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CSE 321 - INTRODUCTION TO ALGORITHMS
HOMEWORK #1

1- I used limit approach for first question parts

a) Assume $f(n) = \log_2 n^2 + 1$, $g(n) = n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\log_2 n^2 + 1}{n} = \lim_{n\to\infty} \frac{\log_2 n^2}{n} + \lim_{n\to\infty} \frac{1}{n} \to 0$$

$$= \lim_{n\to\infty} \frac{\log_2 n^2}{n} = \frac{\infty}{\infty} \quad \text{We use L'Hopital}$$

$$= \lim_{n\to\infty} \frac{f'(x)}{g'(x)} = \frac{2n}{n^2 . \ln 2} = \frac{2}{2n} = \frac{1}{n} = 0$$

(We use L'Hopital again).

* So if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ that means $f(n) \in o(g(n))$

* And $f(n) \in o(g(n))$ iff $f(n) \in O(g(n))$ → We said that in properties.

* Finally it is $\boxed{true}$

b) Assume $f(n) = \sqrt{n(n+1)}$, $g(n) = n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\sqrt{n(n+1)}}{n} = \frac{\sqrt{n . n \left(1+\frac{1}{n}\right)}}{n} = \frac{n . \sqrt{\left(1+\frac{1}{n}\right)}}{n} = \sqrt{\left(1+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \sqrt{\frac{1}{n}} = 1 + 0 = 1$$

* If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ that means $f(n) \in \sim(g(n))$

* $f(n) \in \sim g(n)$ iff $f(n) \in \Theta(g(n))$ we said in properties
* $f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$
* So it is $\boxed{true}$

c) Assume $f(n)=n^{n-1}$, $g(n)=n^n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{n^{n-1}}{n^n} = \frac{1}{n} = 0$$

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ that means $f(n) \in o(g(n))$

And if $f(n) \in o(g(n))$ then $f(n) \notin \Theta(g(n))$
So this statement is $\boxed{\text{false}}$

d) Assume $f(n)= 2^n+n^3$, $g(n)=4^n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{2^n+n^3}{4^n} = \frac{2^n}{4^n} + \frac{n^3}{4^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n + \underbrace{\lim_{n\to\infty} \frac{n^3}{4^n}}_{2^n > n^2 \text{ (Speed of Functions Approaching Infinity)}}$$

$$= 0 + 0$$
$$= 0$$

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ that means $f(n) \in o(g(n))$

$f(n) \in o(g(n))$ iff $O(f(n)) \subset O(g(n))$ → We said that in properties
So this statement is $\boxed{\text{true.}}$

e) Assume $f(n) = 2\log_3 \sqrt[3]{n}$, $g(n) = 3\log_2 n^2$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{2\log_3 \sqrt[3]{n}}{3\log_2 n^2} = \frac{2}{3} \lim \frac{\log_3 n^{\frac{1}{3}}}{\log_2 n^2} = \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot \lim_{n\to\infty} \frac{\log_3 n}{\log_2 n}$$

$$= \frac{1}{9}\cdot \lim_{n\to\infty} \frac{\log_3 n}{\log_2 n} \Rightarrow \text{Apply L'Hospital} \Rightarrow \frac{1}{9}\cdot \lim_{n\to\infty} \left(\frac{\frac{1}{n\ln 3}}{\frac{1}{n\ln 2}}\right)$$

$$= \frac{1}{9}\cdot\frac{\ln 2}{\ln 3} \cong 0.07$$

* If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ (where $c>0$), that means $f(n) \in \Theta(g(n))$

* If limit was 0, then we could say $O(f(n)) \subset O(g(n))$ but since limit is c, then the complexity cannot be little oh. Because the complexity Θ (theta) is not equal o (little oh).

