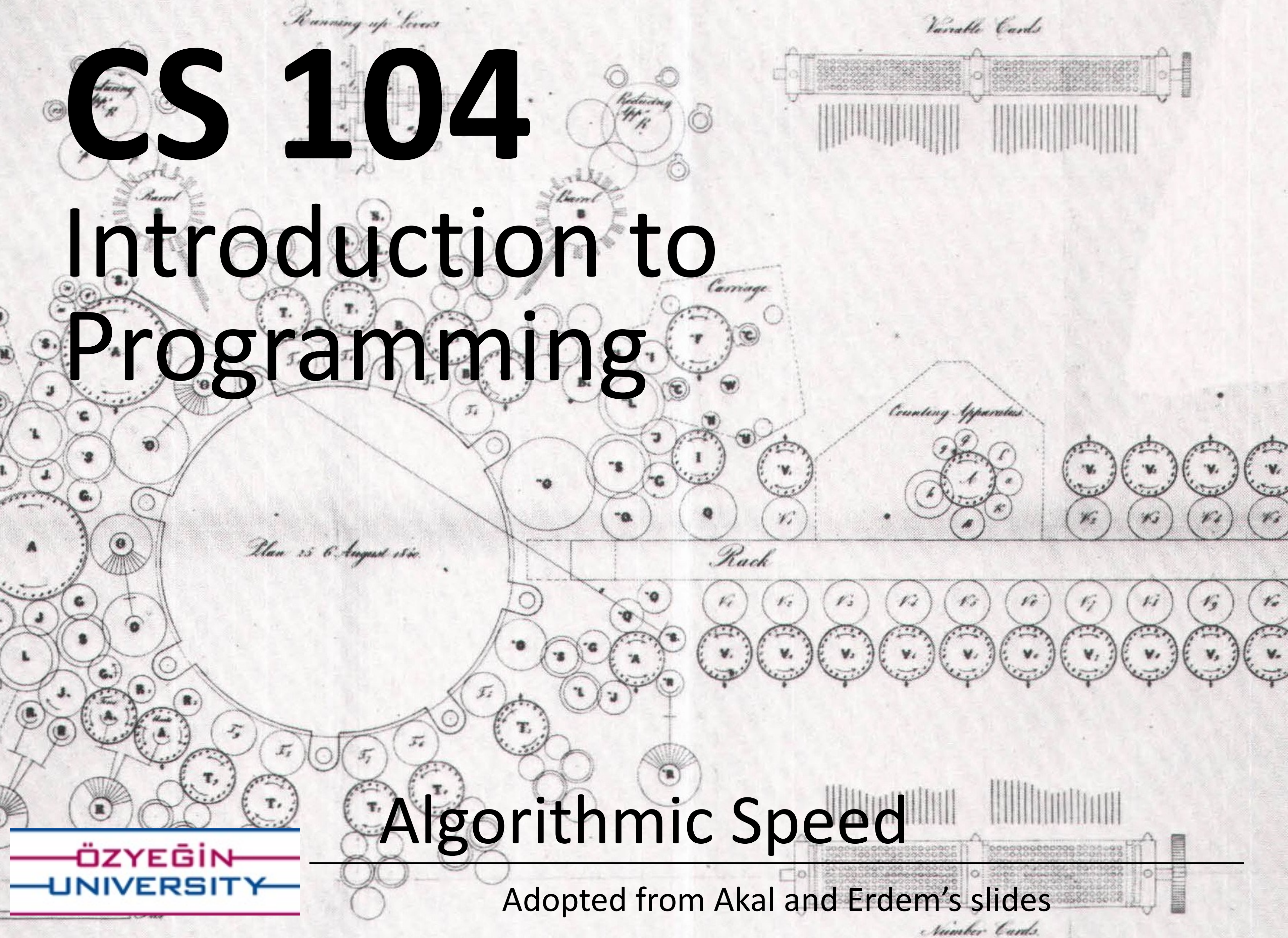
Drawing of Analytical Engine. Image: University of Cambridge



CS 104

Introduction to Programming

Algorithmic Speed

Adopted from Akal and Erdem’s slides

ÖZYEĞİN UNIVERSITY

Lecture Overview

- Algorithmic Complexity

Slides based on material prepared by E. Grimson, J. Guttag and C. Terman in MITx 6.00.1x

Computational complexity

- How much time will it take a program to run?
- How much memory will it need to run?

- Need to balance minimizing computational complexity with conceptual complexity
  - Keep code simple and easy to understand, but where possible optimize performance

Measuring complexity

- Goals in designing programs
  1. It returns the correct answer on all legal inputs
  2. It performs the computation efficiently

- Typically (1) is most important, but sometimes (2) is also critical, e.g., programs for collision detection, avionic systems, drive assistance etc.

- Even when (1) is most important, it is valuable to understand and optimize (2)

How do we measure complexity?

