CATEGORIFICATION OF WEBS

EMINE YILDIRIM - UNIVERSITY OF LEEDS

JOINT WORK WITH IAN LE

- ALGEBRA SEMINAR AT YORK, February 20th, 2025 -

MOTIVATION

- Classical invariant theory studies rings of $SL(V)$ invariants of collection of vectors and linear forms.

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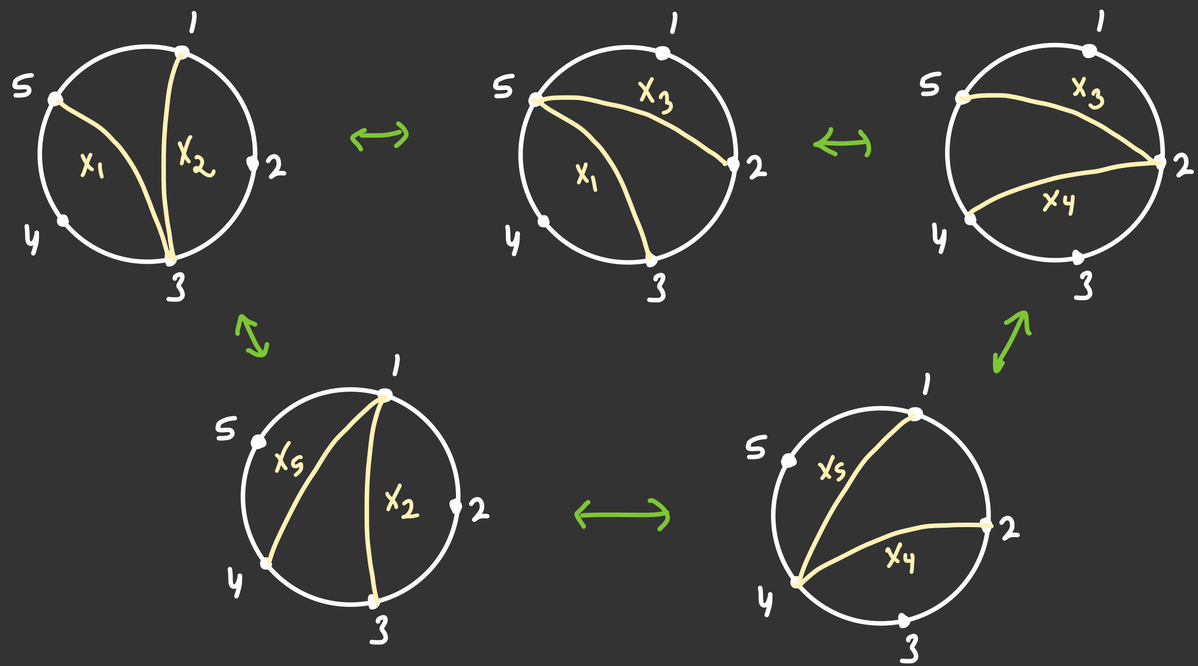
Conjecture (Fomin - Pylyavskyy):

Every such ring carries a cluster algebra structure.

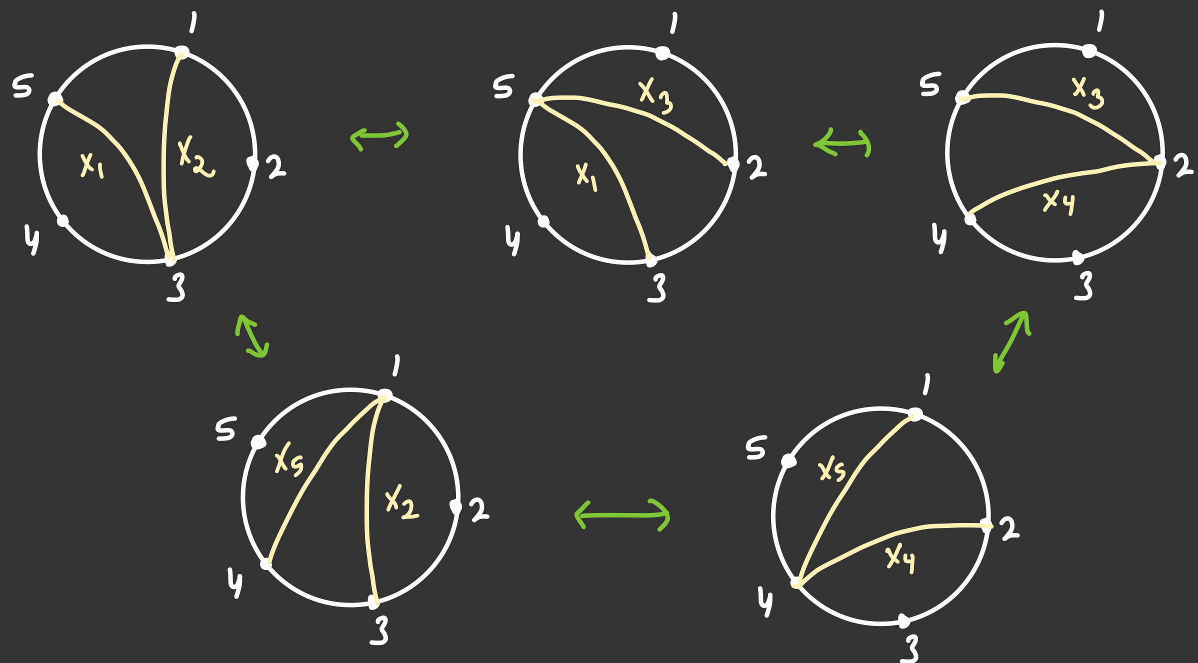
A cluster algebra



A cluster algebra

"Ptolemy Rule"

