MATH3195/M5195 EXERCISE SHEET 3

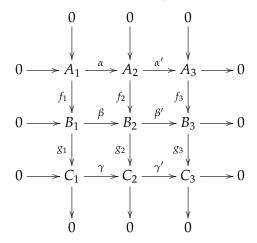
DUE: MARCH 11, 2024

Problem 1. (a) Let $R = \mathbb{Q}[[x,y]]$ and let $J = \langle xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2y \rangle$ be an ideal in R. Show that J is minimally generated by two elements in R.

(b) Let R = K[t] and consider $M = K[t, t^{-1}]$ as R-module and let I = tR be an ideal in R. Show that M = IM but $M \neq 0$. Why does this example not contradict Nakayama's lemma?

Problem 2. Prove the isomorphism theorems for modules.

Problem 3. Prove the 3×3 -lemma: Let R be a ring. Assume that



is a commutative diagram of *R*-modules and all columns and the middle row is exact. Show that the top row is exact if and only if the bottom row is exact.

Problem 4. (Localisation of a module) Let R be a ring and $A \subset R$ be multiplicatively closed. Let M be an R-module. Assume we know that $(m,a) \sim (n,b)$ if and only if mbc = nac for some $c \in A$ defines an equivalence relation on $M \times A$. (Note: recall the definition of an equivalence relation.)

(a) Writing $A^{-1}M$ for the set of equivalence classes of \sim , and $\frac{m}{a}$ for the class containing (m,a), show that the operation

$$\frac{m}{a} + \frac{n}{b} = \frac{bm + an}{ab}$$

is well defined and hence that $A^{-1}M$ is an abelian group.

(b) By defining an appropriate multiplication rule, show that $A^{-1}M$ has the structure of an $A^{-1}R$ -module.

Problem 5. Let *R* be a ring and $A \subset R$ be multiplicatively closed.

- (a) Suppose that $\phi: M \to N$ is a homomorphism of R modules. Show ϕ induces an $A^{-1}R$ -homomorphism $A^{-1}M \to A^{-1}N$.
- (b) Suppose $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules. Show that $0 \to A^{-1}L \to A^{-1}M \to A^{-1}N \to 0$, with the induced maps from (i), is an exact sequence of $A^{-1}R$ -modules. (*Remark*: This means that localization is an exact functor from the category of R-modules to the category of $A^{-1}R$ -modules.)