- Given a function, would like to answer: “How long will this take to run?”

- Could just run on some input and time it.

- Problem is that this depends on:
  1. Speed of computer
  2. Specifics of Programming Language implementation
  3. Value of input

- Avoid (1) and (2) by measuring time in terms of number of basic steps executed

Measuring basic steps

- Use a **random access machine (RAM)** as model of computation
  - Steps are executed sequentially
  - Step is an operation that takes constant time
    - Assignment
    - Comparison
    - Arithmetic operation
    - Accessing object in memory

- For point (3), measure time in terms of size of input

But complexity might depend on value of input?

```
def linearSearch(L, x):
  for e in L:
     if e==x:
        return True
  return False
```

- If `x` happens to be near front of `L`, then returns `True` almost immediately
- If `x` not in `L`, then code will have to examine all elements of `L`
- Need a general way of measuring

Cases for measuring complexity

- **Best case:** minimum running time over all possible inputs of a given size
  - For `linearSearch` – constant, i.e. independent of size of inputs
- **Worst case:** maximum running time over all possible inputs of a given size
  - For `linearSearch` – linear in size of list
- **Average (or expected) case:** average running time over all possible inputs of a given size
- We will focus on worst case – a kind of **upper bound** on running time

Example

```
def fact(n):
    answer = 1
    while n > 0:
        answer *= n
        n -= 1
    return answer
```

- Number of steps
  1 (for assignment)
  5*n (1 for test, plus 2 for first assignment, plus 2 for second assignment in while; repeated n times through while)
  1 (for return)
- 5*n+2steps
- But as n gets large, 2 is irrelevant, so basically 5*n steps

Example

- What about the multiplicative constant (5 in this case)?

- We argue that in general, multiplicative constants are not relevant when comparing algorithms

Example

```
def sqrtExhaust(x, eps):  linear search
    step = eps**2
    ans = 0.0
    while abs(ans**2 − x) >= eps and ans <= max(x, 1):
        ans += step
    return ans
```

- If we call this on 100 and 0.0001, will take one billion iterations of the loop
  – Have roughly 8 steps within each iteration

Example

```
def sqrtBi(x, eps):
    low = 0.0
    high = max(1, x)
    ans = (high + low)/2.0
    while abs(ans**2 - x) >= eps:
        if ans**2 < x:
            low = ans
        else:
            high = ans
        ans = (high + low)/2.0
    return ans
```

binary search

lower bound

upper bound

Logarithmic input size

- If we call this on 100 and 0.0001, will take thirty iterations of the loop
  - Have roughly 10 steps within each iteration
- 1 billion or 8 billion versus 30 or 300 – it is size of problem that matters

Measuring complexity

- Given this difference in iterations through loop, multiplicative factor (number of steps within loop) probably irrelevant

- Thus, we will focus on measuring the complexity as a function of input size
  - Will focus on the largest factor in this expression
  - Will be mostly concerned with the worst-case scenario

Asymptotic notation

- Need a formal way to talk about relationship between running time and size of inputs

- Mostly interested in what happens as size of inputs gets very large, i.e. approaches infinity

Example

```
def f(x):
    for i in range(1000):
        ans = i
    for i in range(x):
        ans += 1
    for i in range(x):
        for j in range(x):
            ans += 1
```

1000 times

1

ans = ans + 132

2x

x times

x times

2x^2

ans = ans + 1

2

O(x^2)

Complexity is $1000 + 2x + 2x^2$, if each line takes one step

Example

- $1000+2x+2x^2$

- If x is small, constant term dominates
  - E.g., x = 10 then 1000 of 1220 steps are in first loop

- If x is large, quadratic term dominates
  - E.g. x = 1,000,000, then first loop takes 0.000000005% of time, second loop takes 0.0001% of time (out of 2,000,002,001,000 steps)!

Example

- So really only need to consider the nested loops (quadratic component)

- Does it matter that this part takes $2x^2$ steps, as opposed to say x^2 steps?

  – For our example (x = 106), if our computer executes 100 million steps per second, difference is ~5.5 hours versus ~2.75 hours (X2 vs. 2*X2)

  – On the other hand, if we can find a linear algorithm, this would run in a fraction of a second (X vs. 2*X)

  – So multiplicative factors probably not crucial, but order of growth is crucial

Rules of thumb for complexity

- Asymptotic complexity
  - Describe running time in terms of number of basic steps
  - If running time is sum of multiple terms, keep one with the largest growth rate
  - If remaining term is a product, drop any multiplicative constants

- Use "Big O" notation (aka Omicron)
  - Gives an upper bound on asymptotic growth of a function

Complexity classes

- ***O(1)*** denotes constant running time
- ***O(log n)*** denotes logarithmic running time
- ***O(n)*** denotes linear running time
- ***O(n log n)*** denotes log-linear running time
- ***O(n^c)*** denotes polynomial running time (*c* is a constant)
- ***O(c^n)*** denotes exponential running time (*c* is a constant being raised to a power based on size of input)