$$x_3 = \frac{1+x_2}{x_1}$$



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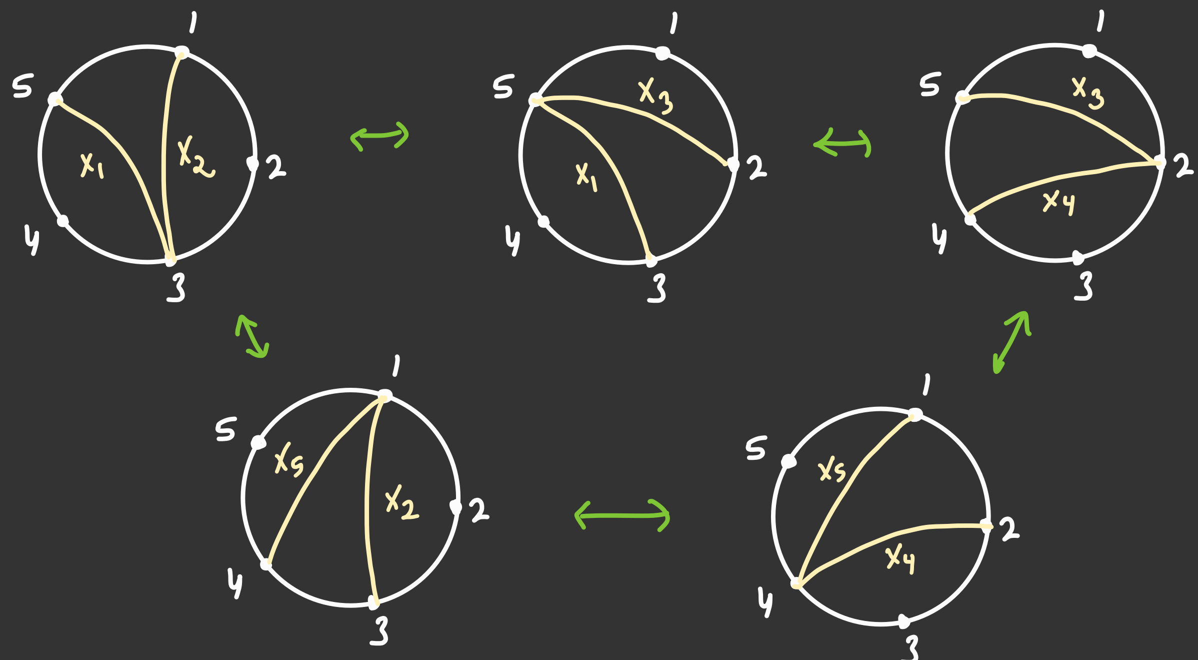
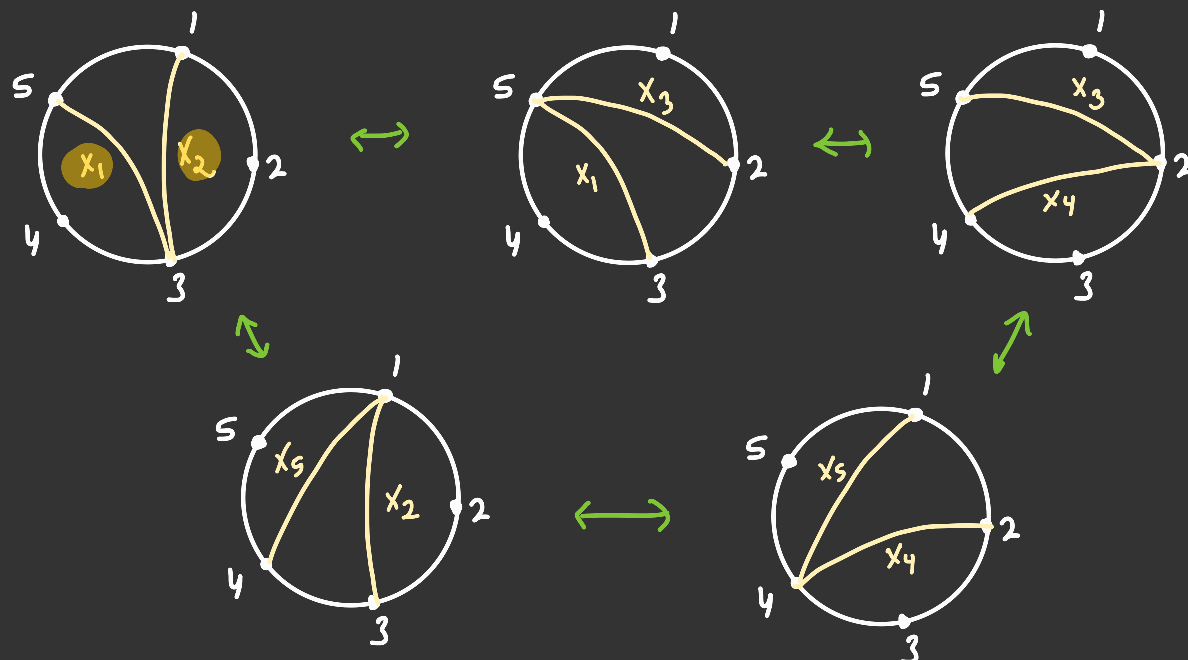


$$x_4 = \frac{1+x_3}{x_1} = \frac{1+\frac{1+x_1}{x_2}}{x_1} = \frac{1+x_1+x_2}{x_1 x_2}$$

A cluster algebra

$$\boxed{x_3 = \frac{1+x_2}{x_1}}$$



$$x_4 = \frac{1+x_3}{x_1} = \frac{1+\frac{1+x_1}{x_2}}{x_1} = \boxed{\frac{1+x_1+x_2}{x_1 x_2}}$$

$$x_5 = \frac{1+x_4}{x_3} = \frac{\frac{1+x_1+x_2}{x_1x_2}+1}{\frac{1+x_1}{x_2}} = \boxed{\frac{1+x_2}{x_1}}$$

$$R_{a,b}(V) := \mathbb{C}[(V^*)^a \times V^b]^{SL(V)}$$

Conjecture (Fomin - Pylyavskyy):

Every such ring carries a cluster algebra structure.

↳ Conjecture proved by FP when $V = \mathbb{C}^3$ using the beautiful Kuperberg web basis, (goes back Reshetikhin-Turaev).

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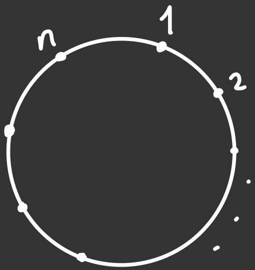
↳ Webs are ways to pictorially represent particular elements.

Let D be a disk with n marked points $1, \dots, n$ labeled clockwise on its boundary.



