WHY THE RETURN TO PICTURES IN ALGEBRA?

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DIRECTED GRAPHS - i.e. QUIVERS

$$Q = (\underset{\substack{\uparrow \\ \text{the set of vertices}}}{Q_0}, \underset{\substack{\uparrow \\ \text{the set of edges}}}{Q_1}, s, t)$$

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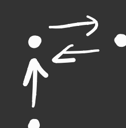
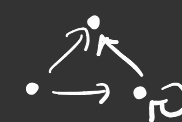
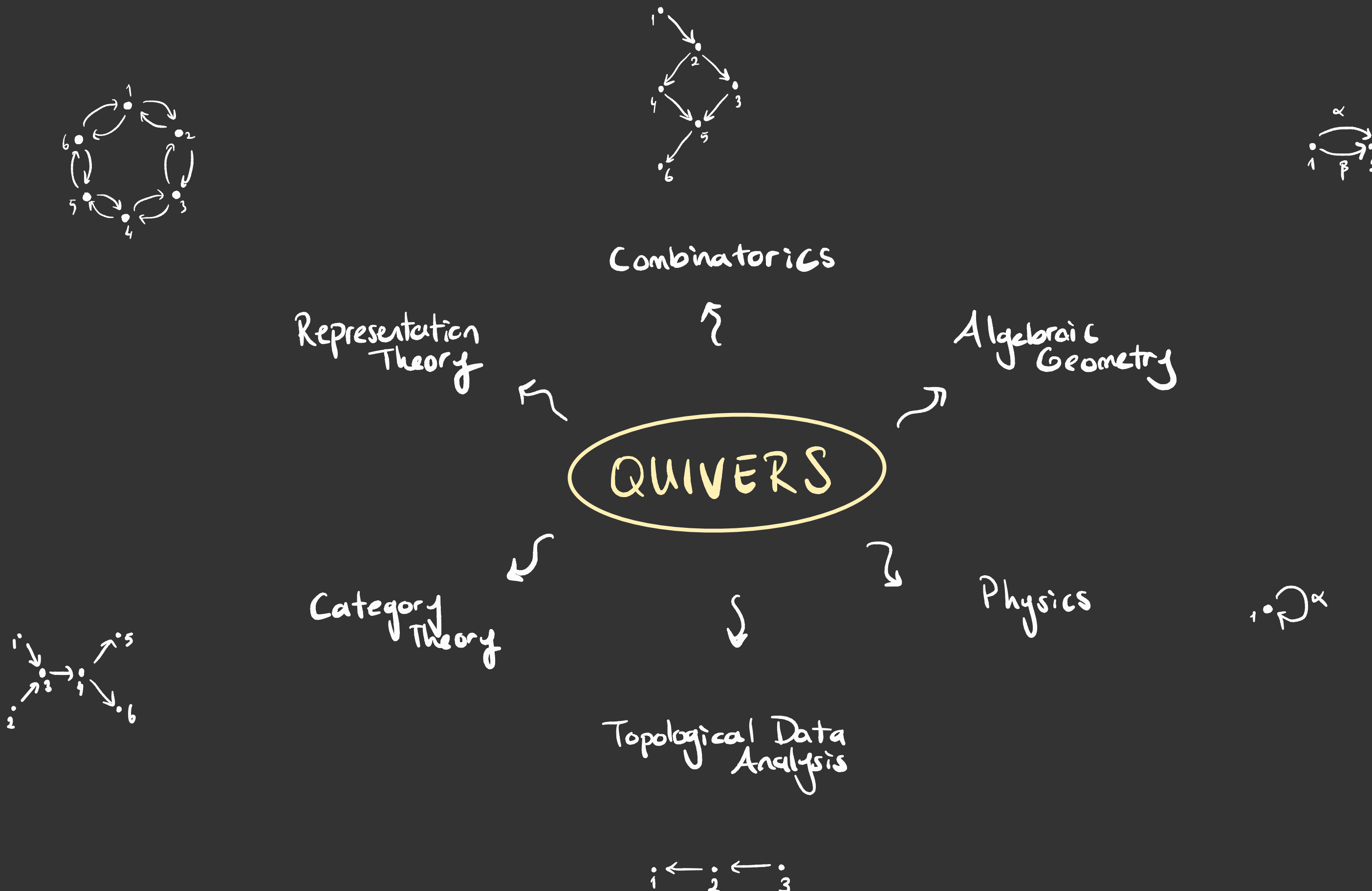
the set of vertices

the set of edges

Examples






QUIVERS
Combinatorics
Representation Theory
Algebraic Geometry
Category Theory
Topological Data Analysis
Physics

PATH ALGEBRAS

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Example: $Q : \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet}$

$$\mathbb{C}Q = \mathrm{Span}_{\mathbb{C}} \{ e_1, e_2, e_3, \alpha, \beta, \beta\alpha \}$$

where e_i : paths of length zero,
α, β : paths of length one,
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The path algebra $\mathbb{C}Q$ in this case is isomorphic to the ring of lower triangular 3×3 matrices.

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Note: $\mathbb{C}Q$ in this case is isomorphic to $\mathbb{C}[x]$.

Example: Q: $1 \bullet \circlearrowright \alpha$ with the relation $\alpha^3 = 0$

$\mathbb{C}Q/I = \mathrm{Span}_{\mathbb{C}}\{ e_1, \alpha, \alpha^2 \}$ a quotient of the path algebra $\mathbb{C}Q$.

THEOREM: Let A be a basic finite dimensional k-algebra. There exists an admissible ideal I of kQ such that

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MODULES over A ?