Constant complexity

- Complexity independent of inputs

- Very few interesting algorithms in this class, but can often have pieces that fit this class

- Can have loops or recursive calls, but number of iterations or calls independent of size of input

Logarithmic complexity O(logn)

- Complexity grows as log of size of one of its inputs
- Example:
  - Bisection search find square root
  - Binary search of a list sorted list

Logarithmic complexity

sorted

```
def binarySearch(alist, item):
    first = 0
    last = len(alist)-1
    found = False

    while first<=last and not found:
        midpoint = (first + last)//2
        if alist[midpoint] == item:
            found = True
        elif item < alist[midpoint]:
            last = midpoint-1
        else:
            first = midpoint+1

    return found
```

Logarithmic complexity

```python
def binarySearch(alist, item):
    first = 0
    last = len(alist)-1
    found = False

    while first<=last and not found:
        midpoint = (first + last)//2
        if alist[midpoint] == item:
            found = True
        elif item < alist[midpoint]:
            last = midpoint-1
        else:
            first = midpoint+1

    return found
```

- Only have to look at loop as no function calls
- Within while loop constant number of steps
- How many times through loop?
  - How many times can one divide indexes to find midpoint?
  - *O(log(len(alist)))*

Linear complexity

- Searching a list in order to see if an element is present
- Add characters of a string, assumed to be composed of decimal digits

```
def addDigits(s):
    val = 0
    for c in s:
        val += int(c)
    return val
```

- *O(len(s))*

Linear complexity

- Complexity can depend on number of recursive calls

```
def fact(n):
    if n == 1:
        return 1
    else:
        return n*fact(n-1)
```

- Number of recursive calls?
  - `fact(n)`, then `fact(n-1)`, etc. until get to `fact(1)`
  - Complexity of each call is constant
  - *O(n)*

Log-linear complexity

- Many practical algorithms are log-linear
- Very commonly used log-linear algorithm is **merge sort**

Polynomial complexity

- Most common polynomial algorithms are quadratic, i.e., complexity grows with square of size of input

- Commonly occurs when we have nested loops or recursive function calls

Quadratic complexity

```
def isSubset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

Quadratic complexity

```
def isSubset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

- Outer loop executed len(L1) times
- Each iteration will execute inner loop up to len(L2) times
- *O(len(L1)*len(L2))*
- Worst case when L1 and L2 same length, none of elements of L1 in L2
- $O(len(L1)^2)$

Quadratic complexity

Find intersection of two lists, return a list with each element appearing only once

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
                tmp.append(e1)
    res = []
    for e in tmp:
        if not(e in res):
            res.append(e)
    return res
```

Quadratic complexity

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
               tmp.append(e1)
    res = []
    for e in tmp:
        if not(e in res):
           res.append(e)
    return res
```

- First nested loop takes *len(L1)*len(L2)* steps
- Second loop takes at most *len(L1)* steps
- Latter term overwhelmed by former term
- *O(len(L1)*len(L2))*

Exponential complexity

- Recursive functions where more than one recursive call for each size of problem
  - Towers of Hanoi
  - Fibonacci series
- Many important problems are inherently exponential
  - Unfortunate, as cost can be high
  - Will lead us to consider approximate solutions more quickly

Exponential Complexity

```
def fib(N):
    if N == 1 or N == 0:
        return N
    else:
        return fib(N-1) + fib(N-2)
```

Exponential Complexity

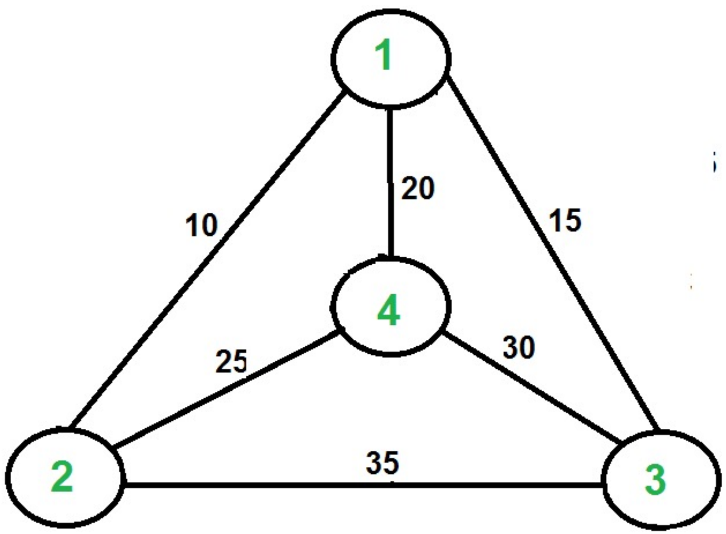
```
def fib(N):
    if N == 1 or N == 0:
        return N
    else:
        return fib(N-1) + fib(N-2)
```

- Assuming return statement is constant time
- Recall the recursive tree
- Complexity of this function is $O(\sim 2^n)$

Factorial Complexity

- The travelling salesperson problem.
- A salesperson has to visit n towns. Each pair of towns is joined by a route of a given length. Find the shortest possible route that visits all the towns and returns to the starting point.

