· FRIEZES ·

EXPLORING COMBINATORIAL WINDMILLS ACROSS ALGEBRAIC LANDSCAPE

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April 2024

COMBINATORICS
"Friezes"

GEOMETRY

ALGEBRA

REPRESENTATION THEORY

COMBINATORICS

"Friezes"

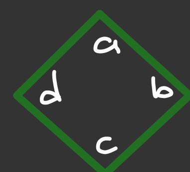
$$\cdots \quad \begin{array}{ccccccccccccccc} 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\ & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \end{array} \quad \cdots$$

FRIEZES

$$\begin{array}{cccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \cdots \\
\cdots & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & &
\end{array}$$

FRIEZES

$$
\begin{array}{ccccccccccccccccc}
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
& 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & \\
& & ? & & ? & & ? & & ? & & ? & & ? & & & &
\end{array}
$$

Rule : 

$\rightsquigarrow \; bd - ac = 1$

FRIEZES

0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 . . . .

. . . 2 2 1 4 1 2 2 2

3 1 3 3 1 3 3

FRIEZES

$$
\begin{array}{ccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & \\
 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & \\
 & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & &
\end{array}
$$

FRIEZES

```
        0   0   0   0   0   0   0   0
          1   1   1   1   1   1   1       ...
...     2   2   1   4   1   2   2   2
          3   1   3   3   1   3   3
        4   1   2   2   2   1   4
          1   1   1   1   1   1   1
        0   0   0   0   0   0   0
```

COMBINATORICS
"Friezes"

$$
\cdots
\begin{array}{cccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & \\
 & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 \\
4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
\cdots
$$

'Conway-Coxeter frieze'

CONWAY - COXETER

finite friezes of width n $\quad \overset{1-1}{\longleftrightarrow} \quad$ triangulations of $(n+3)$-gon

CONWAY - COXETER

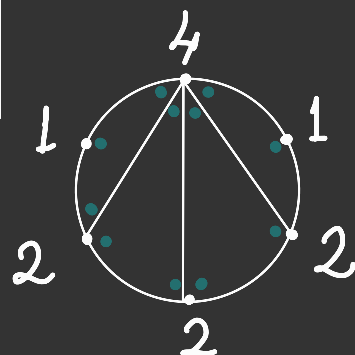
finite friezes of width n $\overset{1-1}{\longleftrightarrow}$ triangulations of $(n+3)$-gon

Symmetries :

$$
\begin{array}{ccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & \\
 & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & & \\
\cdots & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & & \cdots & \\
 & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & & &
\end{array}
$$

COMBINATORICS
"Friezes"

GEOMETRY
"triangulations"



$\longleftrightarrow$

$$
\begin{array}{ccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & & \cdots \\
 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & \\
 & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
$$

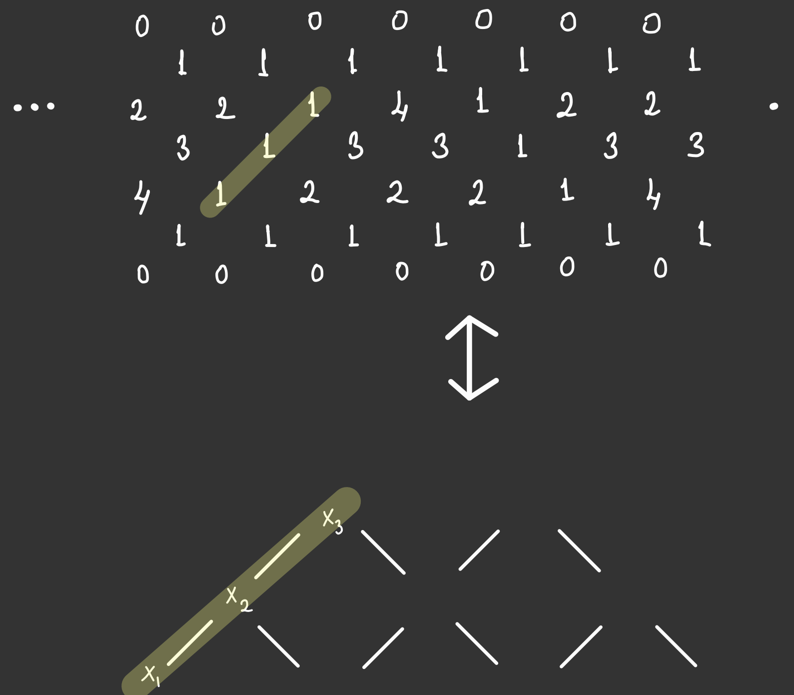
COMBINATORICS
"Friezes"

$$\begin{array}{ccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
\cdots & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & \cdots \\
 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 \\
 & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

ALGEBRA
"cluster algebras"

COMBINATORICS

"Friezes"