QUIVER REPRESENTATIONS

A quiver representation $V = (V_i, \varphi_\alpha)$ for all $i \in Q_0$ & $\alpha \in Q_1$.

vector spaces

linear maps

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↳ There are exactly 3 <u>indecomposable</u> representations:

$$k \leftarrow 0 \quad , \quad k \xleftarrow{\sim} k \quad , \quad 0 \leftarrow k$$

Example: $\quad Q : \quad \underset{1}{\bullet} \longrightarrow \underset{3}{\bullet} \longleftarrow \underset{2}{\bullet}$

A representation: $\quad V = \quad k^2 \xrightarrow[\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}]{} k^2 \xleftarrow[\begin{bmatrix} 1 \\ 1 \end{bmatrix}]{} k$

Example:

$$Q: \quad 1 \longrightarrow 3 \longleftarrow 2$$

A representation:

$$V = k^2 \xrightarrow[\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}]{} k^2 \xleftarrow[\begin{bmatrix} 1 \\ 1 \end{bmatrix}]{} k$$

Example:

$$1 \circlearrowleft \alpha \qquad \text{with} \qquad \alpha^3 = 0.$$

A representation:

$$\mathbb{C}^3 \circlearrowleft \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

where notice that $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = 0.$

Another way to describe a quiver representation:

$\rightsquigarrow$ Think Q as a category $\mathcal{C}$ $\begin{cases} \text{objects} : \text{vertices} \\ \text{morphisms} : \text{paths} \end{cases}$

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$\rightsquigarrow$ Then V is just a functor from $\mathcal{C}$ to finite dimensional vector spaces:

$$V : \mathcal{C} \longrightarrow \text{mod}\, k$$

Example : $Q: \quad 1 \longrightarrow 3 \longleftarrow 2$

A representation : $V = k^2 \xrightarrow[\begin{bmatrix}1&1\\0&1\end{bmatrix}]{} k^2 \xleftarrow[\begin{bmatrix}1\\1\end{bmatrix}]{} k$

Another one : $V' = k \xrightarrow{1} k \longleftarrow 0$

Example:

$$Q : \quad \underset{1}{\bullet} \longrightarrow \underset{3}{\bullet} \longleftarrow \underset{2}{\bullet}$$

A representation:

$$\begin{tikzcd}
V \arrow[r, phantom, "="] & k^2 \arrow[r, "{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}"'] & k^2 & k \arrow[l, "{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}"] \\
V' \arrow[u] \arrow[r, phantom, "="] & k \arrow[u, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] \arrow[r, "1"] & k \arrow[u, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"'] & 0 \arrow[u, "0"'] \arrow[l, "0"']
\end{tikzcd}$$

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V' = \arrow[u] & k \arrow[u, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"'] \arrow[r, "1"] & k \arrow[u, "\begin{bmatrix} 1 \\ 0 \end{bmatrix}"'] & 0 \arrow[l, "0"'] \arrow[u, "0"']
\end{tikzcd}$$

$\rightsquigarrow$ $\mathrm{Rep}(Q)$ or $\mathrm{Rep}(Q, I)$ is the category of quiver representations.

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- $\operatorname{Mod} A \longleftrightarrow \operatorname{Rep}_k(Q, I)$
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Representation Category of quivers

AUSLANDER-REITEN QUIVERS

– depicting the representation category again with quivers.

Example: $Q: \quad 1 \leftarrow 2 \leftarrow 3$ (Dynkin type A_3)

$\operatorname{rep} Q$:

$$
\begin{tikzcd}[column sep=small]
& & k \leftarrow k \leftarrow k \arrow[dr] & & \\
& k \leftarrow k \leftarrow 0 \arrow[ur] \arrow[dr] & & 0 \leftarrow k \leftarrow k \arrow[dr] & \\
k \leftarrow 0 \leftarrow 0 \arrow[ur] & & 0 \leftarrow k \leftarrow 0 \arrow[ur] & & 0 \leftarrow 0 \leftarrow k
\end{tikzcd}
$$

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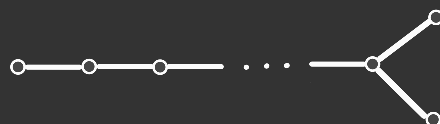
\begin{tikzcd}
 & & \bullet \arrow[dr] & & \\
 & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[dr] & \\
\bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet
\end{tikzcd}
$$

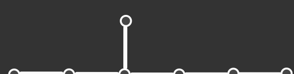
THEOREM (GABRIEL '72)

Q is a Dynkin diagram iff kQ has finitely many indecomposable/iso.

A_n : 

D_n : 

E_6 : 

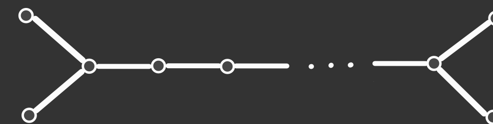
E_7 : 

E_8 : 

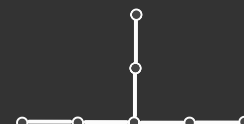
TAME HEREDITARY ALGEBRAS

$\tilde{A}_n$:

$\tilde{D}_n$:



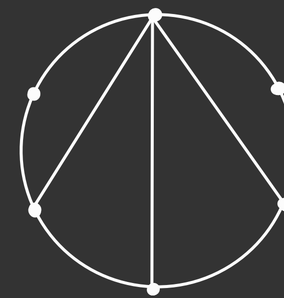
$\tilde{E}_6$:



$\tilde{E}_7$:

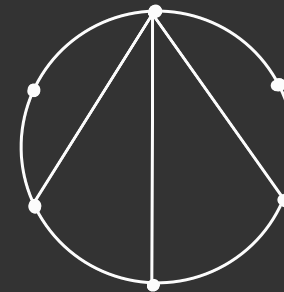
$\tilde{E}_8$:

SURFACES



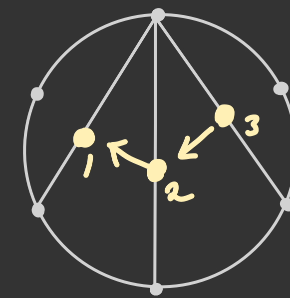
SURFACES

$\theta_n : \quad \overset{\cdot}{1} \leftarrow \overset{\cdot}{2} \leftarrow \overset{\cdot}{3} \qquad \longleftrightarrow$



