Calculus I

– to Noelle

"Strive hard to do hard things"

About Author: Dr. Andrew Misseldine is originally from Boise, Idaho. He received his B.S. and M.S. in Mathematics from Boise State University in 2008 and 2010, respectively, and his Ph.D. in Mathematics in 2014 from Brigham Young University. He accepted a Lecturer position at Southern Utah University within the Mathematics Department in Fall 2014. Andrew later became an Assistant Professor in 2016 and received tenure and promotion to Associate Professor in 2022. In 2024, Andrew became Chair of the Department of Mathematics at SUU, where he currently serves.

Andrew has served on several committees within and without the Mathematics Department. He served eight years as a mathematics judge for the Sterling Scholar competition and served four years on the Faculty Senate, with two years on the Faculty Senate Executive Committee. Andrew was a Provost Fellow in 2022. He is currently serving on the CLEP College Mathematics Development Committee for ETS. He publishes scholarly papers regularly in the topics of Algebraic Combinatorics and Representation Theory. He has been a proponent for Open Educational Resources for many years, advocating for them in the classroom, publishing articles about their usage in mathematics education, and even self-publishing a few open mathematical texts himself (such as the very one you are reading now). Andrew hosts a Youtube channel (youtube.com/@Misseldine) about undergraduate mathematics, receiving about 25,000 views per month. For "fun," Andrew teaches middle and high school students cryptography to excite them about mathematics, as the standard K12 curriculum, which is designed to prepare students for Calculus one day, turns many talented students off to mathematics.

Andrew resides in Cedar City, UT, with his wife, Heather, and seven childern. In his free time, Andrew loves playing games (board, card, or video) with his friends and family, playing his cello, reading fantasy novels, watching "Bluey" (with or without his kids), and taking his family to Disneyland. Andrew lived in South Korea during 2004-2006, serving as a missionary for his church. Even all these decades later, he still cherishes the language, the culture, and the food of Korea. This gives context to the many artifacts that decorate his office, including iconography from the Legend of Zelda, Pokémon, Lord of the Rings, South Korea, many family photos, and, of course, mathy things.

Dr. Misseldine loves to hear from his readers. If you have any questions, concerns, or feedback, please contact Dr. Misseldine at andrewmisseldine@suu.edu. You may also visit his website at `website` or the aforementioned YouTube channel for more information about Dr. Misseldine's other textbooks, lecture videos, and scholarly papers.

Calculus I

Andrew Misseldine

2025 Edition

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1.1 Functions

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. The set D is called the **domain** of f and consists of all the elements for which f is defined on, that is, D is the collection of all possible inputs of the function. The set E is called the **range** of f and consists of all the elements of the form $f(x)$, that is, E is the collection of all possible outputs of the function.

With very few exceptions, the sets D and E will be the set of real numbers.

There are four common ways to represent a function:

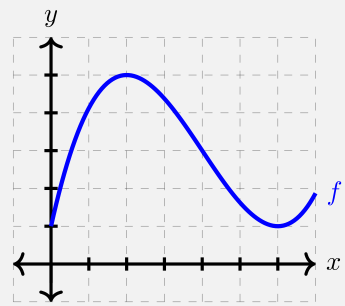
- Verbally (by a description in words)
- Numerically (by a table of values)
- Visually (by a graph)
- Algebraically (by a explicit formula)

It is often useful to switch between one representation of a function to another as new insights can be discovered. Sometimes a function is easy to describe in one representation but more practical to use in a different representation. Much of Precalculus is learning to go back and forth between these different representations.

Remark 1.1.2 (Domain Convention)**.** If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number. If a function is given by a graph or table, then the domain is the set of all numbers which are represented on the graph or table. ▼

Example 1.1.3. The graph of a function f is shown below.

(a) Find the values of $f(2)$ and $f(4)$.



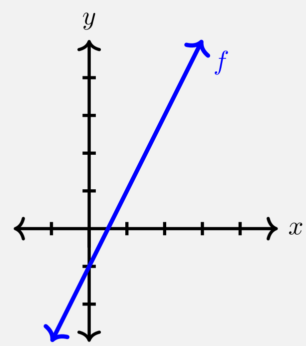
By inspecting the graph, we determine that the points $\boxed{(2,5)}$ lies on the graph. Thus, $f(2) = 5$. Likewise, the point $(4,3)$ lies on the graph. Thus, $\boxed{f(4) = 3}$.

(b) What are the domain and range of f?

The domain is the collection of all possible x-values. By inspecting the graph, the horizontal values range between 0 and 7. Thus, $\operatorname{dom} f = [0,7]$. The range is the collection of all possible y-values. By inspecting the graph, the vertical values range between 1 and 5. Thus, $\operatorname{ran} f = [1,5]$.

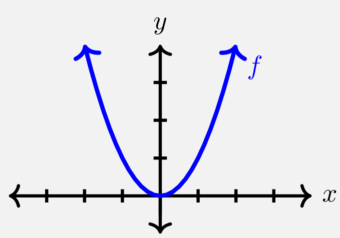
Example 1.1.4. Sketch the graph and find the domain and range of each function.

(a) $f(x) = 2x - 1$.



The graph of f is a line with slope 2 and y-intercept -1. We see that $\boxed{\operatorname{dom} f = (-\infty,\infty)}$ and $\boxed{\operatorname{ran} f = (-\infty,\infty)}$.

(b) $g(x) = x^2$.



The graph of g is a parabola with vertex at the origin and concaves upward. We see that $\boxed{\operatorname{dom} g = (-\infty,\infty)}$ and $\boxed{\operatorname{ran} g = [0,\infty)}$.

Example 1.1.5. Find the domain of each function.

(a) $f(x) = \sqrt{x+2}$.

The radicand must be nonnegative, so we solve the inequality $x + 2 \geq 0$. This implies that $x \geq -2$. Therefore, $\boxed{\operatorname{dom} f = [-2,\infty)}$.

(b) $g(x) = \dfrac{1}{x^2 - x}$.

Here, we must determine which choices of x make the denominator equal to zero. The domain of g is every real number except these values. We thus solve the equation $x^2 - x = 0$. This involves factoring.

$$x^2 - x = x(x-1) = 0.$$

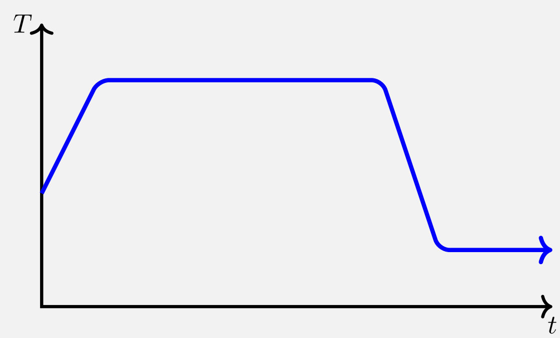
Thus, the solutions are $x = 0, 1$. Therefore, $\boxed{\operatorname{dom} g = \{x \mid x \neq 0, 1\} = (-\infty, 0) \cup (0, 1) \cup (1, \infty)}$.

Example 1.1.6. If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$.

$$\begin{aligned}
\frac{f(a+h) - f(a)}{h} &= \frac{[2(a+h)^2 - 5(a+h) + 1] - [2a^2 - 5a + 1]}{h} \\
&= \frac{[2(a^2 + 2ah + h^2) - 5(a+h) + 1] - [2a^2 - 5a + 1]}{h} \\
&= \frac{[\cancel{2a^2} + 4ah + 2h^2 - \cancel{5a} - 5h + \cancel{1}] - [\cancel{2a^2} - \cancel{5a} + \cancel{1}]}{h} = \frac{4ah + 2h^2 - 5h}{h} \\
&= \frac{h(4a + 2h - 5)}{h} = \boxed{4a + 2h - 5}.
\end{aligned}$$

Example 1.1.7. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t:



Definition 1.1.8. If a function f satisfies $f(-x) = f(x)$ for all x in the domain of f, then f is called an **even function**. If a function f satisfies $f(-x) = -f(x)$ for all x in the domain of f, then f is called an **odd function**.

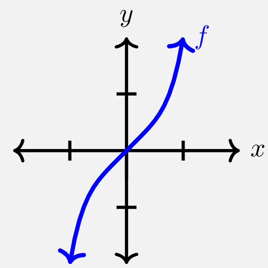
These above definitions are an algebraic way of capturing a geometric notation of symmetry. We give now these geometric definitions. A function f is even if its graph is symmetric about the y-axis and a function f is odd if its graph is symmetric about the origin, that is, its graph is left unchanged by a rotation of π radians (180°) around the origin.

Example 1.1.9. Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$

$f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$

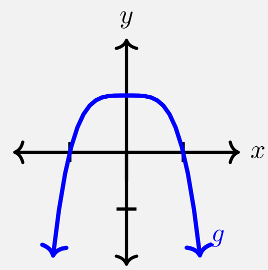
Thus, $\boxed{f \text{ is odd}}$.



(b) $g(x) = 1 - x^4$

$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

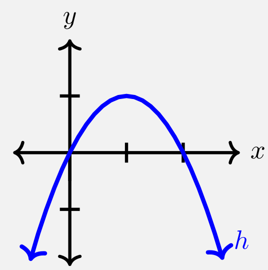
Thus, $\boxed{g \text{ is even}}$.



(c) $h(x) = 2x - x^2$

$h(-x) = 2(-x) - (-x)^2 = -2x - x^2 \neq -h(x), h(-x)$

Thus, $\boxed{h \text{ is neither even nor odd}}$.



Definition 1.1.10. Let f be a function defined on some interval. Then for any two numbers x_1 and x_2 in the interval, f is **increasing** on the interval if

$$f(x_1) < f(x_2), \quad \text{whenever} \quad x_1 < x_2,$$

and f is **decreasing** on the interval if

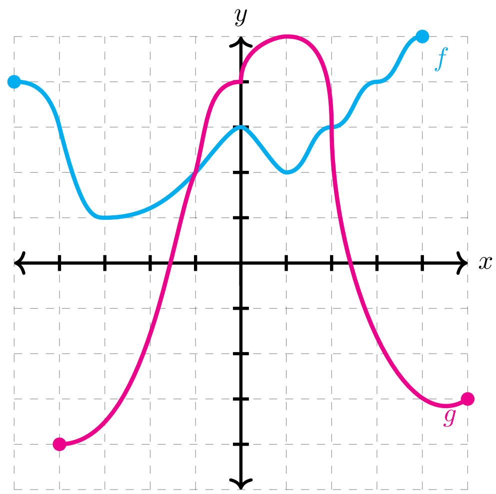
$$f(x_1) > f(x_2), \quad \text{whenever} \quad x_1 < x_2.$$

[i]See §1.1 Review of Functions in OpenStax Calculus Volume 1 for additional reading.

Exercises

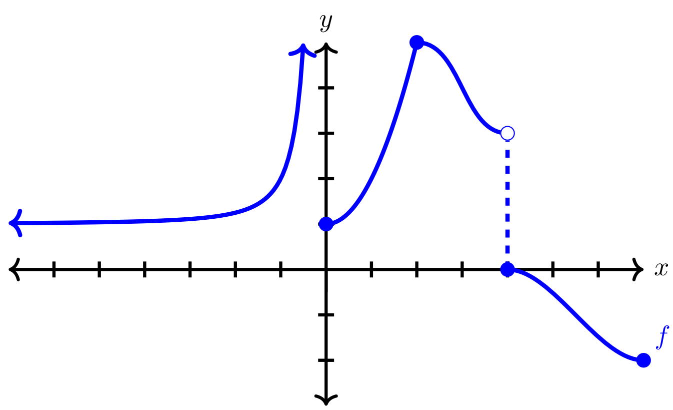
(Go to Solutions)

For Exercises 1–6, the graphs of f and g are illustrated.



1. Find the domain and range of f.
2. Find the domain and range of g.
3. Evaluate $g(0)$.
4. Solve $f(x) = 3$.
5. Solve $f(x) = g(x)$.
6. On what intervals is f decreasing?

♠ 7. In the function f illustrated below, on what intervals is f increasing?



For Exercises 8–21, determine analytically if the function is even, odd, or neither. Justify your response.

8. $f(x) = x^5 - x$
9. $g(x) = x^4 - 1$
♠ 10. $h(x) = \sqrt[3]{x}$
♠ 11. $f(x) = \sqrt[3]{2x^2 + 1}$
♠ 12. $g(x) = x^{1/2}$
13. $h(x) = x^{1/3}$
14. $f(x) = x^{1/4}$
15. $g(x) = \dfrac{1}{x}$
16. $h(x) = \dfrac{1}{x^2}$
17. $f(x) = x + \dfrac{4}{x}$
18. $f(x) = \dfrac{x}{x^2+1}$
♠ 19. $g(x) = \dfrac{x}{x^2+x+1}$
♠ 20. $h(x) = x|x|$
21. $f(x) = \dfrac{2x}{|x|}$

For Exercises 22–32, find the domain of the function.

22. $f(x) = \dfrac{2x+1}{x^2-1}$
23. $f(x) = \dfrac{(3x-1)(2x+7)}{(x+2)(5x+4)}$
24. $f(x) = \dfrac{(x+2)(5x+4)}{(3x-1)(2x+7)}$
25. $f(x) = \dfrac{x^2+3x+2}{x^2+4x+4}$
26. $f(x) = \dfrac{x^2+2x+1}{x^2-1}$
27. $f(x) = \sqrt{2x+1}$

28. $g(x) = \sqrt{3-x} - \sqrt{2+x}$

♠ 29. $f(x) = \sqrt{2 - \sqrt{x}}$

30. $h(x) = \dfrac{4}{\sqrt{x^2 - 6x}}$

♠ 31. $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$

32. $f(x) = \dfrac{\sqrt{2x+1}}{x^2 - 1}$

For Exercises 33–40, given the function f, evaluate and simplify the difference quotient $\dfrac{f(a+h) - f(a)}{h}$, where $h \neq 0$.

33. $f(x) = x^2$

34. $f(x) = 2x^2 + 1$

♠ 35. $f(x) = 5 - 9x$

♠ 36. $f(x) = 4 + 2x - x^2$

♠ 37. $f(x) = 5x^2 - x + 3$

38. $f(x) = 2x^2 + x + 1$

♠ 39. $f(x) = \dfrac{7}{x+4}$

40. $f(x) = x^3$

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For Exercises 41–42, factor the polynomial.

41. $x^2 + 2x - 8$

42. $2x^2 + 5x - 3$

For Exercises 43–44, solve the equation.

43. $2x^3 + 3x^2 - 2x - 3 = 0$

44. $2x^{10/3} + 3x^{7/3} - 2x^{4/3} - 3x^{1/3} = 0$

For Exercises 45–50, simplify the expression, e.g., rationalize the deonominator, clear compounded fractions, reduce to lowest terms.

45. $\sqrt{72x^3y^{10}}$

46. $\sqrt[3]{16x^7y^3z^{11}}$

47. $\dfrac{2}{\sqrt{5}+2}$

48. $\dfrac{h}{\sqrt{4+h}-2}$

49. $\dfrac{\frac{1}{y^2} - \frac{1}{xy} - \frac{2}{x^2}}{\frac{1}{y^2} - \frac{3}{xy} + \frac{2}{x^2}}$

50. $\dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$

"A wise man can learn more from a foolish question than a fool can learn from a wise answer." – Bruce Lee

Lecture Videos

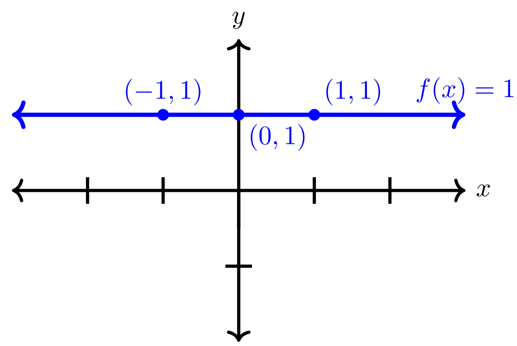
Power Functions | Radical Functions | Reciprocal Functions and their Graphs | Piece-wise Functions

1.2 The Library of Algebraic Functions

Calculus is very much a study of continuous functions. To be best equipped for this study, we need to review basic algebraic functions that are used to build more complicated ones.

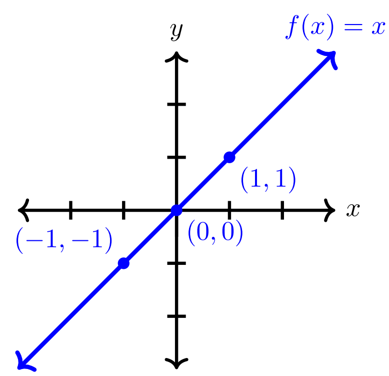
Constant Function $f(x) = 1$

dom f	:	$(-\infty, \infty)$
ran f	:	$\{1\}$
y-intercept	:	1
x-intercepts	:	$\emptyset$
symmetry	:	y-axis
monotonicity	:	constant on $(-\infty, \infty)$



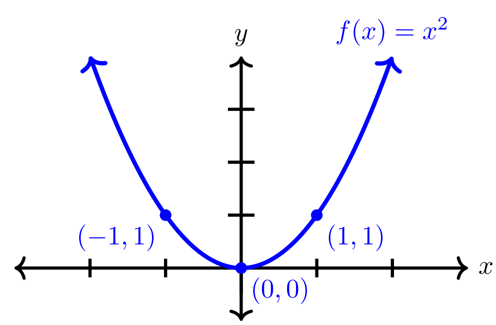
Identity Function $f(x) = x$

dom f	:	$(-\infty, \infty)$
ran f	:	$(-\infty, \infty)$
y-intercept	:	0
x-intercepts	:	$\{0\}$
symmetry	:	origin
monotonicity	:	increasing on $(-\infty, \infty)$



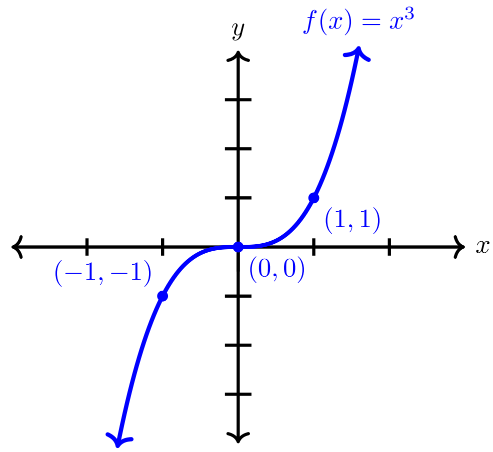
Square Function $f(x) = x^2$

dom f	:	$(-\infty, \infty)$
ran f	:	$[0, \infty)$
y-intercept	:	0
x-intercepts	:	$\{0\}$
symmetry	:	y-axis
monotonicity	:	decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$



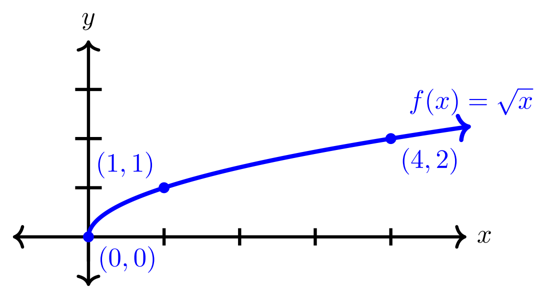
Cube Function $f(x) = x^3$

dom f	:	$(-\infty, \infty)$
ran f	:	$(-\infty, \infty)$
y-intercept	:	0
x-intercepts	:	$\{0\}$
symmetry	:	origin
monotonicity	:	increasing on $(-\infty, \infty)$



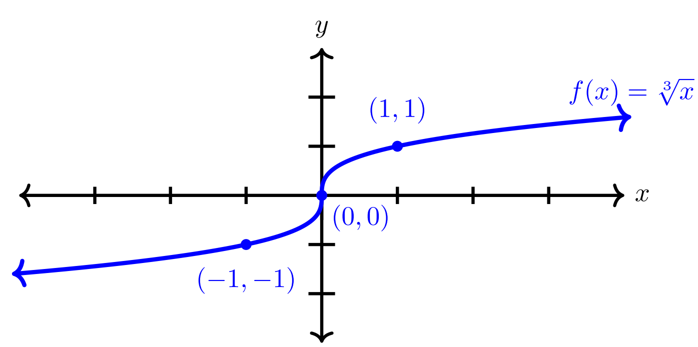
Square Root Function $f(x) = \sqrt{x}$

dom f	:	$[0, \infty)$
ran f	:	$[0, \infty)$
y-intercept	:	0
x-intercepts	:	$\{0\}$
symmetry	:	none
monotonicity	:	increasing on $(0, \infty)$



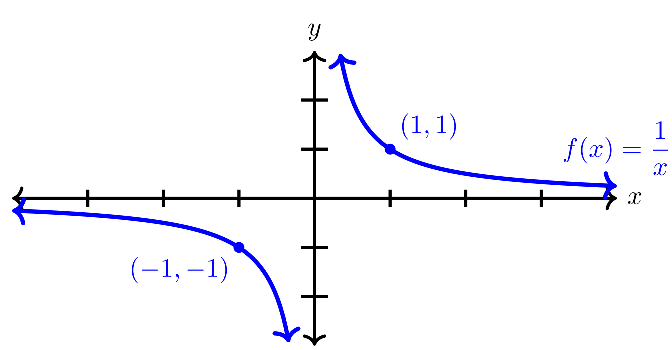
Cube Root Function $f(x) = \sqrt[3]{x}$

dom f	:	$(-\infty, \infty)$
ran f	:	$(-\infty, \infty)$
y-intercept	:	0
x-intercepts	:	$\{0\}$
symmetry	:	origin
monotonicity	:	increasing on $(-\infty, \infty)$



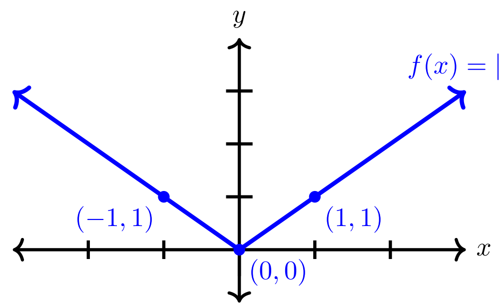
Reciprocal Function $f(x) = \dfrac{1}{x}$

dom f	:	$(-\infty, 0) \cup (0, \infty)$
ran f	:	$(-\infty, 0) \cup (0, \infty)$
y-intercept	:	none
x-intercepts	:	none
symmetry	:	origin
monotonicity	:	decreasing on $(-\infty, 0) \cup (0, \infty)$



Absolute Value Function $f(x) = |x|$

$$\begin{array}{rcl} \operatorname{dom} f & : & (-\infty, \infty) \\ \operatorname{ran} f & : & [0, \infty) \\ y\text{-intercept} & : & 0 \\ x\text{-intercepts} & : & \{0\} \\ \text{symmetry} & : & y\text{-axis} \\ \text{monotonicity} & : & \text{decreasing on } (-\infty, 0); \\ & & \text{increasing on } (0, \infty) \end{array}$$



From these important functions, we can build many more!

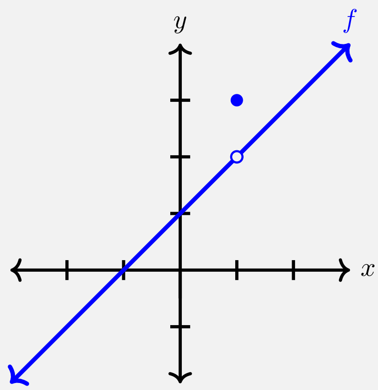
Definition 1.2.1. A **piecewise function** is a function broken into two or more pieces. When reading a piecewise function, the first column provides the function formula and the second determines which x-values use which formula.

Example 1.2.2. Graph the following piecewise functions.

(a) $f(x) = \begin{cases} x+1, & x \neq 1 \\ 3, & x = 1. \end{cases}$

Note that

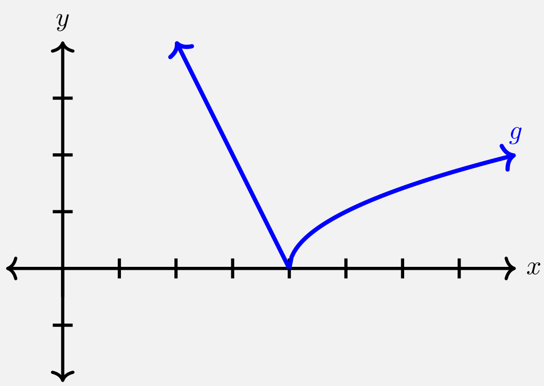
$$\begin{array}{rcl} f(0) & = & (0) + 1 = 1; \\ f(1) & = & 3; \\ f(2) & = & (2) + 1 = 3; \\ f(3) & = & (3) + 1 = 4. \end{array}$$



(b) $g(x) = \begin{cases} \sqrt{x-4}, & x \geq 4 \\ 8 - 2x, & x < 4, \end{cases}$

Note that

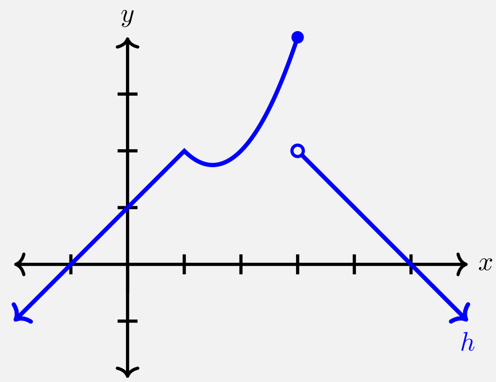
$$\begin{array}{rcl} g(0) & = & 8 - 2(0) = 8; \\ g(3) & = & 8 - 2(3) = 2; \\ g(4) & = & \sqrt{(4) - 4} = 0; \\ g(8) & = & \sqrt{(8) - 4} = 2. \end{array}$$



(c) $h(x) = \begin{cases} x+1, & x < 1 \\ x^2 - 3x + 4, & 1 \le x \le 3 \\ 5 - x, & x > 3 \end{cases}$

Note that

$$\begin{aligned} h(0) &= (0) + 1 = 1; \\ h(1) &= (1)^2 - 3(1) + 4 = 2; \\ h(3) &= (3)^2 - 3(3) + 4 = 4. \end{aligned}$$



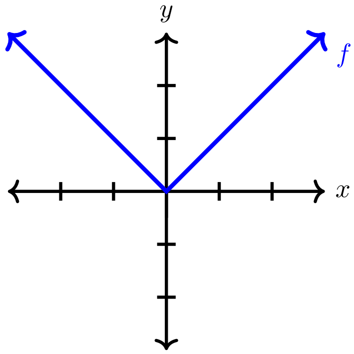
[i]See §1.2 Basic Classes of Functions in OpenStax Calculus Volume 1 for additional reading.

Exercises

(Go to Solutions)

For Exercises 1–11, which function has its graph illustrated to the right?

1.



