Mathematical Modelling Coursework

Emilia Podgorski

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Introduction

This project explores the dynamics of a non-linear supply and demand model, proposed in the paper "Landscape and flux theory of non-equilibrium open economy" by Zhang and Whang in 2017. The model proposed is:

$$\frac{dP}{dt} = (-1+a)P + c - Q$$
$$\frac{dQ}{dt} = P - (d + bQ^2)Q$$

where Q=Quantity and P=Price

This is different to the standard linear model where both supply and demand are two straight lines. In this model the supply curve has a cubic element, which can lead to interesting behaviours in the system, such as bi-stable states and limit cycle oscillations.

By exploring the dynamics of the non-linear system with different parameter values and modifications added, we can see how the goods market will behave under different conditions and outside factors. The phase space can represent both the monopoly/oligopoly and competition states of the market, and show the relationship between them. It can also determine the stability of the market.

Question 1

The first question explores the behaviour of the system when supply is linear (b=0). With the parameters a=-1, b=0, c=1 and d=-1 the equations simplify to:

$$\frac{dP}{dt} = -2P + 1 - Q$$
$$\frac{dQ}{dt} = P + Q$$

The fixed points can be found by looking at the null-clines:

$$\frac{dP}{dt} = 0 \implies Q = 1 - 2P$$

$$\frac{dQ}{dt} = 0 \implies Q = -P$$

So there's a fixed point at $P = 1$, $Q = -1$

The stability of this fixed point can be found by calculating the Jacobian for the system. Here Q has been chosen to be the x value and P to be the y, as this was the decision made in figure 1 of the paper. Setting $f = \frac{dQ}{dt}$ and $g = \frac{dP}{dt}$,the Jacobian for this dynamical system is:

$$\begin{pmatrix} \frac{\partial f}{\partial Q} & \frac{\partial f}{\partial P} \\ \frac{\partial g}{\partial Q} & \frac{\partial g}{\partial P} \end{pmatrix} = \begin{pmatrix} -d - 3bQ^2 & 1 \\ -1 & -1 + a \end{pmatrix}$$

In the code for question 1 the fsolve function verifies this fixed point and plots it on a phase portrait along with the nullclines. The limits of the graph are set at -4 and 4 in both axis to emulate figure 1 in the paper. Then a function for the Jacobian is used to calculate the Jacobian and its eigenvalues for the fixed point $(-1, 1)$, which classifies it as a saddle point as it has both a negative and positive eigenvalue (0.618 and -1.618). These results are displayed in the command window.

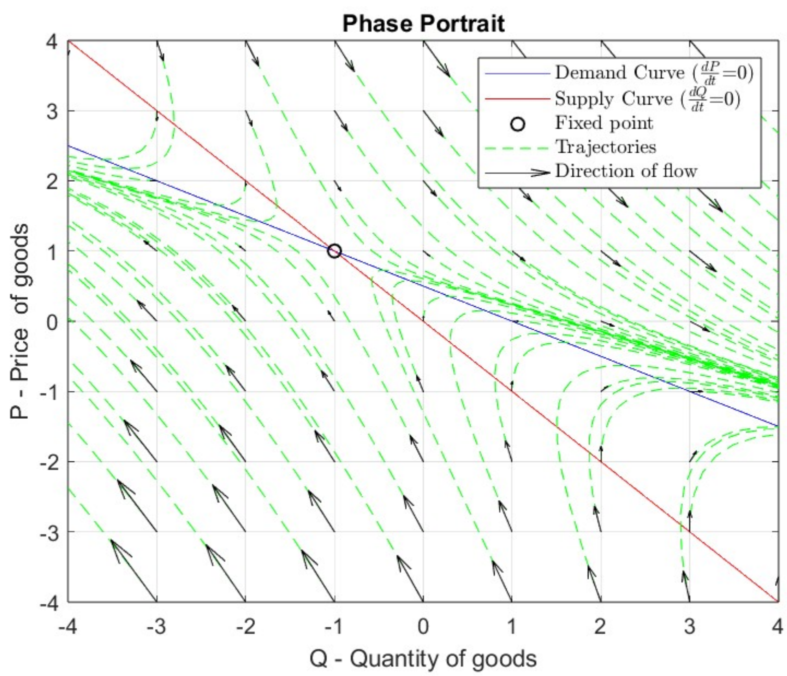

Figure 1: Phase portrait showing the dynamics of the system when a=-1, b=0, c=1 and d=-1. Both the supply and demand curve are downward sloping straight lines that intersect at a saddle point, which all the trajectories of the system move away from.