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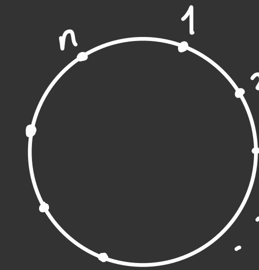

$\boxed{\text{for this talk : boundary vertices colored black}}$

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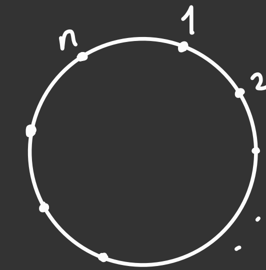

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A **tensor diagram** T is a finite bipartite graph drawn in D, with a fixed bipartition of its vertex set into black and white color sets, subject to the followings

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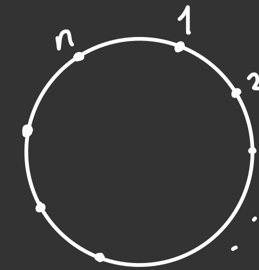

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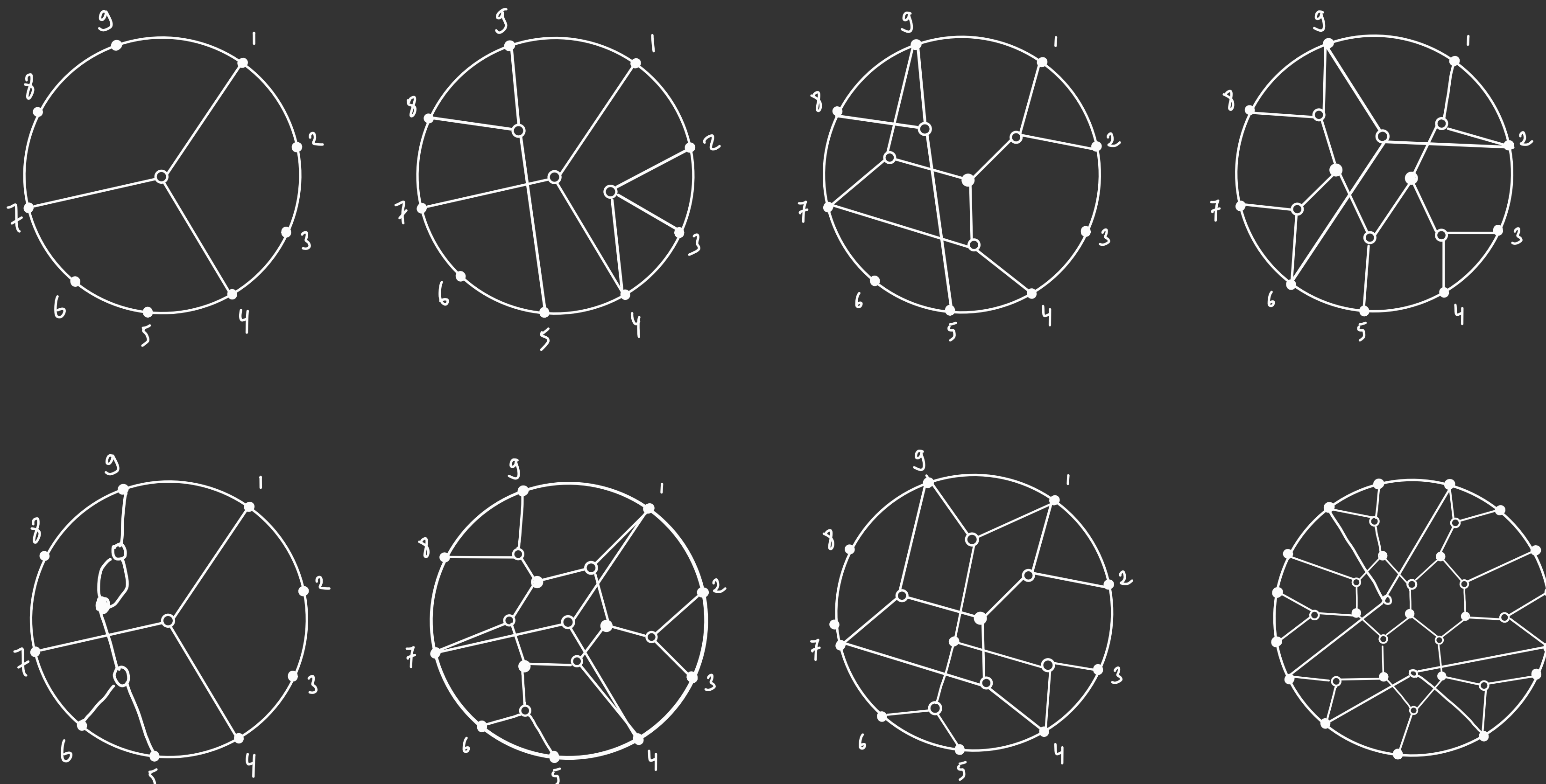
A **tensor diagram** T is a finite bipartite graph drawn in D, with a fixed bipartition of its vertex set into black and white color sets, subject to the followings

- interior vertices are trivalent.
- cyclic ordering of edges incident to trivalent vertices.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First Fundamental Theorem: Any $SL(V)$ invariant can be represented (non-uniquely) as a linear combination of invariants associated with tensor diagrams obtained by superposition of tripods and edges.

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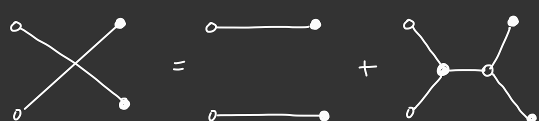
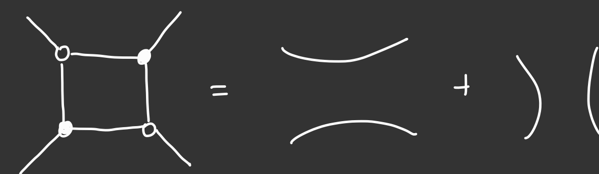
Skein relations:

1)

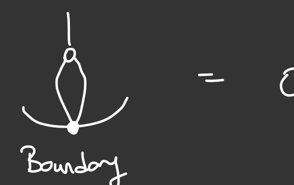


2)



3)

4)



5)

Definition: A tensor diagram is called a "web" if it is planar.