1. Consider city 1 as the starting and ending point.
2. Generate all (n-1)! Permutations of cities.
3. Calculate cost of every permutation and keep track of minimum cost permutation.
4. Return the permutation with minimum cost.



Complexity classes

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- ***O(c^n)*** denotes exponential running time (*c* is a constant being raised to a power based on size of input)
- ***O(n!)*** denotes factorial running time

Comparing complexities

- So, does it really matter if our code is of a particular class of complexity?

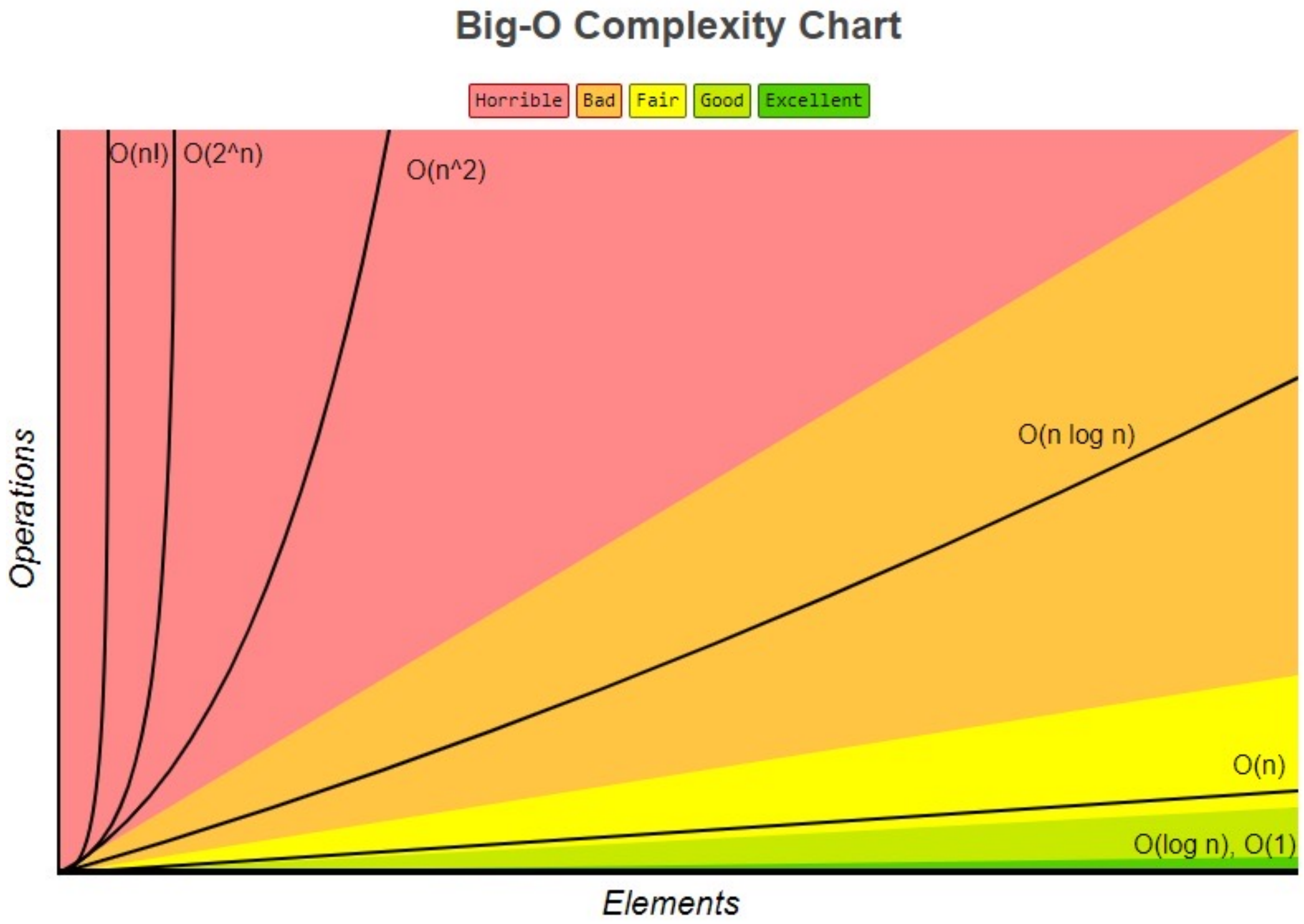
- Depends on size of problem, but for large scale problems, complexity of worst case makes a difference

Comparing complexities - example

- There are alternative approaches with differing algorithm comlexities for doing *something* on a list of *n* elements.
- Now you want to compare them. Assume that computer makes three billion calculations per second. Lets look for the running time of the algorithms.