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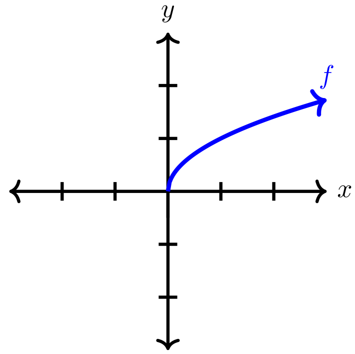

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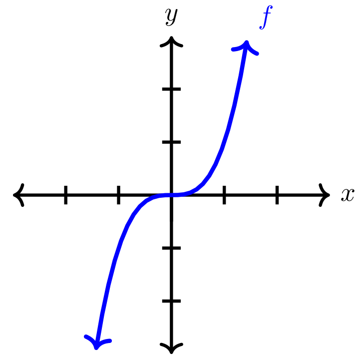
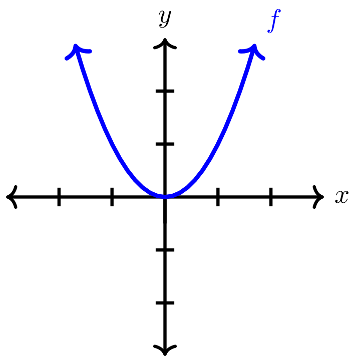
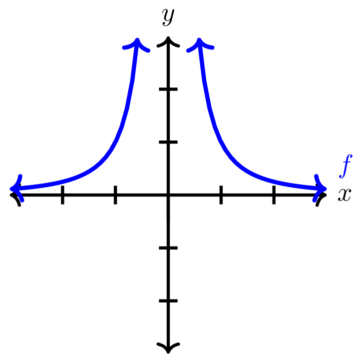


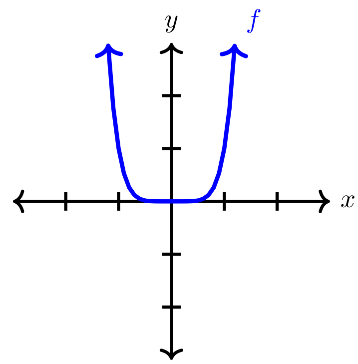
♠ 4.



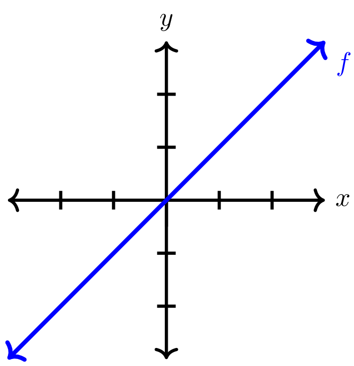
♠ 5.



♠ 6.



7.



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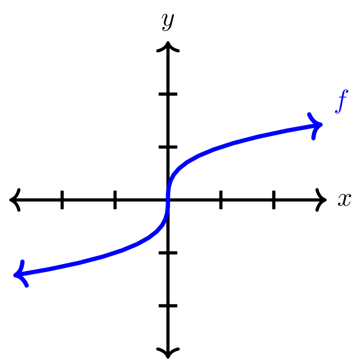

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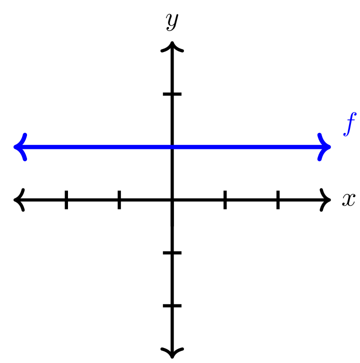

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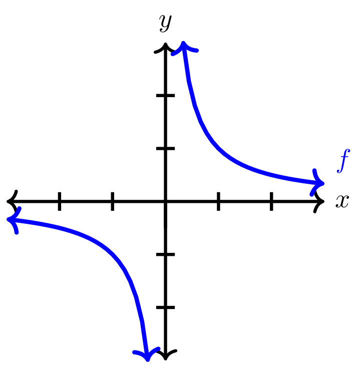

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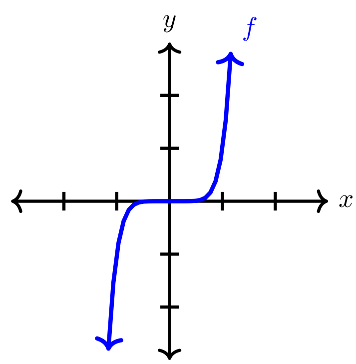
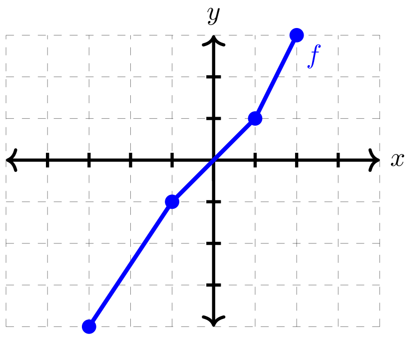


For Exercises 12–17, the graph of f is provided below. Find a piece-wise function definition for f which matches the graph provided.

12.



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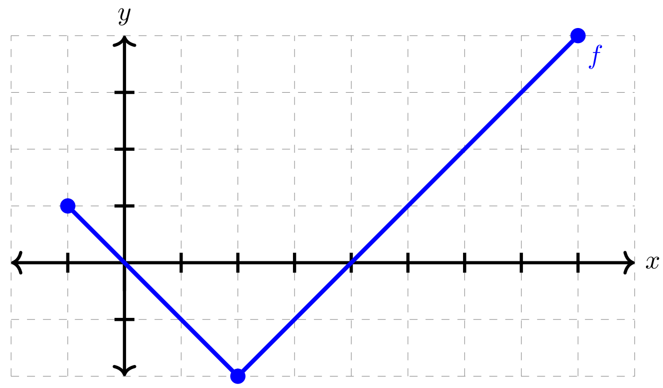

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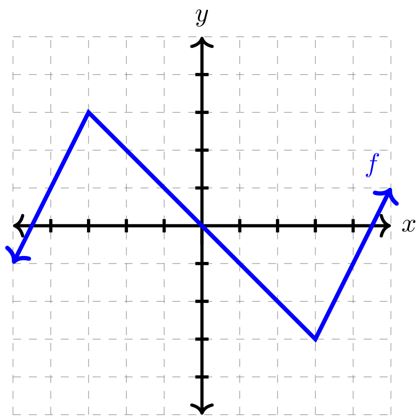

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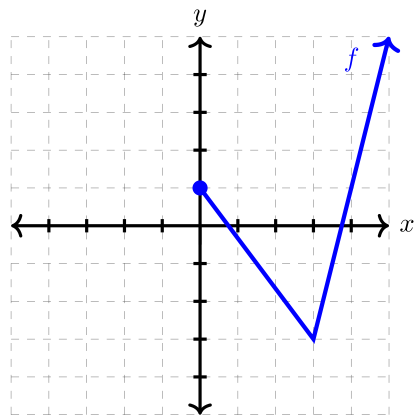

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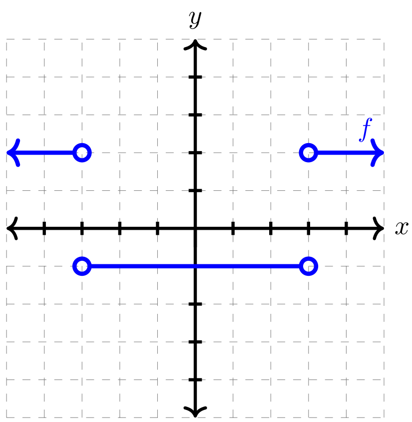

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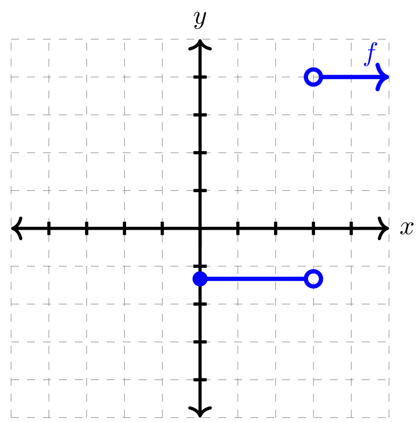


"Our character is basically a composite of our habits. Because they are consistent, often unconscious patterns, they constantly, daily, express our character." – Stephen Covey

Lecture Videos

Graph Transformations

Algebra of Functions

Function Composition

Composition of Rational Functions

Composition of Square Root Functions

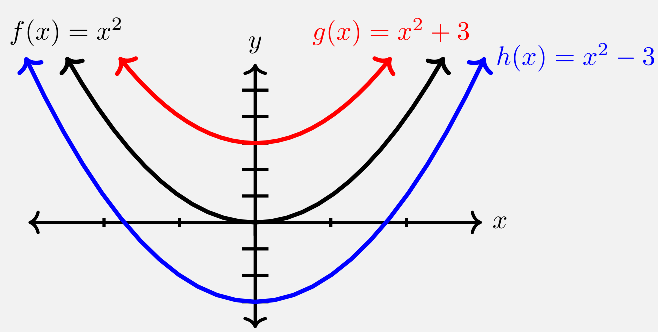
Function Decomposition

1.3 Function Composition

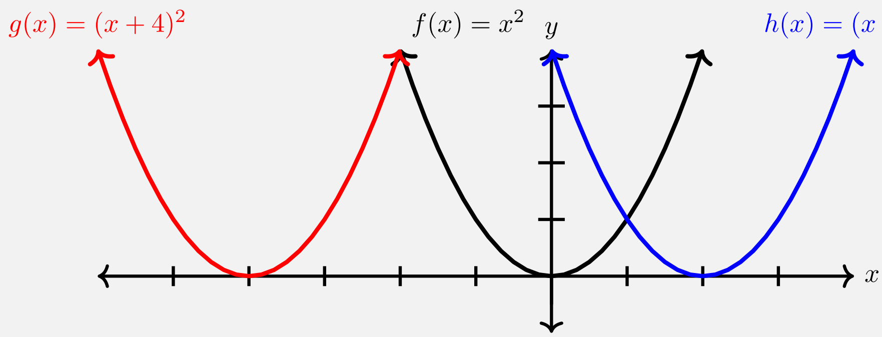
Proposition 1.3.1 (Shifts). *Suppose $c > 0$. To obtain the graph of*

- $y = f(x) + c$, *shift the graph of* $y = f(x)$ *a distance* c *units upward,*
- $y = f(x) - c$, *shift the graph of* $y = f(x)$ *a distance* c *units downward.*
- $y = f(x - c)$, *shift the graph of* $y = f(x)$ *a distance* c *units to the right,*
- $y = f(x + c)$, *shift the graph of* $y = f(x)$ *a distance* c *units to the left.*

Example 1.3.2. Let $f(x) = x^2$. Sketch the graphs for $g(x) = x^2 + 3$ and $h(x) = x^2 - 3$.



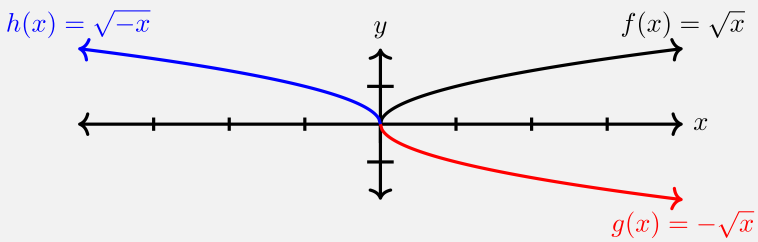
Example 1.3.3. Let $f(x) = x^2$. Sketch the graphs for $g(x) = (x + 4)^2$ and $h(x) = (x - 2)^2$.



Proposition 1.3.4 (Reflections)**.** *To obtain the graph of*

- $y = -f(x)$, *reflect the graph of* $y = f(x)$ *about the* x*-axis,*
- $y = f(-x)$, *reflect the graph of* $y = f(x)$ *about the* y*-axis,*
- $y = -f(-x)$, *rotate the graph of* $y = f(x)$ *about the origin* π *radians (*$180°$*).*

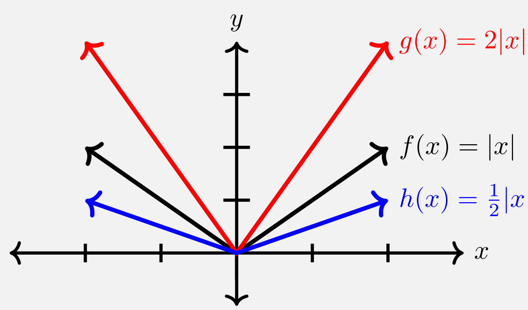
Example 1.3.5. Let $f(x) = \sqrt{x}$. Sketch the graphs for $g(x) = -\sqrt{x}$ and $h(x) = \sqrt{-x}$.



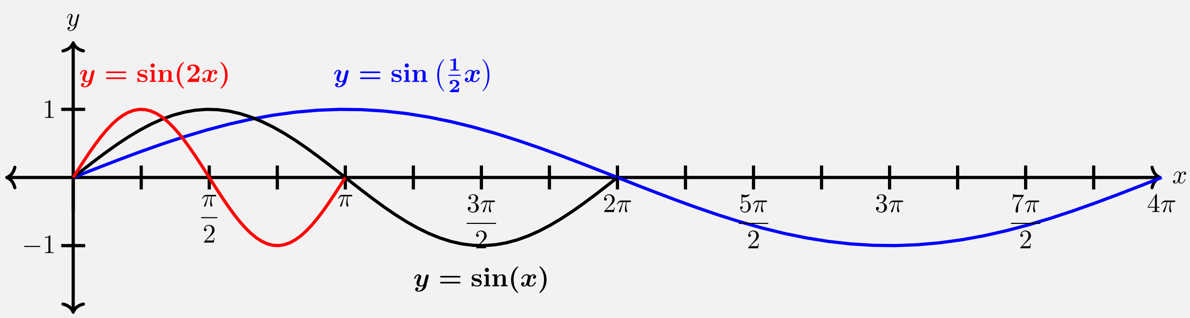
Proposition 1.3.6 (Stretches)**.** *Suppose that* $c > 1$*. To obtain the graph*

- $y = cf(x)$, *stretch the graph of* $y = f(x)$ *vertically by a factor of* c*,*
- $y = (1/c)f(x)$, *shrink the graph of* $y = f(x)$ *vertically by a factor of* c*,*
- $y = f(cx)$, *shrink the graph of* $y = f(x)$ *horizontally by a factor of* c*,*
- $y = f(x/c)$, *stretch the graph of* $y = f(x)$ *horizontally by a factor of* c*.*

Example 1.3.7. Let $f(x) = |x|$. Sketch the graphs for $g(x) = 2|x|$ and $h(x) = \frac{1}{2}|x|$.



Example 1.3.8. Let $f(x) = \sin(x)$. Sketch the graphs for $g(x) = \sin(2x)$ and $h(x) = \sin\left(\frac{1}{2}x\right)$.



Definition 1.3.9. Let f and g be functions. Then

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) && \operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g \\
(f-g)(x) &= f(x)-g(x) && \operatorname{dom}(f-g) = \operatorname{dom} f \cap \operatorname{dom} g \\
(fg)(x) &= f(x)g(x) && \operatorname{dom}(fg) = \operatorname{dom} f \cap \operatorname{dom} g \\
(f/g)(x) &= f(x)/g(x) && \operatorname{dom}(f/g) = (\operatorname{dom} f \cap \operatorname{dom} g) \setminus \{x \mid g(x) = 0\}
\end{aligned}$$

Example 1.3.10. Let $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{x}{x-1}$. Then $\operatorname{dom} f = \{x \mid x \neq -2\}$ and $\operatorname{dom} g = \{x \mid x \neq 1\}$.

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) = \frac{1}{x+2} + \frac{x}{x-1} = \frac{1}{x+2}\left(\frac{x-1}{x-1}\right) + \frac{x}{x-1}\left(\frac{x+2}{x+2}\right) \\
&= \frac{(x-1)+x(x+2)}{(x-1)(x+2)} = \frac{(x-1)+(x^2+2x)}{(x-1)(x+2)} = \boxed{\frac{x^2+3x-1}{(x-1)(x+2)}} \\
(f-g)(x) &= f(x)-g(x) = \frac{(x-1)-x(x+2)}{(x-1)(x+2)} = \frac{(x-1)-(x^2+2x)}{(x-1)(x+2)} = \boxed{-\frac{x^2+x+1}{(x-1)(x+2)}} \\
(fg)(x) &= f(x)g(x) = \left(\frac{1}{x+2}\right)\left(\frac{x}{x-1}\right) = \boxed{\frac{x}{(x-1)(x+2)}} \\
(f/g)(x) &= \frac{f(x)}{g(x)} = \frac{1/(x+2)}{x/(x-1)} = \left(\frac{1}{x+2}\right)\left(\frac{x-1}{x}\right) = \boxed{\frac{x-1}{x(x+2)}}
\end{aligned}$$

The domains of $f+g$, $f-g$, and fg are $\{x \mid x \neq 1, -2\}$. The domain of f/g is $\{x \mid x \neq 1, -2, 0\}$.

Definition 1.3.11. Given two functions f and g, the **composite function** $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f, that is, $\operatorname{dom} f \circ g = \{x \in \operatorname{dom} g \mid g(x) \in \operatorname{dom} f\}$.

Example 1.3.12. Let $f(x) = x^2 + 1$ and $g(x) = x - 3$. Then compute:

(a) $(f \circ g)(x) = f(g(x)) = f(x-3) = (x-3)^2 + 1 = (x^2 - 6x + 9) + 1 = \boxed{x^2 - 6x + 10}$.

Since the domain of both f and g are all real numbers, the domain of $f \circ g$ is also all real numbers.

(b) $(g \circ f)(x) = g(f(x)) = f(x) - 3 = (x^2 + 1) - 3 = \boxed{x^2 - 2}$.

Since the domain of both f and g are all real numbers, the domain of $g \circ f$ is also all real numbers.

Example 1.3.13. Let $f(x) = \sqrt{x}$ and let $g(x) = \sqrt{2-x}$. Compute the following composites and their domains:

(a) $(f \circ g)(x) = \sqrt{g(x)} = \sqrt{\sqrt{2-x}} = \boxed{\sqrt[4]{2-x}, \quad \operatorname{dom} f \circ g = (-\infty, 2]}$.

(b) $(g \circ f)(x) = \sqrt{2 - f(x)} = \boxed{\sqrt{2 - \sqrt{x}}, \quad \operatorname{dom} g \circ f = [0, 4]}$.

(c) $(g \circ g)(x) = g(g(x)) = \sqrt{2 - g(x)} = \boxed{\sqrt{2 - \sqrt{2-x}}, \quad \operatorname{dom} g \circ g = [-2, 2]}$.

Example 1.3.14. Let $F(x) = \sqrt[4]{x+9}$. Find two functions f and g such that $F(x) = (f \circ g)(x)$.

Let $g(x) = x + 9$ and $f(x) = \sqrt[4]{x}$. Then $F = f \circ g$.

Notice also that if $h(x) = \sqrt{x+9}$ and $k(x) = \sqrt{x}$, then $F = k \circ h$. Thus, composition factors are not necessarily unique.

Example 1.3.15. Find functions f and g such that $f \circ g = H$ if $H(x) = \dfrac{1}{x+1}$.

Let $g(x) = x + 1$ and $f(x) = 1/x$. Then $(f \circ g)(x) = H(x)$.

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[i]See §1.1 Review of Functions and §1.2 Basic Classes of Functions in OpenStax Calculus Volume 1 for additional reading.

Exercises

(Go to Solutions)

For Exercises 1–5, the graphs of f and g are illustrated. Evaluate the expression.

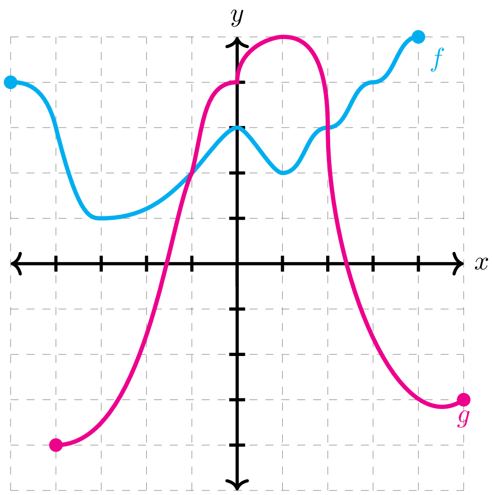
1. $f(g(-4))$

♠ 2. $g(f(-4))$

♠ 3. $(g \circ f)(1)$

♠ 4. $(f \circ g)(1)$

5. $(f \circ f)(2)$



For Exercises 6–6, for functions f and g, find $f+g$, $f-g$, fg, and f/g and their respective domains.

6. $f(x) = x^3 + 3x^2$, $g(x) = 5x^2 - 3$

For Exercises 7–12, explain how the graph is obtained from the graph of $y = f(x)$.

7. $y = 4f(x)$

8. $y = f(4x)$

♠ 9. $y = f(x) + 4$

♠ 10. $y = f(x+4)$

♠ 11. $y = -f(x) - 1$

12. $y = 4f\left(\frac{1}{4}x\right)$

For Exercises 13–15, graph the function $y = f(x)$. Indicate three points on the graph. Indicate which transformations were applied to basic function from Section 1.2 to obtain f.

13. $f(x) = -2(x-2)^3 + 3$

♠ 14. $f(x) = 2\sqrt{x-2} + 3$

15. $f(x) = \frac{1}{2}(x-1)^5 - 2$

For Exercises 16–20, for functions f and g, find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their respective domains.

16. $f(x) = 2x + 4$, $g(x) = 7x + 2$

♠ 17. $f(x) = x^2 - 1$, $g(x) = 2x + 4$

♠ 18. $f(x) = \sqrt{x}$, $g(x) = \sqrt[3]{9-x}$

♠ 19. $f(x) = x + \dfrac{1}{x}$, $g(x) = \dfrac{x+11}{x+2}$

20. $f(x) = x^2 + 4$, $g(x) = \sqrt{x-2}$

For Exercises 21–27, for the function F, find functions f and g such that $F(x) = (f \circ g)(x)$.

21. $F(x) = \sqrt[4]{x^2+1}$

♠ 22. $F(x) = (2x + x^2)^4$

♠ 23. $F(x) = \sqrt[3]{\dfrac{x}{6+x}}$

♠ 24. $F(x) = \sqrt{(x^3+6)^2 - 1}$

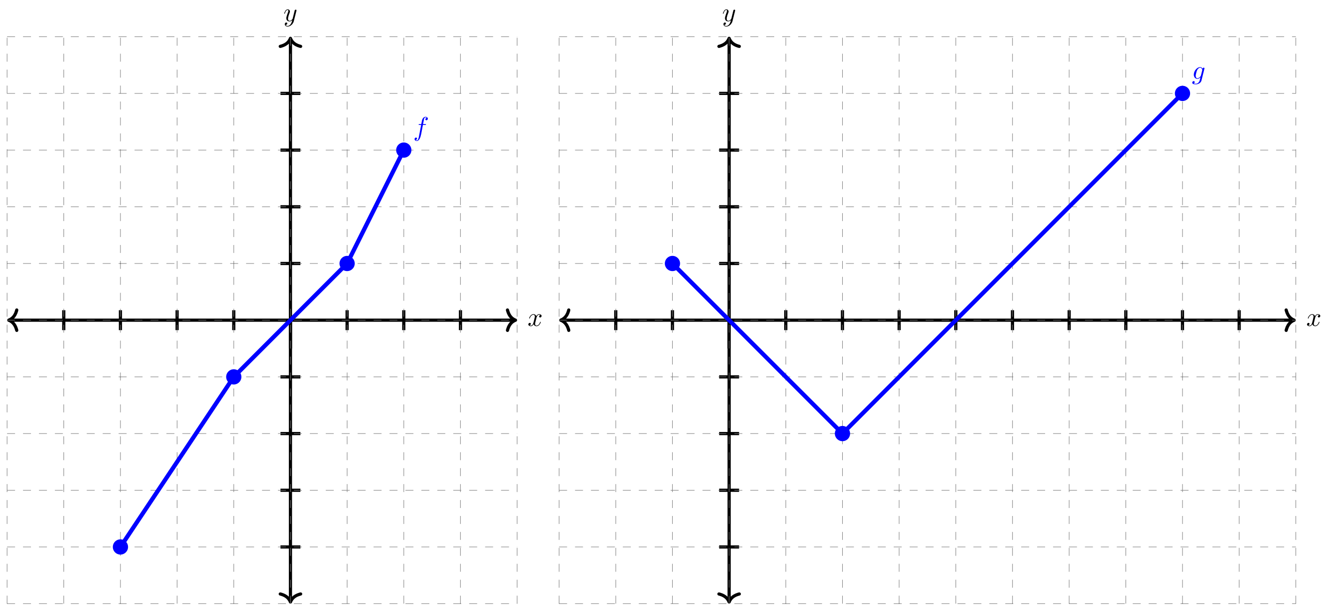
25. $F(x) = 1 - \sqrt{x^2+1}$

26. $F(x) = \dfrac{1}{x^2+1}$

27. $F(x) = \dfrac{x^2}{x^2+1}$

Deeper Dive

The graphs of f and g are given below:



28. Find piece-wise function definitions for f and for g which match the graphs provided above.

For Exercises 29–31, for the function F, find functions f and g such that $F(x) = (f \circ g)(x)$.

29. $(g \circ f)(2)$

30. $(g \circ f)(-1)$

31. $(f \circ g)(6)$

32. Graph $h(x) = -2f(x+1) + 2$.

"The difference between happiness and misery ... often comes down to an error of only a few degrees."
– Dieter F. Uchtdorf

Lecture Videos

Radian Measure

Arc Length

Definitions of the Six Trigonometric Ratios

Right Triangle Trigonometry

The Unit Circle Diagram

The Graph of Sine

1.4 Trigonometric Functions

Definition 1.4.1. Trigonometry is the algebraic study of angles, triangles, circles, and their relationships with each other. An **angle** is a measurement of how much two intersecting line segments (or rays) differ by rotation.

There are two common ways to measure angles: Degrees and Radians. Comparing degrees with radians, we have the identity $360° = 2\pi$ radians. Solving for radians, we have

$$1 \text{ radian} = \left(\frac{180°}{\pi}\right). \tag{1.4.1}$$

Solving for degrees, we have

$$1° = \frac{\pi}{180} \text{ radians} \tag{1.4.2}$$

Example 1.4.2. Convert degree measures to radians and radian measures to degrees.

(a) $45° = 45\left(\frac{\pi}{180}\right) = \frac{45\pi}{180} = \boxed{\frac{\pi}{4} \text{ radians}}$. (b) $\frac{9\pi}{4}$ radians $= \frac{9\pi}{4}\left(\frac{180°}{\pi}\right) = \boxed{405°}$.

Although radian measure seems more complicated at first glance compared to degree measure, the next example illustrates some of the advantages radians offer.

Example 1.4.3. Suppose that we want to calculate the length of an **arc** of a circle. The arc is bounded by two radii of the circle. These line segments define an angle, which we denote as θ. If the circle has radius length r, then the length of the arc, denoted as s, follows from the simple formula

$$s = \theta r. \tag{1.4.3}$$

Please note that this formula only holds for radian measure!

