

• FRIEZES •

EXPLORING COMBINATORIAL WINDMILLS ACROSS ALGEBRAIC LANDSCAPE

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April 2024

COMBINATORICS
"Friezes"

GEOMETRY

ALGEBRA



REPRESENTATION THEORY

COMBINATORICS
"Friezes"

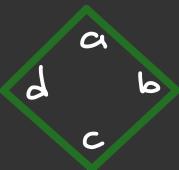
... 0 0 0 0 0 0 0 0 ...
 | | | | | | | | |

FRIESES

... 0 0 0 0 0 0 0 0 0 ...
... 1 1 1 1 1 1 1 1 1 ...
2 2 1 4 1 2 2 2 2 ...

FRIEZES

	0	0	0	0	0	0	0	0	0	0	...
...	1	1	1	1	1	1	1	1	1	1	...
2	2	1	4	1	2	2	2	2	2	2	...
?	?	?	?	?	?	?	?	?	?	?	...

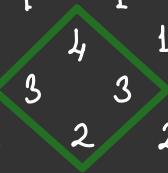
Rule :  $\rightarrow bd - ac = 1$

FRIESES

	0	0	0	0	0	0	0	0	0	...
...	1	1	1	1	1	1	1	1	1	...
2	2	1	4	1	2	2	2	2	2	...
3	1	3	3	1	3	3	3	3	3	...

FRIESES

	0	0	0	0	0	0	0	0	0	...
...	1	1	1	1	1	1	1	1	1	...
2	2	1	4	1	2	2	2	2	2	...
3	1	3	3	3	1	3	3	3	3	...
4	1	2	2	2	1	1	1	4	4	...



FRIEZES

COMBINATORICS
"Friezes"

...

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	1	4	1	2	2	
3	1	3	3	1	3	3	...
4	1	2	2	2	1	4	
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0

'Conway-Coxeter frieze'

CONWAY - COXETER

finite friezes
of width n

| - |
↔

triangulations of $(n+3)$ -gon

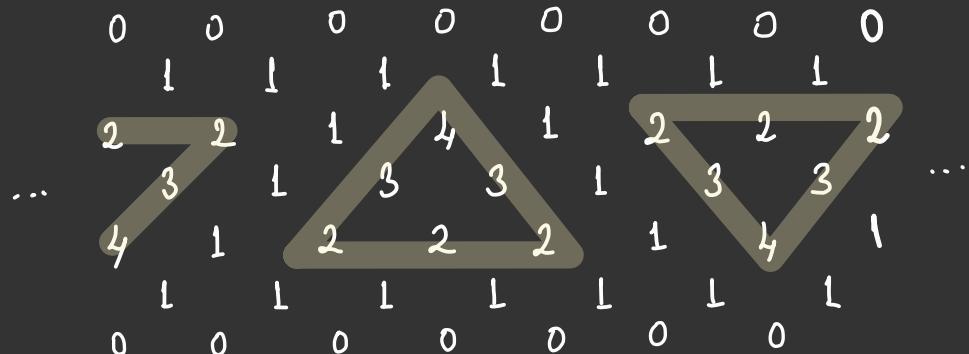
CONWAY - COXETER

finite friezes
of width n

\longleftrightarrow

triangulations
of $(n+3)$ -gon

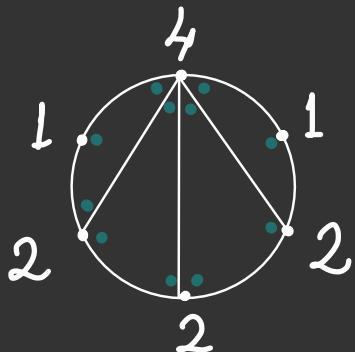
Symmetries :



COMBINATORICS "Friezes"

GEOMETRY

"triangulations"



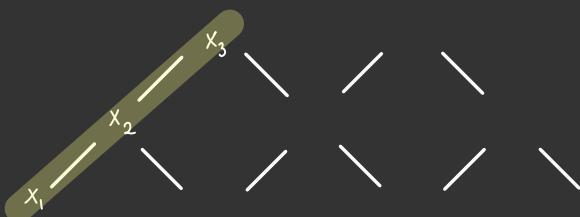
COMBINATORICS
"Friezes"