* So this statement is $\boxed{\text{false}}$

f) Assume $f(n) = \log_2 \sqrt{n}$, $g(n) = (\log_2 n)^2$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\log_2 \sqrt{n}}{(\log_2 n)^2} = \frac{1}{2}\cdot \lim_{n\to\infty} \frac{\log_2 n}{(\log_2 n)^2} = \frac{1}{2}\cdot \lim_{n\to\infty} \frac{1}{\log_2 n} = \frac{1}{2}\cdot 0 = 0$$

* If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ that means $f(n) \in o(g(n)) \rightarrow$ In properties

* And if $f(n) \in o(g(n))$ then $f(n) \notin \Theta(g(n))$

* And $f(n)$ and $g(n)$ have the same order if $f(n) \in \Theta(g(n))$

* So this statement is $\boxed{\text{false}}$

2- I compare these numbers in pairs. Also, firstly we can say polynomial functions smaller than exponential functions. The comparisons are below:

* $\lim_{n\to\infty} \frac{n^2}{n^3} = 0$ so ① $\boxed{(n^2) < (n^3)}$ (If $f(n) \in o(g(n))$, $f(n) < g(n)$)

* $\lim_{n\to\infty} \frac{n^2 \log n}{\sqrt{n}} = \infty$ Therefore $n^2\log n$ faster than $\sqrt{n}$, ② $\boxed{\sqrt{n} < n^2 \log n}$

* $\lim_{n\to\infty} \frac{\log n}{10^n} = 0$, ③ $\boxed{\log n < 10^n}$

* $\lim_{n\to\infty} \frac{2^n}{8^{\log n}} = \infty$ ④ $\boxed{8^{\log n} < 2^n}$

Now we can compare these 4 groups. First we start with logarithmic functions (2. and 3. groups)

* $\lim_{n\to\infty} \frac{\log n}{n^2 \log n} = 0$, $\boxed{\log n < n^2 \log n}$ * $\lim_{n\to\infty} \frac{\log n}{\sqrt{n}} = 0$, $\boxed{\log n < \sqrt{n}}$

We can say $\boxed{\log n < \sqrt{n} < n^2 \log n}$

Now we will compare 1. and above groups.

* $\lim_{n\to\infty} \frac{n^2}{n^2 \log n} = 0$, $\boxed{n^2 < n^2 \log n}$

* $\lim_{n\to\infty} \frac{\sqrt{n}}{n^2} = 0$, $\boxed{\sqrt{n} < n^2}$

Now the order

$\boxed{\log n < \sqrt{n} < n^2 < n^2 \log n < 10^n}$

Now we can guess and compare for 3 left functions.

* $\lim_{n\to\infty} \frac{n^2 \log n}{n^3} = 0$, $\boxed{n^2 \log n < n^3}$

* $\lim_{n\to\infty} \frac{8^{\log n}}{n^3} = 1$, $\boxed{8^{\log n} = n^3}$

* $\lim_{n\to\infty} \frac{n^3}{2^n} = 0$, $\boxed{n^3 < 2^n}$

The last order is:

$$\boxed{\log n < \sqrt{n} < n^2 < n^2 \log n < 8^{\log n} = n^3 < 2^n < 10^n}$$

3-a) Best case : Now we have 3 situations and I will analyse it.
First situation : Code checks if statement in every way. So if this condition is true, then 2 assignment will be done and loop will finish for that cycle.
Second situation : Code checks if statement. If this condition is false, then checks else if condition. If, else if condition is true, then follow this block and check if condition inside. So this situation contains 1 condition and 2 assignment. It will be done after these operations and loop will finish for that cycle.
Third situation : If and else if conditions may be false. So just check 2 conditions and the loop will finish for that cycle.

Every situation contains constant operations. So these are $3n+1$, $4n+1$, $2n$. There are same in time complexity which is $\Omega(n)$.

Worst Case : Worst case is the second situation Because this contains maximum operations But it doesn't matter because this complexity is also $(4n+1) = O(n)$.

As a result, hence best case and worst case equal each other, the average case is $\Theta(n)$ (Theta).

3-b) The i number increases like that:

2 3 7 43 ....

These are:

2^1, 2^2-1, 2^3-1, 2^6-21, 2^{11}....

So I said, 2^{2^k} approximately. And if $2^{2^k}=n$, then $k = \log\log n$

And it is $O(\log\log n)$ because the constants decrease the k, so in infinity $\log\log n$ is upper bound for that.