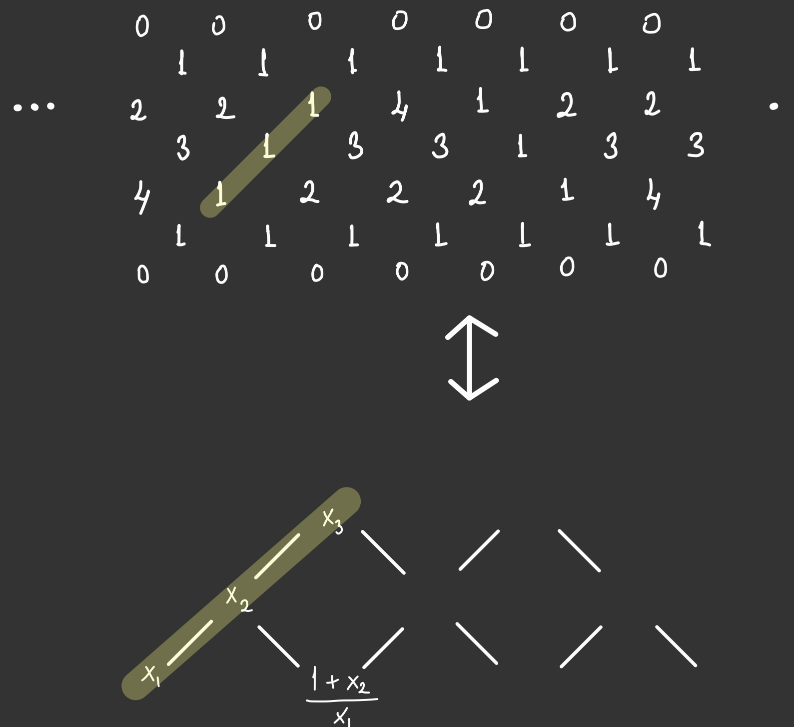


ALGEBRA

"cluster algebras"

COMBINATORICS

"Friezes"



ALGEBRA

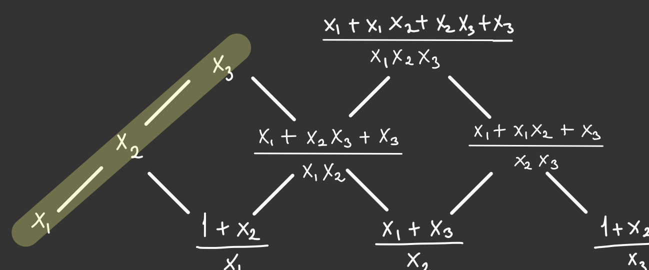
"cluster algebras"

COMBINATORICS
"Friezes"

$$\begin{array}{ccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & \cdots & \\
 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & \\
 & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}$$

$\updownarrow$

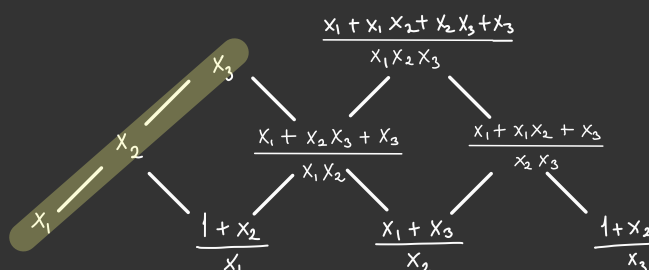
ALGEBRA
"cluster algebras"



COMBINATORICS
"Friezes"

0 0 0 0 0 0 0
1 1 1 1 1 1 1
2 2 1 4 1 2 2
3 1 3 3 1 3 3
4 1 2 2 2 1 4
1 1 1 1 1 1 1
0 0 0 0 0 0 0

$\updownarrow$



ALGEBRA
"cluster algebras"

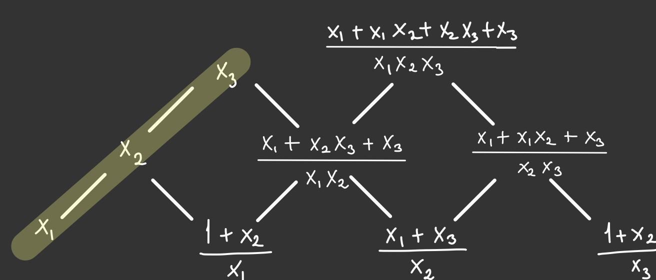
'A_3 cluster algebra'
: subring of the field of fractions $\mathbb{Q}(x_1,x_2,x_3)$

COMBINATORICS
"Friezes"

0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	1	4	1	2	2
3	1	3	3	1	3	3
4	1	2	2	2	1	4
1	1	1	1	1	1	1
0	0	0	0	0	0	0

$\updownarrow$

(Set $x_1 = x_2 = x_3 = 1$)



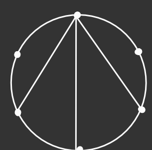
ALGEBRA
"cluster algebras"

COMBINATORICS
"Friezes"

$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & \\
 & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & \\
4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

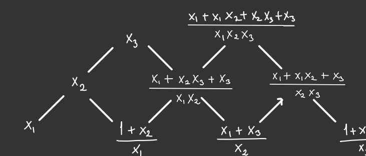
'Conway-Coxeter frieze'

GEOMETRY
"triangulations"



'A_3 triangulation'

ALGEBRA
"cluster algebras"



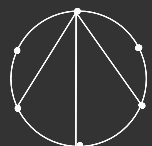
'A_3 cluster algebra'

COMBINATORICS
"Friezes"

0		0		0		0		0		0		0	
	1		1		1		1		1		1		1
2		2		1		4		1		2		2	
	3		1		3		3		1		3		3
4		1		2		2		2		1		4	
	1		1		1		1		1		1		1
0		0		0		0		0		0		0	

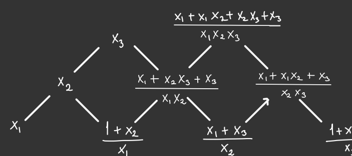
'Conway-Coxeter frieze'

GEOMETRY
"triangulations"

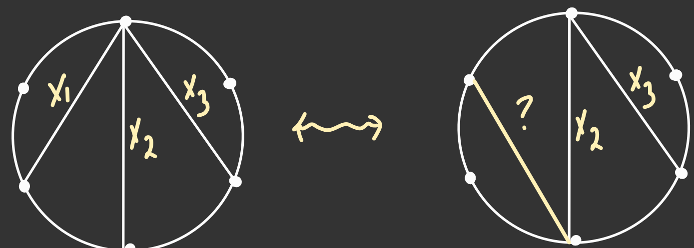


'A_3 triangulation'