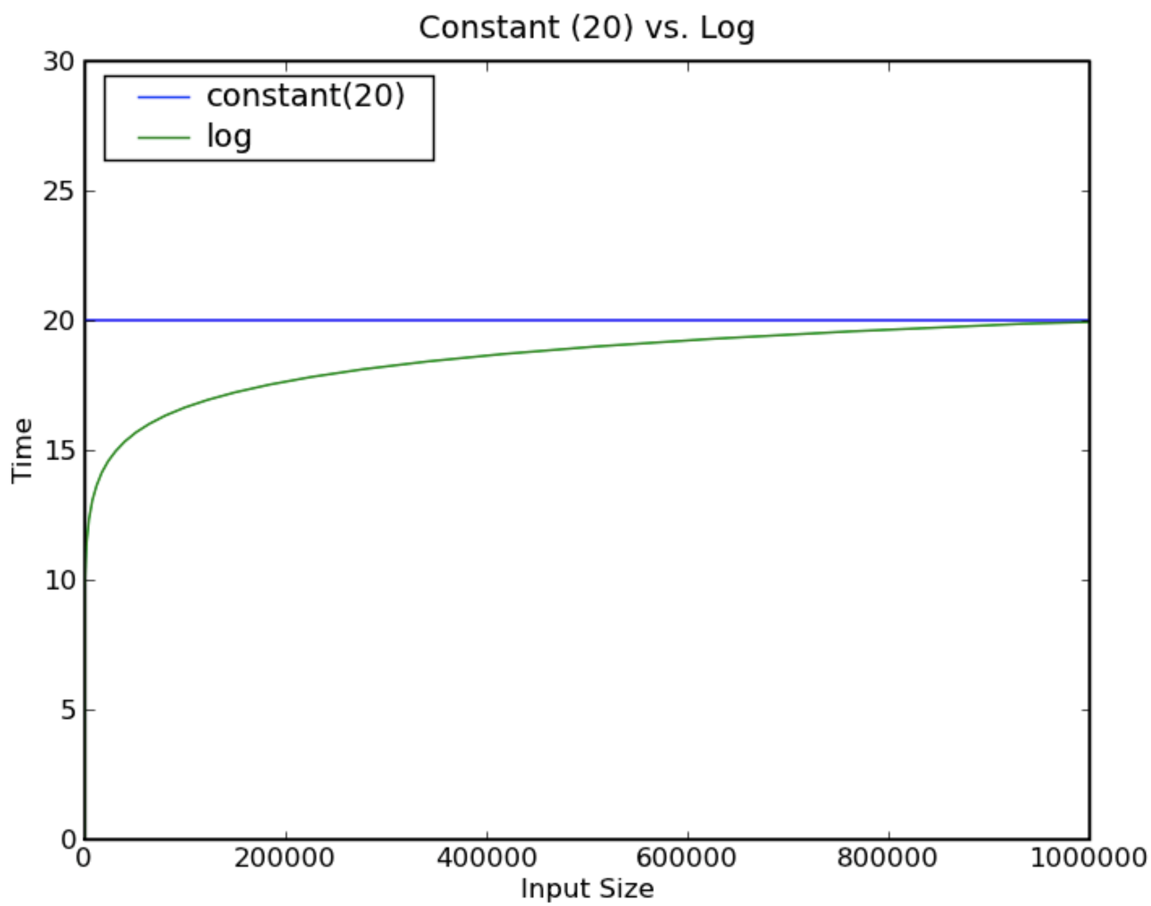
Complexity	n=10	n=1000	n=10^5	n=10^10
O(logn)	< 1msec	< 1msec	< 1msec	< 1msec
O(n)	< 1msec	< 1msec	< 1msec	< 1 min
O(nlogn)	< 1msec	< 1msec	< 1 sec	< 2 min
$O(n^2)$	< 1msec	< 1msec	< 1 min	~1000 year
$O(2^n)$	< 1 sec	<1000 year	<1000 year	<1000 year
O(n!)	< 1 sec	<1000 year	>1000 year	>1000 year