To find the trajectories of this system, ode45 was used to solve the dynamical system over a time period of 12 months, with starting points placed evenly around the grid. The quiver function was then used to plot the direction of flow in the graph. The graph created is displayed in Figure 1.

The fixed point is at (-1,1), with negative quantity and positive price. This is likely to represent the monopoly/oligopoly state of the market, as individual companies thrive when supply is low and they can increase the price.

The trajectories and arrows show the behaviour of the system. Here it can be seen that the trajectories move away from the saddle point, and never touch it. Some get close, but ultimately over time they move far off to the left or right. This suggests that the market never reaches this equilibrium, no matter the initial conditions. There also aren't any stable oscillations between price and quantity. The system either veers strongly towards negative quantity and positive price, or the other way round. Therefore this is an unstable market.

Question 2

This question looks at non-linear versions of the system, with b $\neq$ 0. This results in the supply curve becoming a cubic equation, meaning the null clines can intersect more than once. The code needed to be modified to plot and classify multiple fixed points. An array was implemented to store every result from the fsolve function, which were then rounded to 4 decimal places with the round function. The unique function was used so that each fixed point only appeared once in the array 'all_fp'. These points were then individually classified and plotted.

2i)

Inputting the parameter values a=-0.5, b=0.3, c=0 and d=-1 gives the graph displayed in Figure 2. It can be seen that there are 3 fixed points. According to the output of the code, points (1.054, -0.703) and (-1.054, 0.703) are stable fixed points, while (0,0) is a saddle point. The trajectories spiral towards the stable fixed points, suggesting these are stable spirals. This is confirmed by the fact that both their eigenvalues have real and imaginary components. The trajectories move close to, but ultimately pass, the saddle point.

This agrees with the results from the paper, which states that when $0 < a < 1$ there are two stable fixed points and one saddle point. This means that it's a bi-stable state.

In the context of this model, bi-stability represents the coexistence of competition and monopoly/oligopoly markets. Some trajectories head towards the competition state, while others head towards the monopoly state. Both these stable states exist simultaneously. The competition state is likely to be at (1.054, -0.703) when the quantity is high and price is low. This is because a market is competitive when there's a high quantity of a product that many people can sell, with a lower production price. The monopoly state is likely to be at (-1.054, 0.703) because when the quantity available is low, the few sellers that have it can increase the price.

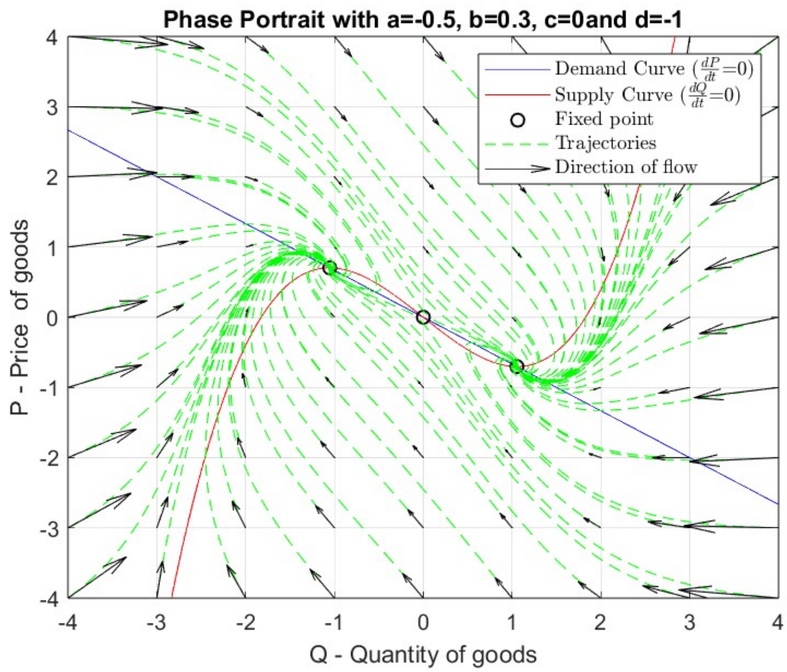

Figure 2: A phase portrait where the supply and demand curve intersect three times, at two stable spirals and one saddle point. The trajectories spiral towards either the monopoly/oligopoly stable state or the competition stable state.