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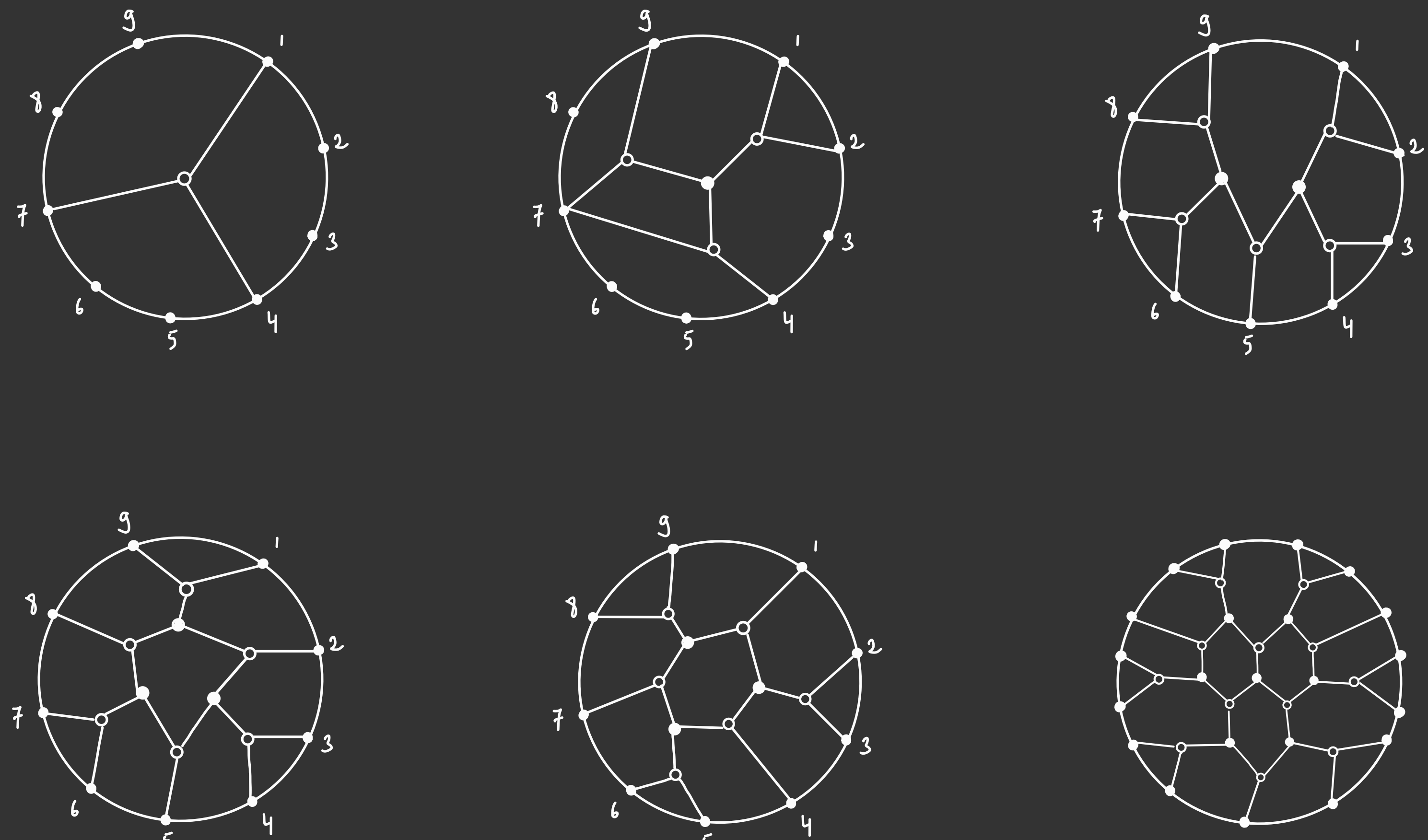


↳ An invariant associated to a non-elliptic* web is called a

web invariant.

Theorem: Web invariants form a linear basis in the ring of invariants.

* A web is called non-elliptic if it has no multiple edges, and no 4-cycles whose all four vertices are internal.

↳ An invariant associated to a non-elliptic* web is called a

<u>web invariant</u>.

<u>Theorem</u>: Web invariants form a linear basis in the ring of invariants.

<u>Remark</u>: Any linear combination of tensor diagrams can be transformed into a linear combination of non-elliptic webs by repeated application of skein relations.

* A web is called <u>non-elliptic</u> if it has no multiple edges, and no 4-cycles whose all four vertices are internal.

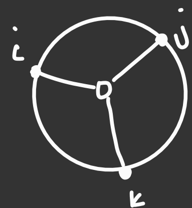
↳ $Gr(k,n)$: the Grassmanian of k-dimensional subspaces of an n-dimensional complex vector space.

$\mathbb{C}[Gr(k,n)]$ homogeneous coordinate ring of Grassmanians

$\cong$ (Plücker embedding)

The ring of $SL(V)$ invariants of n-tuples of vectors in a a k-dimensional vector space V

FACT: Every tensor diagram $[T]$ with n boundary vertices defines an invariant $T \in \mathbb{C}[\widehat{Gr(k,n)}]$.



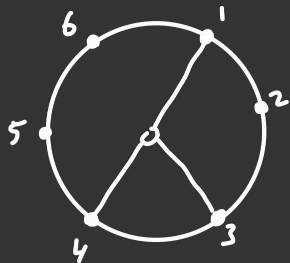
$\leftrightsquigarrow \quad P_{ijk}$ "Plücker coordinates"

Ex: Consider $Gr(3,6)$:

$$\begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{bmatrix}_{3\times 6} \end{array}$$

e.g. $P_{134} = \det \begin{bmatrix} v_1 & v_3 & v_4 \end{bmatrix}$

How does this tripod compute determinant?