	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1
...	2	2	1	4	1	2	2	..
	3	3	3	3	1	3	3	
4	1	2	2	2	1	4		
	1	1	1	1	1	1	1	
0	0	0	0	0	0	0	0	

ALGEBRA
"cluster algebras"

COMBINATORICS
"Friezes"

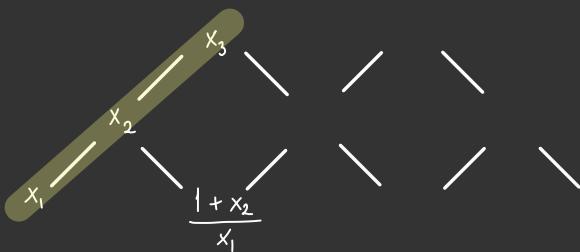
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
...	2	2	1	4	1	2	2
3	3	1	3	3	1	3	3
4	1	2	2	2	1	4	
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0



ALGEBRA
"cluster algebras"

COMBINATORICS "Friezes"

	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1
...	2	2	1	4	1	2	2	..
	3	3	3	3	1	3	3	
	4	1	2	2	2	1	4	
	1	1	1	1	1	1	1	
	0	0	0	0	0	0	0	



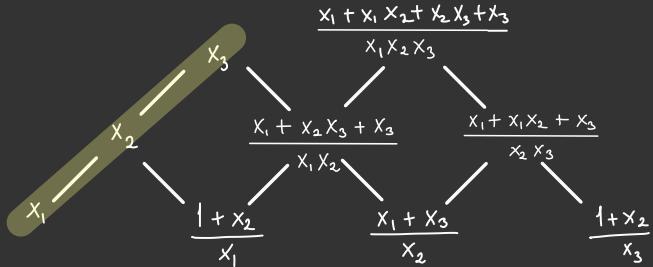
ALGEBRA "cluster algebras"

COMBINATORICS
"Friezes"

	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1
...	2	2	1	4	1	2	2	..
	3	1	3	3	1	3	3	
4	1	2	2	2	1	4		
	1	1	1	1	1	1	1	
0	0	0	0	0	0	0	0	

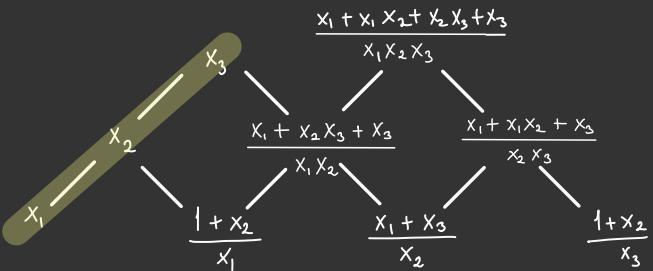


ALGEBRA
"cluster algebras"



COMBINATORICS
"Friezes"

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	1	4	1	2	2	
3	3	3	3	1	3	3	
4	1	2	2	2	1	4	
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0



ALGEBRA
"cluster algebras"

' A_3 cluster algebra'
: subring of the field
of fractions $\mathbb{D}(x_1, x_2, x_3)$

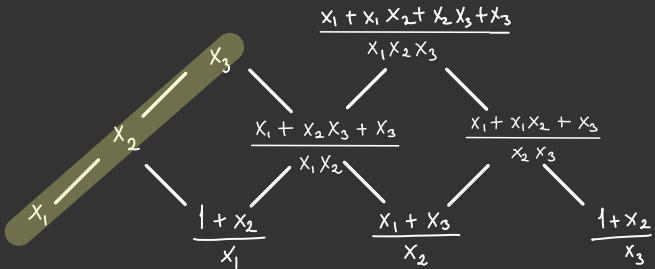
COMBINATORICS
"Friezes"