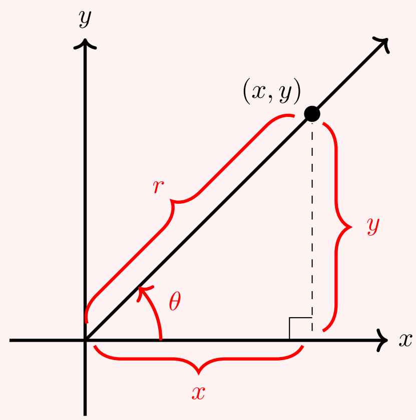
Suppose we have a circle with radius length 1, which is called the **unit circle**. The **circumference** of the circle is the arc length of one complete rotation around the circle, that is, the length around the whole circle. Using formula (1.4.3), the circumference of the unit circle is $C = 2\pi \cdot 1 = 2\pi$. Furthermore, if a circle has radius r, then the formula for circumference is given as

$$C = 2\pi r. \tag{1.4.4}$$

Theorem 1.4.4 (Pythagorean Theorem).[i] *In a triangle with a right angle, if the longest side of the triangle (called the hypotenuse) is r and the shorter sides are x and y (called the legs), then*

$$r^2 = x^2 + y^2.$$

Definition 1.4.5. Given the angle θ, consider the diagram, illustrated to the right, where the initial side is the x-axis. By the Pythagorean Theorem, we have that $r = \sqrt{x^2 + y^2}$. This denotes the distance between the point (x, y) and the origin. From these values: x, y, and r, we define the six **trigonometric functions**:



$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

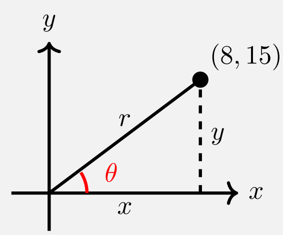
$$\csc\theta = \frac{r}{y} = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{r}{x} = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}$$

If $x = 0$, then $\sec\theta$ and $\tan\theta$ are undefined. Likewise, if $y = 0$, then $\csc\theta$ and $\cot\theta$ are undefined.

Example 1.4.6. Consider the following right triangle, illustrated to the right. Substituting the known values $x = 8$ and $y = 15$ in the Pythagorean equation gives

$$r = \sqrt{8^2 + 15^2} = \sqrt{64 + 225} = \sqrt{289} = 17.$$

We have $x = 8$, $y = 15$, and $r = 17$. The values of the six trigonometric functions of angle θ are found by using the definitions:



$$\sin\theta = \frac{y}{r} = \frac{15}{17} \qquad \cos\theta = \frac{x}{r} = \frac{8}{17} \qquad \tan\theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{15} \qquad \sec\theta = \frac{r}{x} = \frac{17}{8} \qquad \cot\theta = \frac{x}{y} = \frac{8}{15}.$$

Now there are a lot of right triangles with angle θ. So, we may simplify the situation by setting $r = 1$. Then the point (x, y) is a point of the unit circle. So, sine measures the y-coordinate of any point on the unit circle with respect to a given angle θ. Likewise, cosine measures the x-coordinate of any point on the unit circle with respect to a given angle θ. This definition of sine and cosine is just as useful as the definition via right triangles.

For your convenience, Figure 1.1[ii] is a diagram of the unit circle with all the special angles labeled in radians and degrees. You will need to memorize this diagram. Using references angles, it suffices to memorize the first quadrant. In the meanwhile, feel free to use this diagram on your homework.

Example 1.4.7. Find the following function values without the using a calculator.

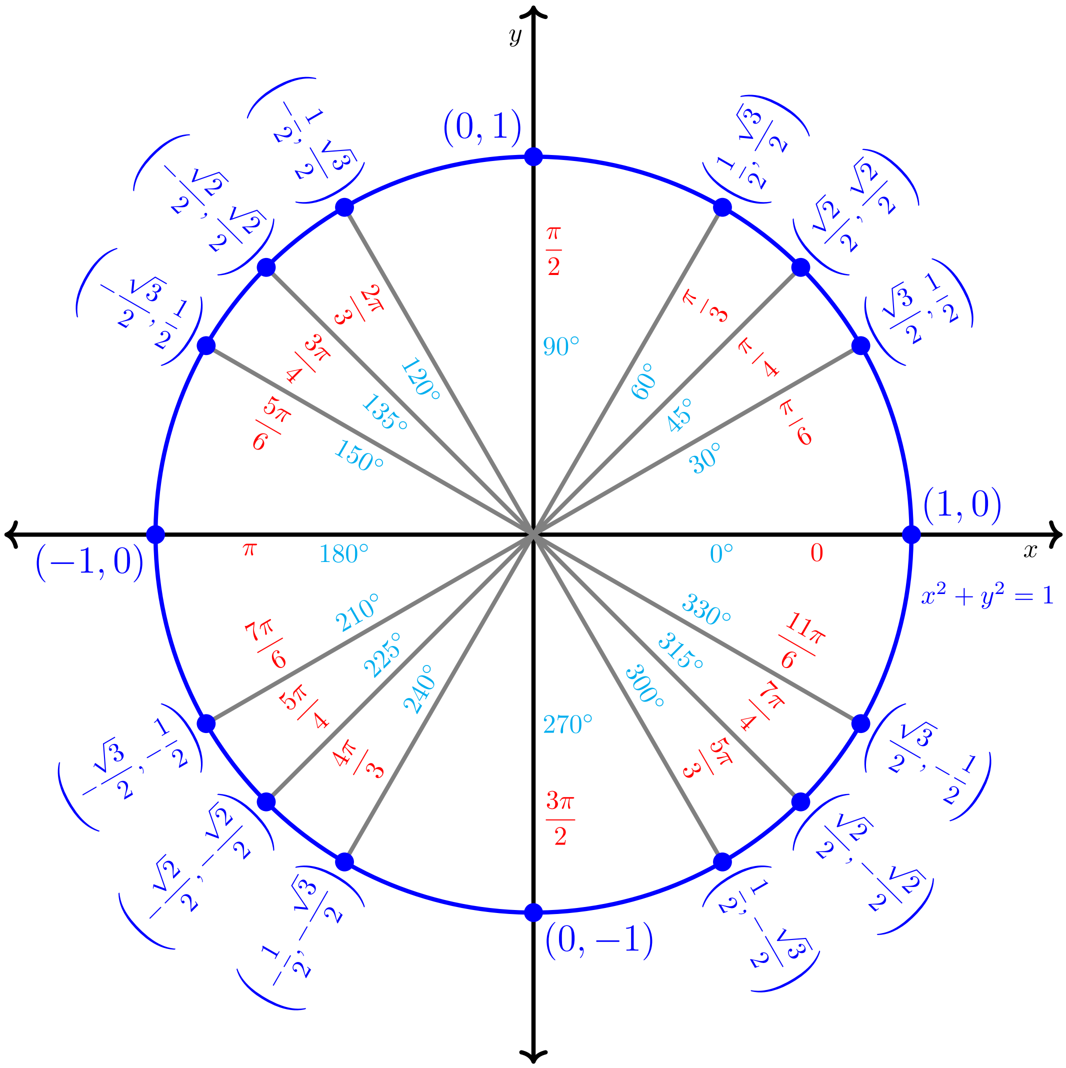


Figure 1.1: The Unit Circle

(a) $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(b) $\tan\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

(c) $\cot\frac{\pi}{3} = \frac{1}{\tan\frac{\pi}{3}} = \frac{1}{\sqrt{3}}$

(d) $\sec\frac{2\pi}{3} = \frac{1}{\cos\frac{2\pi}{3}} = \frac{1}{-1/2} = -2$

It will also be necessary for us to remember trigonometric identities and the graphs of the trigonometric functions. Some identities you must know are:

$$\sin^2 x + \cos^2 x = 1 \tag{1.4.5}$$
$$\sin(-x) = -\sin(x) \tag{1.4.6}$$
$$\cos(-x) = \cos(x) \tag{1.4.7}$$
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \tag{1.4.8}$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{1.4.9}$$

Additional important trigonometric identities can be found in the appendix.

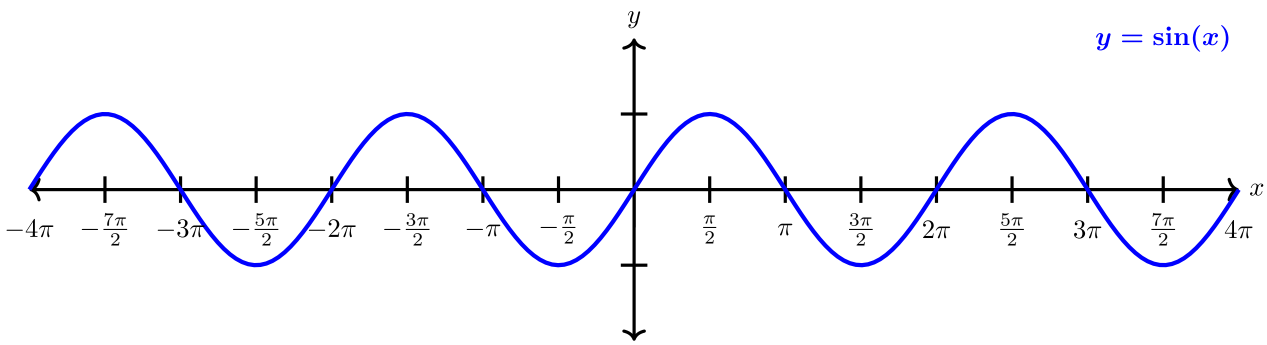
Example 1.4.8. Prove the identity:

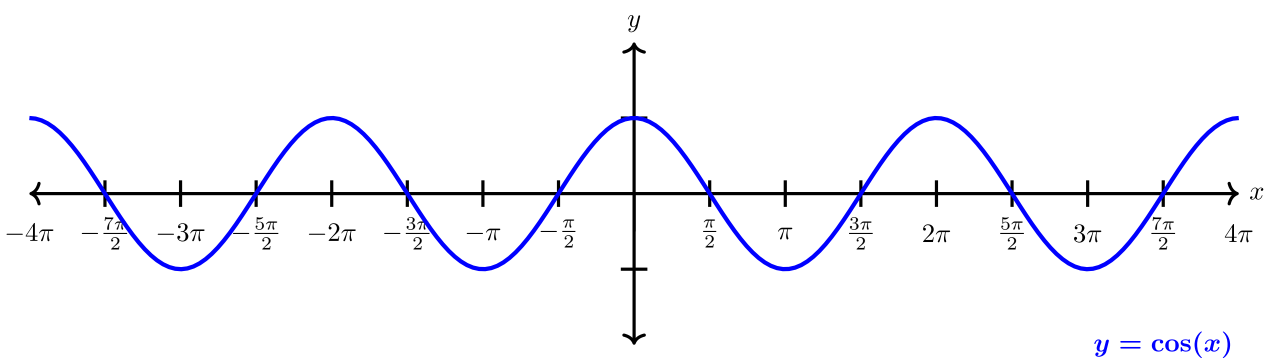
$$\tan^2\alpha - \sin^2\alpha = \tan^2\alpha\sin^2\alpha.$$

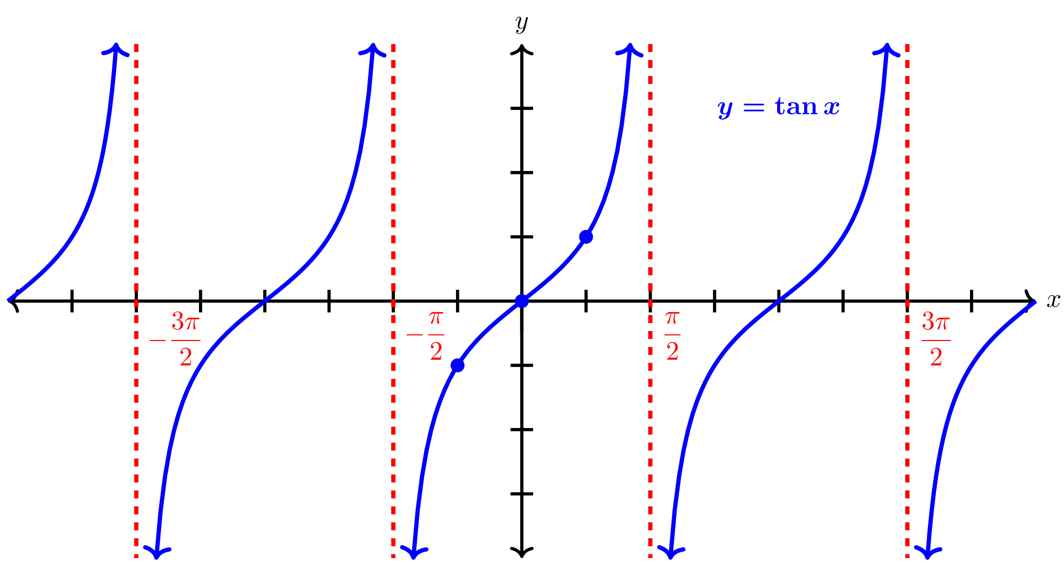
When in doubt, one can convert the trigonometric expression into sines and cosines. Starting with the left-hand side we have

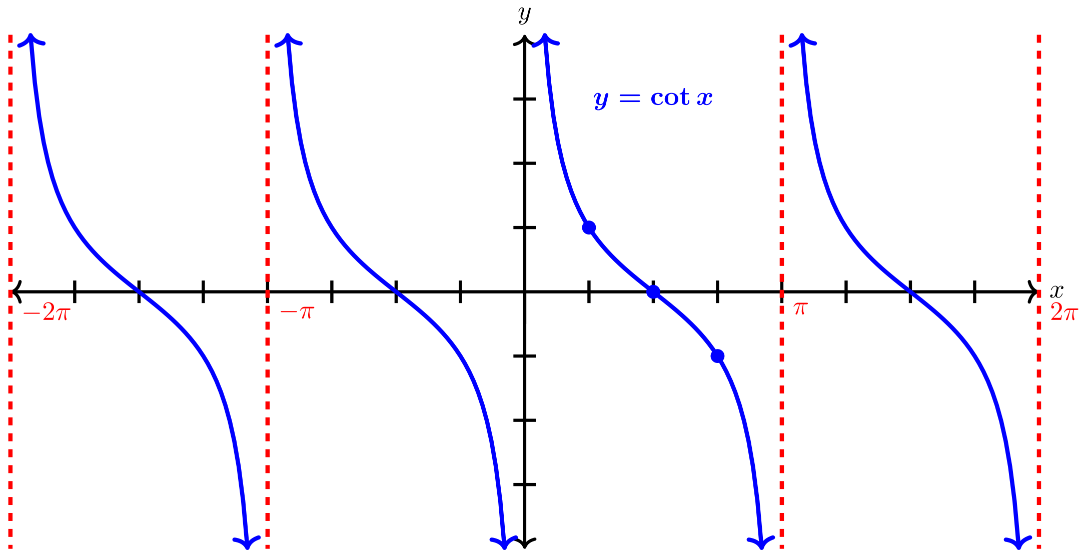
$$\begin{aligned}\tan^2\alpha - \sin^2\alpha &= \frac{\sin^2\alpha}{\cos^2\alpha} - \sin^2\alpha = \sin^2\alpha\left(\frac{1}{\cos^2\alpha} - 1\right)\\ &= \sin^2\alpha(\sec^2(\alpha) - 1) = \sin^2\alpha\tan^2\alpha.\end{aligned}$$

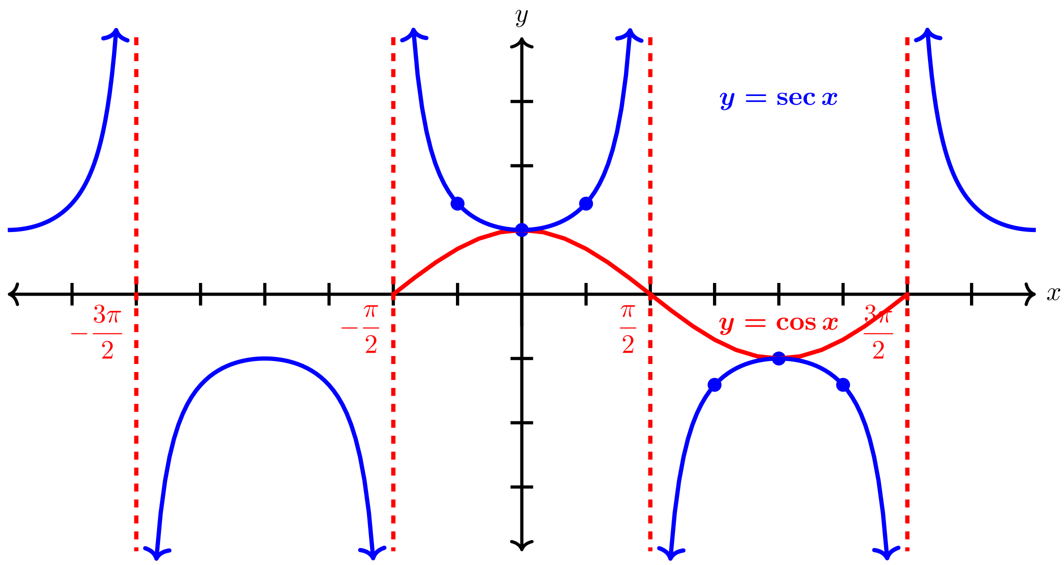
You will also need to know the graphs of the six trigonometric functions, which we add to our library of functions. You will also need to transform them as we did in Section 1.2.

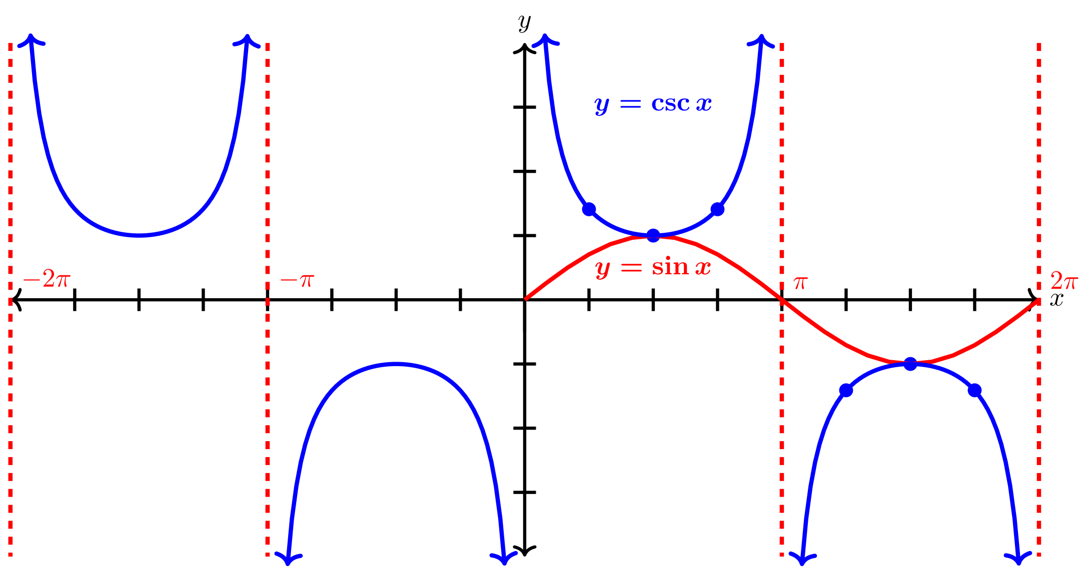












[i]In the Chinese tradition, the Pythagorean Theorem is better known as the *Gougu Rule* (勾股定理).
[iii]See §1.3 Trigonometric Functions in OpenStax Calculus Volume 1 for additional reading.

Exercises

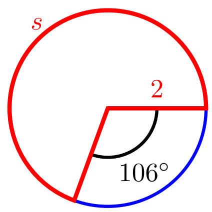
(Go to Solutions)

For Exercises 1–7, convert each angle from angles to radians or vice versa.

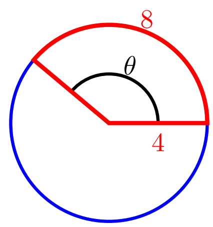
1. $108°$
2. $-210°$
3. ♠ $3960°$
4. ♠ $67.5°$
5. ♠ $\dfrac{\pi}{6}$
6. ♠ $\dfrac{7\pi}{3}$
7. 3.2

For Exercises 8–13, the graphs of f and g are illustrated. Evaluate the expression.

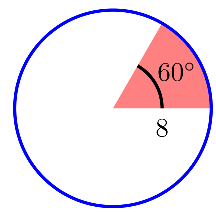
8. Find s below.



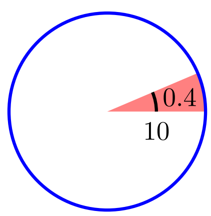
9. ♠ Find θ below.



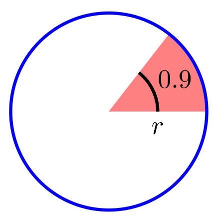
10. Find the area of the shaded region below.



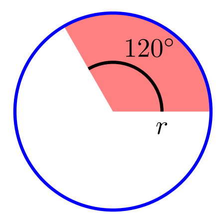
11. ♠ Find the area of the shaded region below.



12. Find the radius r if the shaded region has area $A = 12$ below.



13. Find the radius r if the shaded region has area $A = 26$ below.



For Exercises 14–19, find the exact value of the trigonometric function without a calculator.

14. $\sin \dfrac{2\pi}{3}$
15. ♠ $\tan\left(-\frac{4\pi}{3}\right)$
16. ♠ $\sec \dfrac{5\pi}{6}$
17. ♠ $\csc\left(-\frac{7\pi}{2}\right)$
18. $\cot \dfrac{7\pi}{6}$
19. $\cos \dfrac{8\pi}{3}$

For Exercises 20–22, find all five remaining trigonometric ratios for θ provided $0 < \theta < \dfrac{\pi}{2}$. Find the radian measure of $\angle A$.

20. $\tan\theta = \dfrac{28}{96}$
21. ♠ $\sin\theta = \dfrac{160}{164}$
22. $\tan\theta = \dfrac{60}{32}$

For Exercises 23–25, graph the given function.

23. $f(x) = \sec\left(x - \dfrac{\pi}{4}\right)$
24. ♠ $g(x) = \dfrac{1}{2} + \sin(x)$
25. $h(x) = 1 + \dfrac{1}{2}\tan x$

"The real energy occurs in each connection between two people, which can bring about exponential returns."
– Tom Rath

Lecture Videos

Exponential Laws | Graphs of Exponential Functions | Curve Fitting Exponential Functions | Exponential Growth

1.5 Exponential Functions

Theorem 1.5.1 (Laws of Exponents)**.** *If m, n, a and b are real numbers and a and b are positive, then*

(a) $a^m a^n = a^{m+n}$

(b) $a^m/a^n = a^{m-n}$

(c) $(a^m)^n = a^{mn}$

(d) $(ab)^n = a^n b^n$

(e) $a^0 = 1$

(f) $a^{-n} = \dfrac{1}{a^n}$

(g) $a^{1/n} = \sqrt[n]{a}$

(h) $a^{m/n} = \sqrt[n]{a^m}$

The above Law of Exponents allow us to compute a^x for any positive real number a and any rational number x. What about $7^{\sqrt{2}}$ or 2^π? Can exponents be computed for irrational numbers?

Notice that $\sqrt{3} \approx 1.73205\ldots$, that is, we can approximate $\sqrt{3}$ by an ascending sequence of rational numbers

$$1.7, 1.73, 1.732, 1.7320, 1.73205, \ldots$$

Similarly, we may approximate $a^{\sqrt{3}}$ by the sequence

$$a^{1.7}, a^{1.73}, a^{1.732}, a^{1.7320}, a^{1.73205}, \ldots$$

With a lack of calculus and without the proper definition of a limit of a sequence, we accept the fact that this sequence approachs a unique number which we call $a^{\sqrt{3}}$. For example,

$$5^{\sqrt{3}} \approx 5^{1.732} \approx 16.2411.$$

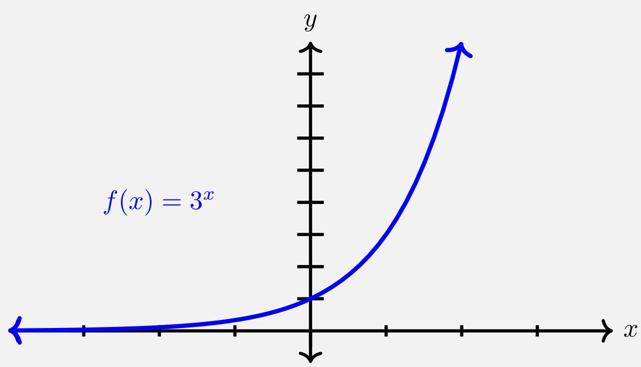
All the rules of exponents apply when the exponents are real (though the proof of this for irrational numbers requires calculus).

Definition 1.5.2. The **exponential function** of base a (where a is a positive number not equal to 1) is defined for all real numbers x by

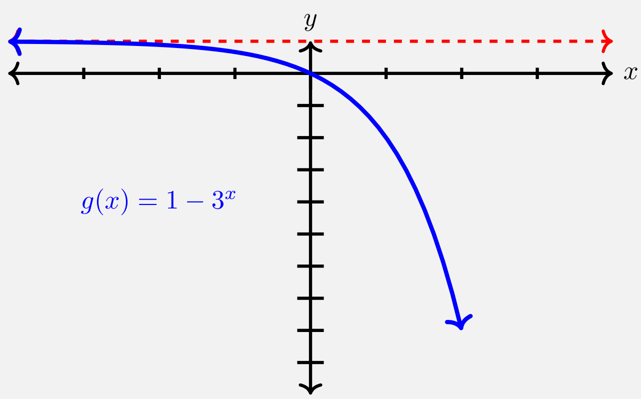
$$f(x) = a^x.$$

Example 1.5.3. Graph the functions.

(a) $f(x) = 3^x$.



(b) $g(x) = 1 - 3^x$.



Example 1.5.4. Find the exponential function satisfying the given conditions.

(a) Find the function $f(x) = a^x$ which passes through $(2, 9)$.

In particular, we have that

$$\begin{aligned} a^2 &= 9 \\ \sqrt{a^2} &= \sqrt{9} \\ a &= 3. \end{aligned}$$

So, $\boxed{f(x) = 3^x}$.

(b) Similarly, find the function $g(x) = a^x$, which passes through $(3, 1/8)$.

$$\begin{aligned} a^3 &= \frac{1}{8} \\ \sqrt[3]{a^3} &= \sqrt[3]{\frac{1}{8}} \\ a &= \frac{1}{2}. \end{aligned}$$

Thus, $\boxed{g(x) = \left(\frac{1}{2}\right)^x}$.

(c) Find $h(x) = Ca^x$ which passes through $(0, 3)$ and $(3, 24)$.

$$Ca^0 = C(1) = C = 3$$

So, $h(x) = 3a^x$. But

$$\begin{aligned} 3a^3 &= 24 \\ a^3 &= 8 \\ a &= 2. \end{aligned}$$

Therefore, $\boxed{h(x) = 3\left(2^x\right)}$.