2ii)

Inputting the parameter values a=-0.1, b=0.3, c=0 and d=-1 gives the graph displayed in Figure 3. Similarly to question 2i, this model has two stable spirals at (-0.551, 0.5) and (0.551, -0.5), as well as a saddle point at (0,0). The behaviour in this system is similar to the last, but the fixed points are closer together and the trajectories seem to move further from their initial conditions before reaching a stable equilibrium. They also spiral more prominently and some trajectories pass very close by the saddle point. This suggests that the market is more volatile. Increasing parameter a means that it's easier for trajectories to move from the competitive market to the monopoly market, and vice versa. However, the trajectories still reach a stable fixed point eventually, so the market is stable.

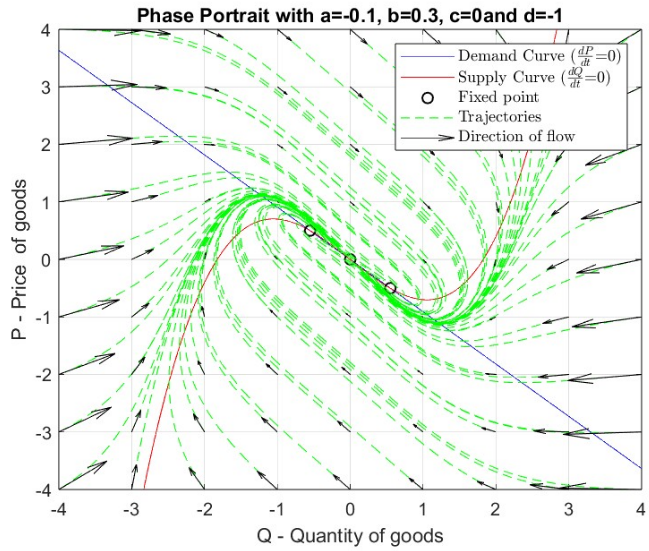

Figure 3: A phase portrait with two stable spirals and a saddle point. The trajectories spiral tightly around the stable spirals and some get very close to the saddle point, but ultimately they reach a stable state.

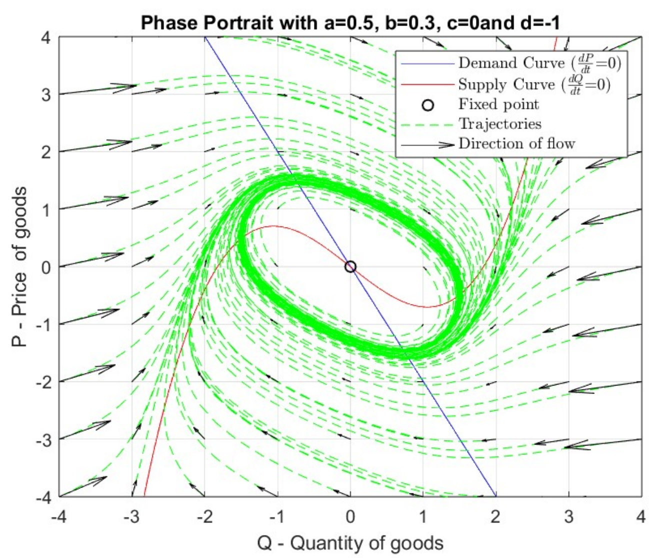

Figure 4: A phase portrait where the supply and demand curve intersect at an unstable fixed point. The trajectories form a limit cycle around this point, moving between the monopoly and competition states.

2iii)

Inputting the parameter values a=0.5, b=0.3, c=0 and d=-1 gives the graph displayed in Figure 4. Here there is just one fixed point at (0,0). The trajectories form a stable limit cycle around this unstable fixed point, which represents the stable periodic oscillation between competition and monopoly states. In between these states the system has a transient steady state. This system doesn't have a stable equilibrium state, but the changes between states are periodic and therefore predictable, so the goods market is stable.

2iv)

Now we look at the same parameters from question 2i, but with c=-0.3. This is the shift parameter, so a negative value of c will shift the demand curve to the left. The phase portrait created is displayed in Figure 5.