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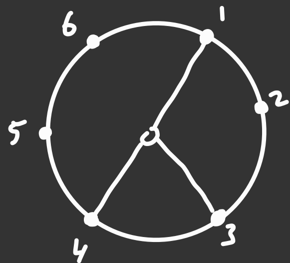

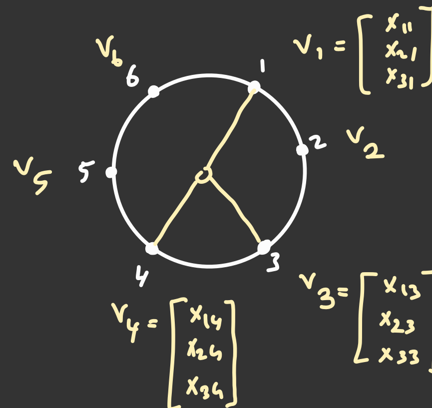

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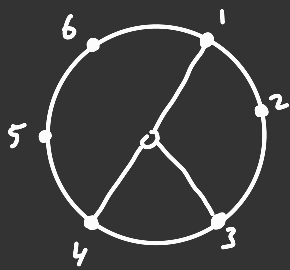
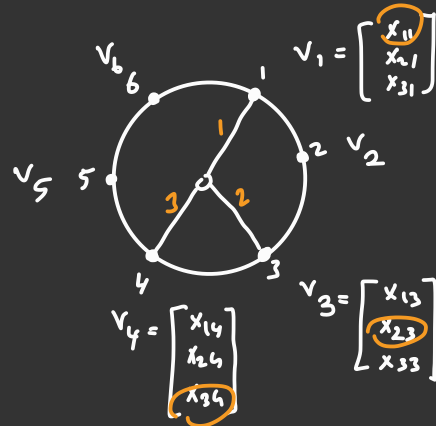
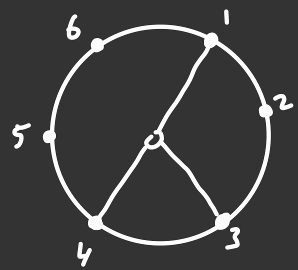
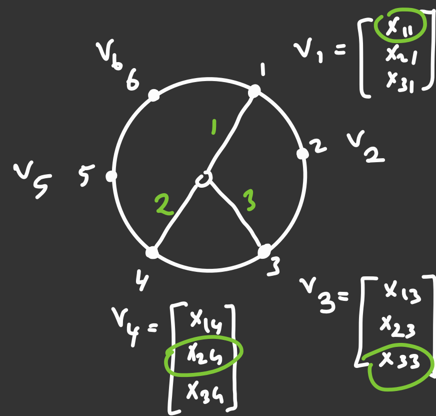




$X_{11}\, X_{23}\, X_{34}$

How does this tripod compute determinant ?





$\rightsquigarrow$ $+x_{11}x_{23}x_{34} - x_{11}x_{33}x_{24}$ + 4 more terms

$\mathrm{sign}(123) = +$ $\quad$ $\mathrm{sign}(132) = -$

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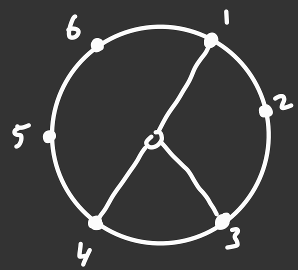
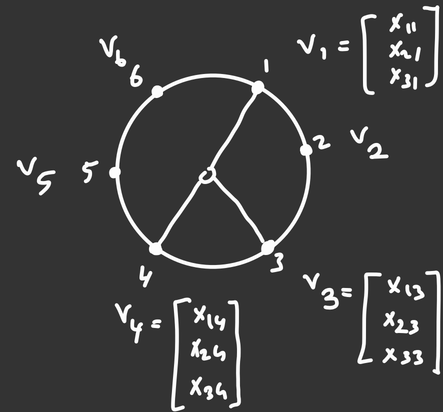


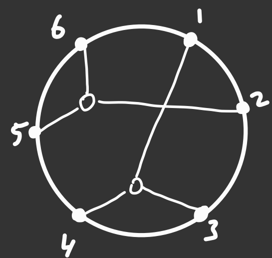


$\rightsquigarrow + x_{11} x_{23} x_{34} - x_{11} x_{33} x_{24} +$ 4 more terms

$\mathrm{sign}(123) = +$ $\qquad$ $\mathrm{sign}(132) = -$

Answer: (i) Find all possible coloring of the edges, we will take the sum over those.

(ii) Write the monomial by choosing ith position for the coloring i for each edge; by putting a sign with respect to the ordering of the coloring.



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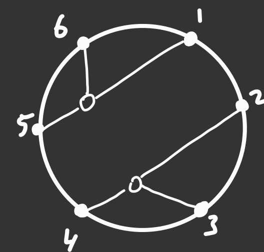

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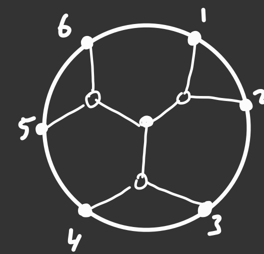
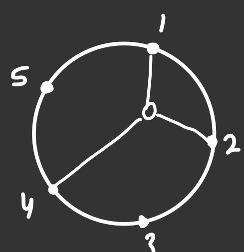


$P_{134} \cdot P_{256} = P_{156} \cdot P_{234} +$ rank 2 web

$$\boxed{\begin{array}{cccccc} Gr(2,n+3) & Gr(3,6) & Gr(3,7) & Gr(3,8) & Gr(3,9) & Gr(4,8) \\ A_n & D_4 & E_6 & E_8 & E_8^{(1,1)} & E_7^{(1,1)} \end{array}}$$