Definition 1.5.5. Let

$$e \approx 2.71828182845904523536\ldots$$

We call the function $f(x) = e^x$ the **Natural Exponential Function**.

We will not go into the details today, but it turns out that $y = e^x$ is the *best* choice for a base of an exponential function. Calculus will reveal this reason. The natural exponential models various natural phenomena, such population growth, radioactive decay, and interest rates to mention a few. In particular, we can model uninhibited growth by the exponential function

$$P(t) = P_0 e^{rt}$$

where P_0 is the initial "population", t is the number of periods of time which have elapsed, and r is the rate of growth with respect to these time units.

Example 1.5.6. The initial bacteria count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the rate of growth is 40% per hour.

(a) Find a function that models the number of bacteria after t hours.

$P_0 = 500, r = 0.40$, and $P(t) = 500e^{0.4t}$.

(b) What is the estimated amount after 10 hours?

$$P(10) = 500e^{0.4\cdot 10} = 500e^4 \approx 500(54.5981) \approx 27,300.$$

So we can expect there to by 27,300 bacteria after 10 hours.

Theorem 1.5.7. *For any positive real number a except 1, $f(x) = a^x$ is a one-to-one function. In particular,*

$$\text{if } a^u = a^v, \text{ then } u = v.$$

Example 1.5.8. Solve the following exponential equations.

(a) $3^{x+1} = 81$.

By Theorem 1.5.7, if we can write both sides of the equation as a power of 3, then we may conclude that the exponents are the same. In other words, we can *cancel out* the bases on both sides of the equation. We note that $81 = 3^4$. Thus,

$$\begin{aligned} 3^{x+1} &= 81 \\ 3^{x+1} &= 3^4 \\ x+1 &= 4, \qquad \text{by Theorem 1.5.7} \\ x &= \boxed{3}. \end{aligned}$$

(b) $e^{-x^2} = (e^x)^2 \cdot \dfrac{1}{e^3}$.

Our strategy will be the same, we want to express both sides of the equation as a power of e.

$$\begin{aligned} e^{-x^2} &= (e^x)^2 \cdot \frac{1}{e^3} \\ e^{-x^2} &= (e^{2x}) \cdot e^{-3} \\ e^{-x^2} &= e^{2x-3} \\ -x^2 &= 2x-3 \\ 0 &= x^2+2x-3 = (x+3)(x-1) \\ x &= \boxed{1, -3.} \end{aligned}$$

[i]See §1.5 Exponential and Logarithmic Functions in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–2, simplify the exponential expressions.

1. $x^8(2x)^7$
2. $\dfrac{(6x^3)^4}{2x^9}$

For Exercises 3–9, find the formula for $y = f(x)$ by applying the given graph transformation to the basic function $y = e^x$.

3. shifting 5 units downward
♠ 4. shifting 4 units to the right
♠ 5. reflecting about the x-axis
6. reflecting about the y-axis
♠ 7. reflecting about the line $y = 6$
♠ 8. reflecting about the line $x = 3$
9. reflecting about the x-axis and the y-axis

For Exercises 10–13, find the domain of the function.

10. $f(x) = \sin(e^{-x})$
♠ 11. $g(x) = \sqrt{1 - 5^x}$
♠ 12. $h(x) = \dfrac{81 - e^{x^2}}{1 - e^{81-x^2}}$
13. $f(x) = \dfrac{4 + x}{e^{\cos x}}$

For Exercises 14–14, graph the function $y = f(x)$. Indicate which transformations were applied to basic function $y = a^x$ to obtain f.

14. $f(x) = -e^{x-1} + 2$

For Exercises 15–15, for the function F, find functions f and g such that $F(x) = (f \circ g)(x)$.

15. $F(x) = 1 - e^{x^2+1}$

♠ 16. Find the exponential function $f(x) = Ca^x$ such that $f(1) = 8$ and $f(3) = 32$.

♠ 17. Under ideal conditions a certain bacteria population is known to double every four hours. Suppose that there are initially 70 bacteria. What is the size of the population after 8 hours? What is the size of the population after t hours? What is the size of the population after 17 hours? Estimate the time for the population to reach 80,000.

♠ 18. A bacteria culture starts with 900 bacteria and doubles in size every half hour. How many bacteria are there after 2 hours? How many bacteria are there after t hours? How many bacteria are there after 40 minutes? Estimate the time for the population to reach 10,000.

♠ 19. If $f(x) = e^x$ and $h \neq 0$, then prove that $\dfrac{f(x+h) - f(x)}{h} = e^x\left(\dfrac{e^h - 1}{h}\right)$.

"My happiness grows in direct proportion to my acceptance, and in inverse proportion to my expectations."
– Michael J. Fox

Lecture Videos

One-to-One Functions

Inverse Functions

The Inverse Function Property

Computing Inverse Functions Algebraically

Finding Inverse Functions of Square Root Functions

Inverses of Linear Fractionals

An Introduction to Logarithms

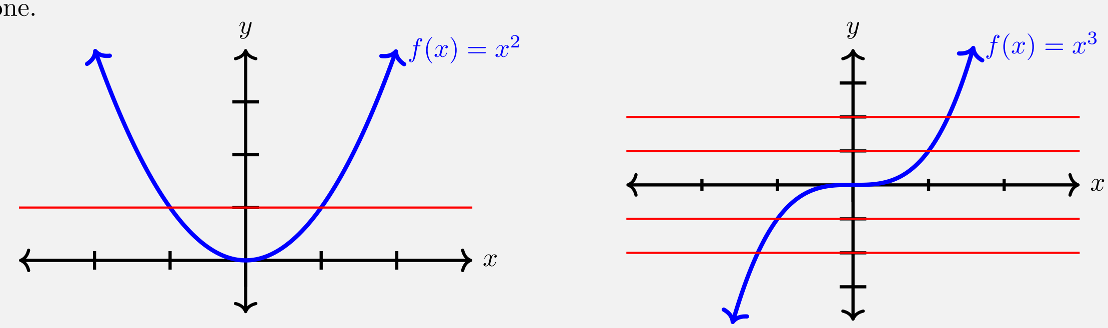
Logarithms ARE the Exponents

1.6 Inverse Functions

Definition 1.6.1. A function f is called **one-to-one** if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

Geometrically, a function is one-to-one if it passes the horizontal line test, that is, for every y-value, there is at most one corresponding x-value.

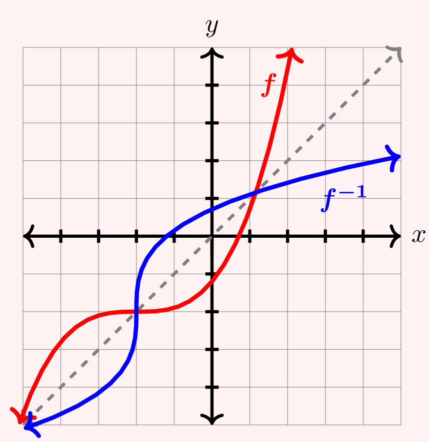
Example 1.6.2. By the Horizontal Line Test, $f(x) = x^2$ is not one-to-one but $g(x) = x^3$ is one-to-one.



Definition 1.6.3. If f is a one-to-one function, define f^{-1}, the **inverse function** of f, by the formula

$$f^{-1}(y) = x \iff f(x) = y.$$

In particular, f^{-1} switches all the ordered pairs (x, y) into (y, x). Geometrically, this corresponds to reflecting the graph of $y = f(x)$ across the diagonal line $y = x$.



An important property about inverse functions is how they relate with function composition.

$$(f^{-1} \circ f)(x) = x, \text{ for all } x \text{ in } \operatorname{dom} f, \text{ and } (f \circ f^{-1})(x) = x, \text{ for all } x \text{ in } \operatorname{ran} f. \tag{1.6.1}$$

Example 1.6.4. Use the function $f(x) = \sqrt[3]{x-2}$ and its inverse function $f^{-1}(x) = x^3 + 2$ to find the following:

(a) $f(10) = \sqrt[3]{(10) - 2} = \sqrt[3]{8} = \boxed{2}$

(b) $(f^{-1} \circ f)(10) = f^{-1}(f(10)) = f^{-1}(2) = (2)^3 + 2 = 8 + 2 = \boxed{10}$

(c) $(f^{-1} \circ f)(x) = f^{-1}(\sqrt[3]{x-2}) = (\sqrt[3]{x-2})^3 + 2 = (x-2) + 2 = \boxed{x}$

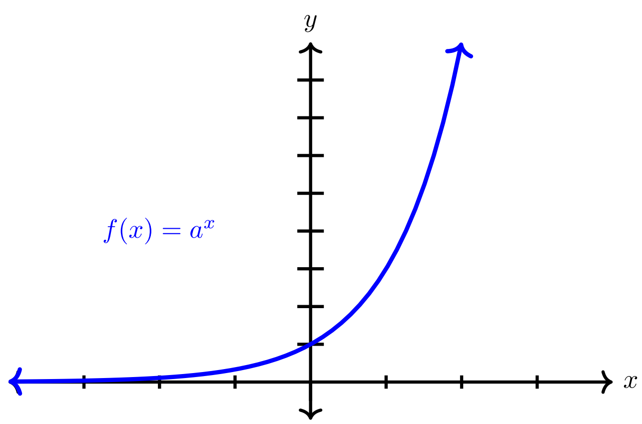
To find the algebraic inverse, we simply have to switch the roles of x and y in the defining formula for f and solve for y.

Example 1.6.5. Find the algebraic inverse of the function $f(x) = \dfrac{4x-1}{2x+3}$.

We first switch x and y and then solve for y.

$$\begin{aligned}
x &= \frac{4y-1}{2y+3} \\
x(2y+3) &= 4y-1 \\
2xy+3x &= 4y-1 \\
2xy-4y &= -1-3x \\
y(2x-4) &= -(3x+1) \\
f^{-1}(x) = y &= \boxed{\frac{-(3x+1)}{2(x-2)}}
\end{aligned}$$

For $a > 0$ and $a \neq 1$, we have



That is, $f(x) = a^x$ is one-to-one. So, f has an inverse function.

Definition 1.6.6. Let $a > 0$ and $a \neq 1$. Then the **Logarithm base** a is the inverse function of $f(x) = a^x$. That is,

$$\log_a(x) = y \iff a^y = x.$$

Theorem 1.6.7 (Properties of Logarthims)**.** *Let $a > 0$ and $a \neq 1$.*

(i) $\log_a 1 = 0$ *(ii)* $\log_a a = 1$ *(iii)* $\log_a a^x = x$ *(iv)* $a^{\log_a x} = x$

We should also mention that $\log_a 0$ is undefined.

Example 1.6.8. Evaluate the logarithms.

(a) $\log_{10} 1000 = \log_{10} 10^3 = \boxed{3}$

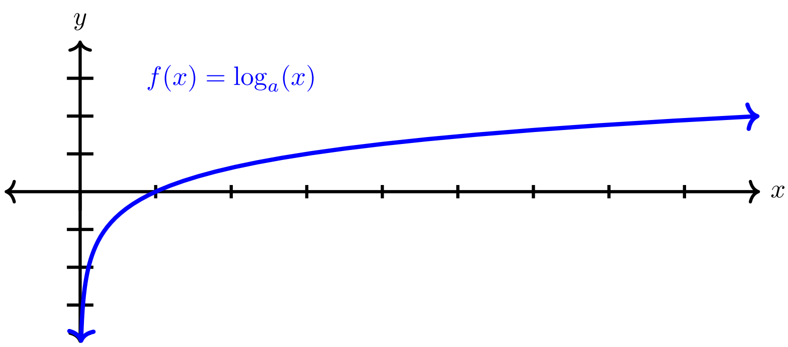
(b) $\log_2 32 = \log_2 2^5 = \boxed{5}$

(c) $\log_{10} 0.1 = \log_{10} 10^{-1} = -\boxed{1}$

(d) $\log_{16} 4 = \log_{16} 16^{1/2} = \boxed{1/2}$

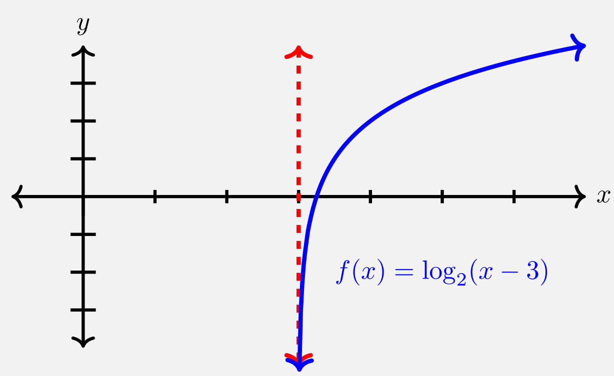
(e) $\log_{\sqrt{2}} 4 = \log_{\sqrt{2}} \sqrt{2}^4 = \boxed{4}$

Since a^x and $\log_a x$ are inverses of each other, we can discover the graph of $f(x) = \log_a x$ by reflecting the graph of $g(x) = a^x$ across the line $y = x$. Thus we have



Example 1.6.9. Sketch the graph of $f(x) = \log_2(x-3)$. Find its domain and range.

This is a horizontal shift right by 3 of $y = \log_2 x$. This is a horizontal change, so vertical attributes of $y = \log_2 x$ would not change as we transform into f. Thus, $\operatorname{dom} f = (3, \infty)$, $\operatorname{ran} f = (-\infty, \infty)$, and there is a vertical asymptote at $x = 3$.



As we saw earlier, the exponential function $y = e^x$ shows up in applications all the time. Its inverse function is just as useful in calculus and other applications.

Definition 1.6.10. The **Natural Logarithm** is defined to be

$$\ln x = \log_e x$$

Most calculators come equipped with a "LN" button,[i] which evaluates the natural logarithm.

[i]The abbreviation ln actually comes from the French "logarithme naturel," which means "natural logarithm."

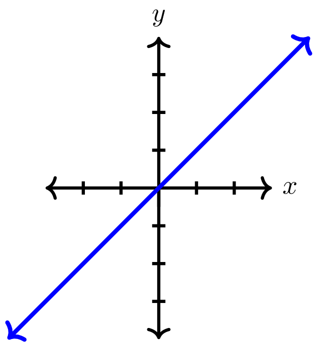
[ii]See §1.4 Inverse Functions and §1.5 Exponential and Logarithmic Functions in OpenStax to find the corresponding sections.

Exercises

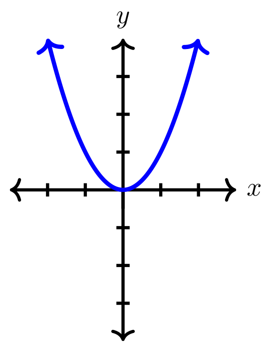
(Go to Solutions)

For Exercises 1–12, which graph correspond to one-to-one function?

1.



2.



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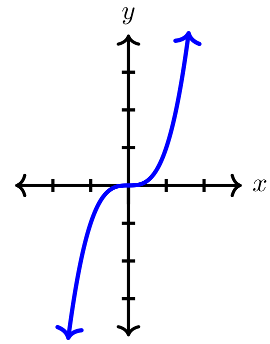

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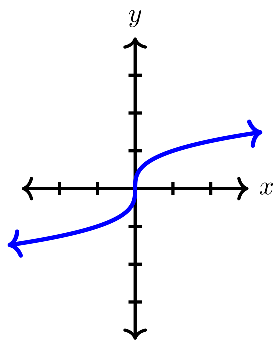

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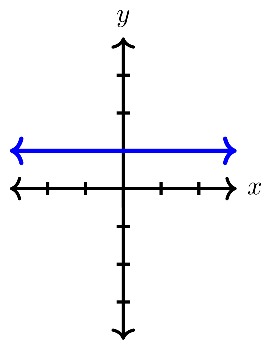

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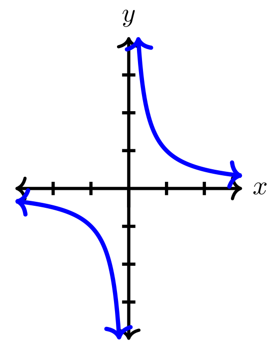
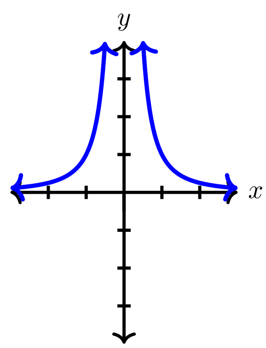
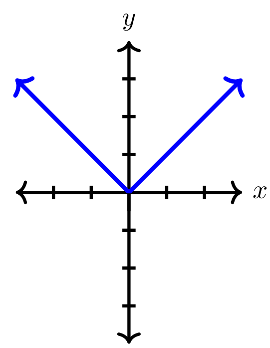


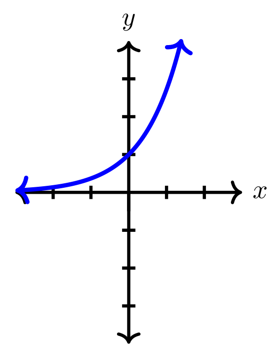
♠ 7.



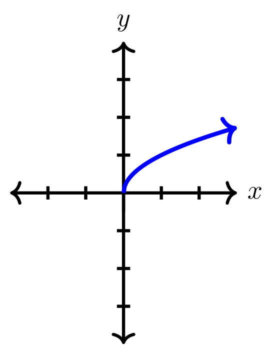
♠ 8.



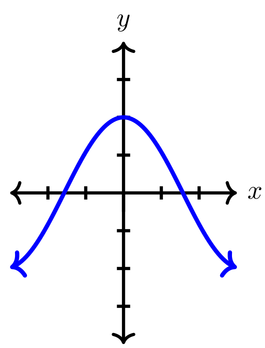
♠ 9.



10.



♠ 11.



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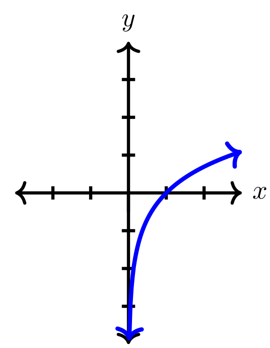


For Exercises 13–16, find the domain of the function.

13. $f(x) = \ln\left(\frac{(x+5)^2}{x^2-4}\right)$

♠ 14. $f(x) = \log_2\left(\frac{(x-1)(x+1)}{x+2}\right)$

15. $f(x) = \ln\left(\frac{\sqrt{2x+1}}{x^2-1}\right)$

16. $f(x) = \frac{1+3x}{1-\ln x}$

For Exercises 17–18, graph the function $y = f(x)$. Indicate which transformations were applied to basic function $y = \log_a(x)$ to obtain f.

17. $f(x) = \ln(-x+2)+2$

18. $f(x) = \ln(x+2)+2$

For Exercises 19–32, find $f^{-1}(x)$, the inverse of f. You may assume that f is one-to-one.

19. $f(x) = 3x^3 - 1$

♠ 20. $f(x) = 3\sqrt[3]{x} - 1$

21. $f(x) = 2\sqrt[3]{x} + 3$

♠ 22. $f(x) = 3 + \sqrt{5+6x}$

♠ 23. $f(x) = \frac{4x-1}{2x+5}$

24. $f(x) = \frac{-3x-4}{x-2}$

25. $f(x) = \frac{3x}{x-2}$

♠ 26. $f(x) = e^{5x-7}$

♠ 27. $f(x) = \ln(x+4)$

♠ 28. $f(x) = -\frac{2x^3}{x^3-1}$

29. $f(x) = \frac{-3x^3-4}{x^3-2}$

30. $f(x) = -\frac{2\sqrt[3]{x}}{\sqrt[3]{x}-1}$

♠ 31. $f(x) = \frac{4\sqrt[3]{x}-2}{3\sqrt[3]{x}+1}$

32. $f(x) = x^4 + 5, x \geq 0$

"Nothing says holidays, like a cheese log." – Ellen DeGeneres

Lecture Videos

Graphs of Logarithms

Laws of Logarithms

The Change of Base Formula

Solving Logarithmic Equations

The Inverse Trigonometric Functions

Computing Inverse Trigonometric Functions

Inverse Trigonometric Expressions and Triangle Diagrams

1.7 Inverse Functions II

We continue to review the notion of inverse functions, continuing the topic of logarithms and considering inverse trigonometric functions.

Theorem 1.7.1 (Laws of Logarithms)**.** *Let $a > 0$ and $a \neq 1$. Let $A, B > 0$ and let $C \in \mathbb{R}$.*

(a) $\log_a(AB) = \log_a A + \log_a B$

(b) $\log_a(A/B) = \log_a A - \log_a B$

(c) $\log_a\left(A^C\right) = C \log_a A$

Example 1.7.2. Us the Laws of Logarithms to expand the given logarithm as much as possible.

(a)
$$\begin{aligned} \log_a\left(x\sqrt{x^2+1}\right) &= \log_a x + \log_a \sqrt{x^2+1} = \log_a x + \log_a (x^2+1)^{1/2} \\ &= \log_a x + \frac{1}{2}\log_a(x^2+1). \end{aligned}$$

(b)
$$\begin{aligned} \ln\left(\frac{x^2}{(x-1)^3}\right) &= \ln(x^2) - \ln((x-1)^3) \\ &= 2\ln x - 3\ln(x-1). \end{aligned}$$

Example 1.7.3. Use the Laws of Logarithms to combine expressions into a single logarithm.

(a)
$$\begin{aligned} 3\log x + \frac{1}{2}\log(x+1) &= \log x^3 + \log(x+1)^{1/2} \\ &= \log\left(x^3\sqrt{x+1}\right) \end{aligned}$$

(b)
$$\begin{aligned} 3\ln s + \frac{1}{2}\ln t - 4\ln\left(t^2+1\right) &= \ln s^3 + \ln\sqrt{t} - \ln\left(\left(t^2+1\right)^4\right) \\ &= \ln\left(s^3\sqrt{t}\right) - \ln\left(t^2+1\right)^4 \\ &= \ln\left(\frac{s^3\sqrt{t}}{(t^2+1)^4}\right). \end{aligned}$$

Theorem 1.7.4 (Change of Base)**.** *For any $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$, we have*

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Notice by the change of base formula, every logarithmic function of any base can be viewed as a vertical stretch/shrink of the natural logarithm. Thus, the idea of different bases is really imaginary.

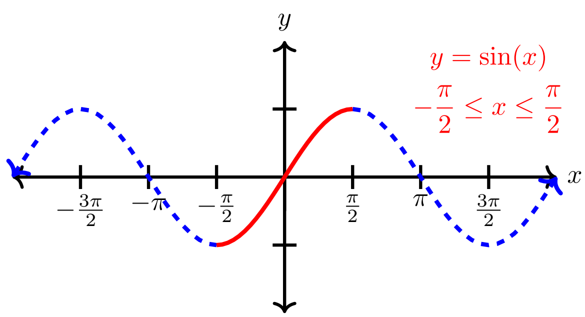
Example 1.7.5. Approximate $\log_5 89$ to four decimal places.

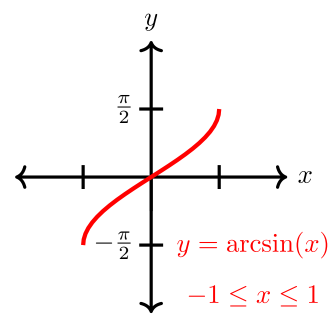
Since most calculators come only equipped with a log base 10 or natural log function we will use the Change of Base formula to approximate the value. Note that

$$\log_5 89 = \frac{\ln 89}{\ln 5}.$$

Now, $\ln 89 \approx 4.48864$ and $\ln 5 \approx 1.60944$. Thus, $\log_5 89 \approx \boxed{2.7889}$.

Coming up with a inverse function for trigonometric functions is not as easy, since not a single trigonometric function is one-to-one. Thus, there is no inverse trigonometric functions, technically. On the other hand, if we restrict the domains, we can produce inverse functions. For example, on the interval $[-\pi/2, \pi/2]$, sine is, in fact, one-to-one.



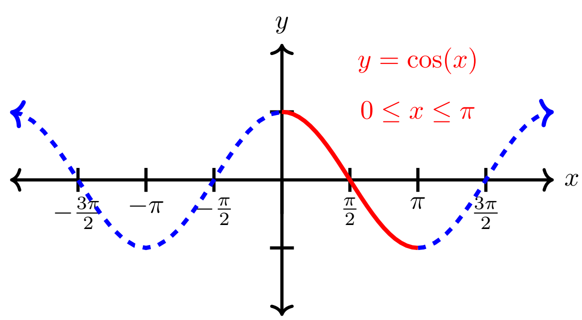


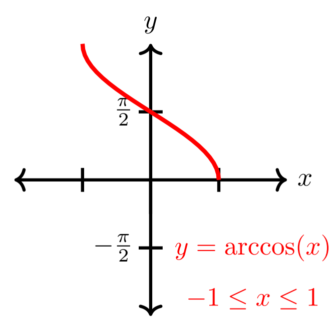
Thus,

$$\sin^{-1}(x) = y \text{ and } -1 \leq x \leq 1 \iff \sin y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}. \tag{1.7.1}$$

Similarly, we let

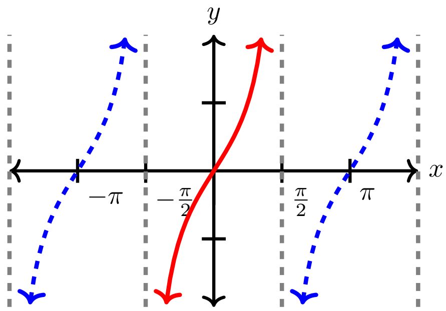
$$\cos^{-1}(x) = y \text{ and } -1 \leq x \leq 1 \iff \cos y = x \text{ and } 0 \leq y \leq \pi. \tag{1.7.2}$$

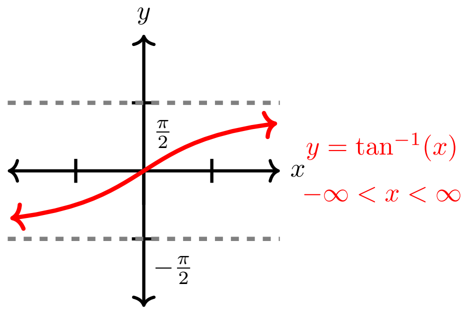




Similarly, we can define inverse tangent. Since the range of tangent is all real numbers, the domain of inverse tangent is also all real numbers. Therefore,

$$\tan^{-1}(x) = y \iff \tan y = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$





Similar definitions of $\sec^{-1}$, $\csc^{-1}$ and $\cot^{-1}$ hold.

Example 1.7.6. Evaluate:

(a) $\sin^{-1}(1/2) = \boxed{\frac{\pi}{6}}$ since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

(b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$ since $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

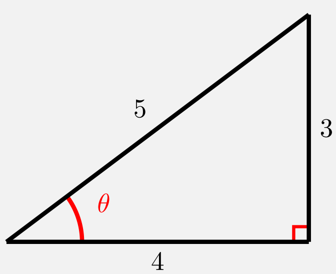
(c) $\tan^{-1}(-1) = \boxed{-\frac{\pi}{4}}$ since $\tan\left(-\frac{\pi}{4}\right) = -1$.