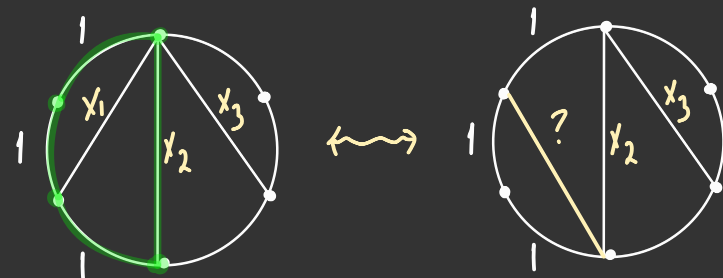
ALGEBRA
"cluster algebras"



'A_3 cluster algebra'



Ptolemy relation :



$$? = \frac{1 \cdot 1 + 1 \cdot x_2}{x_1}$$

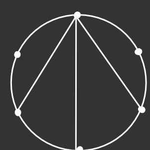
$$= \frac{1 + x_2}{x_1}$$

COMBINATORICS
"Friezes"

0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	1	4	1	2	2
3	1	3	3	1	3	3
4	1	2	2	2	1	4
1	1	1	1	1	1	1
0	0	0	0	0	0	0

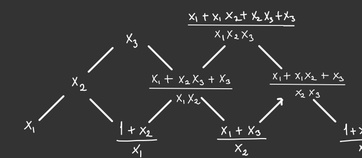
'Conway-Coxeter frieze'

GEOMETRY
"triangulations"

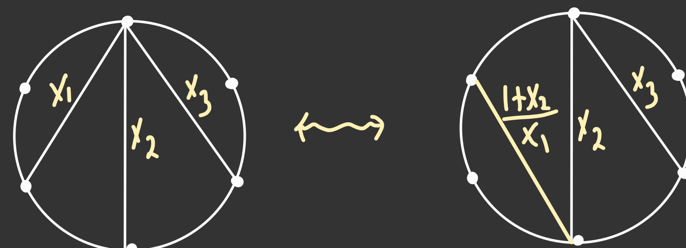


'A_3 triangulation'

ALGEBRA
"cluster algebras"



'A_3 cluster algebra'

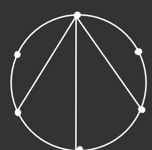


COMBINATORICS
" Friezes "

$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & \\
& 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & \\
4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

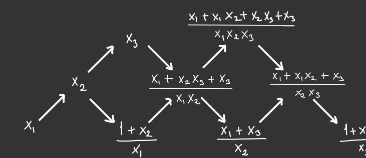
'Conway-Coxeter frieze'

GEOMETRY
" triangulations "



'A_3 triangulation'

ALGEBRA
"cluster algebras"



'A_3 cluster algebra'

REPRESENTATION THEORY
"cluster categories"

GEOMETRY

"triangulations"

REPRESENTATION THEORY

"cluster categories"

$$Q = \quad 1 \leftarrow 2 \leftarrow 3$$

kQ = the path algebra of Q

$$\operatorname{mod} kQ \simeq \operatorname{rep} Q$$

$Q = \overset{1}{\bullet} \xleftarrow{\alpha} \overset{2}{\bullet} \xleftarrow{\beta} \overset{3}{\bullet}$

- A quiver representation : $V_1 \xleftarrow{\varphi_\alpha} V_2 \xleftarrow{\varphi_\beta} V_3$

  ↳ vector spaces at vertices,

  ↳ linear map for arrows.

$$Q = \begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] & \underset{3}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}$$

•

$$k \leftarrow k \leftarrow k$$

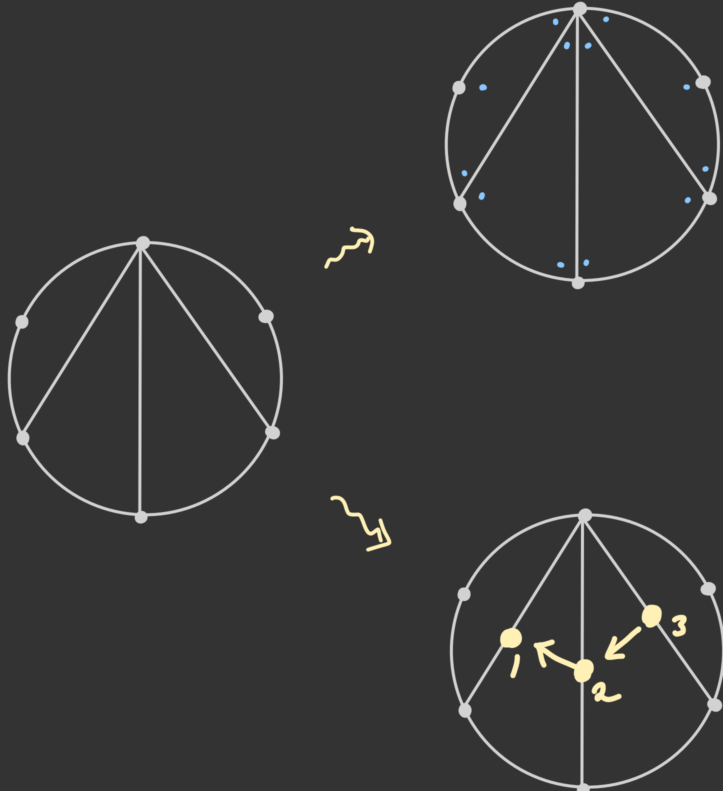
$$k \leftarrow k \leftarrow 0 \qquad 0 \leftarrow k \leftarrow k$$

$$k \leftarrow 0 \leftarrow 0 \qquad 0 \leftarrow k \leftarrow 0 \qquad 0 \leftarrow 0 \leftarrow k$$