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	1	4	1	2	2	2
3	1	3	3	1	3	3	3
4	1	2	2	2	1	4	1
1	1	0	1	0	1	0	1
0	0	0	0	0	0	0	0

(Set $x_1 = x_2 = x_3 = 1$)



ALGEBRA
"cluster algebras"



COMBINATORICS

" Friezes "

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	1	4	1	2	2	2
3	1	3	3	1	3	3	3
4	1	2	2	2	1	4	1
1	1	0	0	0	1	1	0
0	0	0	0	0	0	0	0

'Conway-Coxeter frieze'

GEOMETRY
" triangulations "



'A₃ triangulation'



ALGEBRA
"cluster algebras"

$$\begin{array}{c}
 \frac{x_1 + x_1x_2 + x_2x_3 + x_3}{x_1x_2x_3} \\
 \swarrow \quad \searrow \\
 x_2 \quad x_3 \\
 \swarrow \quad \searrow \\
 \frac{1+x_2}{x_1} \quad \frac{x_1+x_3+x_3}{x_1x_2} \\
 \swarrow \quad \searrow \\
 x_1x_2 \quad \frac{x_1+x_3}{x_2} \\
 \swarrow \quad \searrow \\
 \frac{x_1+x_2+x_3}{x_2x_3} \quad \frac{1+x_2}{x_3}
 \end{array}$$

'A₃ cluster algebra'

COMBINATORICS

" Friezes "

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	1	4	1	2	2	2
3	3	1	3	3	1	3	3
4	1	2	2	2	1	4	1
1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

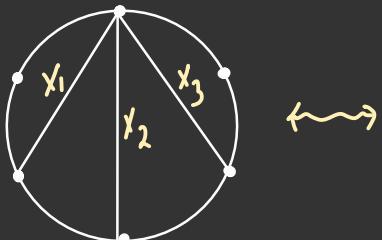
'Conway-Coxeter frieze'

GEOMETRY

" triangulations "



'A₃ triangulation'



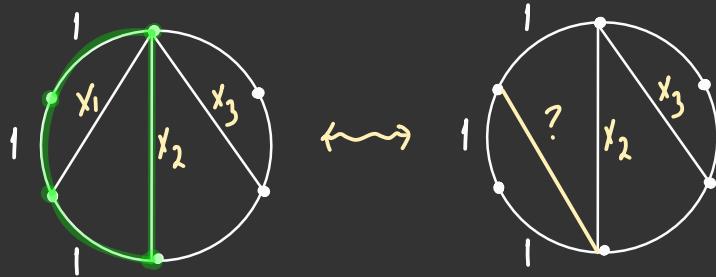
ALGEBRA

"cluster algebras"

$$\begin{array}{c}
 \frac{x_1 + x_1x_2 + x_2x_3 + x_3}{x_1x_2x_3} \\
 \swarrow \quad \searrow \\
 x_2 \quad x_3 \\
 \swarrow \quad \searrow \\
 \frac{1+x_2}{x_1} \quad \frac{x_1+x_3}{x_2} \quad \frac{x_1+x_2+x_3}{x_3} \\
 \end{array}$$

'A₃ cluster algebra'

Ptolemy relation:



$$? = \frac{1 \cdot 1 + 1 \cdot x_2}{x_1}$$

$$= \frac{1 + x_2}{x_1}$$

COMBINATORICS

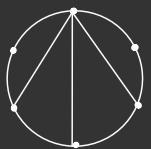
" Friezes "

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	1	4	1	2	2	2
3	3	1	3	3	1	3	3
4	1	2	2	2	1	4	1
1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

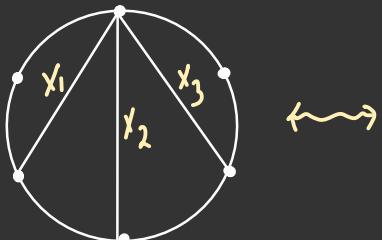
'Conway-Coxeter frieze'

GEOMETRY

" triangulations "



'A₃ triangulation'