Example 1.7.7. Evaluate:

(a) $\sin\left(\sin^{-1}\frac{1}{5}\right) = \boxed{\frac{1}{5}}$

(b) $\sin^{-1}\left(\sin\frac{3\pi}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$ since $\sin^{-1}$ always returns an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

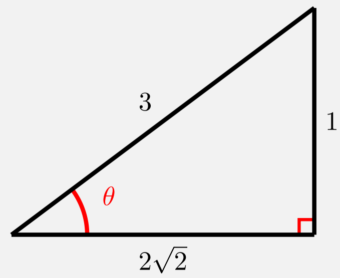
Example 1.7.8. Evaluate $\sin(\tan^{-1}(3/4))$.



Let $\theta = \tan^{-1}(3/4)$. Since $\tan\theta = \frac{\text{opp}}{\text{adj}}$ and $\tan\theta = \tan(\tan^{-1}(3/4)) = 3/4$, we set opp $= 3$ and adj $= 4$. Thus, we can construct a right triangle by letting hyp $= \sqrt{\text{opp}^2 + \text{adj}^2} = \sqrt{9+16} = \sqrt{25} = 5$. Since $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, we conclude that

$$\sin(\tan^{-1}(3/4)) = \sin\theta = \boxed{\frac{3}{5}}.$$

Example 1.7.9. Evaluate $\tan(\sin^{-1}(1/3))$.



Let $\theta = \sin^{-1}(1/3)$. Since $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ and $\sin\theta = \sin(\sin^{-1}(1/3)) = 1/3$, we set opp $= 1$ and hyp $= 3$. Thus, we can construct a right triangle by letting adj $= \sqrt{\text{hyp}^2 - \text{opp}^2} = \sqrt{9-1} = \sqrt{8} = 2\sqrt{2}$. Since $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$, we conclude that

$$\tan(\sin^{-1}(1/3)) = \tan\theta = \boxed{\frac{1}{2\sqrt{2}}}.$$

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[i]See §1.4 Inverse Functions and §1.5 Exponential and Logarithmic Functions in OpenStax to find the corresponding sections.

Exercises

(Go to Solutions)

For Exercises 1–8, simplify the exponential/logarithmic expression without a calculator.

1. $\log_5(125)$
2. $\log_2\left(\frac{1}{8}\right)$
♠ 3. $\ln\left(\frac{1}{e}\right)$
♠ 4. $e^{-2\ln(10)}$
♠ 5. $\ln\left(\ln\left(e^{e^5}\right)\right)$
♠ 6. $\log_2(6) - \log_2(15) + \log_2(20)$
7. $\log_3(90) - \log_3(27) - \log_3(30)$
8. $\log_{15}(\sqrt{15})$

For Exercises 9–17, condense or expand the logarithmic expression.

9. $\log_5\left(\frac{\sqrt[3]{x^2+1}}{x^2-1}\right)$
♠ 10. $\log\left(\frac{x^3\sqrt{x+1}}{(x-2)^2}\right)$
♠ 11. $\ln\left(\frac{x^2\sqrt[3]{x+1}}{(x+1)^2}\right)$
12. $\ln(x\sqrt{1+x^2})$
♠ 13. $\log_2\left(\frac{1}{x}\right) + \log_2\left(\frac{1}{x^2}\right)$
♠ 14. $\log(x^2-3x+2) - 2\log(x+1)$
15. $2\log_3 x - \log_3 y$
16. $\frac{1}{3}\log(x^3+1) - \frac{1}{2}\log(x^2+4)$
17. $\frac{1}{3}\ln\left((x+2)^3\right) + \frac{1}{2}\left(\ln x - \ln\left((x^2+3x+2)^2\right)\right)$

For Exercises 18–29, solve the exponential/logarithmic equation.

18. $e^{7-4x} = 9$
♠ 19. $\log_6(x+9) + \log_6(x) = 2$
♠ 20. $\ln(x+1) - \ln x = 2$
21. $\log_4(x^2-9) - \log_4(x+3) = 3$
22. $\log x + \log(x-3) = 1$
23. $\log_2(x) + \log_2(x-3) = 2$
24. $\log_5(x+1) - \log_5(x-1) = 2$
25. $\log_3(x+15) - \log_3(x-1) = 2$
26. $\log_9(x-5) + \log_9(x+3) = 1$
27. $\log_4(x^2-9) - \log_4(x+3) = 3$
28. $\log_2(x+1) + \log_2(x-1) = 3$
29. $\ln(3x-12) = 8$

For Exercises 30–32, simplify the inverse trigonometric expression without a calculator.

30. $\cos(\tan^{-1}(\sqrt{3}))$
31. $\tan(\sec^{-1}(5))$
32. $\sin\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$

For Exercises 33–48, rewrite the trigonometric expression as an equivalent algebraic expression.

33. $\tan(\sin^{-1}(x))$
♠ 34. $\tan(\cos^{-1}(x))$
♠ 35. $\sin(\cos^{-1}(x))$
36. $\csc(\cos^{-1}(x))$
37. $\cot(\cos^{-1}(x))$
♠ 38. $\sin(\tan^{-1}(x))$
39. $\cos(\tan^{-1}(x))$
40. $\sec(\tan^{-1}(x))$
41. $\csc(\tan^{-1}(x))$
42. $\cos(\sin^{-1}(x))$
♠ 43. $\sec(\sin^{-1}(x))$
44. $\cot(\sin^{-1}(x))$
45. $\sin(\sec^{-1}(x))$
46. $\tan(\sec^{-1}(x))$
47. $\csc(\sec^{-1}(x))$
48. $\cot(\sec^{-1}(x))$

Chapter 2

Limits

"To rise from error to truth is rare and beautiful." – *Victor Hugo*

Lecture Videos

Error and Allowance

An Example of Computing Delta for a Function Given an Epsilon

The Precise Definition of the Limit

2.1 Error and Tolerance

Example 2.1.1. A crystal growth furnace[i] is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose that relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.

(a) How much power is needed to maintain the temperature at 200° C?

We wish to solve the equation

$$T(w) = 0.1w^2 + 2.155w + 20 = 200 \Longleftrightarrow 0.1w^2 + 2.155w - 180 = 0,$$

which can be achieved by the quadratic formula. Note that

$$\begin{aligned} w &= \frac{-2.155 + \sqrt{2.155^2 - 4 \cdot 0.1 \cdot (-180)}}{2 \cdot 0.1} = \frac{-2.155 + \sqrt{4.64403 + 72}}{0.2} \\ &= \frac{-2.155 + 8.75466}{0.2} = \frac{6.59966}{0.2} \\ &= 32.9983 \approx 33W. \end{aligned}$$

(b) If the temperature is allowed to vary from 200° C by up to $\pm 1^\circ$ C, what range of wattage is allowed for the input power?

If the temperature is allowed to fluctuate by a single degree, the acceptable interval of temperature is 199° to 201°. By performing similar computations as above, we compute values of w such that $T(w) = 199$ and $T(w) = 201$, that is, $w \approx 32.8839$ and $w \approx 33.1124$, respectively. In other words, if the power wattage of the furnace is between $32.8839\ W$ and $33.1124\ W$, then the temperature of the furnace will be between $199^\circ C$ and $201^\circ C$.

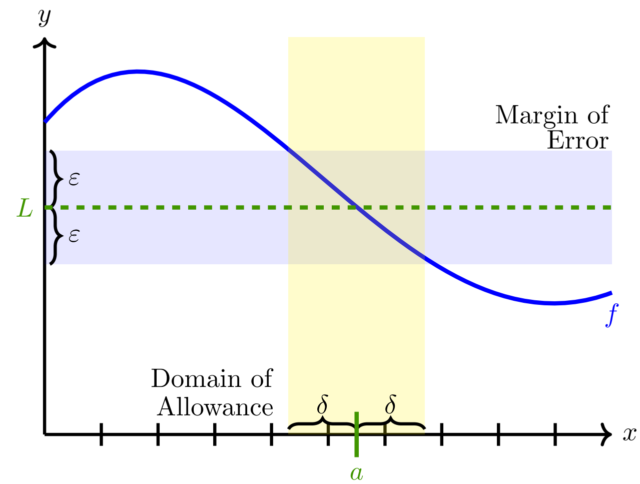
The previous example is a good example of the concept of *error*. Although, $200^\circ C$ is the optimal temperature to grow crystals, it is very unreasonable to expect the temperature to be perfectly set at 200° at all times during the growth, since fluctuations in environmental temperature and power issues from the power supply could cause marginal, maybe even microscopic, defects in maintaining the temperature. So instead, we should be concerned with how close to 200° is close enough, that is, how much *error* is considered allowable.[ii]

When working with functions, we can control the range of outputs by controlling the domain of the input. Continuing with the example from above, if we choose any $32.8839 \leq w \leq 33.1124$ then $199 \leq T(w) \leq 201$.

We will let the Greek letter ε denote our numerical *error tolerance*, that is, the marginal amount our actual output may acceptably differ from the desired output. In general, ε will be small in comparison to the desired output. From the example above, $\varepsilon = 1$. From context, it is nonsense to allow $\varepsilon < 0$ and is impractical to require $\varepsilon = 0$. Thus, we will always assume that $\varepsilon > 0$, although a specific positive value may be specified. Let L denote the desired output of a function, then the interval $(L - \varepsilon,\ L + \varepsilon)$ defines the **margin of error.**

Now that a margin of error is given, it is desirable to have an interval of inputs which will guarantee that the outputs land within the margin of error. This interval is called the **domain of allowance**, denoted $(a - \delta,\ a + \delta)$. For example, if f is a function with desired output L and $f(a) = L$, how far off from a can we be to guarantee that $L - \varepsilon < f(x) < L + \varepsilon$, that is, $|f(x) - L| < \varepsilon$. We will actually search for an interval centered around a, that is, we want to find a real number δ, such that if $a - \delta < x < a + \delta$ then $L - \varepsilon < f(x) < L + \varepsilon$, that is,

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$



Example 2.1.2. In the previous example, we saw that if $32.8839 < w < 33.1124$ then $199 < T(w) < 201$, where $\varepsilon = 1$ and $a = 32.9983$. Notice that $|a - 32.8839| = 0.1144$ and $|a - 33.1124| = 0.1141$. Therefore, if we let $\boxed{\delta = 0.1141}$, then

$$32.8842 < w < 33.1124 \Longrightarrow 199 < T(w) < 201.$$

In other words, $\boxed{(32.8842, 33.1124)}$ is our domain of allowance.

Example 2.1.3. Use a graph to find a number δ such that

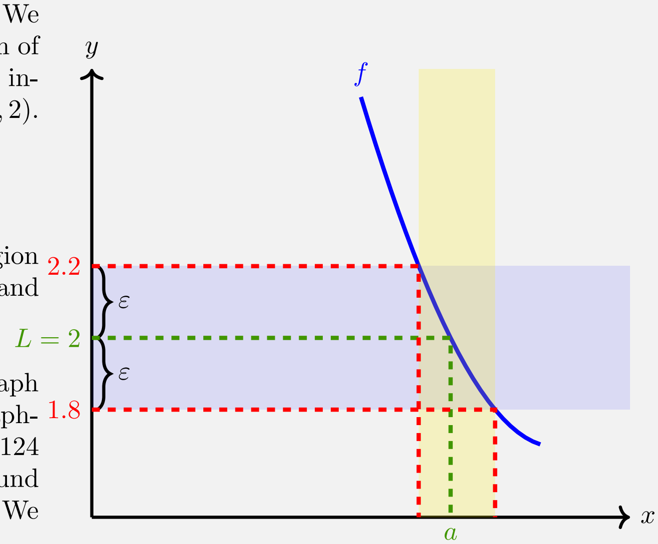
$$|x - 1| < \delta \Longrightarrow |(x^3 - 5x + 6) - 2| < 0.2.$$

Using the same notation from above, we have that $\varepsilon = 0.2$, $f(x) = x^3 - 5x + 6$, $a = 1$, and $L = 2$. We also notice that the point $(1, 2)$ is on the graph of f, that is, $f(1) = 2$ $(f(a) = L)$. We are thus interested in the neighborhood near the point $(1, 2)$. The inequality $|f(x) - 2| < 0.2$ implies that

$$1.8 < f(x) < 2.2.$$

Thus we focus our attention on the region bounded by the horizontal lines $y = 1.8$ and $y = 2.2$.

We next need to determine when does the graph of f intersect these two horizontal lines. A graphing utility shows that $f(x) = 1.8$ when $x \approx 1.124$ and $f(x) = 2.2$ when $x \approx 0.911$. Let us round these answers to eliminate calculator error. We then have that



$$0.92 < x < 1.12 \Longrightarrow 1.8 < f(x) < 2.2.$$

When rounding the previous results, make sure ALWAYS to round toward a, otherwise the implication may be false. Therefore, the interval $(0.92, 1.12)$ maps into the margin of error. To construct the domain of allowance and hence find an appropriate δ, we will shrink the side farthest from a. Since $|a - 0.92| = 0.08$ and $|a - 1.12| = 0.12$, we choose $\boxed{\delta = 0.08}$ and our domain of allowance is $\boxed{(0.92, 1.08)}$.

Now it should be mentioned that our choice of δ is not unique. For example, choosing a smaller δ will also place the output within the acceptable margin of error. Thus, a smaller δ generally guarantees a more precise output, but on the other hand, a smaller δ may be more difficult to implement. Referring to the first example, theoretically $\delta = 10^{-100}$ is an acceptable allowance for δ but may be difficult to monitor with such precision as the current instruments might not be able to measure the power so finely.

In the study of calculus, this idea of error is closely related to the concept of a *limit*, as explained in the next definition.

Definition 2.1.4. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon,$$

or in other words, if $x \neq a$ and $a - \delta < x < a + \delta$ then

$$L - \varepsilon < f(x) < L + \epsilon.$$

Notice that $|x - a|$ measures the distance between x and a on the x-axis and $|f(x) - L|$ measures the distance between $f(x)$ and L on the y-axis. This $\varepsilon - \delta$ definition of the limit is telling us that no matter how small a margin of error is chosen, there is a small domain of allowance of x-values which map to y-values near L.

Example 2.1.5. Prove that $\lim_{x\to 3}(4x - 5) = 7$.

Since ε can be any positive real number, we don't have a lot of control on what ε can be. So, we must come up with a formula for δ with respect to ε. In fact, we want that

$$0 < |x - 3| < \delta \Longrightarrow |(4x - 5) - 7| < \varepsilon.$$

Now, $|(4x - 5) - 7| = |4x - 12| = 4|x - 3|$. Therefore, $4|x - 3| < \varepsilon$ if and only if $|x - 3| < \varepsilon/4$. Thus, we will let $\delta = \varepsilon/4$. Now this isn't technically a proof yet. What follows is:

Proof. Let $\varepsilon > 0$. Then choose $\delta = \varepsilon/4$. If $0 < |x - 3| < \delta$ then $|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < 4(\varepsilon/4) = \varepsilon$. Thus, $0 < |x - 3| < \delta \Longrightarrow |(4x - 5) - 7| < \varepsilon$. Therefore, $\lim_{x\to 3}(4x - 5) = 7$. $\square$

[i] This example, and the whole section frankly, is based upon and inspired by Exercises 2.4.11 and .12 from Stewart Calculus: Early Transcendentals 8th Edition.

[ii] As another example of error, students in general desire to pass their calculus course. This of course means that they need to get good grades. A passing grade is any grade above 73%. For some students, passing simply is not enough. Instead, these students may desire to ace calculus, which means their final grade needs to be at least 93%. Therefore, an A student has a much smaller margin of error than a mere passing student.

[iii] See §2.5 The Precise Definition of a Limit in OpenStax to find the corresponding section.

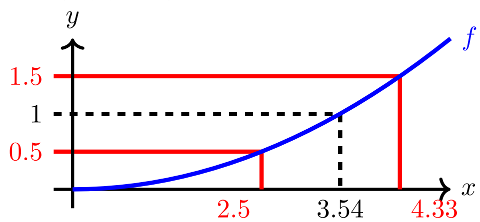
Exercises

(Go to Solutions)

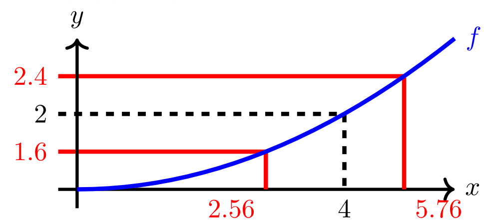
For Exercises 1–12, find the largest number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

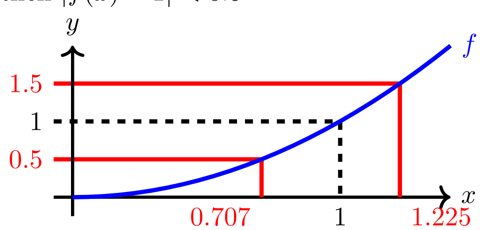
1. if $0 < |x - 3.54| < \delta$
then $|f(x) - 1| < 0.5$



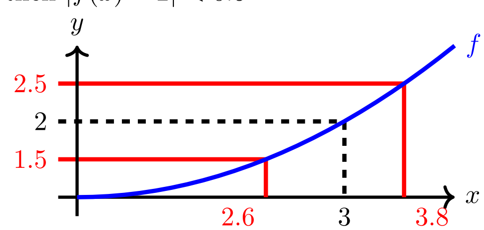
2. if $0 < |x - 4| < \delta$
then $|f(x) - 2| < 0.4$



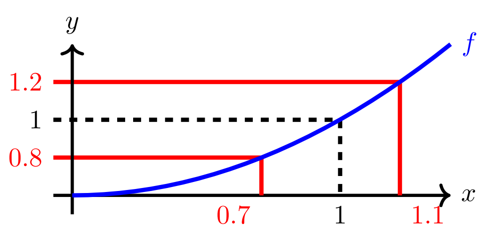
♠ 3. if $0 < |x - 1| < \delta$
then $|f(x) - 1| < 0.5$



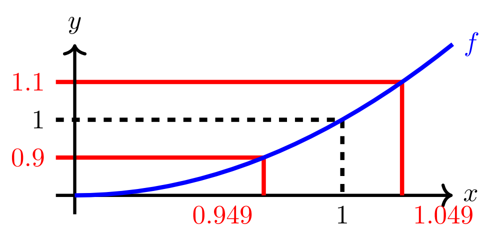
♠ 4. if $0 < |x - 3| < \delta$
then $|f(x) - 2| < 0.5$



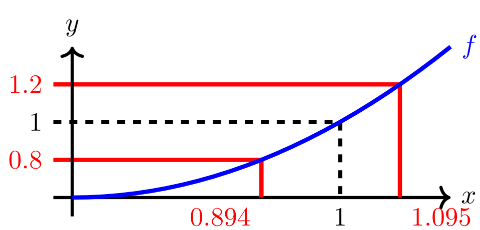
5. if $0 < |x - 1| < \delta$
then $|f(x) - 1| < 0.2$



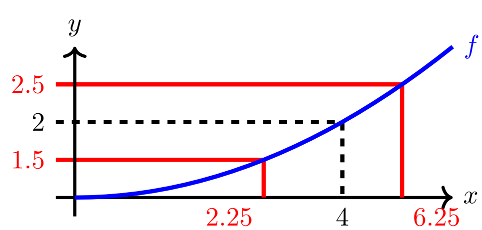
6. if $0 < |x - 1| < \delta$
then $|f(x) - 1| < 0.1$



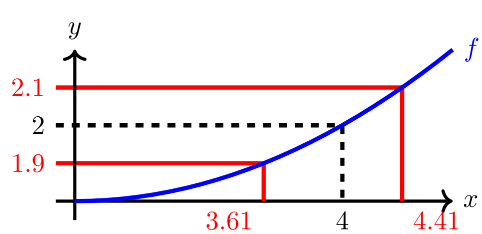
7. if $0 < |x - 1| < \delta$
then $|f(x) - 1| < 0.2$



8. if $0 < |x - 4| < \delta$
then $|f(x) - 2| < 0.5$



9. if $0 < |x - 4| < \delta$
then $|f(x) - 2| < 0.1$



♠ 10. if $0 < |x-2| < \delta$ then $|(2x+1)-5| < 1/10$

♠ 11. if $0 < |x+2| < \delta$ then $|(4-5x)-14| < 1/4$

12. if $0 < |x+1| < \delta$ then $|(3-4x)-7| < 1/4$

♠ 13. A crystal growth furnace[iii] is used in research to determine how best to manufacture crystals used in electric components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts. How much power is needed to maintain the temperature at 199°C? If the temperature is allowed to vary from 199°C by up to ±1°C, what range of wattage is allowed for the input power? Given then $\varepsilon = 1$°C, find the largest number δ such that

$$\text{if} \quad 0 < |w-a| < \delta \quad \text{then} \quad |T(W) - 199| < \varepsilon,$$

where $w = a$ is the solution to the first question.

The following Desmos calculator may prove helpful

♠ 14. A machinist is required to manufacture a circular metal disk[iv] with area 1700 cm^2. What radius produces such a disk? If the machinist is allowed an error tolerance of ±10 cm^2 in the area of the disk, how close to the ideal radius must the machinist control the radius? Given then $\varepsilon = 10$ cm^2, find the largest number δ such that

$$\text{if} \quad 0 < |r-a| < \delta \quad \text{then} \quad |A(r) - 1700| < \varepsilon,$$

where $r = a$ is the solution to the first question and $A(r) = \pi r^2$.

The following Desmos calculator may prove helpful

[iv]Exercise 2.4.12 from Stewart Calculus: Early Transcendentals 8th Edition
[v]Exercise 2.4.11 from Stewart Calculus: Early Transcendentals 8th Edition

"You can't put a limit on anything. The more you dream, the farther you get." – Michael Phelps

Lecture Videos

The Intuitive Definition of a Limit

Computing Limits from the Graph of a Function

Why Do We Need a Precise Definition of a Limit?

2.2 Limits of a Function

Example 2.2.1. We can find the value of of the function defined by

$$f(x) = \frac{x^2 - 4}{x - 2}$$

when $x = 1$ by substitution:

$$f(1) = \frac{1^2 - 4}{1 - 2} = \frac{-3}{-1} = 3.$$

It is also true that when x is a number *very close* to 1 then $f(x)$ is a number *very close* to 3, that is, if we accept a small amount of error above or below $y = 3$ then there is a small amount of allowance to the left and right of $x = 1$ which keeps $y = f(x)$ inside that margin of error.

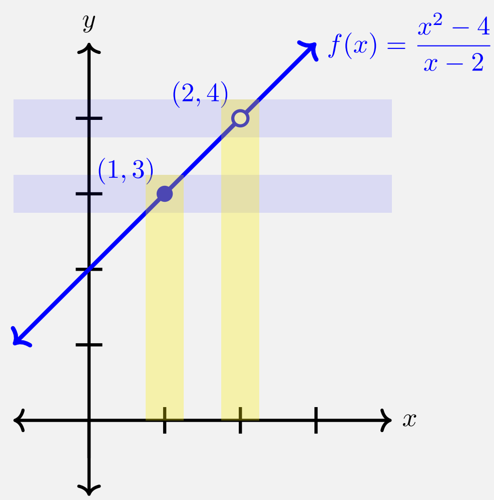
At $x = 2$, the situation is different: $f(2)$ is not defined because the denominator is 0 when $x = 2$. But we can still ask, what happens to $f(x)$ when x is *very close* to (but not equal to) 2? By approximation, we see that $f(x)$ is very close to 4, when x is very close to 2. Put another way, given any amount of error around $y = 4$, there is an allowance around $x = 2$ such that any x-value inside this domain of allowance (except $x = 2$) will produce by f a y-value within the margin of error. As stated before, we say "the limit of $f(x)$ as x approaches 2 is 4," which is denoted as

$$\boxed{\lim_{x \to 2} f(x) = 4}.$$

Our first example showed that

$$\boxed{\lim_{x \to 1} f(x) = 3},$$

because $f(x)$ got closer and closer to 3 as x got closer and closer to 1.[i]



Definition 2.2.2. The phrase "x approaches a from the left" is written $x \to a^-$. Similarly, the phrase "x approaches a from the right" is written $x \to a^+$.
Let $f(x)$ be a function. Then the **limit from the left**, or **left-handed limit**, of $f(x)$ is the value $f(x)$ approaches as $x \to a^-$, which is denoted $\lim\limits_{x \to a^-} f(x)$.
Similarly, the **limit from the right**, or **right-handed limit** of $f(x)$ is the value $f(x)$ approaches

as $x \to a^+$, which is denoted $\lim_{x\to a^+} f(x)$.
The phrase "x approaches a" is written $x \to a$ and means x approaches a from the left or right. Then the **limit** of $f(x)$ is the value $f(x)$ approaches as $x \to a$, which is denoted $\lim_{x\to a} f(x)$.

First note that the limit of a function might not exists at a certain x. We will see an example shortly. Likewise, the left- (right-) handed limit might not exist for some functions, also. On the other hand, if the left-handed and right-handed limits exist, then the limit exists if and only if the left-handed and right-handed limits are the same value.

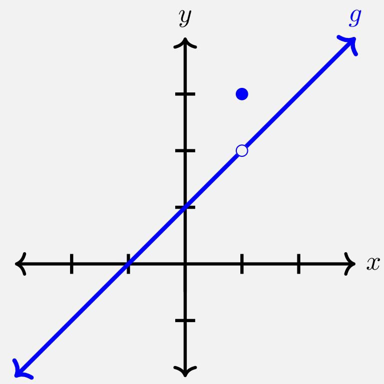
Example 2.2.3. The graph of g is depicted below. Compute the following limits:

(a) $\lim_{x\to 0} g(x)$.

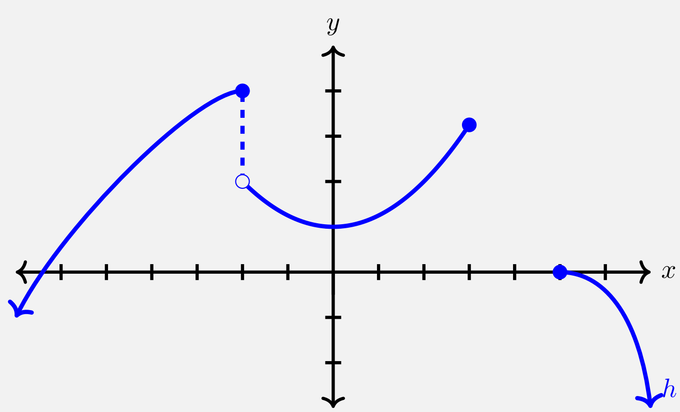
Note that $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = \lim_{x\to 0} g(x) = \boxed{1}$, by inspection of the graph. Likewise, $g(x) = 0$.

(b) $\lim_{x\to 1} g(x)$.

