Problem 5. Let R be a ring and $A \subset R$ be a multiplicatively closed subset.

(a) Suppose that $\phi : M \to N$ is an R-mod. homomorphism. Show that ϕ induces an $A^{-1}R$ homo. $A^{-1}M \to A^{-1}N$.

Define
$$\psi : A^{-1}M \to A^{-1}N$$
$$a^{-1}m \mapsto a^{-1}\phi(m)$$

- ψ is well-defined: If $a_1^{-1}m_1 = a_2^{-1}m_2$, then we have

$$\psi(a_1^{-1}m_1) = a_1^{-1}\phi(m_1) \overset{*}{=} \phi(a_1^{-1}m_1)$$ *since ϕ is an R-mod homomorphism
$$= \phi(a_2^{-1}m_2)$$
$$\overset{*}{=} a_2^{-1}\phi(m_2)$$
$$= \psi(a_2^{-1}m_2).$$

- (i) $$\psi(a_1^{-1}m_1 + a_2^{-1}m_2) = \psi(a_1^{-1}a_2^{-1}(a_2m_1 + a_1m_2))$$
$$= \psi((a_1a_2)^{-1}(a_2m_1 + a_1m_2))$$
$$= (a_1a_2)^{-1}\phi(a_2m_1 + a_1m_2)$$
*since ϕ is an R-mod homomorphism
$$\overset{*}{=} (a_1a_2)^{-1}(a_2\phi(m_1) + a_1\phi(m_2))$$
$$= a_1^{-1}\phi(m_1) + a_2^{-1}\phi(m_2)$$
$$= \psi(a_1^{-1}m_1) + \psi(a_2^{-1}m_2)$$

(ii) $r\psi(a^{-1}m) = r(a^{-1}\phi(m))$
$$= (ra^{-1})\phi(m)$$
$$= (a^{-1}r)\phi(m)$$
$$= a^{-1}(r\phi(m))$$
$$= a^{-1}\phi(rm)$$
$$= \psi(a^{-1}(rm))$$
$$= \psi(r(a^{-1}m))$$

we always use ϕ is an R-mod homo. and the module str. in R.

(b) Suppose $0 \to L \xrightarrow{\phi_1} M \xrightarrow{\phi_2} N \to 0$ is an exact sequence of R-modules. Show that

$0 \to A^{-1}L \xrightarrow{\psi_1} A^{-1}M \xrightarrow{\psi_2} A^{-1}N \to 0$ with induced maps from (i) is an exact sequence of $A^{-1}R$-mod.

- ψ_1 is injective:

$$\psi_1(a_1^{-1}l_1) = \psi_1(a_2^{-1}l_2)$$

$$\Rightarrow \quad a_1^{-1}\phi_1(l_1) = a_2^{-1}\phi_1(l_2)$$

$$\Rightarrow \quad \phi_1(a_1^{-1}l_1) = \phi_1(a_2^{-1}l_2)$$

$\Rightarrow \quad a_1^{-1} l_1 = a_2^{-1} l_2$ since ϕ_1 is injective

- ψ_2 is surjective

For every $a^{-1}n \in A^{-1}N$, there exists $a^{-1}m \in A^{-1}M$ such that

$$\psi_2(a^{-1}m) = a^{-1}n$$

since $\psi_2(a^{-1}m) = a^{-1}\phi_2(m) = a^{-1}n$

because ϕ_2 is surjective.

- $\operatorname{Ker} \psi_2 = \operatorname{Im} \psi_1$

(i) $\operatorname{Ker} \psi_2 \subseteq \operatorname{Im} \psi_1$

Take $a^{-1}m \in \operatorname{Ker} \psi_2$

$\Rightarrow \quad \psi_2(a^{-1}m) = 0$

$\Rightarrow \quad a^{-1}\phi_2(m) = 0$

$\Rightarrow \quad \phi_2(a^{-1}m) = 0$

$\Rightarrow a^{-1}m \in \operatorname{Ker} \phi_2$

$\Rightarrow a^{-1}m \in \operatorname{Im} \phi_1$ since $\operatorname{Ker} \phi_2 = \operatorname{Im} \phi_1$

$\Rightarrow a^{-1}m = \phi_1(a_*^{-1}\ell)$ for some $a_*^{-1}\ell \in A^{-1}L$

$\Rightarrow a^{-1}m = a_*^{-1}\phi_1(\ell)$

$\Rightarrow a^{-1}m = \psi_1(a_*^{-1}\ell)$

$\Rightarrow a^{-1}m \in \operatorname{Im} \psi_1$

(ii) $\operatorname{Im} \psi_1 \subseteq \operatorname{Ker} \psi_2$

Take $a^{-1}m \in \operatorname{Im} \psi_1$

$\Rightarrow \psi_1(a_*^{-1}\ell) = a^{-1}m$ for some $a_*^{-1}\ell$

$\Rightarrow a_*^{-1}\phi_1(\ell) = a^{-1}m$

$\Rightarrow \phi_1(a_*^{-1}\ell) = a^{-1}m$

$\Rightarrow a^{-1}m \in \operatorname{Im} \phi_1$

$\Rightarrow \quad a^{-1}m \in \operatorname{Ker} \phi_2$ since $\operatorname{Ker}\phi_2 = \operatorname{Im}\phi_1$

$\Rightarrow \quad \phi_2(a^{-1}m) = 0$

$\Rightarrow \quad a^{-1}\phi_2(m) = 0$

$\Rightarrow \quad \psi_2(a^{-1}m) = 0$

$\Rightarrow \quad a^{-1}m \in \operatorname{Ker} \psi_2$.