ALGEBRA

" cluster algebras "

$$\begin{array}{c}
 \frac{x_1 + x_1 x_2 + x_2 x_3 + x_3}{x_1 x_2 x_3} \\
 \swarrow \quad \searrow \\
 x_2 \quad x_3 \\
 \swarrow \quad \searrow \\
 \frac{1 + x_2}{x_1} \quad \frac{x_1 + x_3}{x_2} \quad \frac{x_1 + x_1 x_2 + x_3}{x_2 x_3} \\
 \swarrow \quad \searrow \\
 \frac{1 + x_2}{x_3}
 \end{array}$$

'A₃ cluster algebra'

COMBINATORICS

" Friezes "

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	1	4	1	2	2	2
3	3	1	3	3	1	3	3
4	1	2	2	2	1	4	1
1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0

'Conway-Coxeter frieze'

GEOMETRY

" triangulations "



'A₃ triangulation'



ALGEBRA

"cluster algebras"

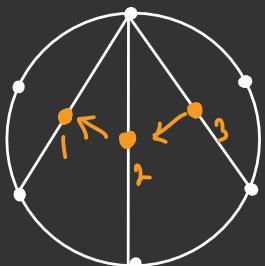
$$\begin{array}{ccccc}
 & & \frac{x_1 + x_1x_2 + x_2x_3 + x_3}{x_1x_2x_3} & & \\
 & \nearrow x_3 & \downarrow & \searrow x_1 + x_2x_3 + x_3 & \\
 x_1 & \nearrow x_2 & \frac{x_1 + x_2}{x_1} & \downarrow & \frac{x_1 + x_2x_3 + x_3}{x_2} \\
 & & \frac{x_1 + x_2}{x_1} & \nearrow x_2x_3 & \searrow \frac{x_1 + x_2}{x_3} \\
 & & & &
 \end{array}$$

'A₃ cluster algebra'

REPRESENTATION THEORY

"cluster categories"

GEOMETRY
" triangulations "



REPRESENTATION THEORY
" cluster categories "

$$\Theta = i_1 \leftarrow i_2 \leftarrow i_3$$

$k\Theta$ = the path algebra of Θ

$\text{mod } k\Theta \simeq \text{rep } \Theta$

$$\theta = \begin{matrix} & \alpha \\ \cdot & \leftarrow & \cdot \\ & \beta & \end{matrix} \begin{matrix} & \cdot \\ \cdot & \leftarrow & \cdot \\ & \gamma & \end{matrix}$$

- A quiver representation : $V_1 \xleftarrow{\epsilon_\alpha} V_2 \xleftarrow{\epsilon_\beta} V_3$

↳ vector spaces at vertices,

↳ linear map for arrows .

$$\theta = \begin{matrix} & \alpha \\ & \swarrow \\ 1 & & 2 & \leftarrow \beta \\ & \searrow & & \\ & & 3 & \end{matrix}$$

$$k \leftarrow k \leftarrow k$$

$$k \leftarrow k \leftarrow o$$

$$o \leftarrow k \leftarrow k$$

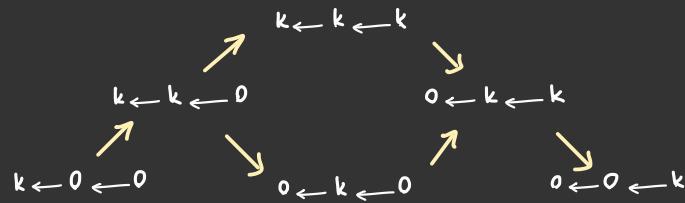
$$k \leftarrow o \leftarrow o$$

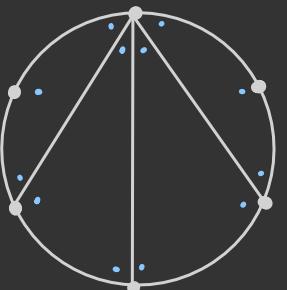
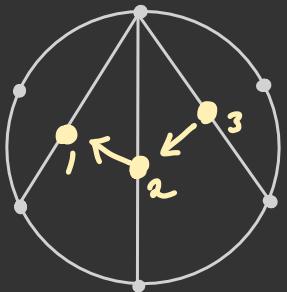
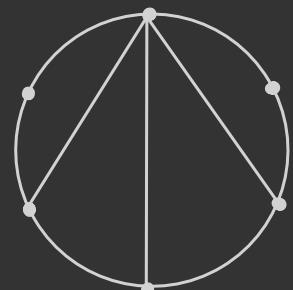
$$o \leftarrow k \leftarrow o$$