Like the previous part, $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) = \lim_{x\to 1} g(x) = \boxed{2}$, by inspection of the graph. On the other hand, $g(x) = 3$.



Example 2.2.4. The graph of h is depicted below. Compute the following limits:



(a) $\lim_{x\to 0} h(x)$.

Note that $\lim_{x\to 0^-} h(x) = \lim_{x\to 0^+} h(x) = \lim_{x\to 0} h(x) = \boxed{1}$. Likewise, $h(x) = 1$.

(b) $\lim_{x\to -2} h(x)$.

Note that $\lim_{x\to -2^-} h(x) = 4$ and $\lim_{x\to -2^+} h(x) = 2$. Thus, $\lim_{x\to -2} h(x)$ $\boxed{\text{does not exist}}$. On the other hand, $h(x) = 4$.

(c) $\lim_{x\to 3} h(x)$.

We can see easy enough that $\lim_{x\to 3^-} h(x) = 3$. But what about $\lim_{x\to 3^+} h(x)$. Note that the domain of h is $\operatorname{dom} h = (-\infty, 3] \cup [5, \infty)$. Thus, it is not possible approach x from the right on the graph of h. In this case, $\lim_{x\to 3} h(x) = \lim_{x\to 3^-} h(x) = \boxed{3}$.

(d) $\lim_{x\to 5} h(x)$.

Like the previous example, since the left-handed limit is not possible, $\lim_{x\to 5} h(x) = \lim_{x\to 5^+} h(x) = \boxed{0}$.

Like in the above example, whenever $x = a$ represents a number on the boundary of the domain of f, we say that $\lim_{x\to a} f(x) = \lim_{x\to a^-} f(x)$ if a is a left-endpoint and $\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x)$ if a is a right endpoint. For example, $x = 3$ is a left-endpoint of the domain of h from the above example, and $x = 5$ is a right-endpoint of its domain.

Example 2.2.5. Find $\lim_{x\to 0} \dfrac{|x|}{x}$.

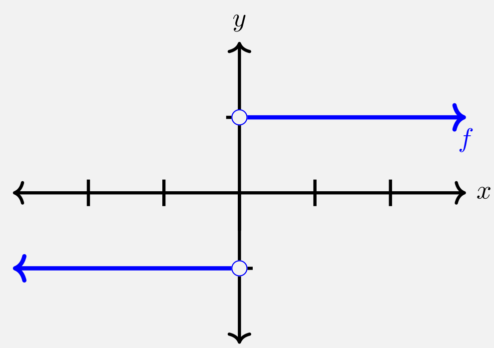
Let $f(x) = \dfrac{|x|}{x}$. We can rewrite $f(x)$ as a piecewise function:

$$f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Then

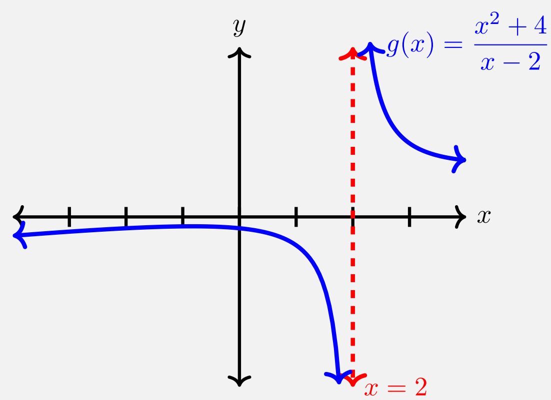
$$\lim_{x\to 0^-} f(x) = -1 \neq 1 = \lim_{x\to 0^+} f(x).$$

Therefore, $\lim_{x\to 0} f(x)$ $\boxed{\text{does not exist}}$.



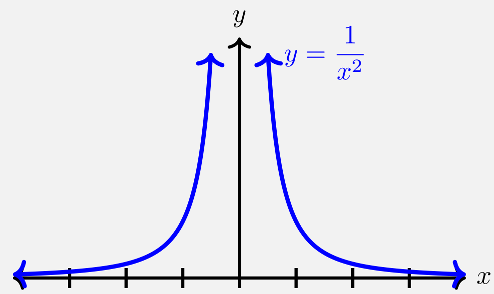
Example 2.2.6. Find $\lim_{x\to 2} g(x)$, where $g(x) = \dfrac{x^2+4}{x-2}$.

We recognize that $g(x)$ has a vertical asymptote at $x = 2$. This recognition can be preformed by using a graphing calculator or remembering properties of rational functions probably learned in a college algebra course. By any method, one recognizes that $\lim_{x\to 2^-} g(x) = -\infty$ and $\lim_{x\to 2^+} g(x) = \infty$. Therefore, $\lim_{x\to 2} g(x)$ $\boxed{\text{does not exist}}$.



Example 2.2.7. Find $\lim_{x\to 0} \frac{1}{x^2}$.

From inspection, one can observe that $1/x^2$ becomes increasingly large as x becomes small. Also, $1/x^2$ is always positive independent whether x is positive or negative. From this, we may determine that $\boxed{\lim_{x\to 0} \frac{1}{x^2} = \infty}$.

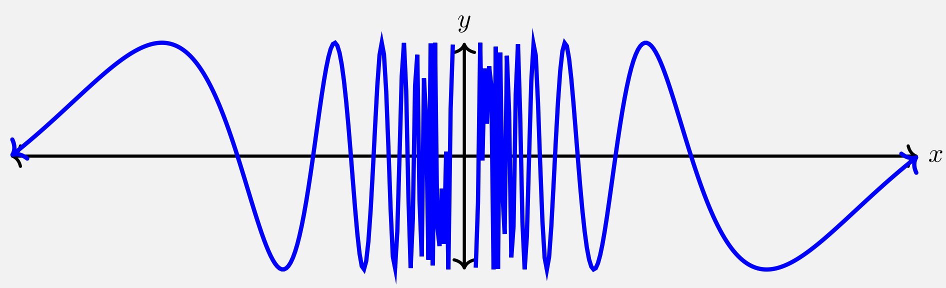


As observed in the previous examples, a vertical asymptote exists whenever a left-hand or right-hand limit is $\pm\infty$.

Example 2.2.8. Investigate $\lim_{x\to 0} \sin \frac{\pi}{x}$. We first notice that $\sin(\pi/x)$ is undefined at $x = 0$. One can check numerically that

$$\begin{aligned} f(1) &= \sin\pi = 0 & f(1/3) &= \sin 3\pi = 0 & f(0.1) &= \sin 10\pi = 0 & f(0.001) &= 0 \\ f(1/2) &= \sin 2\pi = 0 & f(1/4) &= \sin 4\pi = 0 & f(0.01) &= \sin 100\pi = 0 & f(0.0001) &= 0 \end{aligned}$$

Although, we may be tempted to guess that the limit is 0, the graph of the function shows that the limit in fact doesn't exist. For reason like this, one should always be cautious about computing limits via numerical means alone.



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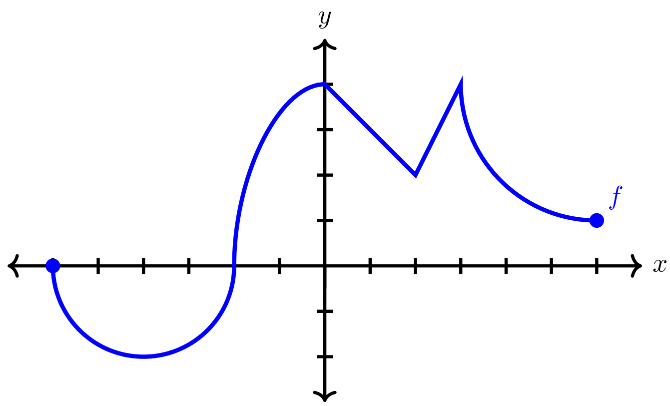
[i]Many students ask the question, "Why can't we divide by zero?" Suppose that we lived in a society which accepted division by zero. What would that world be like? Well, certainly, $0 = 0$. There is no dispute going on there. But also, $1 \cdot 0 = 0 \cdot 0$. Since we can divide by zero, we have $\frac{1\cdot 0}{0} = \frac{0\cdot 0}{0} \Rightarrow 1 = 0$. What? $1 = 0$? What kind of world is this? That certainly can't be, can it? If $1 = 0$, then similarly reasoning also gives that every number is equal to 0, that is, 0 is the only number at all. So, in this world, your bank account is always 0, your 401K will never increase, the local speed limit is 0 mph, your number of friends on Facebook is 0, etc. In a nutshell, any world which allows division by zero is a world where NOTHING happens, because nothing is the only thing which exists.

[ii]See §2.2 The Limit of a Function in OpenStax to find the corresponding section.

Exercises

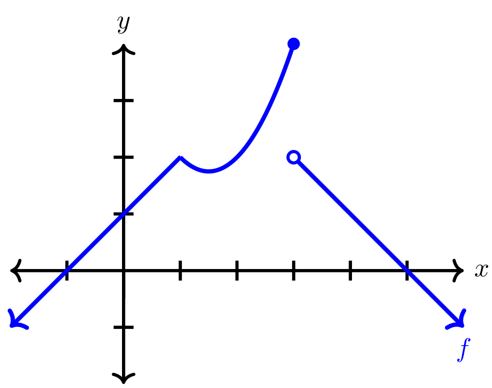
(Go to Solutions)

For Exercises 1–6, the function f is illustrated below. Compute the given limit.



1. $\lim_{x\to -6} f(x)$ ♠ 2. $\lim_{x\to -2} f(x)$ ♠ 3. $\lim_{x\to 0} f(x)$ ♠ 4. $\lim_{x\to 2} f(x)$ 5. $\lim_{x\to 3} f(x)$ 6. $\lim_{x\to 6} f(x)$

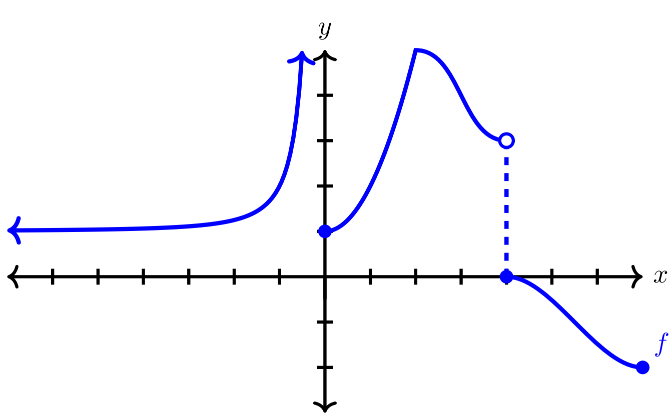
For Exercises 7–14, the function f is illustrated below. Compute the given limit.



7. $\lim_{x\to 5^+} f(x)$ ♠ 8. $\lim_{x\to 1^-} f(x)$ ♠ 9. $\lim_{x\to 1^+} f(x)$ ♠ 10. $\lim_{x\to 1} f(x)$

♠ 11. $\lim_{x\to 3^-} f(x)$ ♠ 12. $\lim_{x\to 3^+} f(x)$ ♠ 13. $\lim_{x\to 3} f(x)$ 14. $\lim_{x\to 0^-} f(x)$

For Exercises 15–24, the function f is illustrated below. Compute the given limit.



15. $\lim_{x\to 7^-} f(x)$

♠ 16. $\lim_{x\to 4^-} f(x)$

♠ 17. $\lim_{x\to 4^+} f(x)$

♠ 18. $\lim_{x\to 4} f(x)$

♠ 19. $\lim_{x\to 2^-} f(x)$

♠ 20. $\lim_{x\to 2^+} f(x)$

♠ 21. $\lim_{x\to 2} f(x)$

♠ 22. $\lim_{x\to 0^+} f(x)$

♠ 23. $\lim_{x\to 0^-} f(x)$

24. $\lim_{x\to 0} f(x)$

♠ 25. Let $f(x) = \dfrac{x^2-5x}{x^2-x-20}$. Evaluate
$f(5.5)$, $f(5.1)$, $f(5.05)$, $f(5.005)$, $f(5.001)$,
$f(4.9)$, $f(4.95)$, $f(4.99)$, $f(4.995)$, and $f(4.999)$.
Use this to estimate $\lim_{x\to 5} f(x)$.

The following Desmos calculator may prove helpful

♠ 26. Let $g(x) = \dfrac{x^2-3x}{x^2-2x-3}$. Evaluate
$g(0)$,$g(-0.5)$,$g(-0.9)$,$g(-0.95)$,$g(-0.99)$,$g(-0.999)$,
$g(-2)$, $g(-1.5)$,$g(-1.1)$,$g(-1.01)$, and $g(-1.001)$.
Use this to estimate $\lim_{x\to -1} g(x)$.

The following Desmos calculator may prove helpful

Deeper Dive

27. Sketch the graph of a function f that satisfies:

$$\lim_{x\to 0^-} f(x) = 2, \qquad \lim_{x\to 0^+} f(x) = 0, \qquad \lim_{x\to 4^-} f(x) = 3,$$
$$\lim_{x\to 4^+} f(x) = 0, \qquad f(0) = 2, \qquad f(4) = 1.$$

28. Using the values of
$$x = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7},$$
estimate the value of the limit $\lim\limits_{x\to 0^+}(1+x)^{1/x}$ to five decimal places. This number is usually given a special label. What is this number usually called?

"Boundaries are to protect life, not to limit pleasures." – Edwin Louis Cole

Lecture Videos

Using Limit Laws to Compute Limits

Computing Limits of a Function using a Simplified Form

Limits of Piece-wise Functions

Simplifying a Limit of a Difference Quotient (Polynomial)

Simplifying a Limit of a Difference Quotient (Radical)

Simplifying a Limit of a Difference Quotient (Rational)

2.3 Properties of Limits

Theorem 2.3.1 (Rules of Limits)**.** *Let a, A, and B be real numbers, and let f and g be functions such that*

$$\lim_{x\to a} f(x) = A \;\text{ and }\; \lim_{x\to a} g(x) = B.$$

(a) If k is a constant, then $\lim_{x\to a} k = k$ and $\lim_{x\to a}[k\cdot f(x)] = k\cdot \lim_{x\to a} f(x) = kA$.

(b) $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A\pm B$.

(c) $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = AB$.

(d) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{A}{B}$, if $B\neq 0$.

Even though this list seems imposing, the properties listed here are quite natural and the student will grow used to them rather quickly. We will also mention that although we can prove the above properties about limits, we will not actually do so. Such proofs are more appropriately postponed until a course in the theory of analysis. That is not to say that we will never prove anything in the course.

Example 2.3.2. Suppose $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = 4$. Use the limit rules to find the following limits.

(a) $\lim_{x\to 2}[f(x)+5g(x)] = \lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x) = 3+5\cdot 4 = \boxed{23}$.

(b) $\lim_{x\to 2}[2f(x)g(x)] = 2\lim_{x\to 2} f(x)\lim_{x\to 2} g(x) = 2(3)(4) = \boxed{24}$.

(c) $\lim_{x\to 2} \dfrac{[f(x)]^2}{\ln g(x)} = \dfrac{\left[\lim_{x\to 2} f(x)\right]^2}{\ln\left(\lim_{x\to 2} g(x)\right)} = \boxed{\dfrac{9}{\ln 4}} \approx 6.492.$

Theorem 2.3.3 (Rules of Limits (Continued))**.**

(e) $\lim_{x\to a} x^n = a^n$ *for all positive integers.*

(f) *For any real number* k*,* $\lim_{x\to a}[f(x)]^k = \left[\lim_{x\to a} f(x)\right]^k = A^k$*, provided this limit exists.*[i]

(g) *For any real number* $b > 0$*,* $\lim_{x\to a} b^{f(x)} = b^{[\lim_{x\to a} f(x)]} = b^A$*.*

(h) *For any real number* $b > 0$ *such that* $b \neq 1$*,* $\lim_{x\to a}[\log_b f(x)] = \log_b \left[\lim_{x\to a} f(x)\right] = \log_b A$ *if* $A > 0$*.*

(i) $\lim_{x\to a} \sin(f(x)) = \sin\left(\lim_{x\to a} f(x)\right)$ *and* $\lim_{x\to a} \cos(f(x)) = \cos\left(\lim_{x\to a} f(x)\right)$*.*

This properties imply that if $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.

Example 2.3.4. Evaluate the following limits and justify each step.

(a) $\lim_{x\to 5}(2x^2 - 3x + 4)$

$$\begin{aligned}
\lim_{x\to 5}(2x^2 - 3x + 4) &= \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5}(4) \quad \text{by } (b) \\
&= 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + 4 \quad \text{by } (a) \\
&= 2(5)^2 - 3(5) + 4 \quad \text{by } (e) \\
&= 50 - 15 + 4 = \boxed{39}
\end{aligned}$$

(b) $\lim_{x\to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Note that by similarly reasoning $\lim_{x\to -2}(x^3 + 2x^2 - 1) = (-2)^3 + 2(-2)^2 - 1 = -8 + 8 - 1 = -1$ and $\lim_{x\to -2}(5 - 3x) = 5 - 3(-2) = 5 + 6 = 11 \neq 0$. Therefore, by (d), we have

$$\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x\to -2}(x^3 + 2x^2 - 1)}{\lim_{x\to -2}(5 - 3x)} = \boxed{\frac{-1}{11}}.$$

Theorem 2.3.5 (Rules of Limits (Continued))**.**

(j) *If* $f(x) = g(x)$ *when* $x \neq a$*, then* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$*, provided the limits exists.*

Example 2.3.6. Determine whether $f(x) = \dfrac{x - 1}{x^2 - 1}$ has a vertical asymptote at $x = 1$.

This is equivalent to asking whether $\lim_{x\to 1^\pm} f(x) = \pm\infty$. Although f is not defined at $x = 1$, notice that $f(x) = \dfrac{x - 1}{x^2 - 1} = \dfrac{x - 1}{(x - 1)(x + 1)}$. Thus, f only differs from the function $y = \dfrac{1}{x + 1}$ by a single point,

$x = 1$. Thus, they have the same limit. By direct substitution, we have $\boxed{\lim_{x\to 1} f(x) = \frac{1}{1+1} = \frac{1}{2}}$. Thus, this is not a vertical asymptote even though 1 is not in the domain of f. This corresponds to a removed point.

Example 2.3.7. If

$$f(x) = \begin{cases} \sqrt{x-4}, & x > 4 \\ 8 - 2x, & x < 4, \end{cases}$$

then determine whether $\lim_{x\to 4} f(x)$ exists.

To the left of $x = 4$, f behaves like the function $y = 8 - 2x$. Thus,

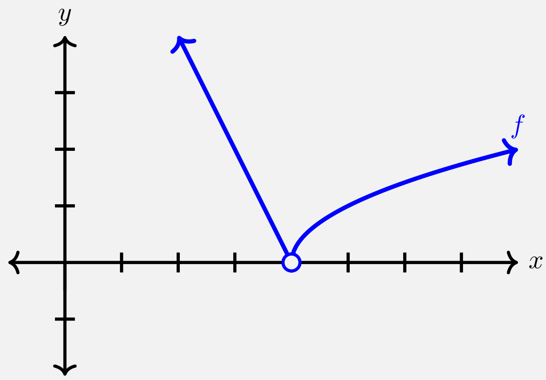
$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 8 - 2x = 8 - 2(4) = 0.$$

Similarly, from the right of $x = 4$, f behaves like the function $y = \sqrt{x-4}$. Thus,

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0.$$

Since the left-hand and right-hand limits agree,

$$\boxed{\lim_{x\to 4} f(x) = 0}.$$



Example 2.3.8. Show that $\lim_{x\to 0} |x| = 0$.

Since

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

we see that

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0, \quad \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0.$$

Therefore, $\boxed{\lim_{x\to 0} |x| = 0}$.

Example 2.3.9. Find $\lim_{h\to 0} \frac{(3+h)^2 - 9}{h}$.

$$\begin{aligned} \lim_{h\to 0} \frac{(3+h)^2 - 9}{h} &= \lim_{h\to 0} \frac{(9 + 6h + h^2) - 9}{h} = \lim_{h\to 0} \frac{6h + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(6+h)}{h} = \lim_{h\to 0} 6 + h = 6 + (0) = \boxed{6} \end{aligned}$$

Example 2.3.10. Find $\displaystyle\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$.

$$\begin{aligned}
\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\left(\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\right) = \lim_{t\to 0}\frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)}\\
&= \lim_{t\to 0}\frac{t^2}{t^2(\sqrt{t^2+9}+3)} = \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3}\\
&= \frac{1}{\sqrt{0^2+9}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}
\end{aligned}$$

Example 2.3.11. Find $\displaystyle\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$.

$$\begin{aligned}
\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\left(\frac{x^2(x+h)^2}{x^2(x+h)^2}\right) = \lim_{h\to 0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}\\
&= \lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2} = \lim_{h\to 0}\frac{-2xh-h^2)}{hx^2(x+h)^2}\\
&= \lim_{h\to 0}\frac{-2x-h)}{x^2(x+h)^2} = \frac{-2x-0)}{x^2(x+0)^2} = \frac{-2x}{x^2x^2} = \boxed{\frac{-2}{x^3}}
\end{aligned}$$

[i] This limit does not exist, for example, when $A < 0$ and $k = 1/2$, or when $A = 0$ and $k \le 0$. The first example is a problem since the limit would be an imaginary number and the second example is the division by zero issue again.

[ii] See §2.3 The Limit Laws in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–6, given that

$$\lim_{x\to 1} f(x) = 1, \quad \lim_{x\to 1} g(x) = -5, \quad \lim_{x\to 1} h(x) = 0,$$

find the limits, if they exist.

1. $\lim_{x\to 1}(f(x) + 4g(x))$
2. $\lim_{x\to 1} g(x)^3$
3. $\lim_{x\to 1} \sqrt{f(x)}$
4. $\lim_{x\to 1} \frac{5f(x)}{g(x)}$
5. $\lim_{x\to 1} \frac{g(x)}{h(x)}$
6. $\lim_{x\to 1} \frac{g(x)h(x)}{f(x)}$

For Exercises 7–12, given that

$$\lim_{x\to -1} f(x) = -1, \quad \lim_{x\to 0} f(x) = 0, \quad \lim_{x\to 1} f(x) = 1, \quad \lim_{x\to 2} f(x) = 2,$$

$$\lim_{x\to -1} g(x) = 0, \quad \lim_{x\to 0} g(x) = 3, \quad \lim_{x\to 1} g(x) \text{ DNE}, \quad \lim_{x\to 2} g(x) = 6,$$

find the limits, if they exist.

7. $\lim_{x\to 2}(f(x) + g(x))$
8. ♠ $\lim_{x\to 1}(f(x) + g(x))$
9. ♠ $\lim_{x\to 0} f(x)g(x)$
10. ♠ $\lim_{x\to -1} \frac{f(x)}{g(x)}$
11. ♠ $\lim_{x\to 2} x^3 f(x)$
12. $\lim_{x\to 1} \sqrt{3 + f(x)}$

For Exercises 13–24, compute the given limit.

13. $\lim_{x\to 2} \frac{x^2 - 3x + 1}{x + 1}$
14. ♠ $\lim_{t\to -2} \frac{t^4 - 8}{2t^2 - 3t + 8}$
15. ♠ $\lim_{z\to -3} \sqrt{z^4 + 4z + 12}$
16. ♠ $\lim_{y\to 8}(2 + \sqrt[3]{y})(4 - 6y^2 + y^3)$
17. ♠ $\lim_{u\to 2} \sqrt{\frac{3u^2 + 4}{5u - 1}}$
18. $\lim_{x\to 3}(5x^3 - 3x^2 + x - 6)$
19. $\lim_{x\to 2} \frac{x - 2}{x^2 + 2x - 8}$
20. ♠ $\lim_{x\to 2} \frac{x^2 - 8x + 12}{x - 2}$
21. ♠ $\lim_{x\to 7} \frac{x^2 - 7x}{x^2 - 6x - 7}$
22. ♠ $\lim_{x\to 5} \frac{x^2 - 10x + 24}{x - 5}$
23. ♠ $\lim_{t\to -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$
24. $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$
25. $\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t}\right)$
26. $\lim_{x\to 0^-} f(x)$ if $f(x) = \begin{cases} \cos x, & x < 0 \\ 0, & x = 0 \\ x^2 + 1, & 0 < x \le 2 \\ 5 - x, & x > 2 \end{cases}$
27. $\lim_{x\to 2^+} f(x)$ if $f(x) = \begin{cases} \cos x, & x < 0 \\ 0, & x = 0 \\ x^2 + 1, & 0 < x \le 2 \\ 5 - x, & x > 2 \end{cases}$
28. ♠ $\lim_{x\to 2} g(x)$ if $g(x) = \begin{cases} \dfrac{x^2 + x - 6}{x - 2}, & x \ne 2 \\ 3, & x = 2 \end{cases}$
29. ♠ $\lim_{x\to 0} g(x)$ if $g(x) = \begin{cases} \dfrac{x^2 + x - 6}{x - 2}, & x \ne 2 \\ 3, & x = 2 \end{cases}$
30. $\lim_{x\to 1^-} h(x)$ if $h(x) = \begin{cases} x^2 + 1, & x \ge 1 \\ -x, & x < 1 \end{cases}$
31. $\lim_{x\to 1^+} h(x)$ if $h(x) = \begin{cases} x^2 + 1, & x \ge 1 \\ -x, & x < 1 \end{cases}$

"Managing is like holding a dove in your hand. Squeeze too hard and you kill it, not hard enough and it flies away" – Tommy Lasorda

Lecture Videos

The Squeeze Theorem

Simplifying a Limit of a Difference Quotient (Exponential)

Simplifying a Limit of a Difference Quotient (Trigonometric)

2.4 Properties of Limits II

Theorem 2.4.1. *If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as $x \to a$, then*

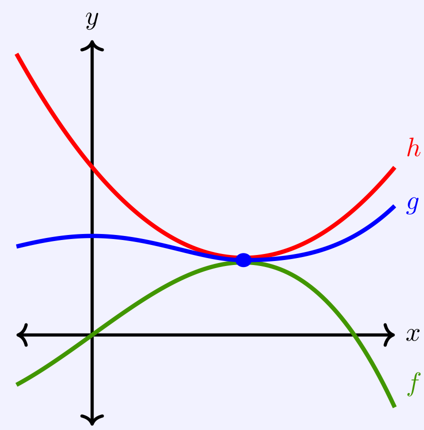
$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Theorem 2.4.2 (The Squeeze Theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ when x is near a and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L.$$

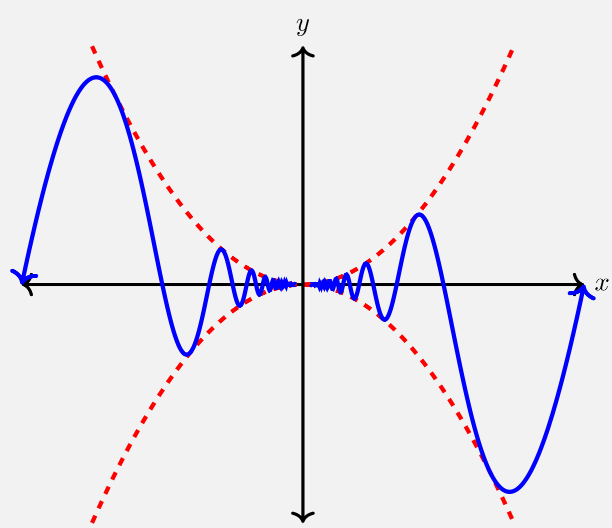


Example 2.4.3. Show that $\lim\limits_{x\to 0} x^2 \sin\dfrac{\pi}{x} = 0$.