SURFACES

$$Q: \quad 1 \leftarrow 2 \leftarrow 3 \qquad \longleftrightarrow$$



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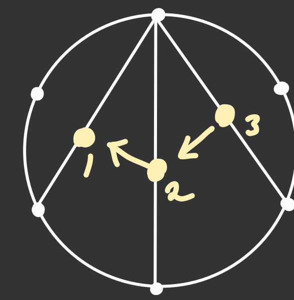
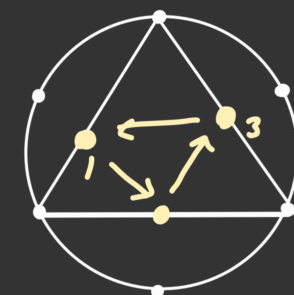
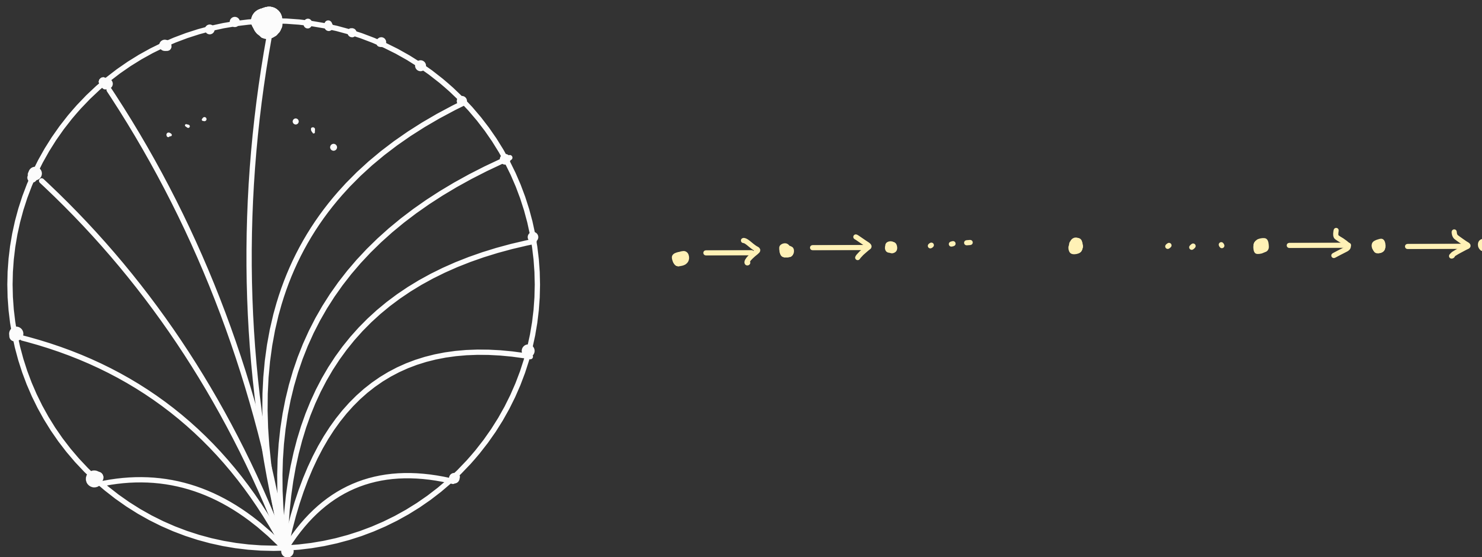
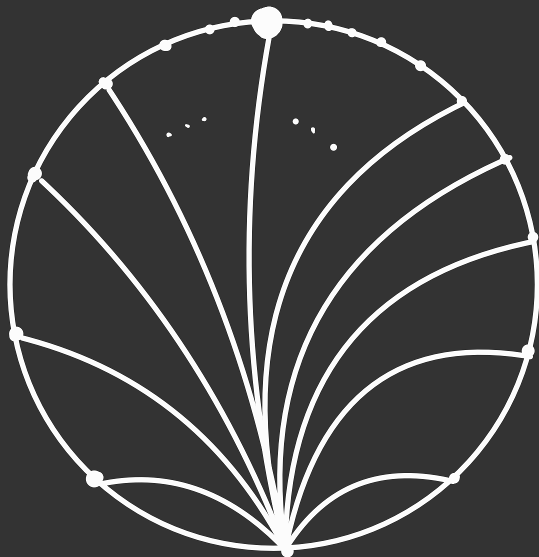


$$Q: \quad \begin{tikzcd} 1 \arrow[dr, "\alpha"'] & & 3 \arrow[ll, "\gamma"'] \\ & 2 \arrow[ur, "\beta"'] & \end{tikzcd} \qquad \longleftrightarrow$$





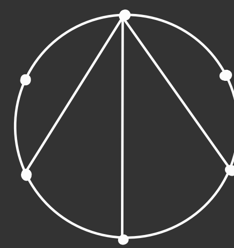


$$\bullet \to \bullet \to \bullet \cdots \quad \bullet \quad \cdots \bullet \to \bullet \to \bullet$$