$$o \leftarrow o \leftarrow k$$

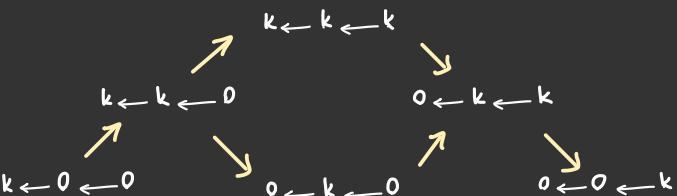
$$\theta = \begin{matrix} & \alpha \\ i & \leftarrow & 2 & \leftarrow & \beta \\ & & 1 & & 3 \end{matrix}$$

• rep θ :



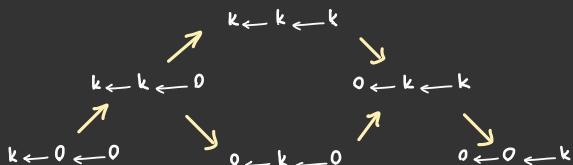
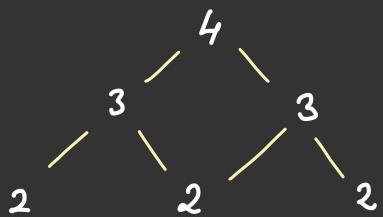
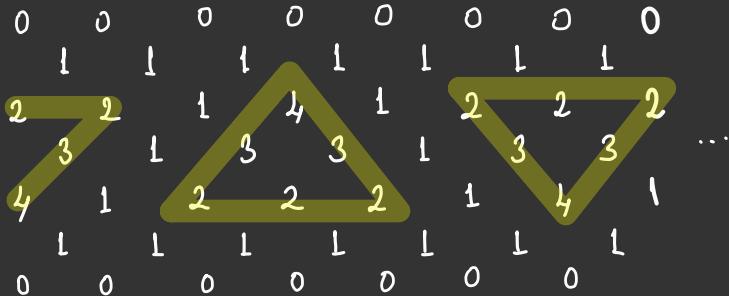


0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	2	1	1	4	1	1	2	2
3	3	1	3	3	3	1	3	3
4	1	2	2	2	2	1	4	1
1	1	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0	0





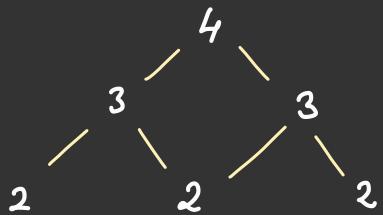
4



$$\begin{matrix} & 4 \\ 2 & 3 & 3 \\ & 2 & 2 & 2 \end{matrix}$$


...

0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	2	4	3	3	2	2	2	2
3	1	1	3	3	1	1	1	1
4	1	2	2	2	1	1	1	1
1	0	0	0	0	0	0	0	0



counting
submodules!



COMBINATORICS

"Friezes"

0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	2	1	4	1	2	2	2
3	1	3	3	1	3	3	3
4	1	2	2	2	1	4	1
1	1	0	0	0	1	1	1
0	0	0	0	0	0	0	0

'Conway-Coxeter frieze'

GEOMETRY

"triangulations"



'A₂ triangulation'