Notice that we cannot use $\lim\limits_{x\to 0} x^2 \lim\limits_{x\to 0} \sin\dfrac{\pi}{x}$ since $\lim\limits_{x\to 0} \sin\dfrac{\pi}{x}$ does not exist. Instead, we can use the Squeeze Theorem. Notice that $-1 \leq \sin\theta \leq 1$, which implies that $-1 \leq \sin\dfrac{\pi}{x} \leq 1$. Thus, $-x^2 \leq x^2 \sin\dfrac{\pi}{x} \leq x^2$. Since $-x^2, x^2$ are polynomials, we compute their limits by direct substitution. The Squeeze Lemma gives us that

$$0 = \lim_{x\to 0} -x^2 \leq \lim_{x\to 0} x^2 \sin\frac{\pi}{x} \leq \lim_{x\to 0} x^2 = 0,$$

that is, $\boxed{\lim\limits_{x\to 0} x^2 \sin\dfrac{\pi}{x} = 0}$.



Example 2.4.4. Prove that $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$.

Since $\lim_{x\to 0^+} \sin(\pi/x)$ does not exist, this limit is not so simple. But let us consider the function $f(x) = e^{\sin(\pi/x)}$. For any function g, it holds that $e^{g(x)} > 0$, for any x. Thus, $f(x) = e^{\sin(\pi/x)} > 0$. Also, $\sin(\pi/x) \leq 1$ for all x. Thus, since e^x is an increasing function, $f(x) = e^{\sin(\pi/x)} \leq e^1 = e$, that is,

$$0 \leq e^{\sin(\pi/x)} \leq e, \quad \text{for all } x.$$

Multiplying this inequality by the positive function $\sqrt{x}$ then gives the inequality we desire for the Squeeze Theorem,

$$0 \leq \sqrt{x}e^{\sin(\pi/x)} \leq e\sqrt{x}, \quad \text{for all } x.$$

Now, $\lim_{x\to 0^+} 0 = \lim_{x\to 0^+} \sqrt{x} = 0$. Therefore, by the Squeeze Theorem, $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$.

Example 2.4.5. If

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

The result follows from the Squeeze Lemma. In particular, $0 \leq f(x) \leq x^2$ for all real numbers x. Thus, $0 = \lim_{x\to 0} 0 \leq \lim_{x\to 0} f(x) \leq \lim_{x\to 0} x^2 = 0$, that is, $\lim_{x\to 0} f(x) = 0$.

Proposition 2.4.6. $\lim_{h\to 0} \dfrac{e^h - 1}{h} = 1$.

Proof. As shown previously, $e = \lim_{h\to 0^+} (1+h)^{1/h}$. In fact, we say that $(1+h)^{1/h} \leq e$ when h is positive and close to zero. Thus,

$$\begin{aligned} (1+h)^{1/h} &\leq e \\ 1+h &\leq e^h \\ h &\leq e^h - 1 \\ 1 &\leq \frac{e^h - 1}{h} \end{aligned}$$

Conversely, $e = \lim_{h\to 0^-} (1+h)^{1/h} = e$ and $e \leq (1+h)^{1/h}$ when h is negative and close to zero. If $h > 0$, then $-h < 0$ and

$$\begin{aligned} e &\leq (1+(-h))^{1/(-h)} = (1-h)^{-1/h} = \left(\frac{1}{1-h}\right)^{1/h} \\ e^h &\leq \frac{1}{1-h} \\ e^h - 1 &\leq \frac{1}{1-h} - 1 = \frac{1}{1-h} - \frac{1-h}{1-h} = \frac{h}{1-h} \\ \frac{e^h - 1}{h} &\leq \frac{1}{1-h} \end{aligned}$$

Therefore, if $h > 0$ and close to zero, then

$$1 \leq \frac{e^h - 1}{h} \leq \frac{1}{1-h}.$$

Note that $\lim_{h \to 0^+} 1 = \lim_{h \to 0^+} \frac{1}{1-h} = 1$. Therefore, by the Squeeze Theorem, $\lim_{h \to 0^+} \frac{e^h - 1}{h} = 1$.

By a similar argument, we can show that if $h < 0$ and close to zero then $\frac{1}{1-h} \leq \frac{e^h - 1}{h} \leq 1$. The Squeeze Theorem then shows that $\lim_{h \to 0^-} \frac{e^h - 1}{h} = 1$. Therefore, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. □

Proposition 2.4.7. $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

Proof. Assume first that $0 < \theta < \pi/2$ and consider the triangle inscribed into the unit circle as in the diagram. By the arc length formula $s = r\theta$, we have that $\theta = |\text{arc } AB|$. By the sine ratio,

$$\sin \theta = \frac{|BC|}{|BO|} = |BC| \leq |\text{arc} AB| = \theta.$$

Thus, $\sin \theta \leq \theta \Longrightarrow \frac{\sin \theta}{\theta} \leq 1$.

On the other hand, if we compare the similar triangles $\triangle OBC$ and $\triangle ODA$, we have that

$$\tan \theta = \frac{|AD|}{|AO|} = |AD| \geq |\text{arc } AB| = \theta.$$

Thus,

$$\theta \leq \tan \theta \Longrightarrow \theta \leq \frac{\sin \theta}{\cos \theta} \Longrightarrow \cos \theta \leq \frac{\sin \theta}{\theta}.$$

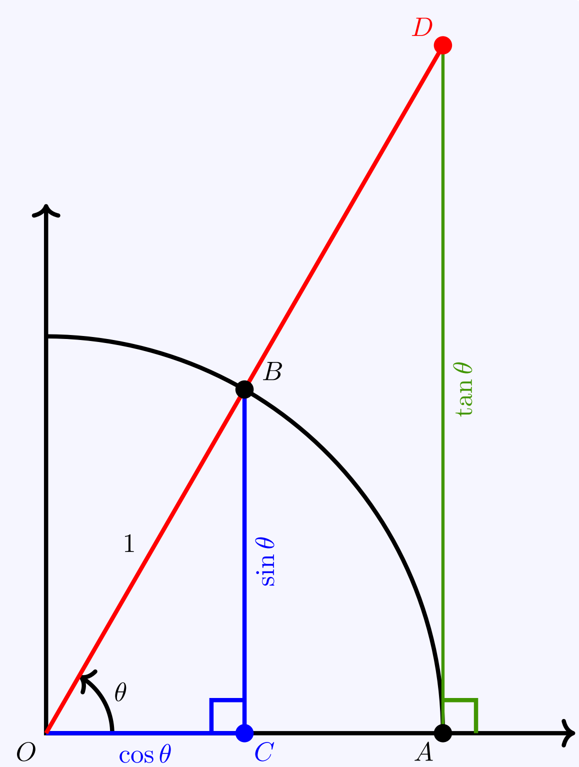
Thus,

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1.$$

Since $\cos \theta$, $\frac{\sin \theta}{\theta}$, and 1 are all even functions, these inequalities hold for the whole interval $(-\pi/2, \pi/2)$. The Squeeze Lemma then implies that

$$1 = \lim_{\theta \to 0} \cos \theta \leq \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \leq \lim_{\theta \to 0} 1 = 1.$$

□



[i]See §2.3 The Limit Laws in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–31, compute the given limit.

1. $\displaystyle\lim_{x\to 3^-} \frac{|x-3|}{x-3}$

2. $\displaystyle\lim_{x\to 3^+} \frac{|x-3|}{x-3}$

3. $\displaystyle\lim_{x\to 3} \frac{|x-3|}{x-3}$

4. $\displaystyle\lim_{x\to 3^-} \frac{|x-3|}{x^2-9}$

5. $\displaystyle\lim_{x\to 3^+} \frac{|x-3|}{x^2-9}$

6. $\displaystyle\lim_{x\to 3^+} \frac{|x-3|}{x^2-9}$

♠ 7. $\displaystyle\lim_{x\to -6} \frac{6-|x|}{6+x}$

8. $\displaystyle\lim_{x\to -3} \frac{\frac{1}{3}+\frac{1}{x}}{x+3}$

9. $\displaystyle\lim_{x\to -3} \frac{\frac{1}{3}-\frac{1}{x}}{x+3}$

♠ 10. $\displaystyle\lim_{x\to -9} \frac{\frac{1}{9}+\frac{1}{x}}{x+9}$

11. $\displaystyle\lim_{t\to 2} \frac{\frac{1}{t-2}-\frac{1}{t+2}}{t-2}$

♠ 12. $\displaystyle\lim_{h\to 0} \frac{(8+h)^{-1}-8^{-1}}{h}$

13. $\displaystyle\lim_{x\to 5} \frac{\sqrt{x-4}-1}{x-5}$

14. $\displaystyle\lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4}$

15. $\displaystyle\lim_{x\to 1^+} \frac{1-\sqrt{x}}{1-x}$

♠ 16. $\displaystyle\lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4}$

17. $\displaystyle\lim_{x\to 0} \frac{3e^{2x}-3e^{x}}{e^{2x}-1}$

♠ 18. $\displaystyle\lim_{h\to 0} \frac{(-2+h)^2-4}{h}$

♠ 19. $\displaystyle\lim_{h\to 0} \frac{(9+h)^3-729}{h}$

♠ 20. $\displaystyle\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$

21. $\displaystyle\lim_{h\to 0} \frac{\frac{1}{5+h}-\frac{1}{5}}{h}$

22. $\displaystyle\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

♠ 23. $\displaystyle\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$

24. $\displaystyle\lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

♠ 25. $\displaystyle\lim_{x\to 3} f(x)$ if $3x-4 \le f(x) \le x^2-3x+5$

♠ 26. $\displaystyle\lim_{x\to 1} g(x)$ if $6x \le g(x) \le 3x^4-3x^2+6$

♠ 27. $\displaystyle\lim_{x\to \pi/2} h(x)$ if $2\sin x \le h(x) \le 2+\left|x-\frac{\pi}{2}\right|$

♠ 28. $\displaystyle\lim_{x\to 0} x^8 \sin\left(\frac{1}{x}\right)$

29. $\displaystyle\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

30. $\displaystyle\lim_{x\to 1} (x-1)^2 \sin\left(\frac{1}{x-1}\right)$

31. $\displaystyle\lim_{x\to 0} x^2 \sin\left(\frac{e}{x}\right)$

Deeper Dive

32. Prove $\lim_{x\to 0^+} \sqrt{x}\sin\left(\frac{\pi}{x}\right) = 0$. Name the major theorem which you will use. Why are we only considering the right-handed limit in this problem?

33. Prove $\lim_{x\to 0} \sqrt{x^3 + x^2}\cos\left(\frac{\pi}{x}\right)$. Name the major theorem which you will use. Why does it make sense to consider the 2-sided limit in this problem but not the previous one?

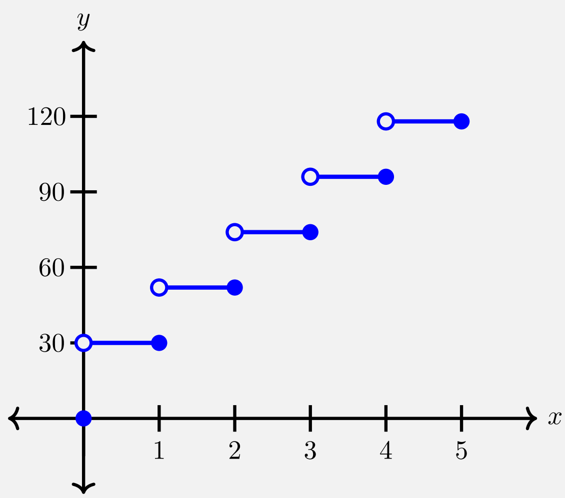
"Excellence is a continuous process and not an accident." – *A. P. J. Abdul Kalam*

Lecture Videos

Continuous Functions | Discontinuities | Continuity of Piece-wise Functions

2.5 Continuity

Example 2.5.1. A trailer rental firm charges a flat \$8 to rent a hitch. The trailer itself is rented for \$22 per day or fraction of a day. Let $C(x)$ represent the cost of renting a hitch and trailer for x days. Looking at the graph, we can see that the function "jumps" at positive integer. In particular the left-handed and right-handed limit disagree when x approaches an integer. So, $\lim\limits_{x\to n} C(x)$ does not exist when n is a positive integer.



Now, these types of "jumps," "rips," or "holes" are graphical ways of representing that the limit does not exist. In this section, we will study the very important class of functions which do not have such "holes." These functions are called *continuous*, which definition we now make more precise.

Definition 2.5.2. A function f is **continuous at $x = c$** if the following three conditions are satisfied:

(a) $f(c)$ is defined, (b) $\lim\limits_{x\to c} f(x)$ exists, (c) $\lim\limits_{x\to c} f(x) = f(c)$.

If f is not continuous at c, it is **discontinuous at $x = c$**.

A function f is **continuous from the right** at $x = c$ if $\lim\limits_{x\to c^+} f(x) = f(c)$. Similarly, a function f is **continuous from the left** at $x = c$ if $\lim\limits_{x\to c^-} f(x) = f(c)$.

Definition 2.5.3. A function is said to be **continuous on an open interval** (a, b) if it is continuous at every x-value in the open interval (a, b). Then we say a function f is **continuous on a closed interval** $[a, b]$ if f is continuous on the open interval (a, b), continuous from the right at $x = a$, and continuous from the left at $x = b$.

Theorem 2.5.4. *We consider the continuity of some familiar functions.*

(a) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function. Then $f(x)$ is continuous at all real numbers x.

(b) Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function, where $p(x)$ and $q(x)$ are polynomials. Then $f(x)$ is continuous at all real numbers x when $q(x) \neq 0$.

(c) Let $f(x) = \sqrt{x}$. Then $f(x)$ is continuous on $[0, \infty)$. Similar properties hold for any other radical function.

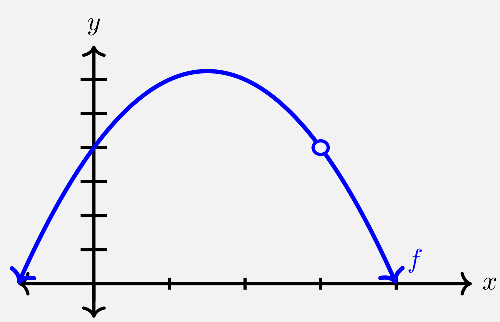
(d) Let $f(x) = a^x$ for $a > 0$. Then $f(x)$ is continuous for all real numbers x.

(e) Let $f(x) = \log_a x$ for $a > 0$ and $a \neq 1$. Then $f(x)$ is continuous on $(0, \infty)$.

(f) Let $f(x)$ be any trigonometric or inverse trigonometric function. Then f is continuous on its domain.

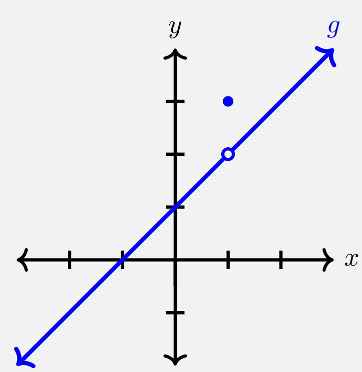
Example 2.5.5. Tell why each function is discontinuous at the indicated x-value.

(a) $f(x) = \dfrac{(-x^2 + 3x + 4)(x - 3)}{x - 3}$ at $x = 3$.



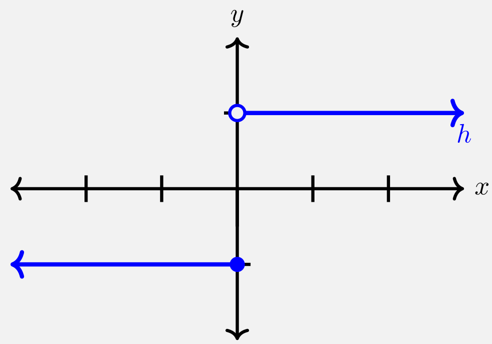
The open circle on the graph of $f(x)$ at the point where $x = 3$ means that $f(3)$ is not defined. Because of this, part 1 of the definition fails.

(b) $g(x) = \begin{cases} x + 1 & x \neq 1, \\ 3 & x = 1. \end{cases}$ at $x = 1$.



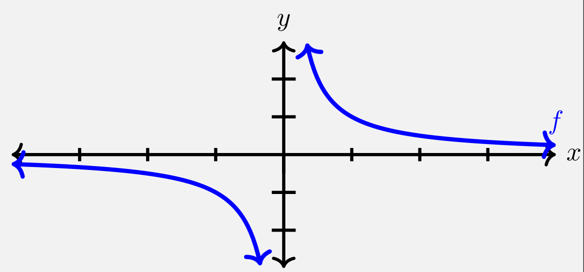
The heavy dot above 1 shows that $g(1)$ is defined. In fact, $g(1) = 3$. The graph also shows, however, that $\lim_{x\to 1} g(x) = 2$. So, $\lim_{x\to 1} g(x) \neq g(1)$, and part 3 of the definition fails.

(c) $h(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$ at $x = 0$.



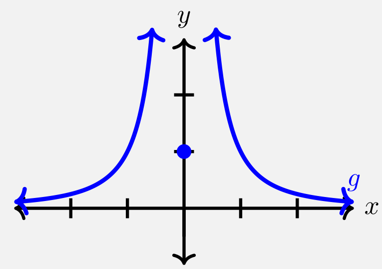
Now, $h(0)$ is defined, that is, $h(0) = -1$. The left-handed and right-handed limits of h also exist at $x = 0$, that is, $\lim\limits_{x \to 0^-} h(x) = -1$ and $\lim\limits_{x \to 0^+} h(x) = 1$, but $\lim\limits_{x \to 0} h(x)$ does not exist. So, $h(x)$ is not continuous at 0, although it is continuous from the left.

(d) $f(x) = \dfrac{1}{x}$ at $x = 0$.



The function f graphed is not defined at $x = 0$, and $\lim\limits_{x \to 0} f(x)$ does not exist. Either of these reasons is sufficient to show that f is discontinuous at 0.

(e) $g(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$



The function has a vertical asymptote at $x = 0$ but the function is still defined at $x = 0$. On the other hand, $\lim\limits_{x \to 0} f(x) = \infty \neq 1$. Thus, $x = 0$ is a discontinuity.

Definition 2.5.6. The discontinuities of examples 2.5.5 (a) and (b) could be repaired by adding a point or moving a point on the graph. These are known as **removable discontinuities**. In this case, the limit exists but the function is undefined or is defined to be some value other than the limit.

The discontinuity of example 2.5.5 (c) is called a **jump discontinuity**. In this case, the left- and right-handed limits exist but do not agree.

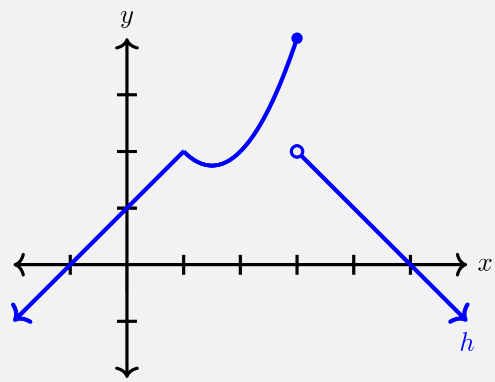
The discontinuities of examples 2.5.5 (d) and (e) are called **vertical asymptotes**, as we have seen already occur when the left- or right-handed limits are infinite.

Example 2.5.7. Find all the values for x where the following function is discontinuous :

$$f(x) = \begin{cases} x + 1 & x < 1 \\ x^2 - 3x + 4 & 1 \leq x \leq 3 \\ 5 - x & x > 3 \end{cases}$$

Because the function acts like a polynomial except at the switch points, f is continuous on the intervals $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$. The only possible jumps (discontinuities) must occur at $x = 1$ and $x = 3$. We check to see that the left and right limits agree at these two values. Because the function acts like a polynomial on all the intervals, we can simply evaluate the polynomials to find the left and right limits, since polynomials are continuous.

We first check the left-handed limit at $x = 1$: $\lim_{x\to 1^-} f(x) = (1) + 1 = 2$. For the right-handed limit at 1, we have $\lim_{x\to 1^+} (1)^2 - 3(1) + 4 = 2$. Thus, $\boxed{\lim_{x\to 1} f(x) = 2 = f(1)}$. Next, we check the left-handed limit at $x = 3$: $\lim_{x\to 3^-} f(x) = (3)^2 - 3(3) + 4 = 4$. For the right-handed limit, we have $\lim_{x\to 3^+} f(x) = 5 - (3) = 2 \neq 4$. Therefore, $\lim_{x\to 3} f(x)$ does not exist. Thus, $\boxed{f(x) \text{ is discontinuous at } x = 3}$.



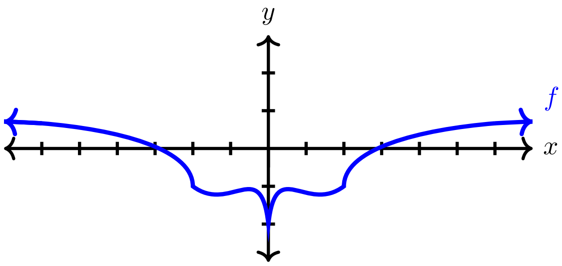
[i]See §2.4 Continuity in OpenStax to find the corresponding section.

Exercises

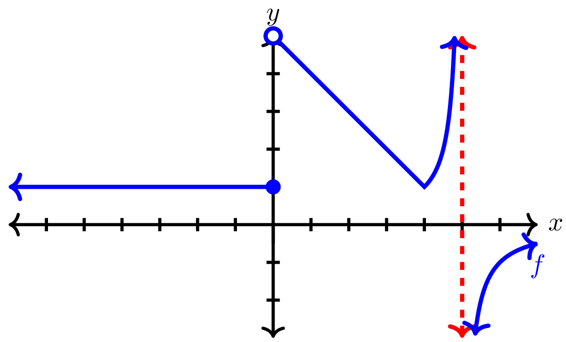
(Go to Solutions)

For Exercises 1–6, the function f is illustrated below. Determine all values of x where f is not continuous.

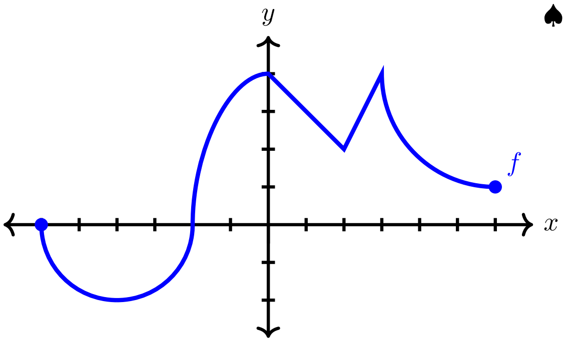
1.



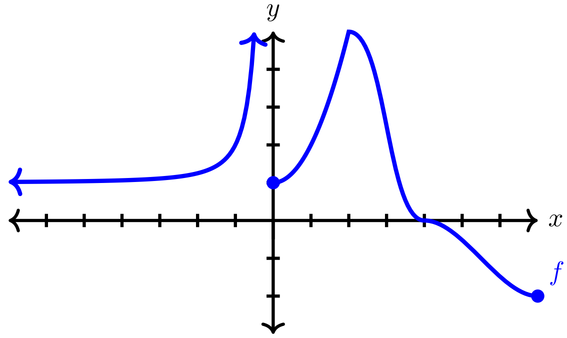
♠ 2.



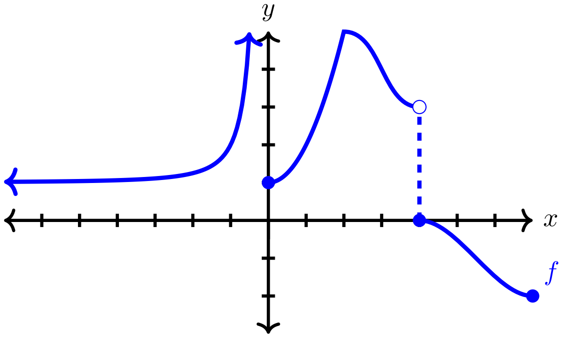
♠ 3.



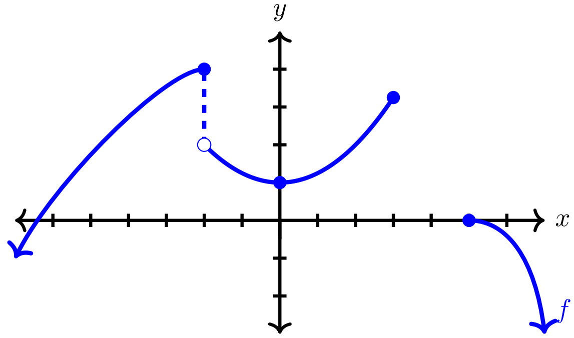
♠ 4.



♠ 5.



6.



For Exercises 7–18, determine the intervals for which the function f is continuous.

7. $f(x) = \sin(x^7)$

8. $f(x) = \ln(1 + \cos x)$

♠ 9. $f(x) = \dfrac{x^2 - 5x - 14}{x - 7}$

♠ 10. $f(x) = \begin{cases} \dfrac{x^2 - 5x - 14}{x - 7}, & x \neq 7 \\ 9, & x = 7 \end{cases}$

11. $f(x) = \begin{cases} \dfrac{x^2 + x - 6}{x - 2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$

12. $f(x) = \begin{cases} \dfrac{1}{x^4}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

13. $f(x) = \begin{cases} x^2 + 1, & x \geq 1 \\ -x, & x < 1 \end{cases}$

♠ 14. $f(x) = \begin{cases} e^x, & x < 1 \\ x^3, & x \geq 1 \end{cases}$

♠ 15. $f(x) = \begin{cases} 2 + x^2, & x \leq 0 \\ 4 - x, & 0 < x \leq 4 \\ (x-4)^2, & x > 4 \end{cases}$

♠ 16. $f(x) = \begin{cases} x + 2, & x \leq 1 \\ \frac{1}{x}, & 1 < x < 2 \\ \sqrt{x-2}, & x \geq 2 \end{cases}$

♠ 17. $f(x) = \begin{cases} x+5, & x<0 \\ e^x, & 0 \le x \le 1 \\ 9-x, & x>1 \end{cases}$

18. $f(x) = \begin{cases} \cos x, & x<0 \\ 0, & x=0 \\ x^2+1, & 0<x\le 2 \\ 5-x, & x>2 \end{cases}$

♠ 19. The function $f(x) = \dfrac{x^2-x-2}{x-2}$ has a removable discontinuity at $x=2$. If we in fact want to remove this discontinuity what should $f(2)$ be?