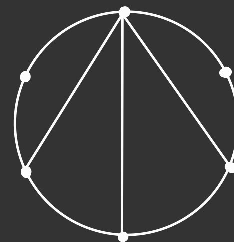
$Q =$ $\mathbb{R}$

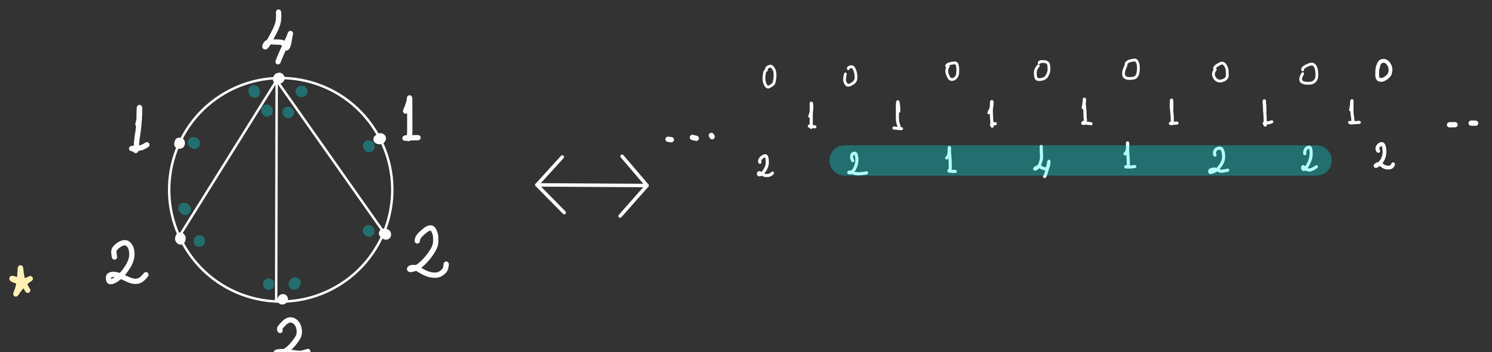
$\mathrm{Rep}(\mathbb{R})$

FRIEZES



FRIEZES





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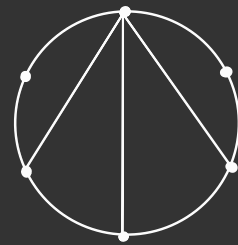

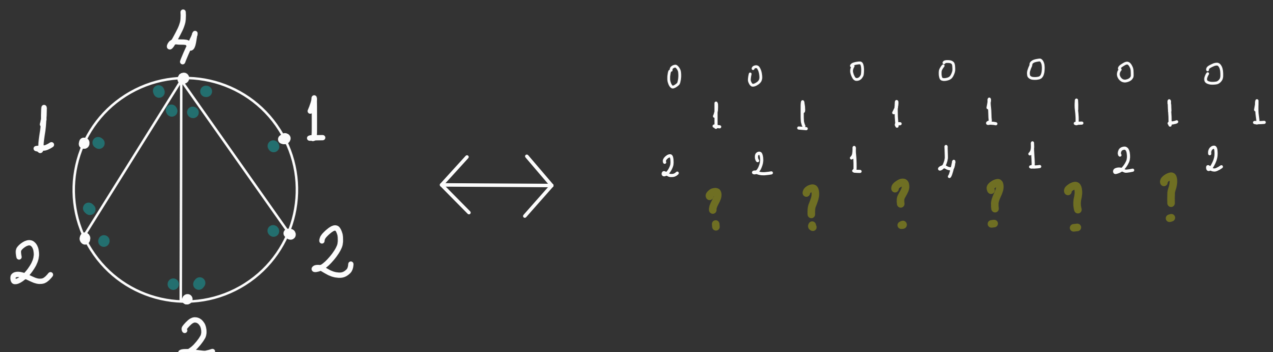

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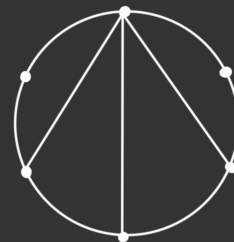
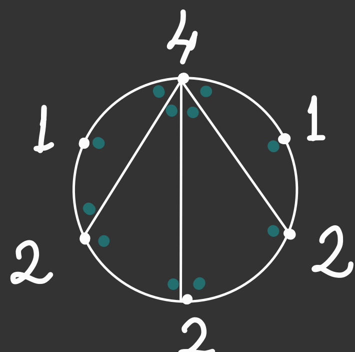




$\longleftrightarrow$

0 0 0 0 0 0 0

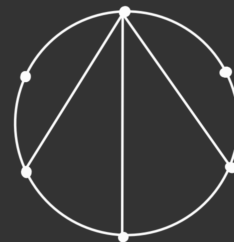
1 1 1 1 1 1 1

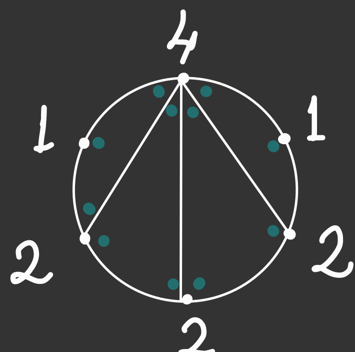
2 2 1 4 1 2 2

? ? ? ? ? ?

Rule : $\begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \rightsquigarrow bc - ad = 1$

FRIEZES

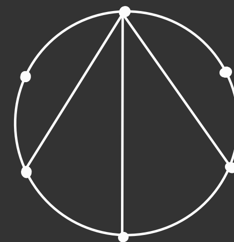


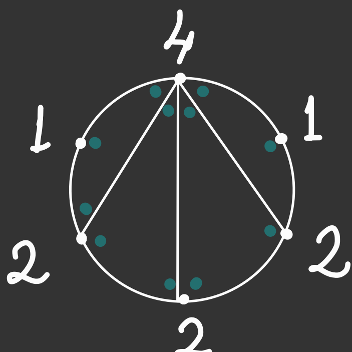


$\longleftrightarrow$

0		0		0		0		0		0		0	
	1		1		1		1		1		1		1
2		2		1		4		1		2		2	
	3		1		3		3		1		3		3

FRIEZES

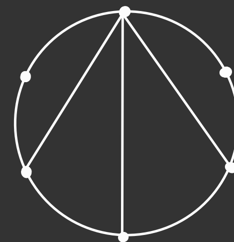


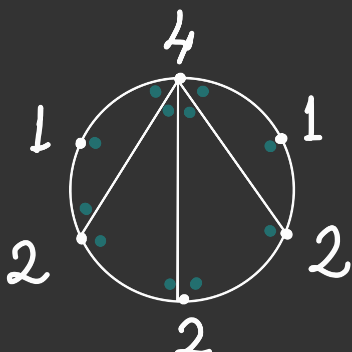


$\longleftrightarrow$

```
0   0   0   0   0   0   0
  1   1   1   1   1   1   1
2   2   1   4   1   2   2
  3   1   3   3   1   3   3
4   1   2   2   2   1   4
```