ALGEBRA

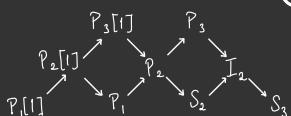
"cluster algebras"

$$\begin{array}{ccccc}
 & & \frac{x_1 + x_1x_2 + x_2x_3 + x_3}{x_1x_2x_3} & & \\
 & \nearrow x_3 & \downarrow & \searrow x_1 + x_2x_3 + x_3 & \\
 x_1 & \nearrow x_2 & \downarrow & \nearrow x_1x_2 & \searrow x_2x_3 \\
 & \downarrow & \frac{1+x_2}{x_1} & \downarrow & \frac{1+x_2}{x_3} \\
 & & x_1x_2 & &
 \end{array}$$

'A₂ cluster algebra'

REPRESENTATION THEORY

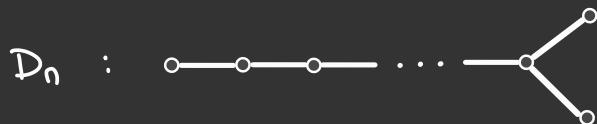
"cluster categories"



'A₂ cluster category'

THEOREM (GABRIEL '72)

α is a Dynkin diagram iff $k\alpha$ has finitely many indecomposable/iso.



TAME HEREDITARY ALGEBRAS

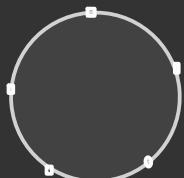
$$\tilde{A}_n \quad : \quad \text{Diagram showing a horizontal chain of nodes connected by straight lines, with a diagonal line connecting the first and last nodes.}$$

$$\tilde{E}_7 \quad : \quad \text{Diagram showing a horizontal line with six open circles. A vertical line segment connects the fourth circle from the left to an open circle above it.}$$

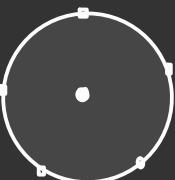
THEOREM (GABRIEL '72)

α is a Dynkin diagram iff $k\alpha$ has finitely many indecomposable/iso.

$$A_n : \circ - \circ - \circ - \cdots - \circ - \circ$$



$$D_n : \circ - \circ - \circ - \cdots - \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array}$$



$$E_6 : \circ - \circ - \circ - \circ - \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array}$$

$$E_7 : \circ - \circ - \circ - \circ - \circ - \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array}$$

$$E_8 : \circ - \begin{array}{c} \circ \\ | \\ \circ \end{array}$$

TAME HEREDITARY ALGEBRAS



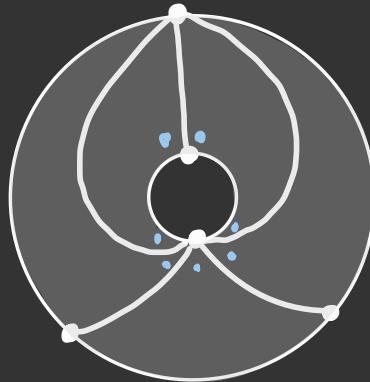
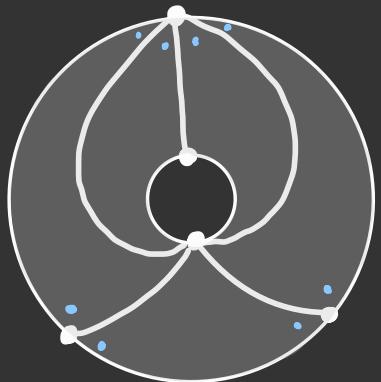
Infinite Friezes



Annulus

cluster algebra of type \tilde{A}

cluster category of \tilde{A}



0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
4	2	2	4	2	2	2	4
...	7	7	3	7	7	3	7
12	10	10	12	10	10	10	