Comparing the Complexities



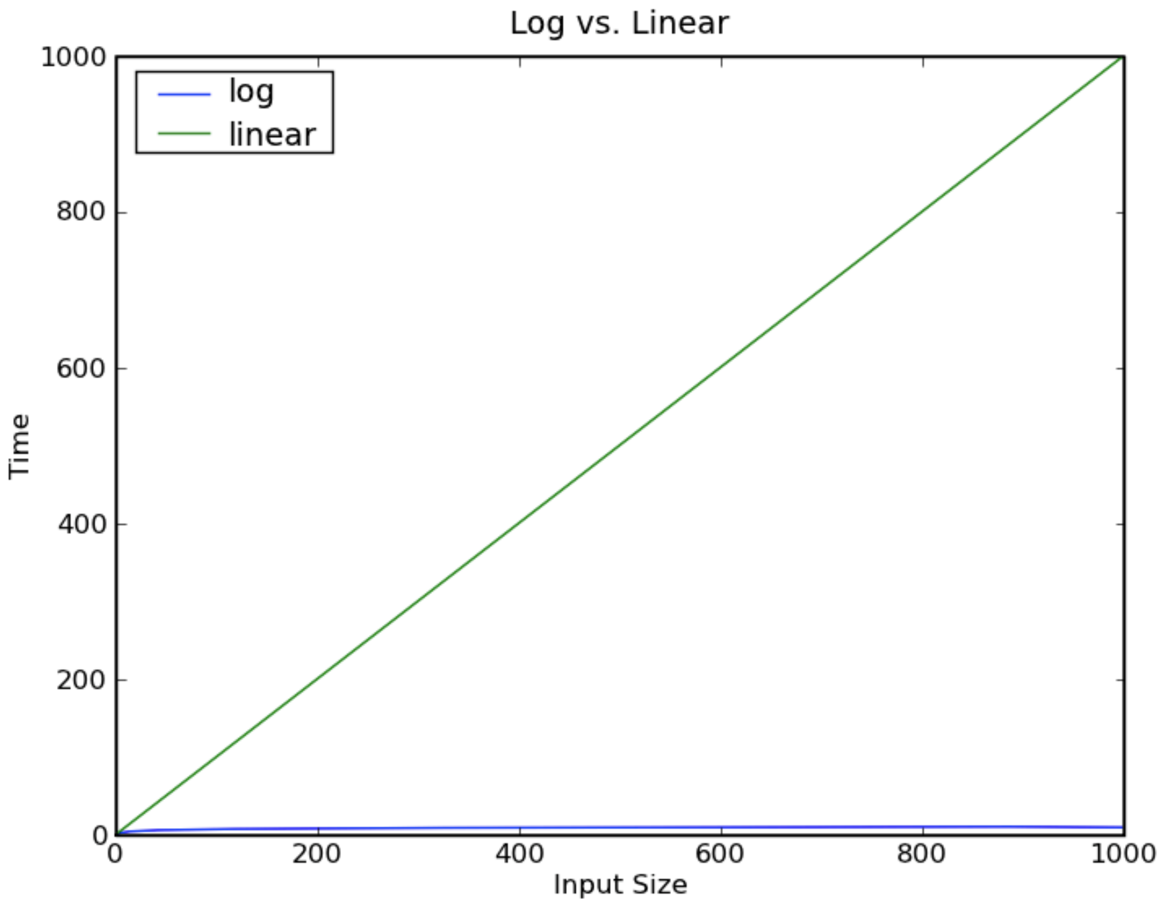
Constant versus Logarithmic

- A logarithmic algorithm is often almost as good as a constant time algorithm
- Logarithmic costs grow very slowly



