Module Title: Commutative rings and algebraic geometry

School of Mathematics

Semester Two

Calculator instructions:

- You are not allowed to use a calculator in this exam.

Dictionary instructions:

- You are not allowed to use your own dictionary in this exam. A basic English dictionary is available to use. Raise your hand and ask an invigilator if you need it.

Exam information:

- There are 5 pages to this examination.
- There will be **2 hours 30 minutes** to complete this examination.
- This examination is worth 100% of the module mark.
- Answer **all** questions.
- The numbers in brackets indicate the marks available for each question.
- You must show and explain all your solutions.
- **You must write all of your answers in the answer booklet provided**. If you require an additional answer booklet, raise your hand so an invigilator can provide one.
- You must clearly state your name and Student ID Number in the relevant boxes on your answer booklet. Other boxes may be left blank.

Turn the page over

Notation: All rings in this paper are commutative with a multiplicative identity 1. We use R for a ring, $I \subseteq R$ for an ideal and K for a field throughout the exam paper.

1. (25 points in total) Write in the booklet your answers to the blank parts of the sentences below. You do not need to write the full sentence.

   (a) (1 pt) A ring is called ______ if every element other than zero has a multiplicative inverse.

   (b) (2 pts) Let R, S be rings. A *ring homomorphism* is a map $\varphi : R \to S$ such that for all $r_1,\ r_2 \in R$, we have the following:

   (i) ______________

   (ii) ______________

   (ii) ______________

   (c) Let φ as above in part (b). Then,

   (i) (1 pt) The kernel of φ, Ker φ, is the set ______________

   (ii) (1 pt) The image of φ, Im φ, is the set ______________

   (d) Let I be a proper ideal in R.

   (i) (1 pt) I is ______ if and only if R/I is an integral domain.

   (ii) (1 pt) I is ______ if and only if R/I is a field.

   (e) (3 pts) Let $S^{-1}R$ be a localization of R. There are bijections

   $\{$ideals in $J \subseteq S^{-1}R\} \leftrightarrow \{$ideals $I \subseteq R$ such that ________________________$\}$

   and

   $\{$prime ideals in $Q \subseteq S^{-1}R\} \leftrightarrow \{$prime ideals $P \subseteq R$ such that ______________$\}$

   (f) (i) (2 pts) The nilradical of R, $nil(R)$, is ______________

   (ii) (2 pts) The Jacobson radical, $J(R)$, is ______________

   (g) Let M, N be R-modules and $\psi : M \to N$ be an R-module homomorphism.

   (i) (2 pts)The set $Hom_R(M, N)$ is an R-module, via the action ______________ for all $r \in R$, $\psi \in Hom_R(M, N)$ and $m \in M$.

   (ii) (2 pts) The cokernel of ψ, $Coker\ \psi$, is the set ______________

   (iii) (2 pts) For $\psi : M \to N$, we can state the first isomorphism theorem as follows

   $$M/\_\_\_\_\_ \cong \_\_\_\_\_\_$$

   (h) (3 pts) We say $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of R-modules if and only if ____.

   (i) (2 pts) If K is algebraically closed, then every maximal ideal of $K[x_1, \cdots, x_n]$ is of the form ____.

2. (25 points in total)

   (a) (6 pts) Let $\sqrt{I} = \{r \in R : r^n \in I$ for some positive integer n$\}$. Show that $\sqrt{I}$ is an ideal and contains I.

   (b) (6 pts) Consider the polynomial ring $R[x, y]$. List and order all monomials with degree less than or equal to 3 with respect to **lex** (lexicographic) and **deglex** (degree lexicographic) orders. What is the leading monomial of the polynomial $p(x, y) = x^4 + x^2y^2 + x^3y + xy^4$ with respect to **lex** and **deglex** orders?

   (c) (6 pts) Write a free resolution of the field K as a $K[x, y]$-module with maps explicitly written and state the exactness at every degree.

   (d) (7 pts) Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of R-modules. Show that for any R-module A, the following

$$0 \to Hom_R(N, A) \xrightarrow{g^*} Hom_R(M, A) \xrightarrow{f^*} Hom_R(L, A)$$

   is exact.

3. (25 points in total)

   (a) (5 pts) Let K be a field. Prove that there is an isomorphism

   $$K[x,y]/I \cong K[x]/\langle x^2 \rangle$$

   where I is the ideal $\langle x^2, y \rangle$.

   (b) (i) (6 pts) Let $R = K[x,y,z,w]$, $I = \langle xy, xz, xw \rangle$ and $J = \langle xy, xz, zw \rangle$. Give minimal primary decompositions of I and J. Explain your solution. (Note that you may assume $J = \langle xy, xz, zw \rangle$ is a radical ideal.)

   (ii) (6 pts) Consider $I = \langle x^2, xy \rangle$ in $K[x,y]$. Take the decompositions

   $$\langle x \rangle \cap \langle x^2, xy, y^2 \rangle = \langle x \rangle \cap \langle x^2, y \rangle$$

   Are they both minimal and primary decompositions for I? Explain. Can you find another one without the component $\langle x \rangle$? Explain.

   (c) (i) (2 pts) Define the vanishing locus $\mathbb{V}(S)$ for a subset S of $K[x_1, \cdots, x_n]$.

   (ii) (6 pts) Consider $I = \langle (x^2 - y^2)(y^2 - z^2) \rangle$ and $J = \langle xy, zy \rangle$ in $K[x,y,z]$. Describe $\mathbb{V}(I)$ and $\mathbb{V}(J)$ in detail, decompose $\mathbb{V}(I)$ into irreducible components and explain; especially explain what it means geometrically.

4. (25 points in total)

   (a) (7 pts in total: each part from (i) to (v) 1 pt, and (vi) 2 pts)

      (i) State the Hilbert Basis Theorem.

      (ii) Give an example of a non-Noetherian ring.

      (iii) Give an example of a Noetherian module over a Noetherian ring.

      (iv) Give an example of a non-Noetherian module over a Noetherian ring.

      (v) If R is Noetherian, is the quotient ring R/I (where I is an ideal as usual) Noetherian? Explain.

      (vi) Is the ring $\mathbb{Z}[i]$ of Gaussian integers a Noetherian $\mathbb{Z}$-module? Is $\mathbb{Z}[i]$ a Noetherian ring? Explain.

   (b) Let I be an ideal of $K[x_1, \cdots, x_n]$ and X be a subset of $\mathbb{A}^n_K$.

      (i) (4 pts) Define $\mathbb{I}(X)$ and explain under what condition(s) we have an equality $X = \mathbb{V}(\mathbb{I}(X))$.

      (ii) (2 pts) When do we have the equality $\sqrt{I} = \mathbb{I}(\mathbb{V}(I))$?

   (c) (6 pts) Show that if every non-empty set of submodules of a module M has a maximal element, then M is Noetherian.

   (d) (6 pts) Show that I is primary if and only if $R/I \neq 0$ and every zero-divisor in R/I is nilpotent.