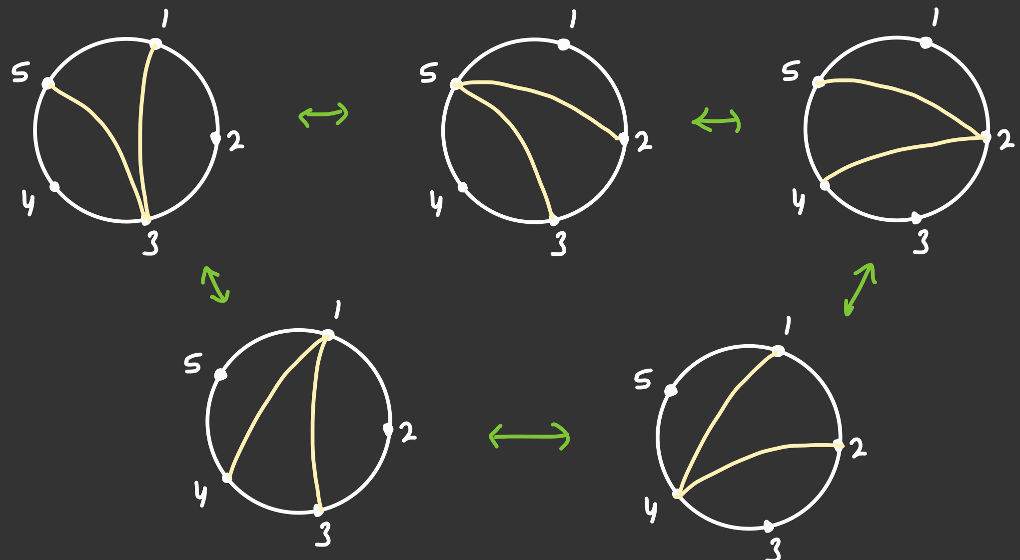
where $E_7^{(1,1)}$ and $E_8^{(1,1)}$ are finite mutation type.

↳ $Gr(3,5) \rightsquigarrow A_2$ (5 mutable + 5 frozen cluster variables)



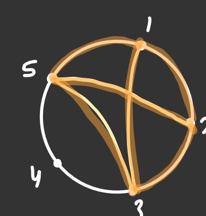
all Plückers $\binom{5}{3} = 10$

↳ $Gr(3,5) \sim Gr(2,5)$



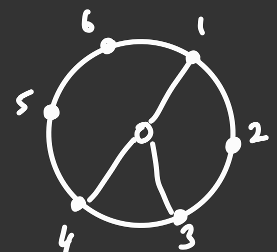
$\rightsquigarrow P_{13}, P_{35}, P_{25}, P_{24}, P_{14}, P_{12}, P_{15}, P_{23}, P_{34}, P_{45}$

$\rightsquigarrow \quad P_{13} \cdot P_{25} = P_{15} P_{23} + P_{12} P_{35}$

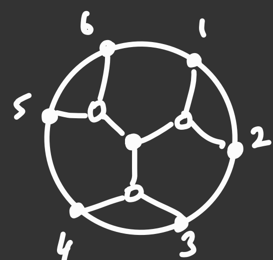


Example: $Gr(3,6) \rightsquigarrow D_4$

(22 cluster variables of which we have 6 frozen.)



Plückers $\binom{6}{3} = 20$



2 non-Plückers

Jensen-King-Su Categorification

Q:

"quiver n vertices"

Jensen-King-Su Categorification

Q:

$$B_{k,n} = \mathbb{C}Q \Big/ \begin{matrix} xy = yx \\ x^k = y^{n-k} \end{matrix}$$

"k fixed"

"n relations"

"n relations"

"quiver n vertices"

Jensen-King-Su Categorification

Q:

$$B_{k,n} = \mathbb{C}Q \Big/ \begin{matrix} xy = yx \\ x^k = y^{n-k} \end{matrix}$$

"quiver n vertices"

"k fixed"

"n relations"

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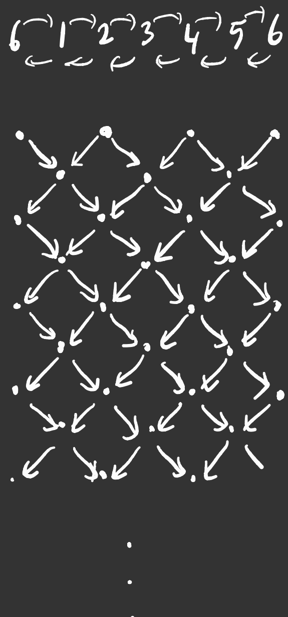
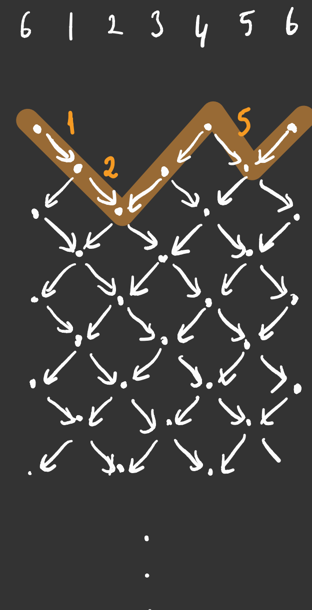
↳ $CM(B_{kn})$ provide a categorification of the cluster structure on Grassmannians.

Rank 1 modules $\xleftrightarrow[1-1]{}$ Plücker coordinates in $\mathbb{C}[Gr(k,n)]$

<u>$k=3 \quad n=6$</u> :

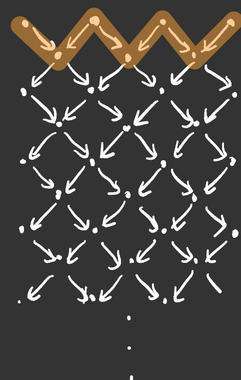




P_{125}

Rank 2 modules

135



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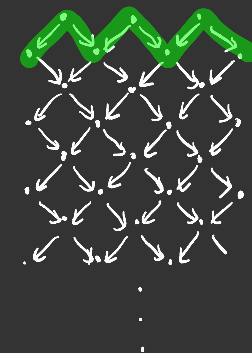
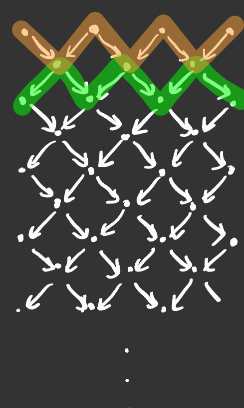


$\frac{135}{246}$

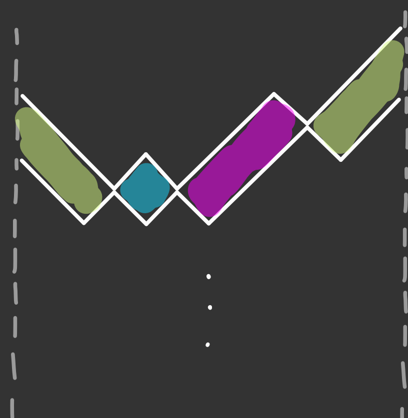


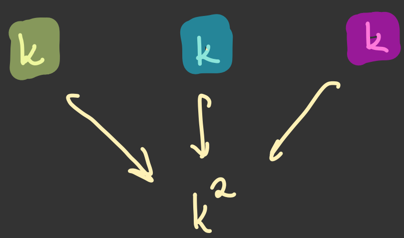
THEOREM (Le - Yildirim)

$$\left\{ \begin{array}{c} \text{rank 2 rigid} \\ \text{indec.s} \\ \text{in JKS} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{certain fncs in } Gr(k,n) \\ \text{which can be pictured} \\ \text{by webs} \end{array} \right\}$$