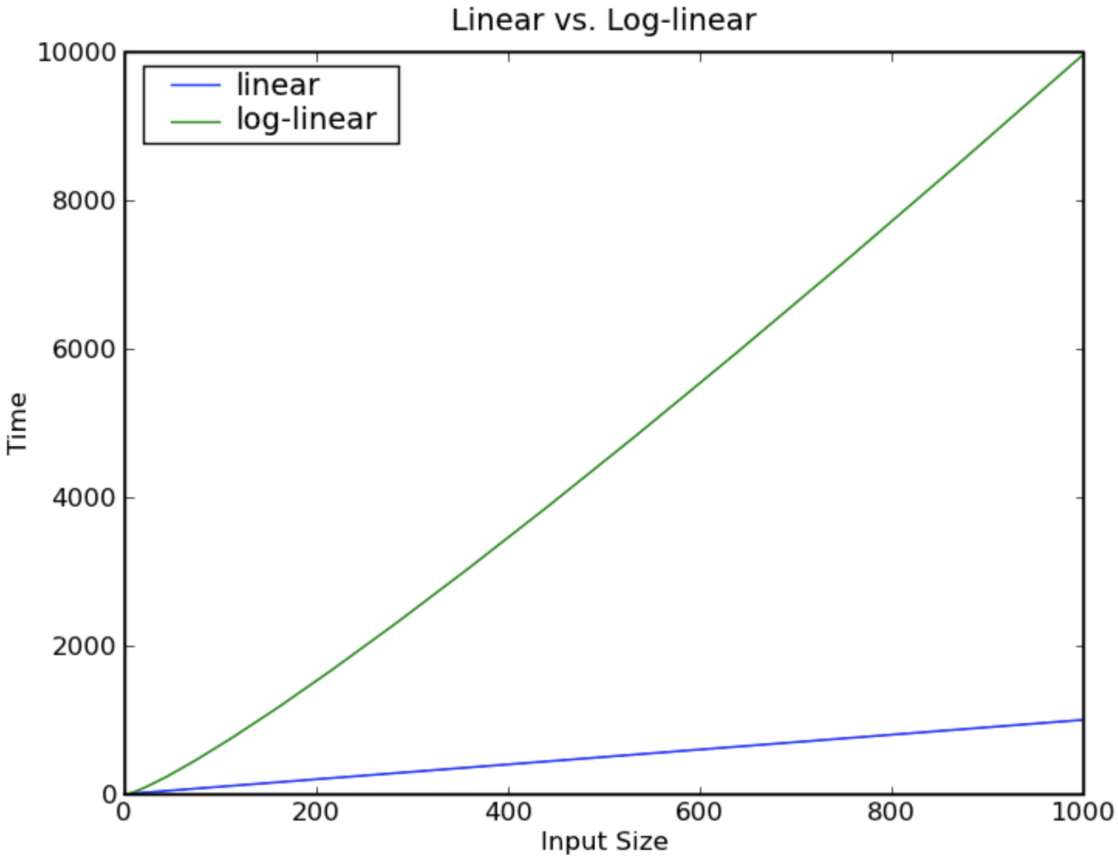
Logarithmic versus Linear

- Logarithmic clearly better for large scale problems than linear
- Does not imply linear is bad, however



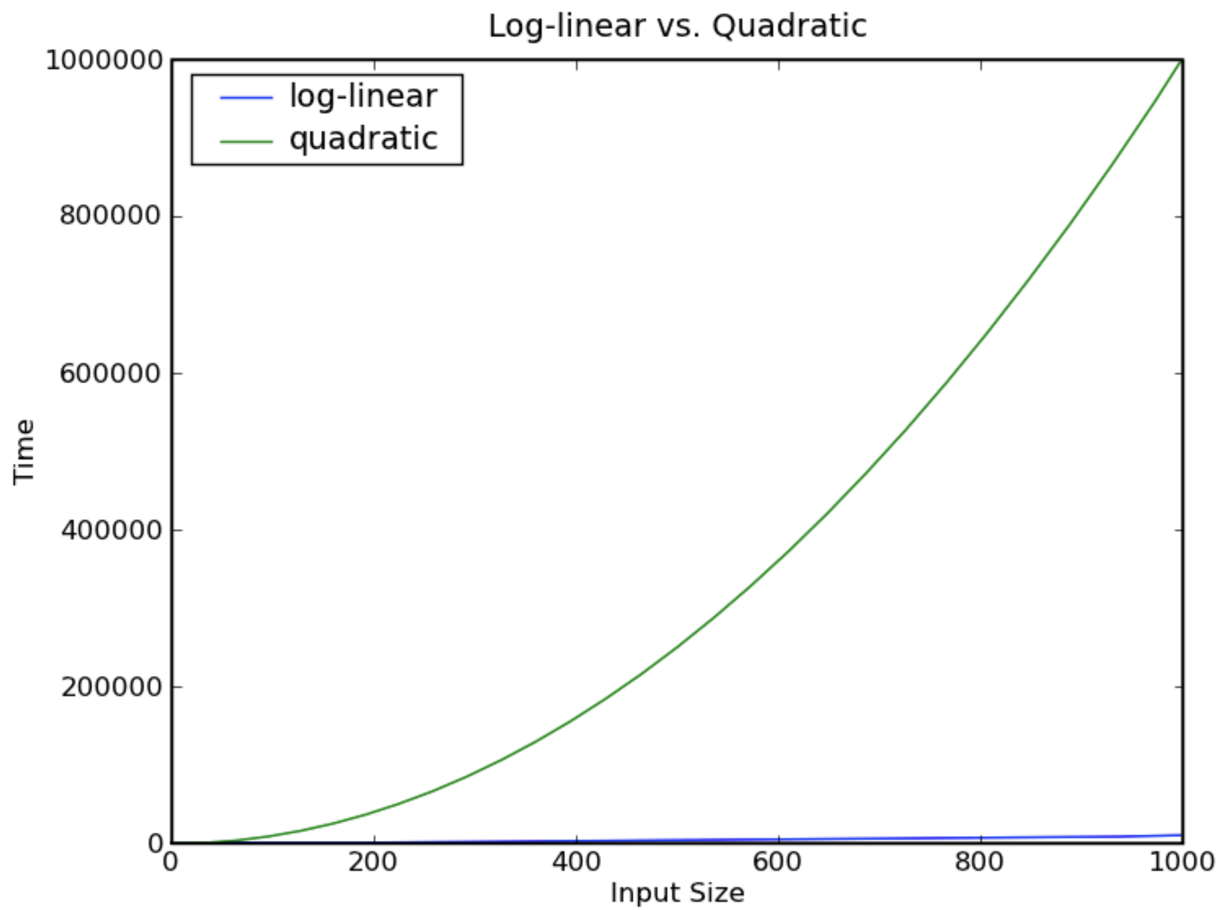
Linear versus Log-linear

- While *log(n)* may grow slowly, when multiplied by a linear factor, growth is much more rapid than pure linear
- *O(n log n)* algorithms are still very valuable.