FRIEZES





$\longleftrightarrow$

0 0 0 0 0 0 0

1 1 1 1 1 1 1

2 2 1 4 1 2 2

3 1 3 3 1 3 3

4 1 2 2 2 1 4

1 1 1 1 1 1 1

0 0 0 0 0 0 0

width

CONWAY - COXETER

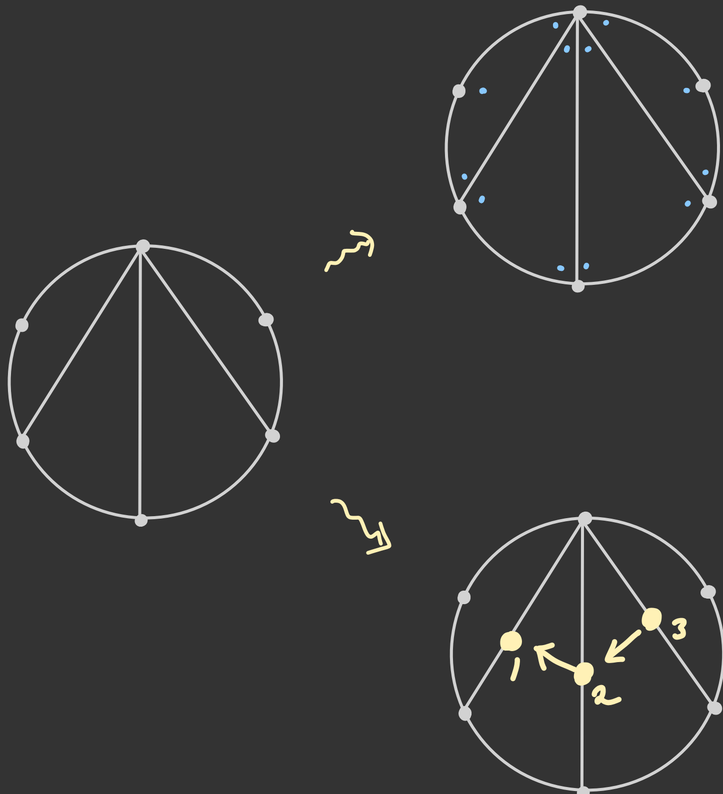
triangulations of $(n+3)$-gon $\overset{1-1}{\longleftrightarrow}$ finite friezes of width n

CONWAY - COXETER

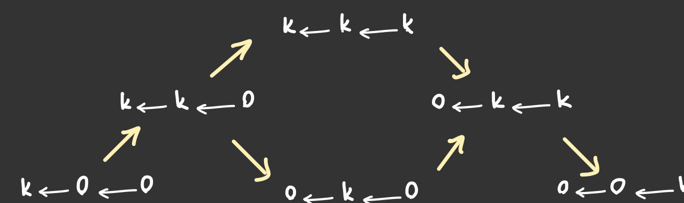
$$\text{triangulations of } (n+3)\text{-gon} \quad \xleftrightarrow{\;1-1\;} \quad \text{finite friezes of width } n$$

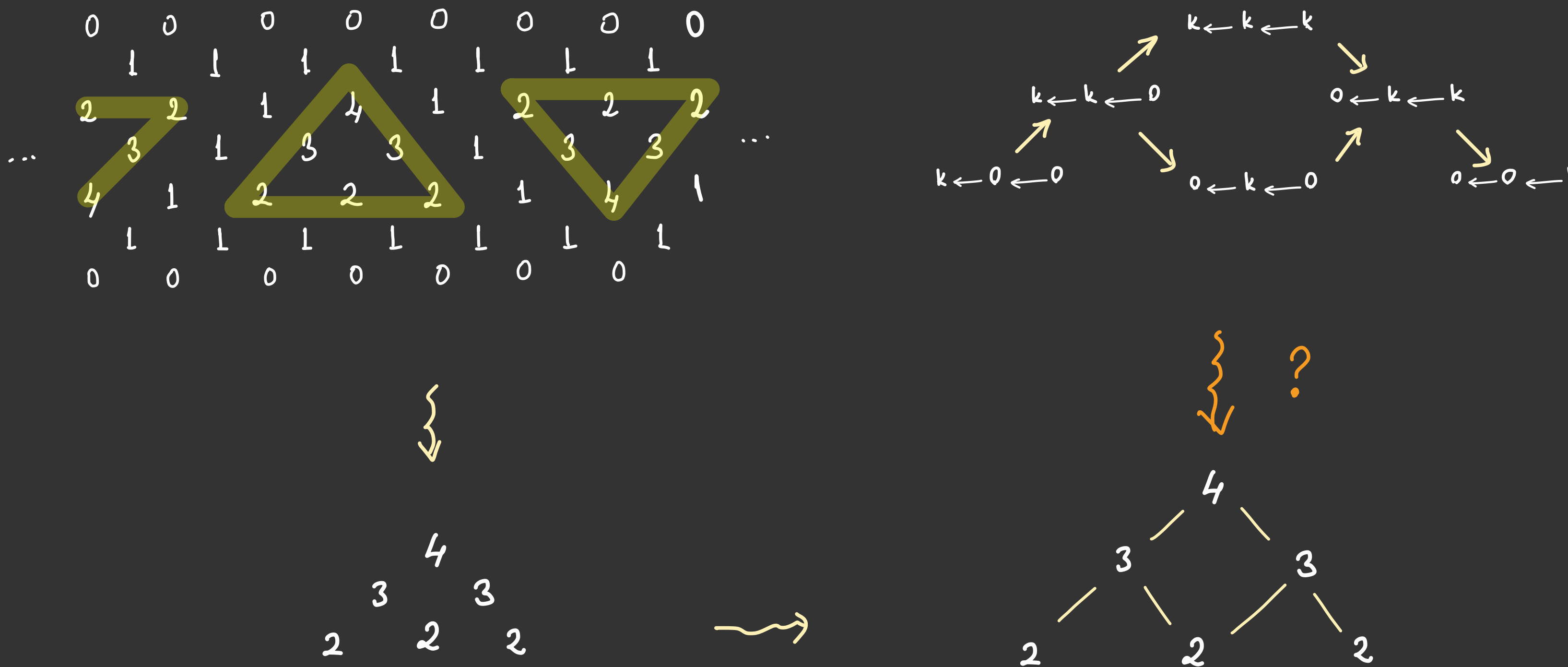
Symmetries :

$$\begin{array}{cccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
 & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & \\
\cdots & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & \cdots & \\
 & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}$$