Compared to Figure 2, this model now only has one fixed point. This is a stable spiral at (-1.278, 0.652). All trajectories eventually end up at the stable monopoly market, meaning it is a mono-stable state. Some of the trajectories pass through the competitive stable state from Figure 2, but then carry on moving to the monopoly state. Just a small shift in the demand curve can change a market from a bi-stable state to a mono-stable one.

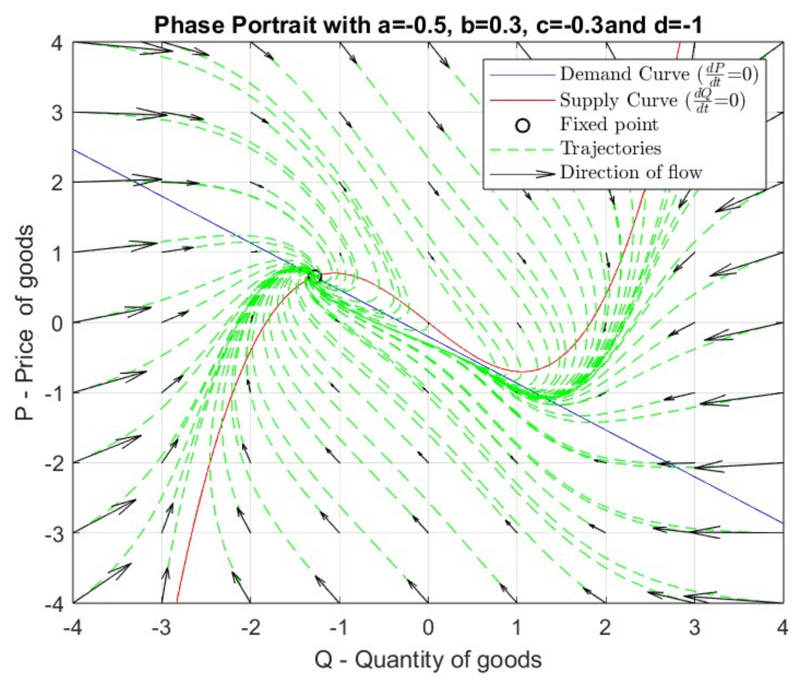

Figure 5: A phase portrait where the supply and demand curve intersect just once, at the stable monopoly state. All trajectories travel towards this state.

2v)

This question applies c=-0.3 to the parameters from question 2iii. The phase portrait created is shown in Figure 6.

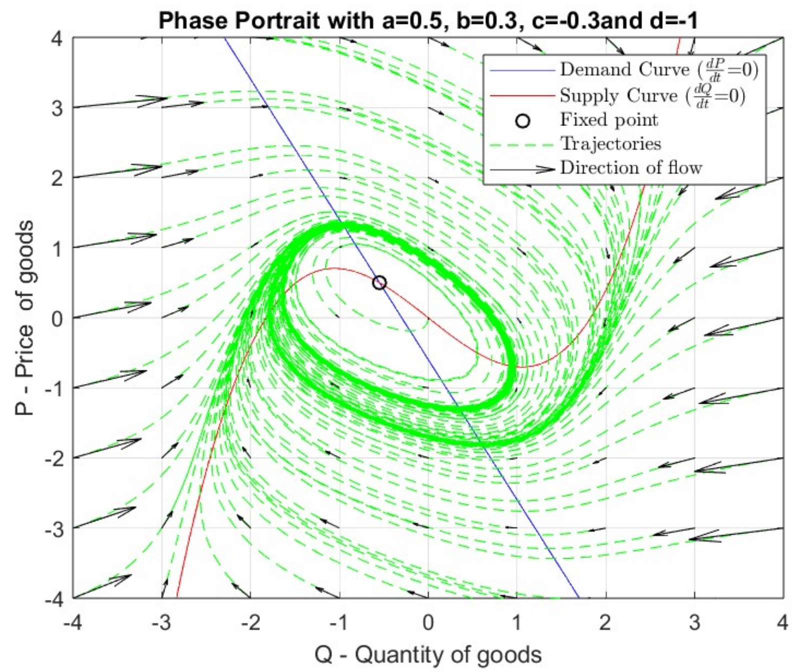

Figure 6: A phase portrait where the fixed point has shifted towards the monopoly state. A limit cycle eventually emerges in the trajectories that slightly leans towards the monopoly state.

Once again the demand curve has shifted to the left. Now the unstable fixed point is at (-0.55, 0.5). The trajectories still eventually reach a stable limit cycle, but it's smaller than before and seems to lean more heavily towards the monopoly market.

