MATH3195/M5195 EXERCISE SHEET 3

DUE: MARCH 11, 2024

Problem 1. (a) Let $R = \mathbb{Q}[[x, y]]$ and let $J = \langle xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2y \rangle$ be an ideal in R. Show that J is minimally generated by two elements in R.

(b) Let $R = K[t]$ and consider $M = K[t, t^{-1}]$ as R-module and let $I = tR$ be an ideal in R. Show that $M = IM$ but $M \neq 0$. Why does this example not contradict Nakayama's lemma?

Problem 2. Prove the isomorphism theorems for modules.

Problem 3. Prove the 3×3-lemma: Let R be a ring. Assume that

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\alpha"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha'"] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "\beta"] \arrow[d, "g_1"'] & B_2 \arrow[r, "\beta'"] \arrow[d, "g_2"'] & B_3 \arrow[r] \arrow[d, "g_3"'] & 0 \\
0 \arrow[r] & C_1 \arrow[r, "\gamma"] \arrow[d] & C_2 \arrow[r, "\gamma'"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a commutative diagram of R-modules and all columns and the middle row is exact. Show that the top row is exact if and only if the bottom row is exact.

Problem 4. (Localisation of a module) Let R be a ring and $A \subset R$ be multiplicatively closed. Let M be an R-module. Assume we know that $(m, a) \sim (n, b)$ if and only if $mbc = nac$ for some $c \in A$ defines an equivalence relation on $M \times A$. (Note: recall the definition of an equivalence relation.)

(a) Writing $A^{-1}M$ for the set of equivalence classes of $\sim$, and $\frac{m}{a}$ for the class containing (m, a), show that the operation

$$\frac{m}{a} + \frac{n}{b} = \frac{bm + an}{ab}$$

is well defined and hence that $A^{-1}M$ is an abelian group.

(b) By defining an appropriate multiplication rule, show that $A^{-1}M$ has the structure of an $A^{-1}R$-module.

Problem 5. Let R be a ring and $A \subset R$ be multiplicatively closed.

(a) Suppose that $\phi : M \to N$ is a homomorphism of R modules. Show ϕ induces an $A^{-1}R$-homomorphism $A^{-1}M \to A^{-1}N$.

(b) Suppose $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules. Show that $0 \to A^{-1}L \to A^{-1}M \to A^{-1}N \to 0$, with the induced maps from (i), is an exact sequence of $A^{-1}R$-modules. (*Remark*: This means that localization is an exact functor from the category of R-modules to the category of $A^{-1}R$-modules.)