $\cos(\alpha + \omega t)$ are constant over surface of loop

$$= B_0 \cos(\alpha + \omega t)(ab)$$

$$\Lambda = N\Phi$$

$$= N B_0 \cos(\alpha + \omega t)(ab)$$

$$V_{ind} = -\frac{d\Lambda}{dt}$$

$$= N B_0 \, ab \sin(\alpha + \omega t) \quad \omega$$

$$V_{ind} = N B_0 ab\omega \sin(\alpha + \omega t)$$

Since $V_{ind}$ is +ve, polarity chosen is correct.

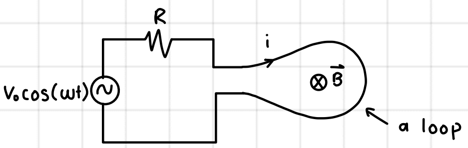
# chapter 13

## INTRODUCTION

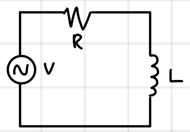
- KVL depends on assumption that there's no fluctuating magnetic field within closed loop
  - ↳ used when there's low freq w/ negligible time variation (i.e. $\frac{d\Phi_m}{dt} = 0$)
    - $\oint \vec{E} \cdot d\vec{\ell} = 0$
- accurate truth is Faraday's Law: $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_m}{dt}$
- when using voltmeters to measure voltage, path matters & readings may change despite placing it across same points
  - ↳ due to non-conservative nature of Faraday's Law

## INDUCTANCE

- consider below circuit:




  - ↳ i varies w/ time so it creates flux through loop (also varies w/ time) & in turn, creates $V_{ind}$ across loop
- $V_{ind} = -\frac{d\Phi_m}{dt}$ but $\Phi$ is proportional to i that's creating field
  - ↳ proportionality constant is called inductance: $\Phi \propto i \longrightarrow \Phi = Li$
  - ↳ $V_{ind} = -L\frac{di}{dt}$
- effect of magnetic flux on current is accounted for by defining circuit element of inductance
  - ↳ above circuit becomes:




  - ↳ can use KVL to analyze
    - still is an apx
- inductance is parameter that depends on geometrical parameters of loop
  - ↳ $L = \frac{\Lambda}{I} = \frac{\Phi_m}{i}$

## SELF AND MUTUAL INDUCTANCE

- consider 2 loops placed next to each other:


  - ↳ $I_1$ creates magnetic field $B_1$ in space around it so enclosed flux on loop 1 itself is $\Phi_1 = \int_{S_1} \vec{B}_1 \cdot d\vec{S}_1$
    - if there's more than 1 turn, $\Lambda_1 = N_1 \Phi_1 = N_1 \int_{S_1} \vec{B}_1 \cdot d\vec{S}_1$
- self-inductance is how time varying flux affects its own current that's generating it
  - ↳ $L_1 = \frac{\Lambda_1}{I_1} = \frac{N_1 \int_{S_1} \vec{B}_1 \cdot d\vec{S}_1}{I_1}$ is self-inductance on loop 1
  - ↳ $L_2 = \frac{\Lambda_2}{I_2} = \frac{N_2 \int_{S_2} \vec{B}_2 \cdot d\vec{S}_2}{I_2}$ is self-inductance on loop 2
- mutual inductance is effect of time varying flux from another coil on current
  - ↳ $I_1$ creates enclosed flux on loop 2 so $\Phi_{1\to2} = \int_{S_2} \vec{B}_1 \, d\vec{S}_2$
    - linkage from 1 to 2 is $\Lambda_{1\to2} = N_2 \Phi_{1\to2} = N_2 \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2$
- mutual effect of $I_1$ on loop 2 is $M_{1\to2} = \frac{N_2 \Phi_{1\to2}}{I_1} = \frac{N_2 \int_{S_2} \vec{B}_1 \, d\vec{S}_2}{I_1}$
  - ↳ similarily, $M_{2\to1} = \frac{N_1 \Phi_{2\to1}}{I_2} = \frac{N_1 \int_{S_1} \vec{B}_2 \cdot d\vec{S}_1}{I_2}$

· by reciprocity theorem, $M_{1\to 2} = M_{2\to 1} = M$

## CALCULATING INDUCTANCE

· to calc self-inductance:
- ↳ apply $I$ to circuit
- ↳ calc magnetic flux density $B$ due to current
- ↳ integrate flux in enclosed surface, which is $\perp$ to current flow
- ↳ calc linkage & divide $\Lambda$ w/ $I$ to get $L$
  - $I$ should cancel

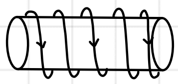
· to calc mutual inductance:
- ↳ $M_{1\to 2} = M_{2\to 1} = M$
- ↳ choose circuit where we can apply Ampere's Law to get $B$
  - call it circuit 1
- ↳ apply $I_1$ on it & calc $B_1$ in space
- ↳ integrate $B_1$ on enclosed surface of loop 2 to get $\Phi_{1\to 2}$
- ↳ calc $\Lambda_{1\to 2}$ & divide by $I_1$ to get $M$