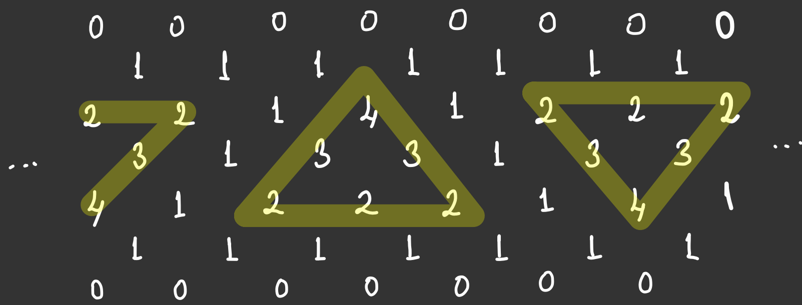


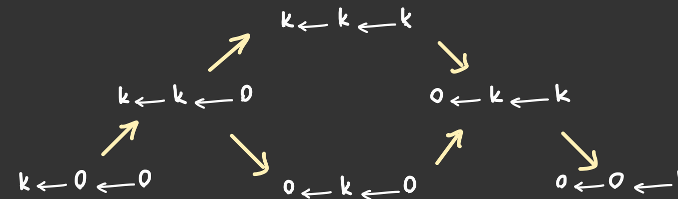
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	
2	2	1	4	1	2	2	2
3	1	3	3	1	3	3	
4	1	2	2	2	1	4	1
1	1	1	1	1	1	1	
0	0	0	0	0	0	0	



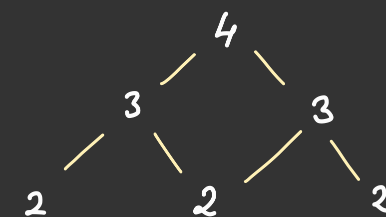


0 0 0 0 0 0 0 0
1 1 1 1 1 1 1
2 2 1 4 1 2 2 2
... 3 1 3 3 1 3 3 ...
4 1 2 2 2 1 4 1
1 1 1 1 1 1 1
0 0 0 0 0 0 0
4
3 3
2 2 2
k ← k ← k
k ← k ← 0
0 ← k ← k
k ← 0 ← 0
0 ← k ← 0
0 ← 0 ← k
?
4
3 3
2 2 2



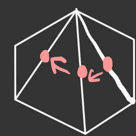


counting submodules



EXTRA :

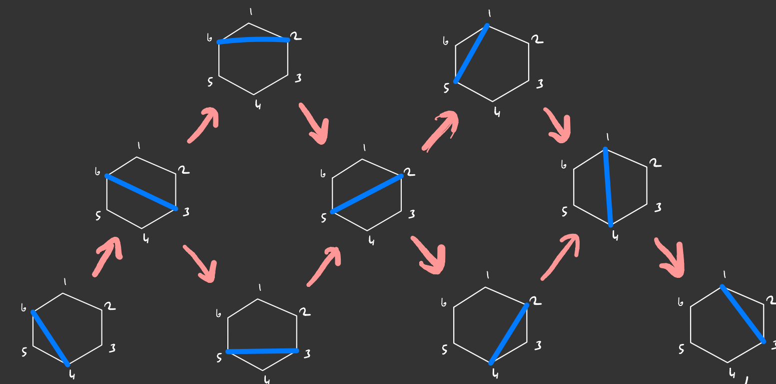
$\mathcal{C}$: hexagon



$$\begin{tikzcd}[column sep=small]
 & & P_3[1] \arrow[dr] & & P_3 \arrow[dr] & & \\
 & P_2[1] \arrow[ur] \arrow[dr] & & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\
P_1[1] \arrow[ur] & & P_1 \arrow[ur] & & S_2 \arrow[ur] & & S_3
\end{tikzcd}$$

'A_3 cluster category'

$\mathcal{C}$:



rep(a)

Cluster category

CONWAY - COXETER

triangulations of $(n+3)$-gon $\overset{1-1}{\longleftrightarrow}$ finite friezes of width n

What about other friezes that never end?

$$\begin{array}{ccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 \\
& 3 & & 3 & & 3 & & 3 & & 3 & & 3 \\
& & 4 & & 4 & & 4 & & 4 & & 4 \\
& & & & & & \vdots
\end{array}$$

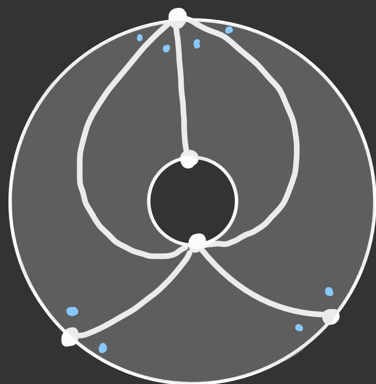
$$\begin{array}{ccccccccccccccc}
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& 4 & & 2 & & 2 & & 4 & & 2 & & 2 & & 4 \\
7 & & 7 & & 3 & & 7 & & 7 & & 3 & & 7 \\
& 12 & & 10 & & 10 & & 12 & & 10 & & 10 \\
& & & & & & & \vdots
\end{array}$$

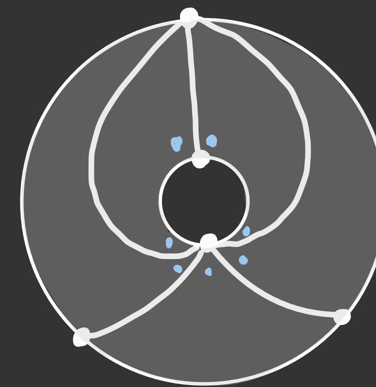
CONWAY - COXETER

triangulations of $(n+3)$-gon $\overset{1-1}{\longleftrightarrow}$ finite friezes of width n

Baur - Parsons - Tschabold '15

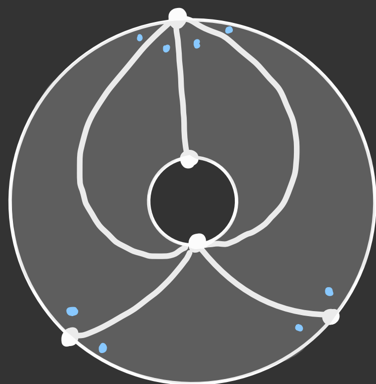
Every periodic infinite frieze comes from a triangulation of annulus.

