Module Title: Commutative rings and algebraic geometry

School of Mathematics

Semester Two 202223

Calculator instructions:

- You are allowed to use a non-programmable calculator in this examination.

Dictionary instructions:

- You are not allowed to use your own dictionary in this exam. A basic English dictionary is available to use. Raise your hand and ask an invigilator if you need it.

Exam information:

- There are 4 pages to this examination.
- There will be **2 hours 30 minutes** to complete this examination.
- This examination is worth 100% of the module mark.
- Answer **four** questions. If you attempt more than four questions, only the best four results will be used in the final marks.
- All questions are worth equal marks.
- You must show all your calculations.
- You must write your answers in the answer booklet provided. If you require an additional answer booklet, raise your hand so an invigilator can provide one.
- You must clearly state your name and Student ID Number in the relevant boxes on your answer booklet. Other boxes may be left blank.

Turn the page over

All rings in this paper are commutative with a multiplicative identity 1.

1. (a) Let R be a ring.

i. Define what it means for a subset $S \subseteq R$ to be **multiplicatively closed**.

ii. Show that the map $\varphi : R \to S^{-1}R$ defined by $\varphi(r) = \frac{r}{1}$ is a ring homomorphism and determine its kernel $\ker(\varphi)$.

iii. Show that for $x \in R$ the following are equivalent:
A. $x = 0$ in R;
B. The image of x in $R_{\mathfrak{p}}$ is 0 for all prime ideals $\mathfrak{p} \subseteq R$;
C. The image of x in $R_{\mathfrak{m}}$ is 0 for all maximal ideals $\mathfrak{m} \subseteq R$.

Hint: For $C \Rightarrow A$ prove that $I_x = \{r \in R : rx = 0\}$ is a proper ideal of R for a nonzero $x \in R$.

(b) i. Define the **Nilradical** $\mathrm{nil}(R)$ and the **Jacobson radical** $J(R)$ of R.

ii. Compute the nilradical and the Jacobson radical for $R = K[x]_{\langle x \rangle}$, where K is a field.

iii. Find an example of a ring R such that $\mathrm{nil}(R) = \mathrm{J}(R)$.

2. (a) Let $R = \mathbb{C}[[x, y, z]]$, and let

$$J = \langle x^2 + \sqrt{2}y^2z^4 - y^2z^2, 2x^2 + 5x^5y, z^4 - 2y^2, y^2z^2 - z^2 - 5x^7 \rangle \ .$$

Show that $J = \langle x^2, y^2, z^2 \rangle$.

(b) Let R be a noetherian ring. Let $I, J \subseteq R$ be ideals. Show that if $J \subseteq \sqrt{I}$, then there exists an element $n \in \mathbb{N}$ such that $J^n \subseteq I$.

(c) Let $R = \mathbb{C}[x, y, z]$ and let $\mathfrak{p} = \langle x + y, y - x, z + x^2 \rangle$.

i. Show that $\mathfrak{p}$ is a prime ideal and describe the maximal ideals in $R_{\mathfrak{p}}$.

ii. Describe the prime ideals and maximal ideals in $R/\mathfrak{p}$.

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3. (a) Let R be a ring and let A, B, C be R-modules. Assume that the sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact.

i. State three equivalent conditions so that this sequence is **split**.

ii. Assume that $C \cong R^n$. Show that there is a map $s : C \to B$ such that $g \circ s = \mathrm{Id}_C$.

(b) Let M be an R-module and assume that F, G are free R-modules (that is, $F \cong R^n$ and $G \cong R^m$ for some m, n). Let

$$0 \to K_1 \xrightarrow{f} F \xrightarrow{g} M \to 0$$

and

$$0 \to K_2 \xrightarrow{\alpha} G \xrightarrow{\beta} M \to 0$$

be short exact sequences of R-modules and define $\gamma : F \oplus G \to M : (x, y) \mapsto g(x) + \beta(y)$ to be the sum of g and β. Show that

$$K_1 \oplus G \cong \ker(\gamma) \ .$$

Hint: Show that there is an inclusion of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r] \arrow[d, hook, "p"] & F \arrow[r] \arrow[d, hook, "q"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ker(\gamma) \arrow[r, hook] & F \oplus G \arrow[r, "\gamma"] & M \arrow[r] & 0 \end{tikzcd} ,$$

where p is induced by the inclusion $K_1 \to F \oplus G : x \mapsto (f(x), 0)$ and $q : F \to F \oplus G$ is the natural inclusion $x \mapsto (x, 0)$. Then use the snake lemma and part (a).

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4. (a) Let R be a noetherian ring. Show that $R[x_1, \dots, x_n]/I$ is a noetherian ring, where $I \subseteq R[x_1, \dots, x_n]$ is an ideal.
Hint: You may find the following fact from the lecture useful: if $0 \to L \to M \to N \to 0$ is an exact sequence of modules over a noetherian ring S, then M is a noetherian S-module if and only if both L and N are noetherian S-modules.

(b) Let R be a noetherian ring and $I \subseteq R$ an ideal. Explain what a **minimal primary decomposition** of I is. You may assume known what a primary deal is.

(c) Let $J = \langle x^2 \rangle \cap \langle y, z \rangle \cap \langle x, y, z \rangle^4$ be an ideal in $R = K[x, y, z]$.

i. Is this a minimal primary decomposition of J? Explain!

ii. For $K = \mathbb{R}$, describe the vanishing set $V(J)$ in $\mathbb{R}^3$. What are the irreducible components of $V(J)$?

5. (a) i. Let R be a ring and S be an R algebra. Define what it means for S to be an **integral extension** of R.

ii. Show that the integral elements of $\mathbb{Q}$ over $\mathbb{Z}$ are the integers.

(b) Let $R = \mathbb{C}[x, y, z]$. Let $f(x, y, z) = xy^3 + y^5 + z^3 \in R$.

i. Let $J = \langle \partial_x(f), \partial_y(f), \partial_z(f) \rangle$ be the Jacobian ideal of f in R. Find $V(J)$.

ii. Argue that $V(J)$ is irreducible and describe $V(J)$ in $\mathbb{C}^3$.

(c) Let K be a field and $\mathbb{A}^n_K$ be affine n-space.

i. Show that if $W \subseteq V$ in $\mathbb{A}^n_K$ are two algebraic sets, then $I(V) \subseteq I(W)$.

ii. Show that $W = V$ if and only if $I(V) = I(W)$.