· can also use energy principles to calc inductance
- ↳ works well when linkage becomes complicated
- ↳ magnetic energy density is $w_m = \frac{1}{2}\frac{B^2}{\mu_0}$
  - total energy in volume of magnetic field is $W_m = \int_{vol} \frac{1}{2}\frac{B^2}{\mu_0}\,dv$
- ↳ energy in inductor is also $W_m = \frac{1}{2}LI^2$
  - $\frac{1}{2}LI^2 = \int_{vol} \frac{1}{2}\frac{B^2}{\mu_0}\,dv$

    $L = \frac{1}{I^2}\int_{vol} \frac{B^2}{\mu_0}\,dv$
  - $B$ is integrated over complete volume of universe so this works well when $B$ is confined within small space (e.g. coaxial cable w/ $B=0$ outside)

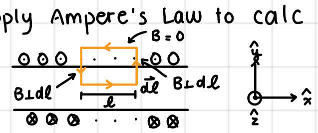
## EXAMPLES

· e.g. very long solenoid has length $\ell$, $N$ turns, & cross-sectional area $A$; calc self-inductance



SOLUTION

Apply Ampere's Law to calc $B$:

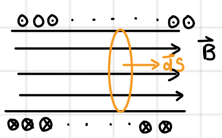



$n = \frac{N}{\ell}$

$\oint \vec{B}\cdot\vec{d\ell} = \mu_0 I_{encl}$

$B\ell = \mu_0 n \ell I$

$\vec{B} = \mu_0 n I\ \hat{x}$




$\Phi_m = \int \vec{B}\cdot\vec{dS}$
$= B\int dS$
$= BA$
$= \mu_0 n I A$

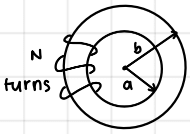
Since all loops are connected in series in same dir:

$\Lambda = N\Phi$
$= N(\mu_0 n I A)$
$= N\mu_0\left(\frac{N}{\ell}\right)IA$
$= \frac{N^2\mu_0 IA}{\ell}$

$L = \frac{\Lambda}{I}$

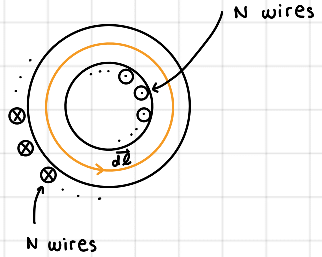
$L = \frac{N^2\mu_0 A}{\ell}$

· e.g. self-inductance of toroid




## SOLUTION

Cut $\perp$ cross-section so we can calc $\vec{B}$ using Ampere's Law:

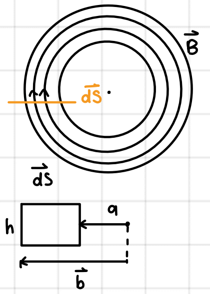



$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl}$$

$$B \oint d\ell = \mu_0 NI$$

$$\vec{B} = \frac{\mu_0 NI}{2\pi r} \hat{\varphi}$$

Inside & outside toroid, $\vec{B} = \vec{0}$ so only consider middle region.




We take $\perp$ cut across $\vec{B}$. Since $\vec{B}$ varies w/r, h stays constant in dS.

$$\Phi = \int \vec{B} \cdot d\vec{S}$$

$$= \int B\, dS$$

$$= \int B h\, dr$$

$$= \int_a^b \frac{\mu_0 NIh}{2\pi} \frac{1}{r} dr$$

$$= \frac{\mu_0 NIh}{2\pi} \ln\left(\frac{b}{a}\right)$$

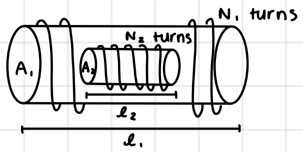
$$\Lambda = N\Phi$$

$$= \frac{\mu_0 N^2 Ih}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L = \frac{\Lambda}{I}$$

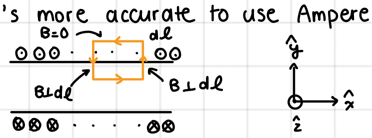
$$\boxed{L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)}$$

· e.g. there's very long solenoid w/ cross section $A_1$, $N_1$ turns, & length $\ell_1$; inside, there's another solenoid w/ cross section $A_2$, $N_2$ turns, & length $\ell_2$; calc mutual inductance btwn 2 solenoids




## SOLUTION

It's more accurate to use Ampere's Law to calc $\vec{B}_1$:




$$n_1 = \frac{N_1}{\ell_1}$$

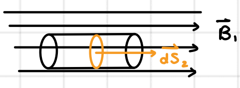
$$\oint \vec{B}_1 \cdot d\vec{\ell} = \mu_0 I_{encl}$$

$$B_1 \oint d\ell = \mu_0 n_1 \ell I_1$$

$$B_1 = \mu_0 n_1 I_1$$

$$\vec{B} = \frac{\mu_0 N_1 I_1}{\ell_1} \hat{x}$$

To calculate $\Phi_{1\to 2}$:




$$\Phi_{1\to 2} = \int \vec{B}_1 \cdot d\vec{S}_2$$

$$= B_1 \int dS_2$$

$$= \frac{\mu_0 N_1 I_1}{\ell_1} (A_2)$$

$$\Lambda_{1\to 2} = N_2 \Phi_{1\to 2}$$

$$= \frac{\mu_0 N_1 N_2 A_2 I_1}{\ell_1}$$

$$M = \frac{\Lambda_{1\to 2}}{I_1}$$

$$\boxed{M = \frac{\mu_0 N_1 N_2 A_2}{\ell_1}}$$