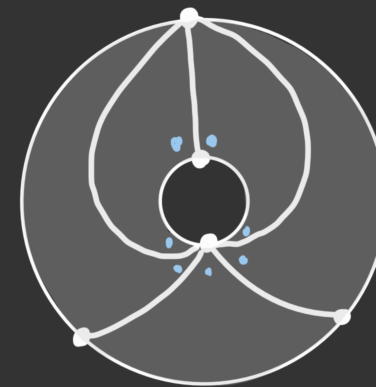




0 0 0 0 0 0 0
1 1 1 1 1 1 1
4 2 2 4 2 2 4
... 7 7 3 7 7 3 7 ...
12 10 10 12 10 10
⋮

0 0 0 0
1 1 1 1
... 2 5 2 5 ...
9 9 9 9
40 16 40
⋮ ⋮





0 0 0 0 0 0 0

1 1 1 1 1 1 1

4 2 2 4 2 2 4

... 7 7 3 7 7 3 7 ...

12 10 10 12 10 10

⋮

0 0 0 0

1 1 1 1

... 2 5 2 5 ...

9 9 9 9

40 16 40

⋮

Baur - Fellner - Parsons - Tschabold

The growth coefficient is the same for both of the friezes in an annulus.

E. Gunawan

L. Bittmann

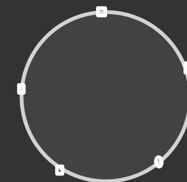
K. Baur

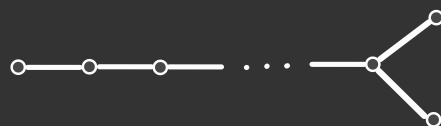
G. Todorov

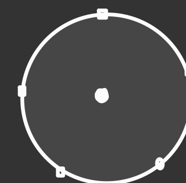
THEOREM (GABRIEL '72)

Q is a Dynkin diagram iff kQ has finitely many indecomposable/iso.

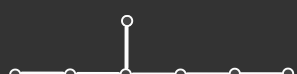
A_n : 



D_n : 



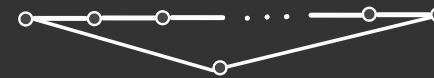
E_6 : 

E_7 : 

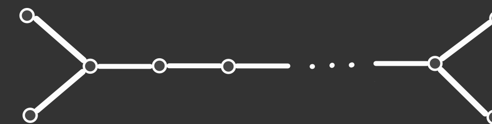
E_8 : 

TAME HEREDITARY ALGEBRAS

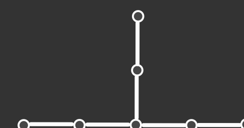
$\tilde{A}_n$:



$\tilde{D}_n$:



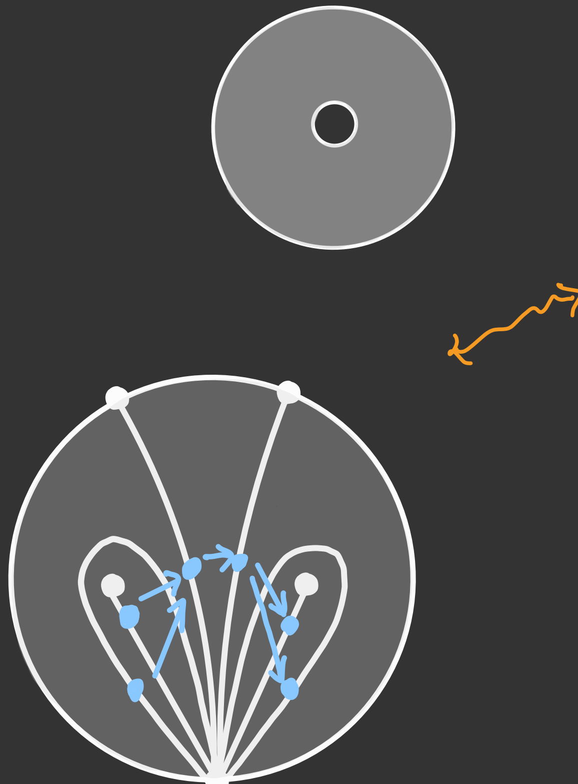
$\tilde{E}_6$:



$\tilde{E}_7$:

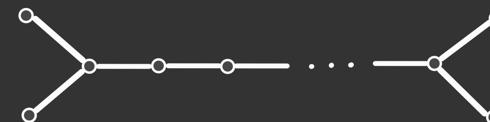
$\tilde{E}_8$:

TAME HEREDITARY ALGEBRAS



$\tilde{A}_n$:

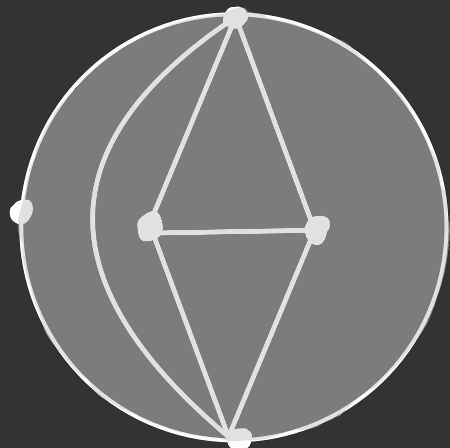
$\tilde{D}_n$:

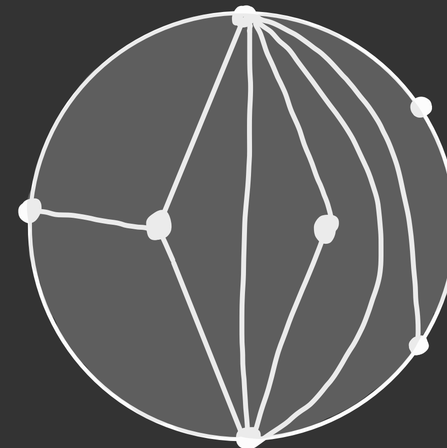


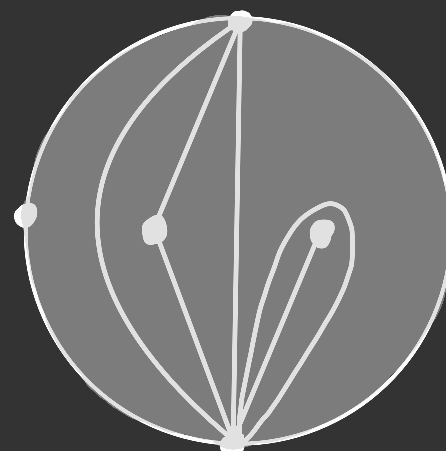
$\tilde{E}_6$:

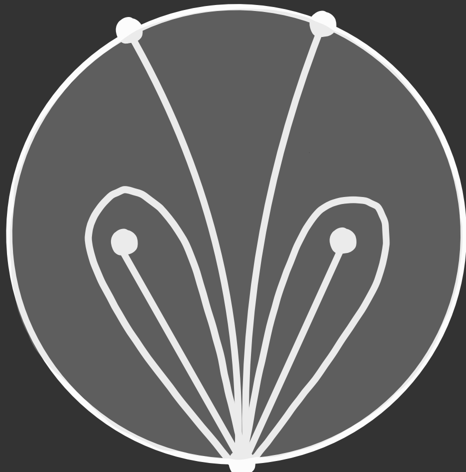
$\tilde{E}_7$:

$\tilde{E}_8$:









. . .

0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1

9 2 2 9 2 2 9 2 . . .

17 3 17 17 3 17 17 3

25 25 32 25 25 32

⋮

. . .

0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1

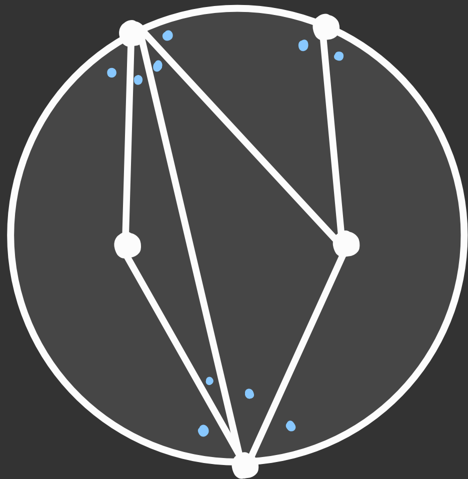
5 5 5 5 5 5 5 . . .

24 24 24 24 24 24 24

115 115 115 115 115 115

⋮

x2



0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1

… 2 4 4 2 4 4 2 …

7 15 7 7 15 7

26 26 24 26 26

45 89 89 45

⋮

0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1

… 2 4 4 2 4 4 2 …

7 15 7 7 15 7

26 26 24 26 26

45 89 89 45

⋮


0 0 0 0 0 0
1 1 1 1 1 1 1
... 4 6 4 6 4 6 ...
23 23 23 23 23
⋮
0 0 0 0 0 0
1 1 1 1 1 1 1
... 2 12 2 12 2 12 ...
23 23 23 23 23
44 44 44 44
⋮

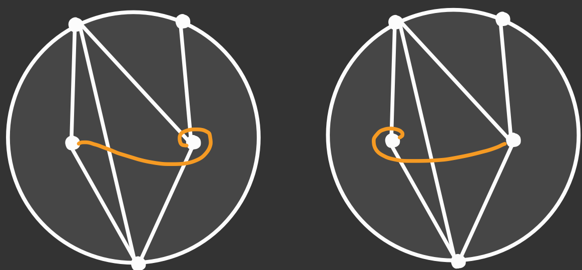
0 0 0 0 0 0

1 1 1 1 1 1 1

. . . 4 6 4 6 4 6 . . .

23 23 23 23 23

⋮



0 0 0 0 0 0

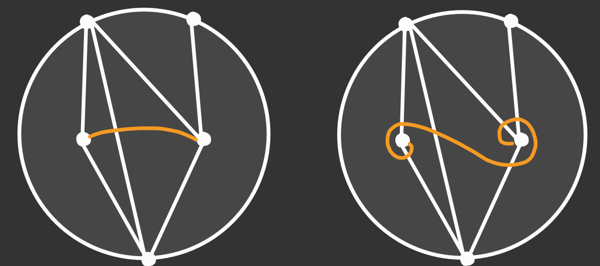
1 1 1 1 1 1 1

. . . 2 12 2 12 2 12 . . .

23 23 23 23 23

44 44 44 44

⋮



$$\begin{array}{ccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & & 4 & & 6 & & 4 & & 6 & & 4 & & 6 & & \cdots \\
 & & & 23 & & 23 & & 23 & & 23 & & 23 & & & \\
 & & & & & & & \vdots & & & & & & &
\end{array}
\qquad
\begin{array}{ccccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & & 2 & & 12 & & 2 & & 12 & & 2 & & 12 & & \cdots \\
 & & & 23 & & 23 & & 23 & & 23 & & 23 & & & \\
 & & & & 44 & & 44 & & 44 & & 44 & & & & \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

THM [BBGTY]

The growth coefficient is the same for all three friezes of twice-punctured disk.

0 0 0 0 0 0 0 0

1 1 1 1 1 1 1

T H A N K Y O U

159 7 13 153 274 374 314

139 90 142 3811 4099 7829

for q=(20, 8, 1, 14, 11, 25, 15, 21)