⋮

0	0	0	0	0
1	1	1	1	1
2	5	2	5	2
9	9	9	9	9
40	16	40	40	

⋮

⋮

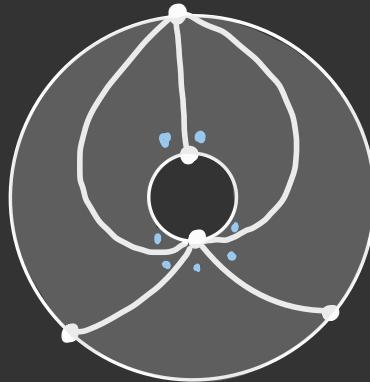
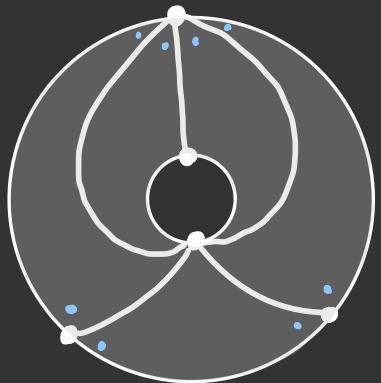
CONWAY - COXETER

$$\begin{matrix} \text{triangulations} \\ \text{of } (n+3)\text{-gon} \end{matrix} \quad \xleftrightarrow{\text{1-1}} \quad \begin{matrix} \text{finite friezes} \\ \text{of width } n \end{matrix}$$

Baur - Parsons - Tschabold '15

Every periodic infinite frieze comes from a triangulation
of annulus.





0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
4	2	2	4	2	2	2	2	4
7	7	3	7	7	7	3	7	7
12	10	10	12	12	10	10	10	10

⋮

0	0	0	0	0
1	1	1	1	1
2	5	2	5	2
9	9	9	9	9
40	16	40	40	40

⋮

⋮

Baur - Fellner - Parsons - Tschabold

The growth coefficient is the same for both of
the friezes in an annulus.

E. Gunawan



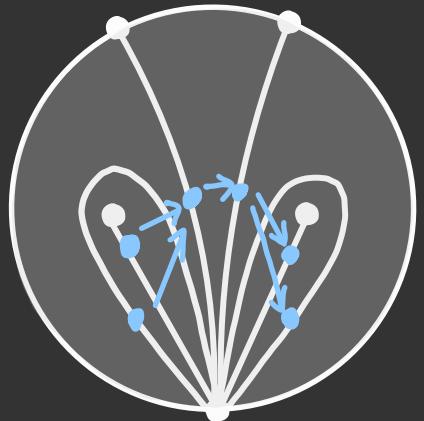
L. Bittmann



G. Todorov

K. Bauer

TAME HEREDITARY ALGEBRAS



$$\tilde{A}_n : \quad \begin{array}{ccccccccc} \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \end{array}$$

$$\tilde{D}_n : \quad \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ & \cdots & \circ & \cdots & \circ \\ \uparrow & & \uparrow & & \uparrow \\ \circ & \cdots & \circ & \cdots & \circ \end{array}$$

$$\tilde{E}_6 : \quad \begin{array}{ccccc} & & \circ & & \\ & & | & & \\ \circ & \cdots & \circ & \cdots & \circ \end{array}$$

$$\tilde{E}_7 : \quad \begin{array}{ccccc} & & & \circ & \\ & & & | & \\ \circ & \cdots & \circ & \cdots & \circ \end{array}$$

$$\tilde{E}_8 : \quad \begin{array}{ccccccccc} & & & & \circ & & & & \\ & & & & | & & & & \\ \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ \end{array}$$

Joint work with K. Baur, L. Billman, E. Gunawan, G. Todorov

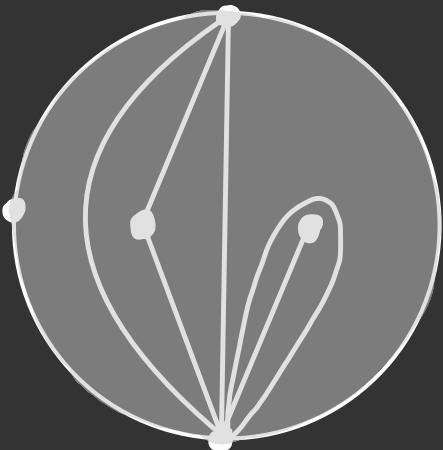
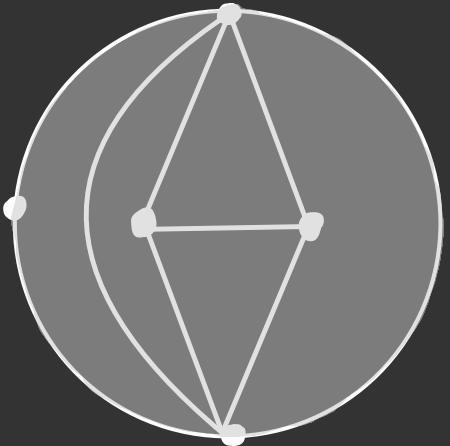
Three pairs of Infinite Friezes