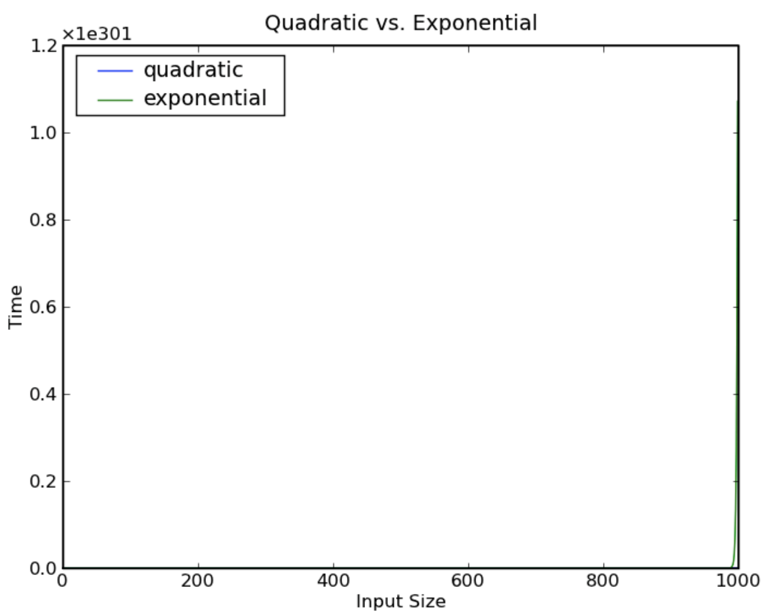
Log-linear versus Quadratic

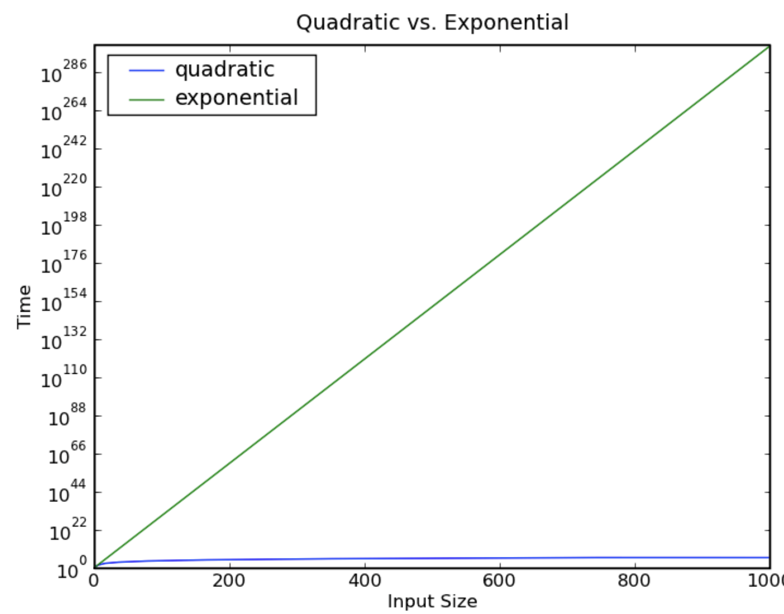
- Quadratic is often a problem, however.
- Some problems inherently quadratic but if possible always better to look for more efficient solutions



Quadratic versus Exponential

- Exponential algorithms very expensive
  - Right plot is on a log scale, since left plot almost invisible given how rapidly exponential grows
- Exponential generally not of use except for small problems





Warning

- Execution time and the algorithm complexity are different paradigms.
- Running time may differ even if two algorithms have the same algorithm complexity (Even when their purposes are the same).

```python
def factIT(n):
    answer = 1
    while n > 0:
        answer *= n
        n -= 1
    return answer
```

```python
def factREC(n):
    if n == 0:
        return 1
    else:
        return n*factREC(n-1)
```

They have same complexity O(n). But their execution times are different.

Tips

- We know that, O(2^n) algorithm complexity is bad. But, if we sure that n won’t be up too high, it won’t be matter.

- When we calculate the big-O, we did not care about constant factors.
  - 5n + 37 -> O(n)

- But, sometimes improving the constants does matter, e.g. in game development
  - 5n+37 → 5n+10 (not worthy, but better than nothing)
  - 5n+37 → 3n+12 (better)