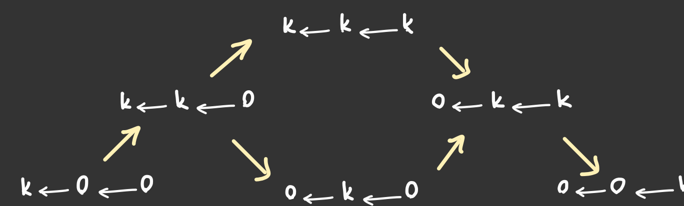
$$Q = \overset{\bullet}{1} \xleftarrow{\alpha} \overset{\bullet}{2} \xleftarrow{\beta} \overset{\bullet}{3}$$

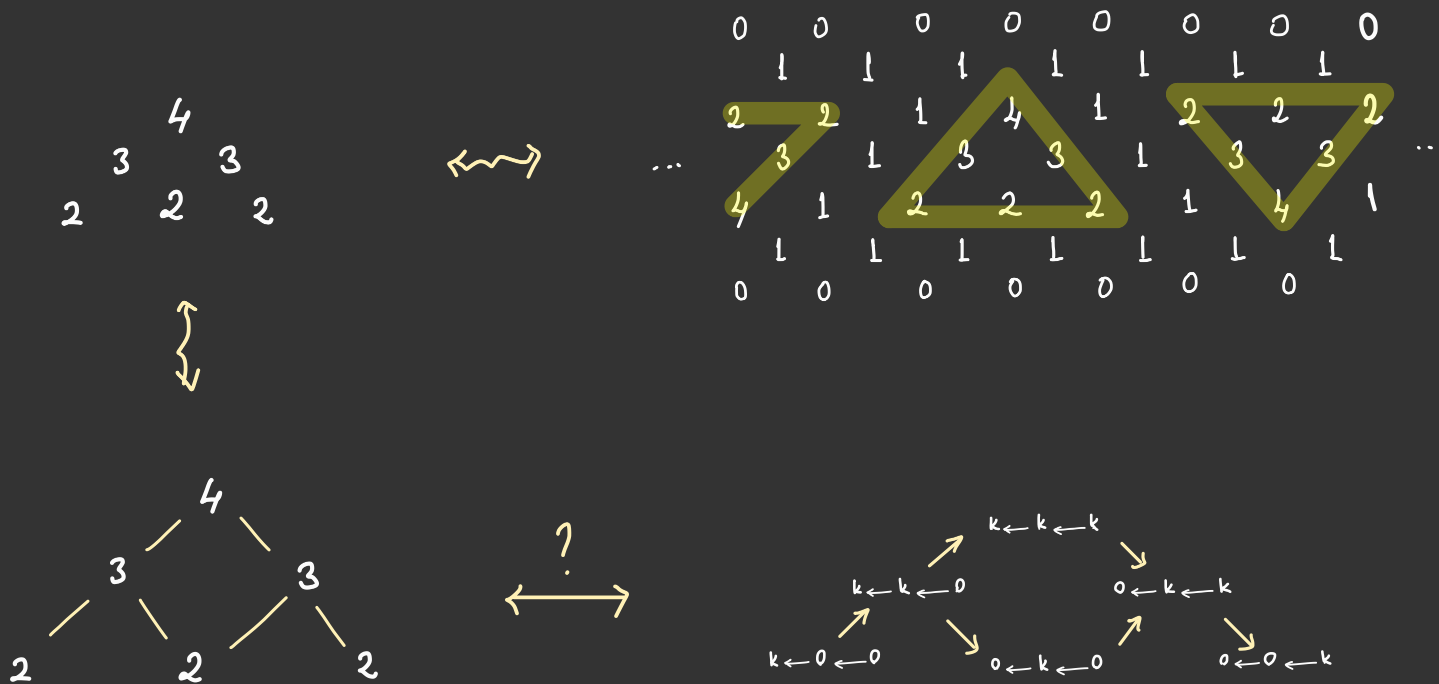
- rep Q =

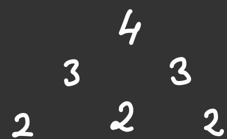
$$\begin{tikzcd}[column sep=small]
 & & k \leftarrow k \leftarrow k \arrow[dr] & & \\
 & k \leftarrow k \leftarrow 0 \arrow[ur] \arrow[dr] & & 0 \leftarrow k \leftarrow k \arrow[dr] & \\
k \leftarrow 0 \leftarrow 0 \arrow[ur] & & 0 \leftarrow k \leftarrow 0 \arrow[ur] & & 0 \leftarrow 0 \leftarrow k
\end{tikzcd}$$

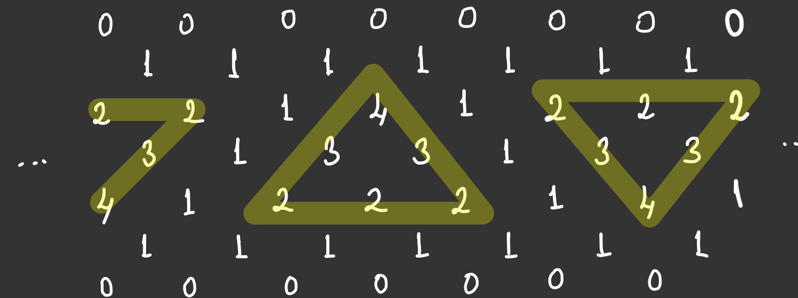

1
2
3

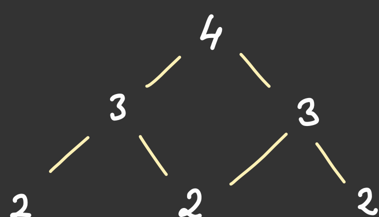

0 0 0 0 0 0 0 0
1 1 1 1 1 1 1
2 2 1 4 1 2 2 2
3 1 3 3 1 3 3
4 1 2 2 2 1 4 1
1 1 1 1 1 1 1
0 0 0 0 0 0 0