Overall, implementing c=-0.3 shifts the market in favour of the monopoly state. This is because monopoly states are more likely when demand decreases and other sellers stop offering the product.

Question 3

This question aims to use an appropriate modification to add the effect of seasonality to the model. A way to do this is by implementing periodic forcing. This is an appropriate method because the forcing can oscillate over a year, meaning it can repeat with the same values every 12 months, mimicking the effect of seasonality.

Seasonality is more likely to affect demand than supply, due to holidays such as Christmas leading to a rise in demand for presents, and changes in the weather (for example, hot weather leads to a rise in demand for ice cream). Though seasonality can also affect supply, for example in the agricultural industry the seasons affect what crops will grow, this is to a lesser degree. Therefore, the forcing should be implemented on the demand curve. The seasonality is an external factor that affects demand, so c can be chosen as the parameter used to implement periodic forcing. This also agrees with the paper, which states that researchers often use parameter c to shift the demand curve and explore the effect when economic conditions change.

Parameter c can be turned into a sine function of time, as this will change periodically. Assume:

$$c(t) = \delta sin(\omega t)$$

where δ is the magnitude of the sine wave and ω is the period. For the value of c to repeat every 12 months, we need to set $\omega = 2\pi/12$. The value of δ is harder to assign. When running the code for different values of δ, it can be found that when $\delta < 0.5$ the forcing hasn't taken effect, as the price and quantity don't oscillate within a 12 month time period. The behaviour is fairly similar between $0.5 < \delta < 1$, so for simplicity we can take $\delta = 1$.

The new function for c was added to the ode equation in the code. The time period that the system's integrated over was increased to 120 months, so as to get a proper view of the effect of the seasonality over 10 years. For this question it felt most useful to plot both the trajectories of the system and the price and quantity over time.

A plot with no forcing (δ=0) is displayed in Figure 7. This is the limit cycle economy seen in question 2iii. Here the values of quantity and price oscillate with similar amplitudes and a period of about 7.5 months, so the natural frequency of this system is around $2\pi/7.5$.

The model with seasonality added (δ=1) is shown in Figure 8. In contrast to the model without seasonality, here the values of price and quantity oscillate with a period of 12 months. This is because it has been entrained to match the period of the forcing. The quantity wave has a larger amplitude than the price wave, as the quantity is the equation that the periodic forcing was added to, so is affected more.

The trajectories initially appear to spiral towards the competition state, perhaps due to the value of the quantity initially being higher than the price, indicating a competition state. However, once the periods of the oscillations become regular the trajectories eventually seem to settle into a stable limit cycle. The long term behaviour of this system can be seen in Figure 9, which shows the trajectories and oscillations between 10 to 20 years after the seasonality being implemented. Here it can clearly be seen that after sufficient time has passed the trajectories form a stable limit cycle.

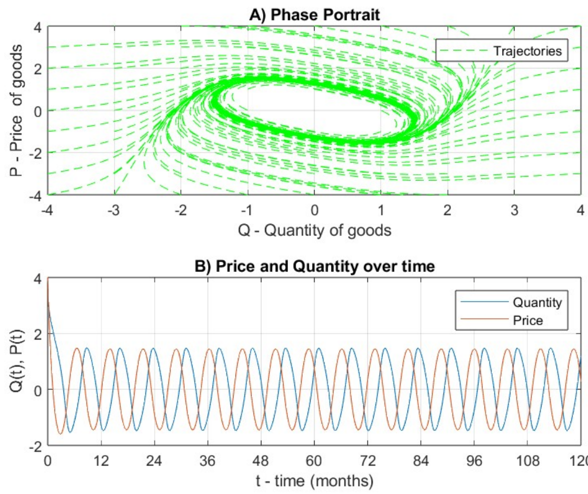

Figure 7: A) Trajectories of the market when a=0.5, b=0.3, c=0 and d=-1. Here there is no periodic forcing so the trajectories move in the stable limit cycle found in question 2iii. B) The values of price and quantity over time. They regularly oscillate with a period of around 7.5 months.