THEOREM (Le - Yildirim)

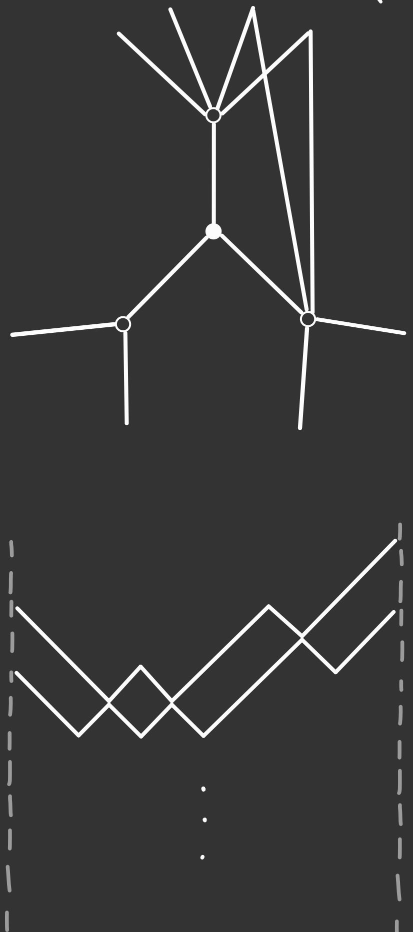
$$\left\{ \text{rank 2 rigid indec.s in JKS} \right\} \xleftrightarrow{1-1} \left\{ \text{certain facs in } Gr(k,n) \text{ which can be pictured by webs} \right\}$$



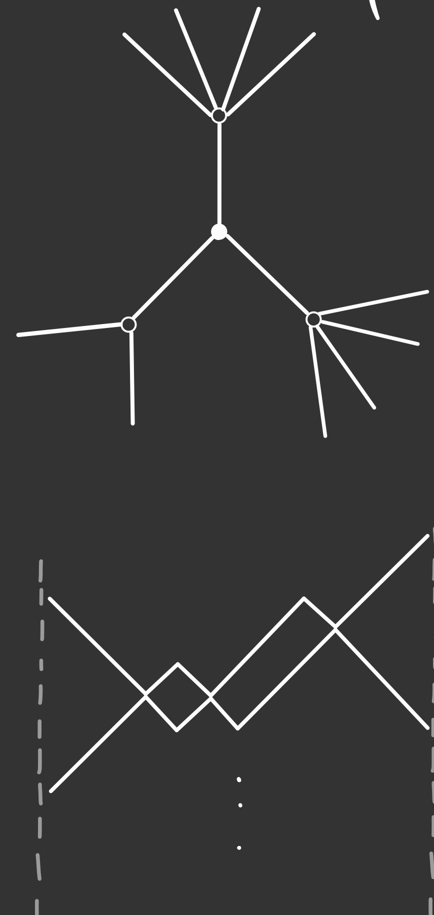


Gr(5, n)

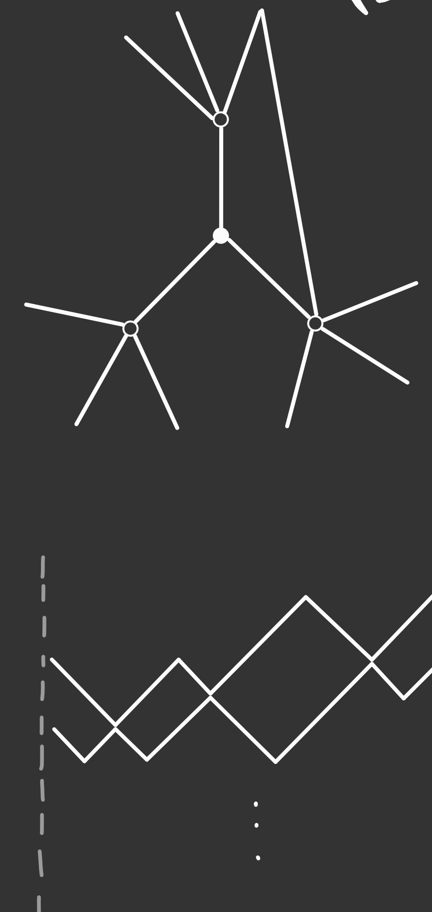
$\left({}_{2}{}^{4,2}{}_{2}\right)$



$\left({}_{2}{}^{4}{}_{4}\right)$



$\left({}_{3}{}^{3,1}{}_{3}\right)$



Baur - Bogdanic - Elsener - Li

There are at most

$$N_{k,n} = \sum_{r=3}^{k} \left(\frac{2r}{3} p_1(r) + 2r\, p_2(r) + 4r\, p_3(r) \right) \binom{n}{2r} \binom{n-2r}{k-r}$$

rigid rank 2 indecomposable modules in $CM(B_{k,n})$ where $p_i(r)$ is the number of partitions $r = r_1 + r_2 + r_3$ such that $r_1, r_2, r_3 \in \mathbb{Z}_{\geq 1}$ and $|\{r_1, r_2, r_3\}| = i$

Conjecture: (BBEL) $N_{k,n}$ = # of rigid indec. rank 2-modules

REFERENCES :

→ I. Le and E. Yildirim "Cluster Categorification of rank 2 webs"