4-a) $i^2 \log i$ is a non decreasing function. Now the computations are below:

$$\int_0^n g(x).dx \leqslant f(n) \leqslant \int_1^{n+1} g(x).dx$$

$$= \int_0^n x^2.\log x.dx \leqslant f(n) \leqslant \int_1^{n+1} x^2.\log x.dx$$

$\int x^2 \log x \Rightarrow$ Integration by parts

$$u = \log x \qquad dv = x^2.dx$$
$$du = \frac{1}{x\ln 2}.dx \qquad v = \frac{x^3}{3}$$

$$\int x^2 \log x = u.v - \int v.du$$

$$= \log x.\frac{x^3}{3} - \int \frac{x^3}{3}\frac{1}{x\ln 2}.dx$$

$$= \log x.\frac{x^3}{3} - \frac{1}{3\ln 2}\int x^2.dx$$

$$= \log x.\frac{x^3}{3} - \frac{x^3}{9\ln 2}$$

Now

$$\left[\frac{x^3}{3}\left(\log x - \frac{1}{3\ln 2}\right)\right]\Bigg|_0^n \leqslant f(n) \leqslant \left[\frac{x^3}{3}\left(\log x - \frac{1}{3\ln 2}\right)\right]\Bigg|_1^{n+1}$$

$$\underbrace{\frac{n^3}{3}\left(\log n - \frac{1}{3\ln 2}\right) - 0}_{\Omega(n^3 \log n)} \leqslant f(n) \leqslant \underbrace{\left[\left(\frac{(n+1)^3}{3}.\left(\log(n+1) - \frac{1}{3\ln 2}\right)\right) - \left(\frac{1}{3}.\left(\log 1 - \frac{1}{3\ln 2}\right)\right)\right]}_{O(n^3 \log n)}$$

$$f(n) \in O(n^3 \log n)$$
$$f(n) \in \Omega(n^3 \log n)$$
$$f(n) \in \Theta(n^3 \log n)$$

b) i^3 is a non - decreasing function so ;

$$\int_0^n g(x).dx \leq f(n) \leq \int_1^{n+1} g(x).dx$$

$$= \int_0^n x^3.dx \leq f(n) \leq \int_1^{n+1} x^3.dx$$

$$= \frac{x^4}{4}\Big|_0^n \leq f(n) \leq \frac{x^4}{4}\Big|_1^{n+1}$$

$$= \underbrace{\frac{n^4}{4}}_{\Omega} \leq f(n) \leq \underbrace{\frac{(n+1)^4-1}{4}}_{O}$$

$f(n) \in \Omega(n^4)$
$f(n) \in O(n^4)$
$f(n) \in \Theta(n^4)$

c) $1/(2\sqrt{i})$ is a non-increasing function so I will use Harmonic Series formula

$$\int_1^{n+1} g(x).dx \leq H(n) \leq \int_0^n g(x).dx$$

$$= \int_1^{n+1} 1/(2\sqrt{x}).dx \leq H(n) \leq \int_0^n 1/(2\sqrt{x}).dx$$

$$= \sqrt{x}\Big|_1^{n+1} \leq H(n) \leq \sqrt{x}\Big|_0^n$$

$$= \underbrace{\sqrt{n+1}-1}_{\Omega} \leq H(n) \leq \underbrace{\sqrt{n}}_{O}$$

$H(n) \in \Omega(\sqrt{n})$
$H(n) \in O(\sqrt{n})$
$H(n) \in \Theta(\sqrt{n})$

d) $\frac{1}{i}$ is a non-increasing function. So I will use Harmonic Series formula

$$\int_{1}^{n+1} \frac{1}{x}\cdot dx \leq H(n) \leq \int_{0}^{n} \frac{1}{x}\cdot dx$$

$$= \underbrace{\ln(n+1) - \ln(1)}_{\text{Lower Bound}} \leq H(n) \leq \ln(n) - \ln(0)$$

So we found lower bound but it didn't work for upper bound.

$$H(n) = 1 + \sum_{i=2}^{n} \frac{1}{i}$$

$$H(n) \leq 1 + \int_{1}^{n} \frac{1}{x}\cdot dx$$

$$= 1 + \ln x \Big|_{1}^{n} = 1 + \ln(n) - \cancel{\ln(1)}^{0}$$

$$= 1 + \ln(n)$$

$$\ln(n+1) \leq H(n) \leq 1 + \ln(n)$$

Both upper and lower bounds are defined in terms of $\ln(n)$

$$H(n) \in Q(\log n)$$

5) - Best Case: If $x = L[1]$ then best case occurs.

$$B(n) = 1 \in \Theta(1)$$

- Worst Case: For the worst case, x must be end of the list or it doesn't exist. So if $x = L[n]$ or does not exist, then worst case occurs.

$$W(n) = n \in \Theta(n)$$