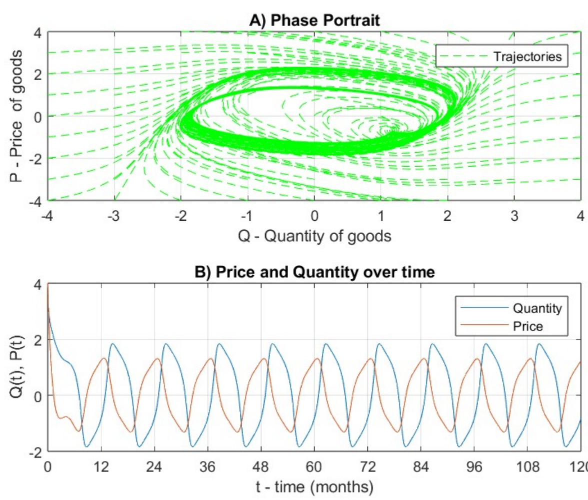

Figure 8: A) Trajectories of the system when a=0.5, b=0.3, d=-1 and seasonality is implemented with c(t)=$sin(2\pi t/12)$. The trajectories are initially quite irregular and tend towards the competition state, but eventually a limit cycle emerges. B) The values of price and quantity settle into oscillations with a period of 12 months, the period of the forcing.

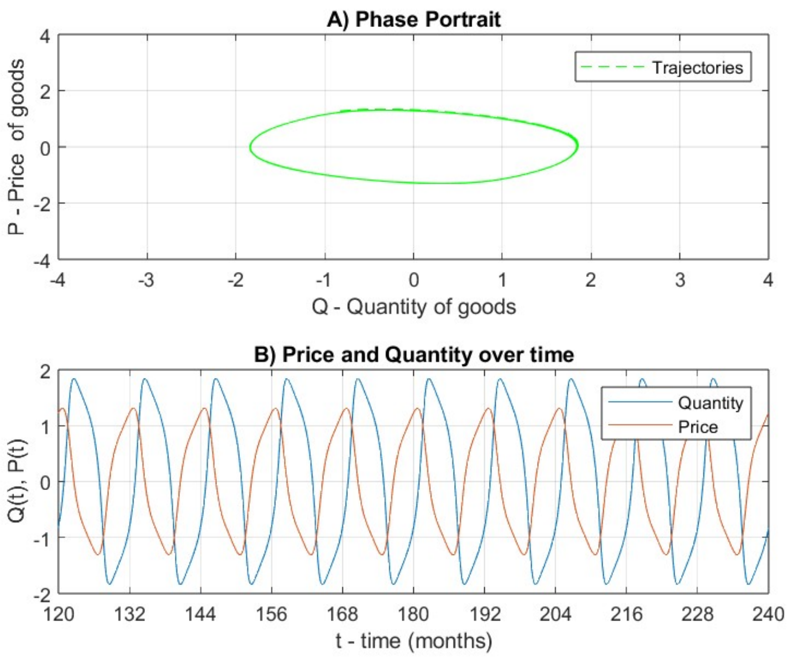

Figure 9: The system with seasonal forcing from t=120 to t=240. The trajectories form a stable limit cycle.

These results suggest that, though implementing seasonality initially makes the market quite unstable, after a few years the market will settle into a stable limit cycle. This could be interpreted as businesses adjusting to and preparing for the regular fluctuations between supply and demand throughout the year, as seasonality is predictable.

Question 4

The system can be turned into stochastic differential equations by adding a random noise term to it. The original equations can be written as:

$$dQ = (P - (d + bQ^2)Q)dt$$
$$dP = ((-1 + a)P + c - Q)dt$$

An element of randomness can be added to the quantity equation, because the quantity of products available can depend on unexpected factors such as consumer trends suddenly making one item popular, or severe weather affecting the transport or production of goods. Therefore, we can add the random element εQdW(t) to the equations, which now become:

$$dQ = (P - (d + bQ^2)Q)dt + \varepsilon Q dW(t)$$
$$dP = ((-1 + a)P + c - Q)dt$$

Here ε is the intensity of the noise, and W(t) is the Weiner process. Also known as Brownian motion, this is a random walk where the steps are drawn from a normal distribution with mean 0 and variance dt. This is what introduces the element of randomness to the noise. The quantity value is also included in the random noise, because the random events will directly affect it.

To implement this into the matlab code, the time period had to be split up into N steps and the Weiner process was simulated. The step sizes were calculated and then the cumulative sum was stored in a vector, which would show the Brownian path taken if plotted.