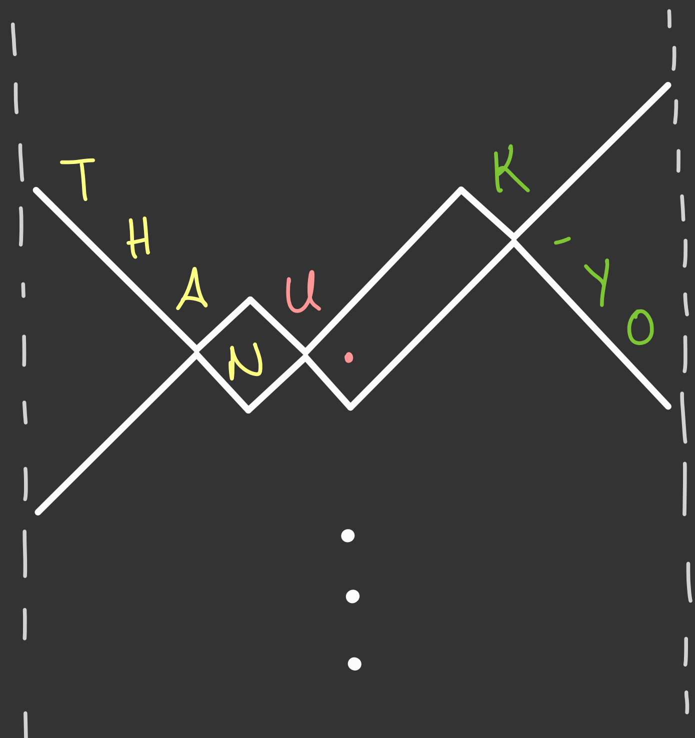
→ S. Fomin and P. Pylyavskyy "Tensor Diagrams and Cluster Algebras"

→ B.T. Jensen - A. King - X. Su "A categorification of Grassmanian cluster algebra."

→ C. Fraser "Braid Group Symmetries of Grassmannian Cluster Algebras"

→ S. Fomin and P. Pylyavskyy "Webs on surfaces, rings of invariants, and clusters"

→ C. Fraser, T. Lam and I. Le "From Dimers to Webs"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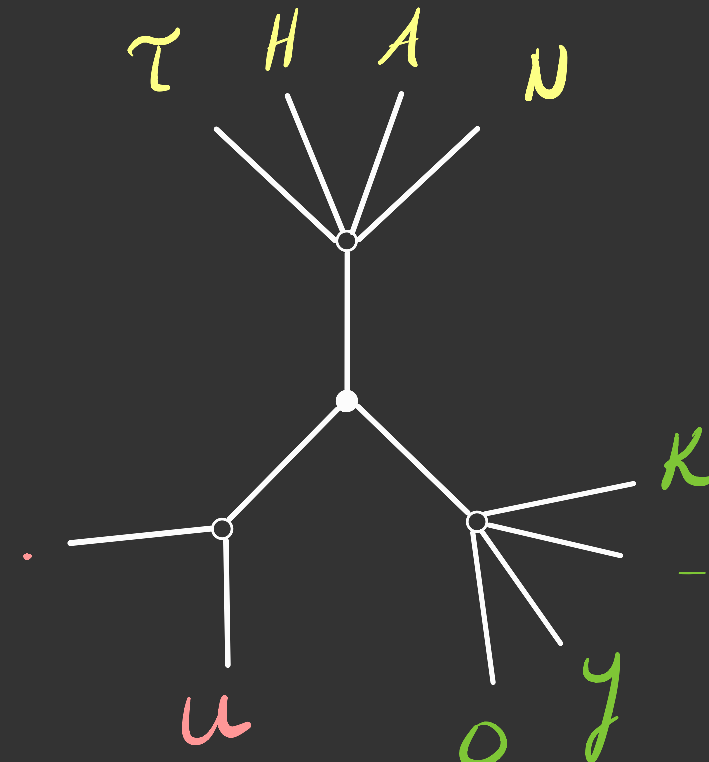
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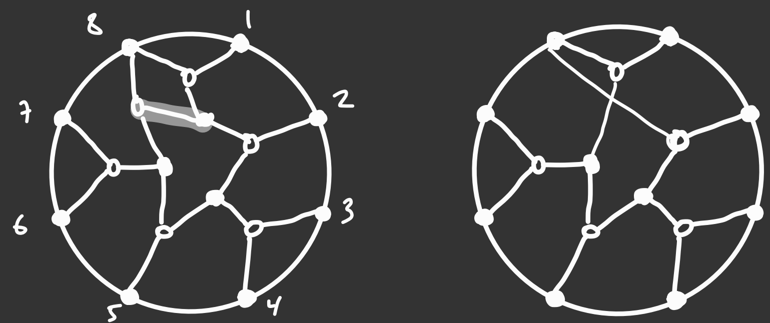
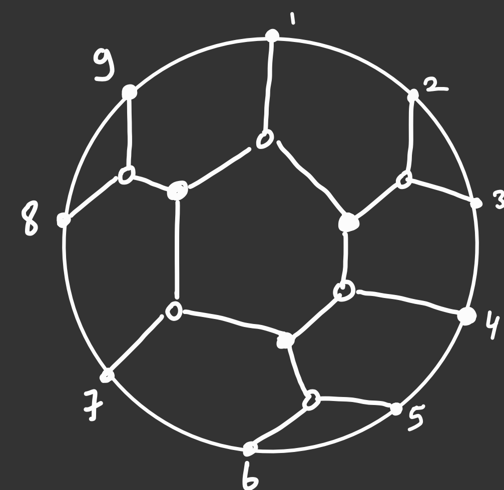
Conjecture. A web invariant z is a cluster variable (or coefficient variable) iff z is indecom. and $z = [D]$ for some tree diagram D.



"arborizable"

Two representation of a non-Plücker cluster variable in $Gr(3,8)$

There are infinitely many indecomposable non-arborizable web invariants.



in $Gr(3,9)$

CONJECTURES: (FP) In the cluster algebra $\mathbb{C}[\widehat{Gr(3,n)}]$:

(i) Two cluster variables lie in the same cluster if and only if they are compatible web invariants.

(ii) The set of cluster (and frozen) variables coincide with the set of indecomposable arborizable web invariants.

(iii) If $n \geq 9$, there are infinitely many indecomposable non-arborizable web invariants.

→ verified in finite type examples for $n < 9$.

Theorem. [C. Fraser] In the cluster algebra $\mathbb{C}[\widehat{Gr(3,9)}]$:

1) Every cluster variable is an indecomposable arborizable web invariant.

2) Every cluster monomial is a web invariant.