k ← k ← k
k ← k ← 0
0 ← k ← k
k ← 0 ← 0
0 ← k ← 0
0 ← 0 ← k

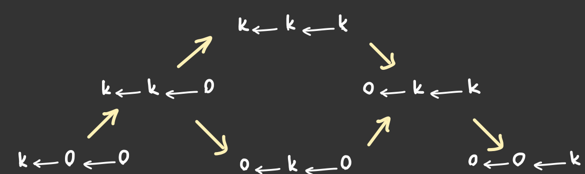








counting submodules!

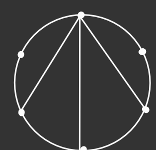


COMBINATORICS
"Friezes"

0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	1	4	1	2	2
3	1	3	3	1	3	3
4	1	2	2	2	1	4
1	1	1	1	1	1	1
0	0	0	0	0	0	0

'Conway-Coxeter frieze'

GEOMETRY
"triangulations"



'A_3 triangulation'

ALGEBRA
"cluster algebras"

$$
\begin{tikzcd}
 & & x_3 \arrow[dr] & & \frac{x_1+x_1x_2+x_2x_3+x_3}{x_1x_2x_3} \arrow[dr] & & \\
 & x_2 \arrow[ur] \arrow[dr] & & \frac{x_1+x_2x_3+x_3}{x_1x_2} \arrow[ur] \arrow[dr] & & \frac{x_1+x_1x_2+x_3}{x_2x_3} \arrow[dr] & \\
x_1 \arrow[ur] & & \frac{1+x_2}{x_1} \arrow[ur] & & \frac{x_1+x_3}{x_2} \arrow[ur] & & \frac{1+x_2}{x_3}
\end{tikzcd}
$$

'A_3 cluster algebra'

REPRESENTATION THEORY
"cluster categories"

$$
\begin{tikzcd}
 & & P_3[1] \arrow[dr] & & P_3 \arrow[dr] & & \\
 & P_2[1] \arrow[ur] \arrow[dr] & & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\
P_1[1] \arrow[ur] & & P_1 \arrow[ur] & & S_2 \arrow[ur] & & S_3
\end{tikzcd}
$$

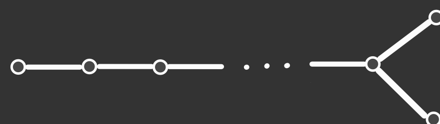
'A_3 cluster category'

THEOREM (GABRIEL '72)

Q is a Dynkin diagram iff kQ has finitely many indecomposable/iso.

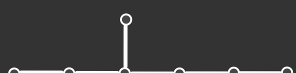
A_n :

D_n :



E_6 :

E_7 :

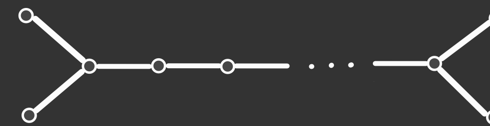


E_8 :

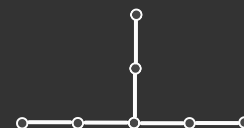
TAME HEREDITARY ALGEBRAS

$\tilde{A}_n$:

$\tilde{D}_n$:



$\tilde{E}_6$:



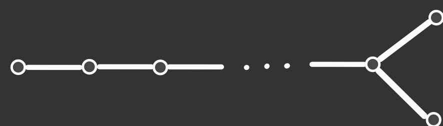
$\tilde{E}_7$:

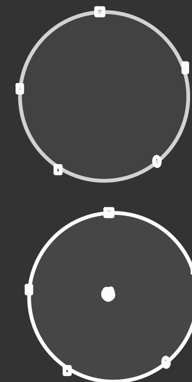
$\tilde{E}_8$:

THEOREM (GABRIEL '72)

Q is a Dynkin diagram iff kQ has finitely many indecomposable/iso.

A_n : 

D_n : 



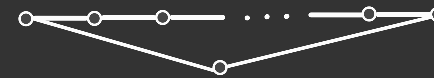
E_6 : 

E_7 : 

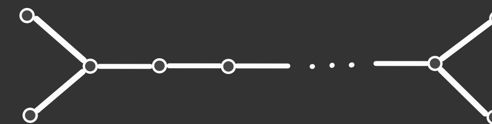
E_8 : 

TAME HEREDITARY ALGEBRAS

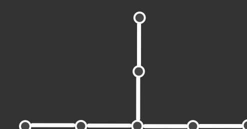
$\tilde{A}_n$:



$\tilde{D}_n$:



$\tilde{E}_6$:



$\tilde{E}_7$:

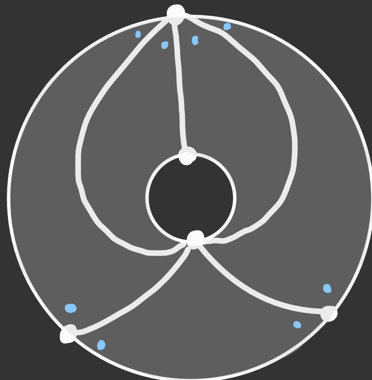
$\tilde{E}_8$:

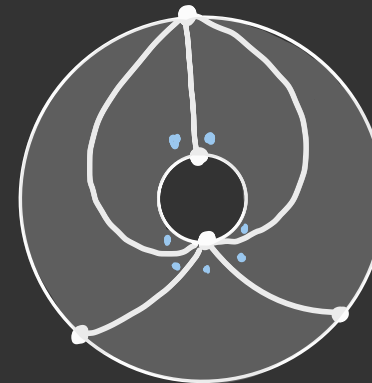
Infinite Friezes

Annulus

cluster algebra of type $\tilde{A}$

cluster category of $\tilde{A}$





0 0 0 0 0 0 0

1 1 1 1 1 1 1

… 4 2 2 4 2 2 4 …

7 7 3 7 7 3 7

12 10 10 12 10 10

⋮

0 0 0 0

1 1 1 1

… 2 5 2 5 …

9 9 9 9

40 16 40

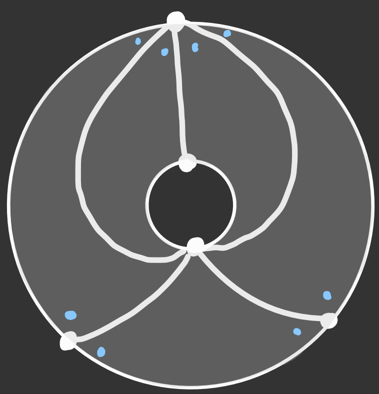
⋮

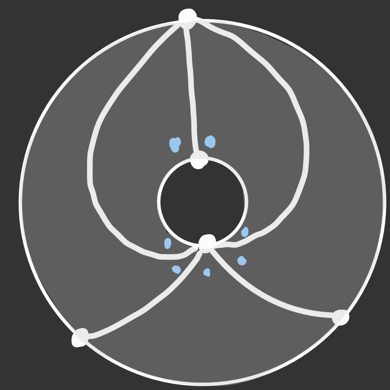
CONWAY - COXETER

triangulations of $(n+3)$-gon $\xleftrightarrow{1-1}$ finite friezes of width n

Baur - Parsons - Tschabold '15

Every periodic infinite frieze comes from a triangulation of annulus.





0 0 0 0 0 0 0

1 1 1 1 1 1 1

4 2 2 4 2 2 4

... 7 7 3 7 7 3 7 ...

12 10 10 12 10 10

⋮

0 0 0 0

1 1 1 1

... 2 5 2 5 ...

9 9 9 9

40 16 40

⋮

Baur - Fellner - Parsons - Tschabold

The growth coefficient is the same for both of the friezes in an annulus.

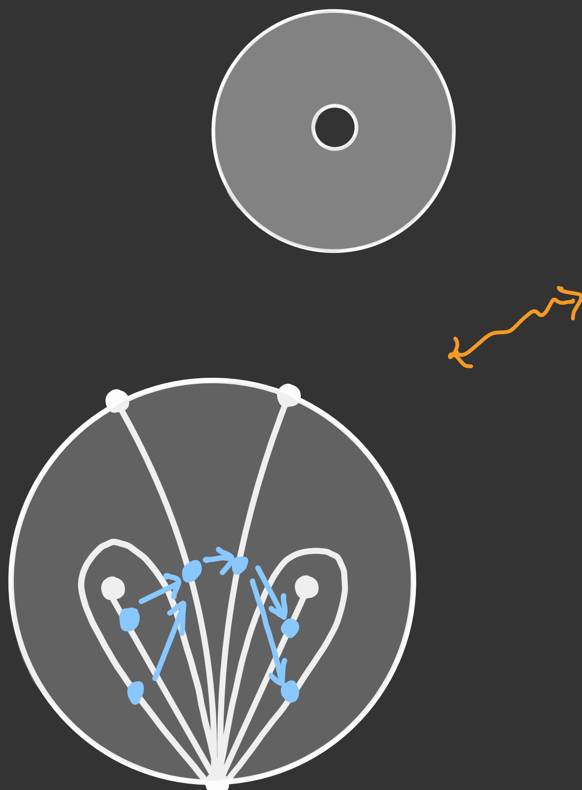
E. Gunawan

L. Bittmann

K. Baur

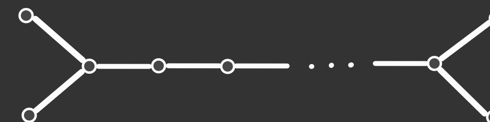
G. Todorov

TAME HEREDITARY ALGEBRAS

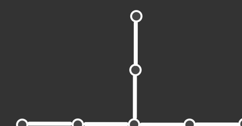


$\tilde{A}_n$:

$\tilde{D}_n$:



$\tilde{E}_6$:

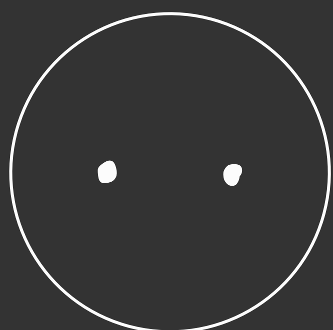


$\tilde{E}_7$:

$\tilde{E}_8$:

Joint work with K. Baur, L. Bittmann, E. Gunawan, G. Todorov

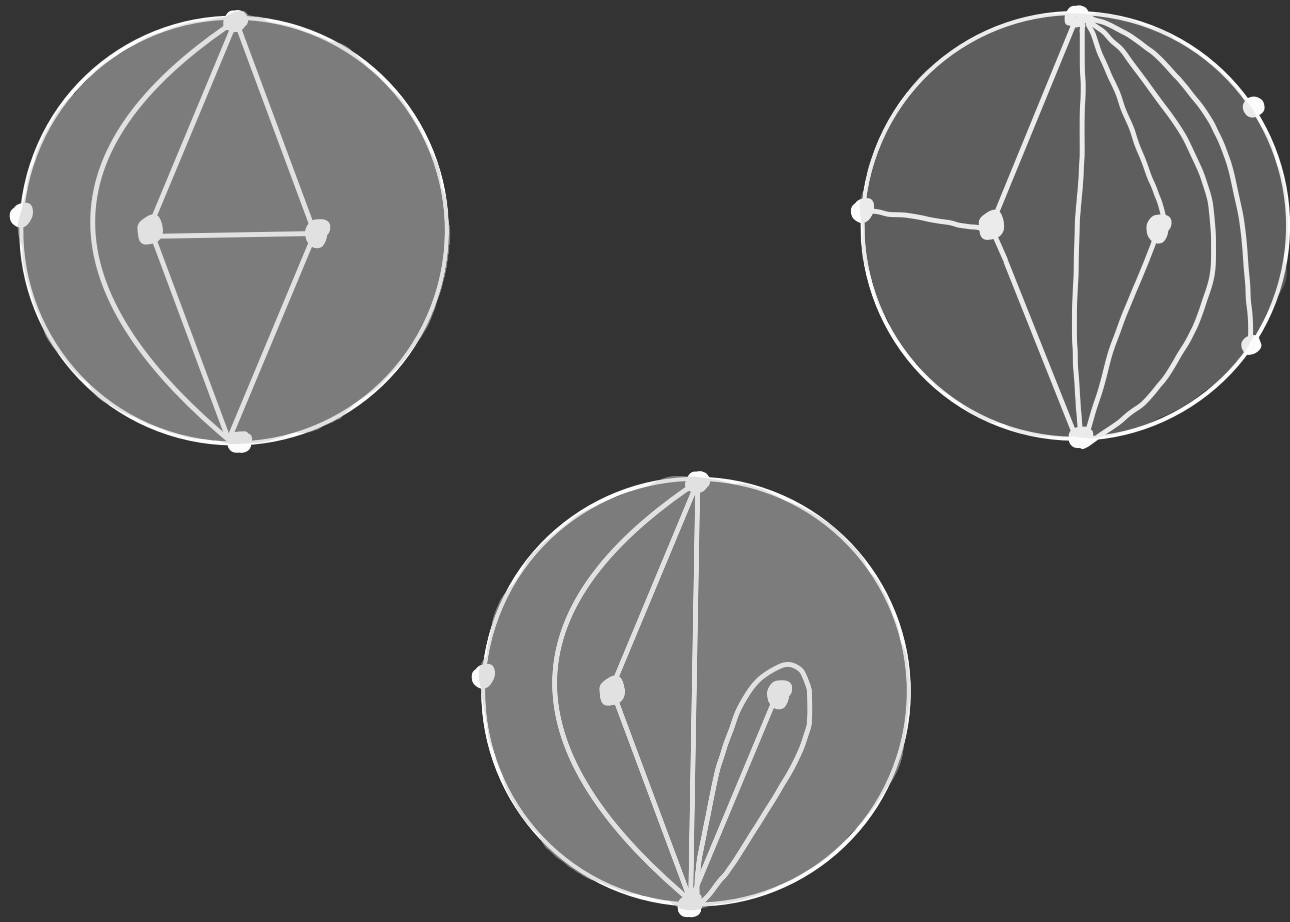
Three pairs of Infinite Friezes

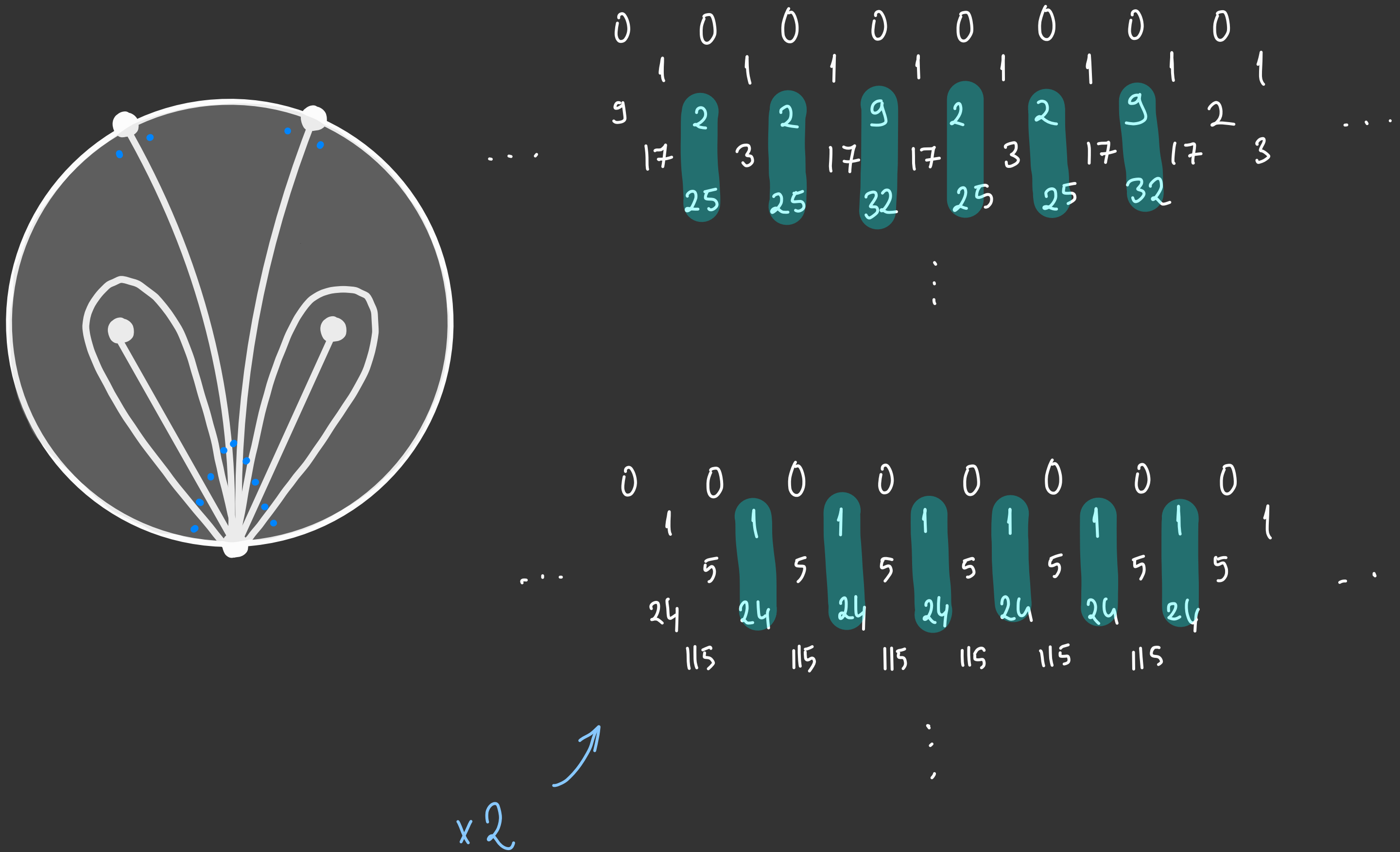


Twice-punctured disk

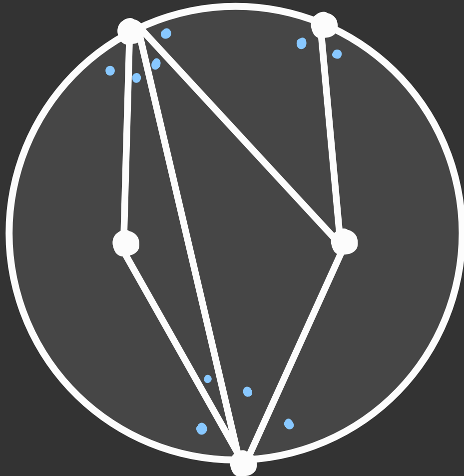
cluster algebra of type $\tilde{D}$

cluster category of $\tilde{D}$





0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1
9 2 2 9 2 2 9 2
17 3 17 17 3 17 17 3
25 25 32 25 25 32
0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1
5 5 5 5 5 5 5
24 24 24 24 24 24 24
115 115 115 115 115 115
x 2



0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1

. . . 2 4 4 2 4 4 2 . . .

7 15 7 7 15 7

26 26 24 26 26

45 89 89 45

⋮

0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1