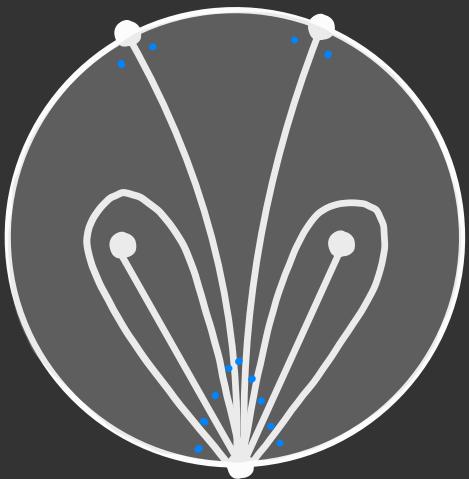


Twice-punctured disk

cluster algebra of type \tilde{D}

cluster category of \tilde{D}





0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
9	2	2	9	2	2	9	2	2	2
...	17	3	17	17	3	17	32	25	3
	25	25	32	25	25	32			

0 0 0 0

1 1 1
5 5 5

24 24 24

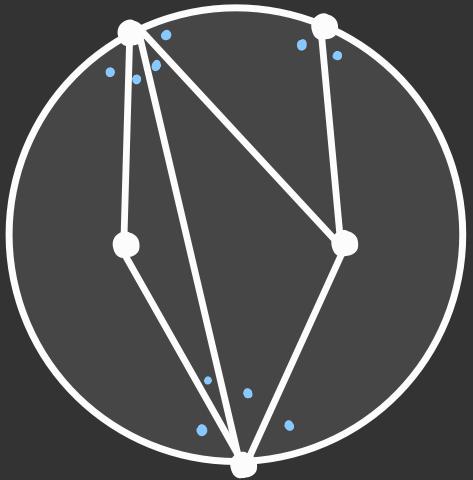
115 115 115

1

5 5 5 5 5 5 5 5
24 24 24 24 24 24 24 24

115 115 115 115 115 115

17



0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1
2	4	4	2	4	4	2	4	4	2	...
7	15	7	7	15	7	7	15	7	7	...
26	26	24	26	26	45	89	89	45	26	...
45	89	89	45	26	45	89	89	45	26	...

:

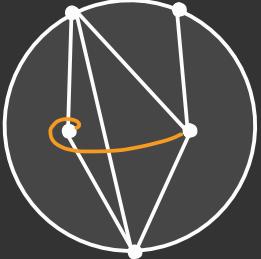
0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1
2	4	4	2	4	4	2	4	4	2	...
7	15	7	7	15	7	7	15	7	7	...
26	26	24	26	26	45	89	89	45	26	...
45	89	89	45	26	45	89	89	45	26	...

:

1 0 0 0 0 0 0 0 1
... 4 6 4 6 4 6 4 6 ...
23 23 23 23 23 23 23

1	0	0	0	0	0	0	1	...
4	1	6	1	4	1	6	1	...
	23	23	23	23	23	23		

:



1	0	0	0	0	0	0	1	...
2	1	12	1	2	1	12	2	...
	23	23	23	23	23	23	23	
	44	44	44	44	44	44	12	

:



THM [BBGTY]

The growth coefficient is the same for all three friezes
of twice-punctured disk.

0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1

T H A N K Y O U

159 7 13 153 274 374 314

• 139 90 142 3811 4099 7829 •

• • • • • • • • • •

• • • • • • • • • •

• • • • • • • • •

for $q = (20, 8, 1, 14, 11, 25, 15, 21)$