As stochastic differential equations can't be solved using ode45, the Euler-Maruyama method had to be implemented. This is an iterative method of solving the equations for one trajectory, again by splitting the time up into smaller steps. The initial condition is chosen to be an equilibrium of the system found in question 2i.

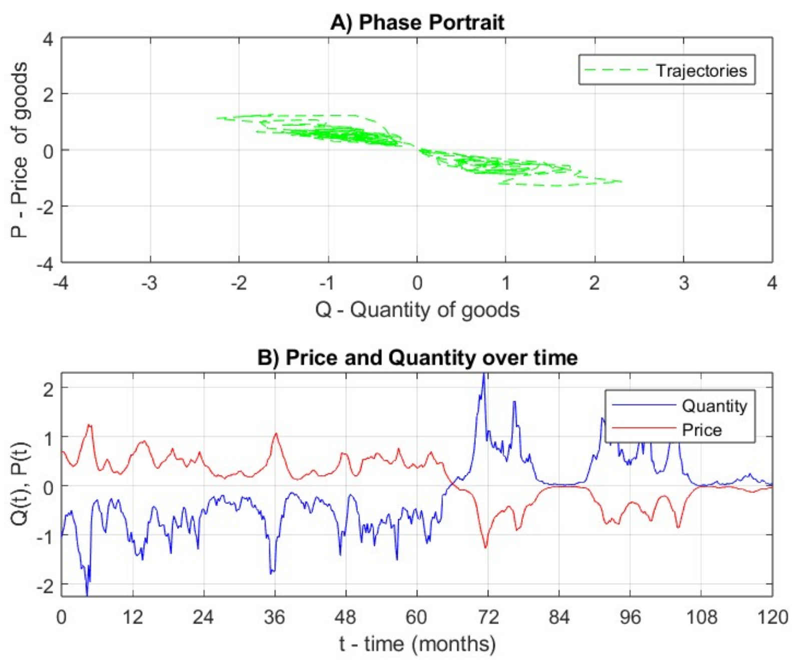

Figure 10: A) A trajectory of the system with a random noise parameter added, with intensity $\varepsilon = 0.5$ and initial conditions at the monopoly equilibrium. The trajectory eventually moves from the monopoly state to the competition state. B) The values of price and quantity over time now fluctuate randomly, but are still negatively correlated.

By choosing the initial condition to be at the monopoly state (-1.054, 0.703), and the noise intensity to be ε=0.5, the graph in Figure 10 can be generated. As the noise generated is random, the graph will look different each time it's generated. With no noise added the system would just stay at the monopoly equilibrium state. In this example, the trajectories start by moving rather erratically around the monopoly state, and at around 66 months the random noise parameter is so large that the system flips to the competition state. This can be seen in the graph for price and quantity over time, where the value of price is initially dominating, and then it flips to quantity being much higher.

This contrasts to the bi-stable state shown in question 2i. Here the random noise has made it possible for a market at equilibrium to flip from one state to another. This may be more applicable to real world scenarios. For example, factors such as consumer trends can lead to fluctuations in demand, and there may suddenly being a higher demand than the monopoly can satisfy. This would lead to other sellers entering the market and the system transforming from a monopoly to a competitive state.