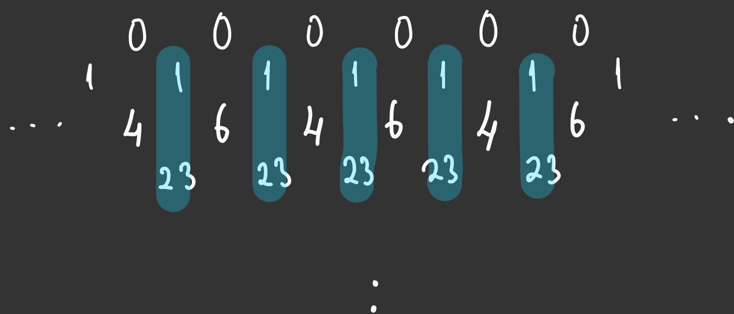
. . . 2 4 4 2 4 4 2 . . .

7 15 7 7 15 7

26 26 24 26 26

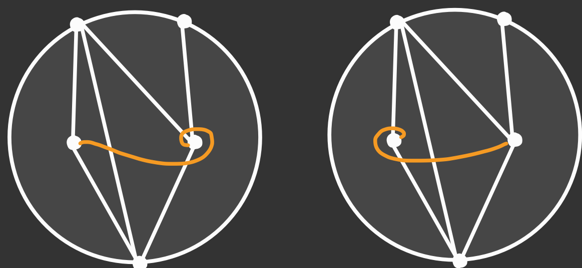
45 89 89 45

⋮

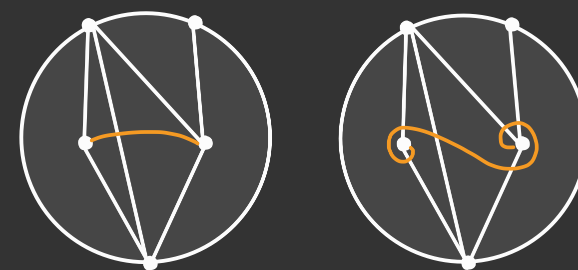

0 0 0 0 0 0
1 1 1 1 1 1 1
. . . 4 6 4 6 4 6 . . .
23 23 23 23 23
:


0 0 0 0 0 0
1 1 1 1 1 1 1
. . . 2 12 2 12 2 12 . . .
23 23 23 23 23
44 44 44 44
:

0 0 0 0 0 0
1 1 1 1 1 1 1
... 4 6 4 6 4 6 ...
23 23 23 23 23
:



0 0 0 0 0 0
1 1 1 1 1 1 1
... 2 12 2 12 2 12 ...
23 23 23 23 23
44 44 44 44
:



THM [BBGTY]

The growth coefficient is the same for all three friezes of twice-punctured disk.


0 0 0 0 0 0 0 0
1 1 1 1 1 1 1
T H A N K Y O U
159 7 13 153 274 374 314
139 90 142 3811 4099 7829
for q=(20, 8, 1, 14, 11, 25, 15, 21)