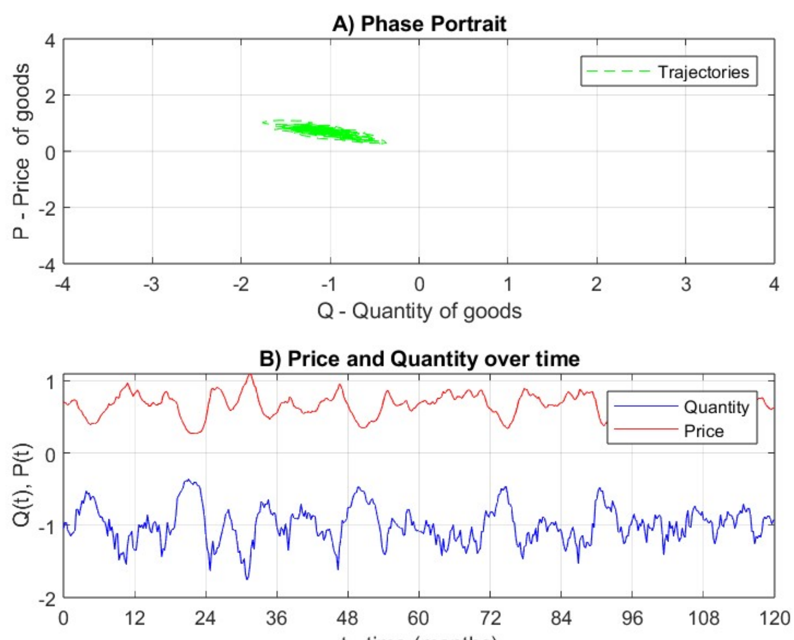

Figure 11: A) A trajectory with random noise intensity $\varepsilon = 0.2$. The trajectory moves away from equilibrium, but still stays at the monopoly state. B) The price is always considerably higher than the quantity, keeping the system in the monopoly state.

Choosing ε=0.2 gives us the graph in Figure 11. Here the intensity of the noise is too low to be able to change the state of the system. The trajectory leaves the stable equilibrium, but remains around the monopoly state.

As shown in Figure 12, setting the intensity of the random noise too high ($\varepsilon = 0.9$) can lead to far too erratic behaviour and effectively crash the system. This behaviour is too volatile to be likely to happen in real life.

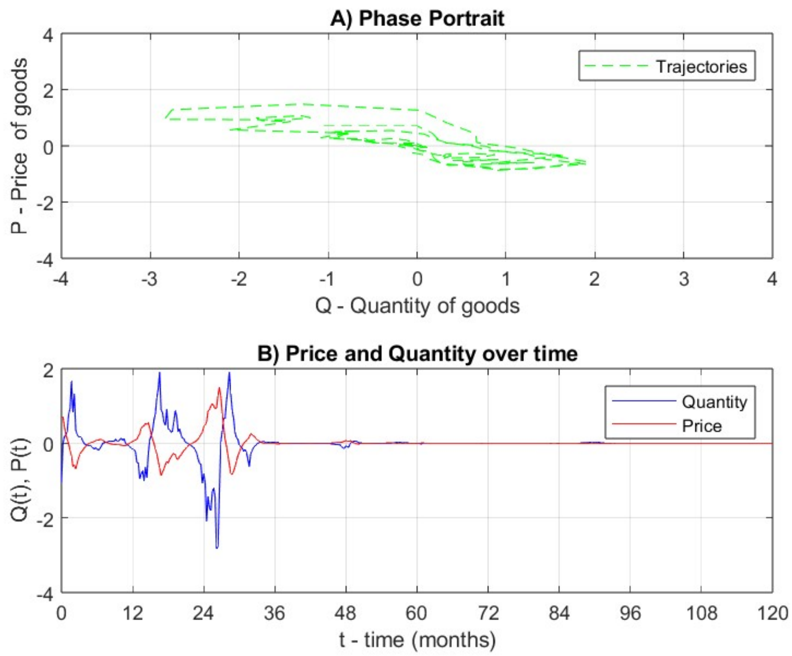

Figure 12: A) A trajectory with random noise intensity ε=0.9. The trajectory moves wildly between the two equilibrium. B) The values of price and quantity fluctuate a lot before eventually flatlining.

These results show that just a little bit of random noise is enough to destabilise the system and move it away from equilibrium. Adding a sufficient amount of random noise allows a market to move from one state to the other. However, the trajectories still don't move far from the stable equilibrium, so it could be argued the market is still fairly stable.

Question 5

The findings of this report can help governmental regulators prepare for transitions between economic states. For example, in the stable limit cycle system shown in question 2iii, the system oscillates between the monopoly market and the competition market. Being prepared for these transition states can help avoid instability. It can also help them prepare for how changes in parameter c, ie. changes in the demand for a product, can change the stability of the system. A negative shift in c will favour the monopoly state, while a positive shift will favour the competition state.

The system becomes unstable when the supply curve is linear with a negative gradient (b=0, d<0). Regulators could put rules into place to ensure that the supply of a product doesn't rely on just one parameter, as there's a risk this parameter will become negative and the market will become unstable. This will help ensure stable market behaviours.

Businesses can prepare for seasonal fluctuations in demand by saving stock for specific times of year. This means they can avoid running out of supply when the demand is high, which is something that could lead to instability or changes in the market.

References

Kun Zhang, Jin Wang, Landscape and flux theory of non-equilibrium open economy, Physica A: Statistical Mechanics and its Applications, Volume 482, 2017, Pages 189-208, ISSN 0378-4371