"Strive for continuous improvement, instead of perfection." – Kim Collins

Lecture Videos

Finding Values to Make Piece-wise Functions Continuous | Combining Continuous Functions | Composition of Continuous Functions | The Intermediate Value Theorem

2.6 Continuity II

Example 2.6.1. Find the value of the constant c that makes the function continuous.

(a) $f(x) = \begin{cases} cx^2, & x \leq 2 \\ x + c, & x > 2 \end{cases}$.

Notice that cx^2 and $x+c$ are continuous functions for any choice of c, although when the graph switches between these two pieces, if c is chosen poorly, the graph will have a jump discontinuity. This jump will be avoided if $cx^2 = x + c$ when $x = 2$, that is,

$$c(2)^2 = (2) + c \quad \Rightarrow \quad 4c = 2 + c \quad \Rightarrow \quad 3c = 2 \quad \Rightarrow \quad c = \boxed{\frac{2}{3}}$$

Therefore, $f(x) = \begin{cases} \frac{2}{3}x^2, & x \leq 2 \\ x + \frac{2}{3}, & x > 2 \end{cases}$ is a continuous function. Note that when $x = 2$, we have $\dfrac{2}{3}x^2 = \dfrac{2}{3}(2)^2 = \dfrac{8}{3}$ and $x + \dfrac{2}{3} = 2 + \dfrac{2}{3} = \dfrac{6}{3} + \dfrac{2}{3} = \dfrac{8}{3}$.

(b) $g(x) = \begin{cases} x^3 + c, & x \leq 3 \\ cx - 5, & x > 3. \end{cases}$

Again, we are concerned about when the piece-wise function switches pieces at $x = 3$. Thus,

$$\begin{aligned} x^3 + c = cx - 5,\ (x = 3) \quad &\Rightarrow \quad (3)^3 + c = c(3) - 5 \quad \Rightarrow \quad 27 + c = 3c - 5 \\ &\Rightarrow \quad 32 = 2c \quad \Rightarrow \quad c = \boxed{16} \end{aligned}$$

Therefore, $g(x) = \begin{cases} x^3 + 16, & x \leq 3 \\ 16x - 5, & x > 3 \end{cases}$ is a continuous function.

Theorem 2.6.2. *If f and g are continuous at a, then the following functions are also continuous at a:*

(a) $f + g$ *(b)* $f - g$ *(c)* fg *(d)* f/g, *if* $g(a) \neq 0$.

Proof. We will prove only the first case. The other proofs are similar and are left for the student. Since f and g are continuous, we have that

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a).$$

Then

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)\\ &= f(a)+g(a) = (f+g)(a).\end{aligned}$$

□

Example 2.6.3. Find $\lim\limits_{x\to -2} \dfrac{x^3+2x^2-1}{5-3x}$.

Since rational functions are continuous on their domains and -2 is in the domain of the function, we can calculate the limit by function evaluation. Thus,

$$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} = \frac{-8+8-1}{5+6} = \boxed{-\frac{1}{11}}$$

Example 2.6.4. Evaluate $\lim\limits_{x\to\pi} \dfrac{\sin x}{2+\cos x}$.

As seen before, sin and cos are continuous functions. So, $2+\cos x$ is likewise continuous. Thus, the function $\dfrac{\sin x}{2+\cos x}$ where ever $2+\cos x \neq 0$. But if $2+\cos x = 0$, then $\cos x = -2$, which is impossible. Therefore, $\dfrac{\sin x}{2+\cos x}$ is continuous on $(-\infty,\infty)$. We can compute the limit by function evaulation:

$$\lim_{x\to\pi} \frac{\sin x}{2+\cos x} = \frac{\sin \pi}{2+\cos\pi} = \frac{0}{2-1} = \boxed{0}.$$

Example 2.6.5. Where is the function $f(x) = \dfrac{\ln x + \tan^{-1}(x)}{x^2-1}$ continuous?

Both $\tan^{-1}(x)$ and x^2-1 are continuous on all real numbers. $\ln(x)$ is continuous on $(0,\infty)$. Also, $f(x)$ is discontinuous when $x^2-1=0 \Leftrightarrow x=\pm 1$. Therefore, f is continuous on the set $\boxed{(0,1)\cup(1,\infty)}$.

Theorem 2.6.6. *If f is continuous at g and $\lim\limits_{x\to a} g(x) = b$, then $\lim\limits_{x\to a} f(g(x)) = f(b)$, that is,*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Example 2.6.7. Evaluate $\lim\limits_{x\to 1} \sin^{-1}\left(\dfrac{1-\sqrt{x}}{1-x}\right)$.

$$\begin{aligned}\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) &= \sin^{-1}\left(\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}\right)\\ &= \sin^{-1}\left(\lim_{x\to 1}\frac{1-x}{(1-x)(1+\sqrt{x})}\right) = \sin^{-1}\left(\lim_{x\to 1}\frac{1}{1+\sqrt{x}}\right)\end{aligned}$$

$$= \sin^{-1}\frac{1}{2} = \boxed{\frac{\pi}{6}}.$$

Theorem 2.6.8. *If g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at a.*

Proof. Since g is continuous at $x = a$, we have that

$$\lim_{x\to a} g(x) = g(a).$$

Thus,

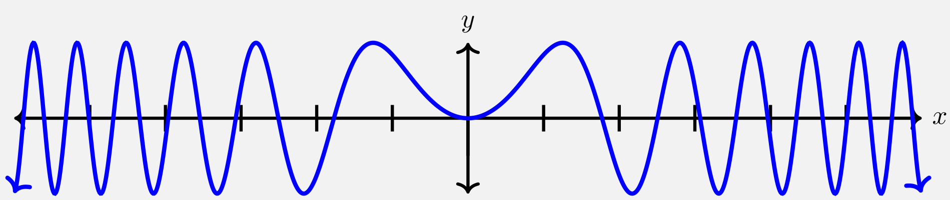
$$\lim_{x\to a}(f\circ g)(x) = \lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(g(a)) = (f\circ g)(a).$$

□

Example 2.6.9. Where are the following functions continuous?

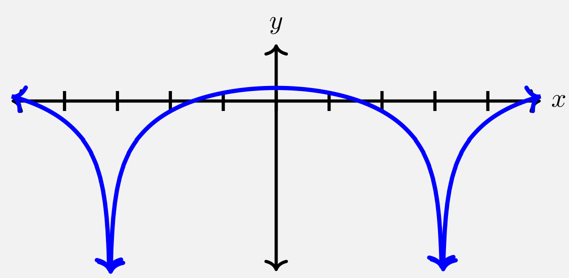
(a) $h(x) = \sin(x^2)$.

Since both $\sin(x)$ and x^2 are continuous with no restrictions on their domains, h is continuous on $\boxed{(-\infty, \infty)}$.

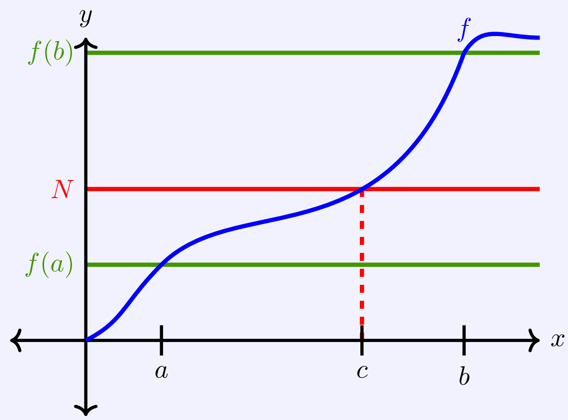


(b) $F(x) = \ln(1 + \cos(x))$.

First, $1 + \cos(x)$ is continuous on $(-\infty, \infty)$. Thus, F is continuous as long as $1 + \cos(x) > 0 \Rightarrow \cos(x) > -1$, which happens as long as $\boxed{x \neq (2k+1)\pi}$ for some integer k.



Theorem 2.6.10 (The Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any value between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.*



Example 2.6.11. Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Proof. Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We first compute

$$\begin{aligned} f(1) &= 4(1)^3 - 6(1)^2 + 3(1) - 2 = 4 - 6 + 3 - 2 = -1 < 0 \\ f(2) &= 4(2)^3 - 6(2)^2 + 3(2) - 2 = 32 - 24 + 6 - 2 = 12 > 0. \end{aligned}$$

The Intermediate Value Theorem with $N = 0$ then implies that there exists some $1 < c < 2$ such that $f(c) = 0$.

Notice also that

$$f(1.2) = -0.128 < 0 \qquad f(1.3) = 0.548 > 0,$$

which implies that a root is between 1.2 and 1.3. Furthermore,

$$f(1.22) = -0.007008 < 0 \qquad f(1.23) = 0.056068 > 0,$$

which implies that a root is between 1.22 and 1.23, and one can continue in this fashion to approximate the root by trial and error. □

[i]See §2.4 Continuity in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–8, use continuity to evaluate the limit.

1. $\lim_{x\to 2} \frac{x^2-1}{x+7}$
2. $\lim_{x\to -2} \sqrt{x^4+3x+6}$
3. ♠ $\lim_{x\to -1} (x+3x^3)^4$
4. ♠ $\lim_{x\to 1} \frac{8+\sqrt{x}}{\sqrt{8+x}}$
5. ♠ $\lim_{x\to \pi} 5\sin(x+\sin x)$
6. ♠ $\lim_{x\to 1} e^{8x^5-8x}$
7. ♠ $\lim_{x\to 4} \tan^{-1}\left(\frac{x^2-16}{7x^2-28x}\right)$
8. $\lim_{x\to 8} (1+\sqrt[3]{x})(2-6x^2+x^3)$

For Exercises 9–17, which value of c makes the following function f continuous?

9. $f(x) = \begin{cases} 3x+1 & x \neq 2 \\ k, & x = 2 \end{cases}$
10. $f(x) = \begin{cases} (2-x)^2, & x \leq 4 \\ \sqrt{c-2x}, & x > 4 \end{cases}$
11. $f(x) = \begin{cases} 2x+c, & x \leq 1 \\ x^2+3, & x > 1 \end{cases}$
12. ♠ $f(x) = \begin{cases} cx^2+8x, & x < 3 \\ x^2-cx, & x \geq 3 \end{cases}$
13. $f(x) = \begin{cases} \frac{x^2-9}{x+3}, & x \neq -3 \\ c, & x = -3 \end{cases}$
14. $f(x) = \begin{cases} cx^3-1, & x \leq 2 \\ cx+3c, & x > 2 \end{cases}$
15. $f(x) = \begin{cases} x^2+cx, & x \geq 3 \\ cx^2-x, & x < 3 \end{cases}$
16. $f(x) = \begin{cases} 2\sin x, & x < \frac{\pi}{4} \\ c\cos x, & x \geq \frac{\pi}{4} \end{cases}$
17. $f(x) = \begin{cases} 3x^2, & x < 1 \\ c\sqrt{x}, & x \geq 1 \end{cases}$
18. ♠ Which values of a and b make $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x < 2 \\ ax^2-bx+3, & 2 \leq x < 3 \\ 4x-a+b, & x \geq 3 \end{cases}$ continuous?

For Exercises 9–17, using the Intermediate Value Theorem, prove that the given equation has a solution.

19. $\cos x = x$
20. ♠ $x^4+x-9=0$
21. $x^4+x-17=0$
22. ♠ $2x^3-4x^2+3x-2=0$
23. $x^5-3x^3-x^2+1=0$
24. ♠ $\sqrt[3]{x} = 1-x$
25. $e^x = 3-2x$
26. $\cos x = x^2-x$
27. ♠ $\cos x = x^3$
28. $\ln x = 3-2x$

Deeper Dive

29. Prove that there is a number that is exactly one more that its cube. Name the major theorem which you will use.

"To Infinity, and Beyond!" – Buzz from Toy Story

Lecture Videos

Vertical Asymptotes | Limits at Infinity | Arithmetic at Infinity | Horizontal Asymptotes

2.7 Limits at Infinity

We will begin this section review limits which approach infinity, that is, function withvertical asymptotes. Remember, a graph has a vertical asymptote if either the left- or right-handed limits approach $\pm\infty$. In particular, if r is any real number, then a limit of the form $\frac{r}{0}$ indicates a vertical asymptote. For example, if n is an integer, then

$$\lim_{x\to 0^+} \frac{1}{x^n} = \infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{x^n} = \begin{cases} \infty, & n = \text{even} \\ -\infty, & n = \text{odd} \end{cases}.$$

From Trigonometry, we know that

$$\lim_{x\to(\pi/2+\pi k)^+} \tan(x) = -\infty, \qquad \lim_{x\to(\pi/2+\pi k)^-} \tan(x) = \infty,$$

$$\lim_{x\to(\pi/2+2\pi k)^+} \sec(x) = -\infty, \quad \lim_{x\to(\pi/2+2\pi k)^-} \sec(x) = \infty, \quad \lim_{x\to(3\pi/2+2\pi k)^+} \sec(x) = \infty, \quad \lim_{x\to(3\pi/2+2\pi k)^-} \sec(x) = -\infty$$

for any integer k, which occurs exactly when $\cos(x) = 0$. Likewise,

$$\lim_{x\to(\pi k)^+} \cot(x) = \infty, \qquad \lim_{x\to(\pi k)^-} \cot(x) = -\infty,$$

$$\lim_{x\to(2\pi k)^+} \csc(x) = \infty, \quad \lim_{x\to(2\pi k)^-} \csc(x) = -\infty, \quad \lim_{x\to(\pi+2\pi k)^+} \csc(x) = -\infty, \quad \lim_{x\to(\pi+2\pi k)^-} \csc(x) = \infty$$

for any integer k, which occurs exactly when $\sin(x) = 0$. Finally, $\ln(x)$ also has a vertical asymptote since

$$\lim_{x\to 0^+} \ln(x) = -\infty.$$

Example 2.7.1. Find the following limits.

(a) $\displaystyle\lim_{x\to 2} \frac{3x-2}{x-2}$.

Our limit has the form $\frac{4}{0}$, which means that the graph has a vertical asymptote at $x = 2$. Now, to compute $\displaystyle\lim_{x\to 2^+} \frac{3x-2}{x-2}$, we choose values of x to the right of 2 to determine the approach toward the asymptote. Note that for $x = 3$, we have the $\frac{3x-2}{x-2} = \frac{3(3)-2}{(3)-2} = \frac{7}{1} > 0$. Thus, $\displaystyle\lim_{x\to 2^+} \frac{3x-2}{x-2} = \infty$. On the other hand, using $x = 1$, we see that $\frac{3x-2}{x-2} = \frac{3(1)-2}{(1)-2} = \frac{1}{-1} < 0$. Thus, $\displaystyle\lim_{x\to 2^+} \frac{3x-2}{x-2} = -\infty$. Even though the graph has a vertical asymptote at $x = 2$, it holds that $\displaystyle\lim_{x\to 2} \frac{3x-2}{x-2}$ $\boxed{\text{does not exist}}$.

(b) $\displaystyle\lim_{x\to 2} \frac{3x-2}{(x-2)^2}$.

Mimicking what we say before, we note that $\lim_{x\to 2^+} \frac{3x-2}{(x-2)^2} = \lim_{x\to 2^-} \frac{3x-2}{(x-2)^2} = \infty$. Therefore, $\boxed{\lim_{x\to 2} \frac{3x-2}{(x-2)^2} = \infty}$.

(c) $\lim_{x\to 2} -\frac{3x^2-7x+2}{(x-2)^3}$.

Note that the limit has the form $\frac{0}{0}$, so it is not clear what the limit will be. We need to factor the numerator. Note that $\lim_{x\to 2} -\frac{3x^2-7x+2}{(x-2)^3} = -\lim_{x\to 2} \frac{(3x-1)(x-2)}{(x-2)^3} = -\lim_{x\to 2} \frac{(3x-1)}{(x-2)^2} = \boxed{-\infty}$.

Often times it is worthwhile to ask what is the end behavior of a function, that is, what is the behavior of the function when x grows arbitrarily large (it grows bigger and bigger) or when x grows arbitrarily small. In terms of limits, we ask what are $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. For example, in Biology, scientists can model population growth of bacteria, bunnies, or humans with functions. But a specific ecosystem can only support a certain volume of population before resources (e.g. food, water) grow too scarce. This is called the *carrying capacity* of the ecosystem. If $f(x)$ is a function which models the growth of a population, then $\lim_{x\to\infty} f(x)$ represents the carrying capacity of the ecosystem.

In general, the limits $\lim_{x\to\pm\infty} f(x)$ are called the **end behavior** of f. Geometrically, if $\lim_{x\to\pm\infty} f(x)$ exists, then $f(x)$ admits a **horizontal asymptote**.

For many functions, the end behavior is infinite. For example, if n is a positive integer,

$$\lim_{x\to\infty} x^n = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} x^n = \begin{cases} \infty, & n = \text{even} \\ -\infty, & n = \text{odd} \end{cases}.$$

For polynomials in general, the leading term determines the end behavior, that is,

$$\lim_{x\to\pm\infty} (a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0) = \lim_{x\to\pm\infty} a_n x^n.$$

Example 2.7.2. Find the following limits.

(a) $\lim_{x\to\infty} x^3 - 2x^2 + 1 = \lim_{x\to\infty} x^3 = \boxed{\infty}$.

(b) $\lim_{x\to\infty} -2x^4 + 3x^2 - x + 2 = \lim_{x\to\infty} (-2x^4) = -2 \lim_{x\to\infty} x^4 = -2(\infty) = \boxed{-\infty}$.

The end behavior for rational functions can be very different than polynomials.

Proposition 2.7.3. $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$ *for* $n > 0$.

Let r be any real number and let $c > 0$. Then

$$\begin{array}{cccc} r + \infty = \infty, & r + (-\infty) = -\infty, & \infty + \infty = \infty, & (-\infty) + (-\infty) = -\infty \\ c\infty = (-c)(-\infty) = \infty, & c(-\infty) = (-c)(\infty) = -\infty, & \infty \cdot \infty = \infty, & (-\infty)(-\infty) = \infty \\ r/\infty = 0, & r/(-\infty) = 0, & c^\infty = \infty, (c > 1), & c^{-\infty} = 0, (c > 1) \end{array}$$

Beware the following indeterminate forms:

$$\infty - \infty, \quad \frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty.$$

On the other hand, expressions like $\dfrac{r}{0}$ are NOT indeterminant forms. If a limit resembles this form, then the graph has a vertical asymptote here as seen before.

The next several examples have the indeterminant form $\dfrac{\infty}{\infty}$.

Example 2.7.4. Find $\displaystyle\lim_{x\to\infty} \frac{8x+6}{3x-1}$.

We can use the result that $\displaystyle\lim_{x\to\infty} \frac{1}{x^n} = 0$ to evaluate the limit. We do this by multiplying the numerator and denominator by $1/x$:

$$\begin{aligned} \lim_{x\to\infty} \frac{8x+6}{3x-1} &= \lim_{x\to\infty} \left(\frac{8x+6}{3x-1}\right) \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{8+\frac{6}{x}}{3-\frac{1}{x}} \\ &= \frac{8 + 6\displaystyle\lim_{x\to\infty}\frac{1}{x}}{3 - \displaystyle\lim_{x\to\infty}\frac{1}{x}} = \frac{8+6\cdot 0}{3-0} = \boxed{\frac{8}{3}}. \end{aligned}$$

Example 2.7.5. Find $\displaystyle\lim_{x\to\infty} \frac{3x+2}{4x^3-1}$.

We'll try the same trick that we used last time, but instead we'll multiply by $\dfrac{1}{x^3}$:

$$\lim_{x\to\infty} \frac{3x+2}{4x^3-1} = \lim_{x\to\infty} \left(\frac{3x+2}{4x^3-1}\right) \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{3}{x^2}+\frac{2}{x^3}}{4-\frac{1}{x^3}} = \frac{3\cdot 0 + 2\cdot 0}{4-0} = \frac{0}{4} = \boxed{0}.$$

Example 2.7.6. Find $\displaystyle\lim_{x\to\infty} \frac{x^2+2}{4x-3}$.

$$\lim_{x\to\infty} \frac{x^2+2}{4x-3} = \lim_{x\to\infty} \left(\frac{x^2+2}{4x-3}\right) \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x+\frac{2}{x}}{4-\frac{3}{x}} = \frac{\infty}{4} = \boxed{\infty}$$

Theorem 2.7.7 (Horizontal Asymptotes of Rational Functions)**.** *Let* $R(x) = \dfrac{P(x)}{Q(x)}$.

(a) If $\deg P > \deg Q$, *then* R *has no horizontal asymptotes.*

(b) If $\deg P < \deg Q$, *then* $\displaystyle\lim_{x\to\pm\infty} R(x) = 0$.

(c) If $\deg P = \deg Q$, *then* $\displaystyle\lim_{x\to\pm\infty} R(x) = p/q$, *where* p *and* q *are the leading coefficients of* P *and* Q, *respectively.*

Example 2.7.8. Evaluate $\lim_{x\to\infty} \ln\left(\frac{x}{x^2+1}\right)$.

Let $t = \frac{x}{x^2+1}$. By what we have seen before, we have $\lim_{x\to\infty} t = \lim_{x\to\infty} \frac{x}{x^2+1} = 0$. Also, since $t = \frac{x}{x^2+1} > 0$ when $x > 0$, we see that $t \to 0^+$ when $x \to \infty$. Since ln is continuous, we have

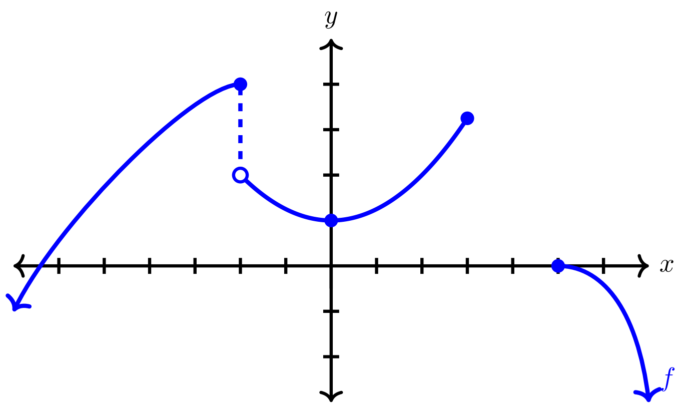
$$\lim_{x\to\infty} \ln\left(\frac{x}{x^2+1}\right) = \lim_{t\to 0^+} \ln(t) = \boxed{-\infty}.$$

[i]See §4.6 Limits at Infinity and Asymptotes in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

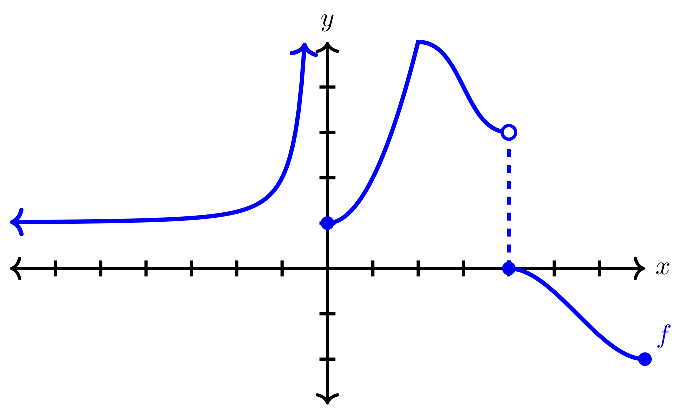
For Exercises 1–2, the function f is illustrated below. Compute the given limit.



♠ 1. $\lim_{x\to-\infty} f(x)$

♠ 2. $\lim_{x\to\infty} f(x)$

For Exercises 3–6, the function f is illustrated below. Compute the given limit.



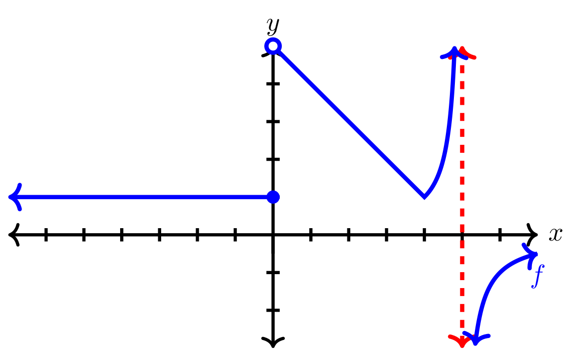
♠ 3. $\lim_{x\to-\infty} f(x)$

♠ 4. $\lim_{x\to0^-} f(x)$

♠ 5. $\lim_{x\to0^+} f(x)$

♠ 6. $\lim_{x\to\infty} f(x)$

For Exercises 7–6, the function f is illustrated below. Compute the given limit.



[i]See §2.4 Continuity in OpenStax to find the corresponding section.

♠ 7. $\lim_{x\to-\infty} f(x)$ ♠ 8. $\lim_{x\to5^-} f(x)$ ♠ 9. $\lim_{x\to5^+} f(x)$ ♠ 10. $\lim_{x\to\infty} f(x)$

For Exercises 11–18, the function f is given as

$$f(x) = \begin{cases} \dfrac{1}{x}, & x < 0 \\ 5+x, & 0 \le x \le 1 \\ 6\sqrt{x}, & x > 1. \end{cases}$$

Compute the given limit.

♠ 11. $\lim_{x\to-\infty} f(x)$ ♠ 12. $\lim_{x\to0^-} f(x)$ ♠ 13. $\lim_{x\to0^+} f(x)$ ♠ 14. $\lim_{x\to0} f(x)$

♠ 15. $\lim_{x\to1^-} f(x)$ ♠ 16. $\lim_{x\to1^+} f(x)$ ♠ 17. $\lim_{x\to1} f(x)$ ♠ 18. $\lim_{x\to\infty} f(x)$

For Exercises 19–44, evaluate the limit.

19. $\lim_{x\to-3^-} \dfrac{3x^2+18x-81}{2x^2+7x+3}$ 20. $\lim_{x\to-3^+} \dfrac{3x^2+18x-81}{2x^2+7x+3}$ 21. $\lim_{x\to-3} \dfrac{3x^2+18x-81}{2x^2+7x+3}$

22. $\lim_{x\to-1/2^+} \dfrac{3x^2+18x-81}{2x^2+7x+3}$ 23. $\lim_{x\to-1/2^-} \dfrac{3x^2+18x-81}{2x^2+7x+3}$ 24. $\lim_{x\to-1/2} \dfrac{3x^2+18x-81}{2x^2+7x+3}$

25. $\lim_{x\to-4} \dfrac{x^2-5x+6}{x^2+2x-8}$ ♠ 26. $\lim_{x\to-2^+} \dfrac{x+1}{x+2}$ ♠ 27. $\lim_{x\to2^+} \dfrac{x^2-2x}{x^2-4x+4}$

♠ 28. $\lim_{x\to-2^-} \dfrac{2x^2+7x-4}{x^2+x-2}$ ♠ 29. $\lim_{x\to2^+} \dfrac{x^2+7x+10}{2x^2-2x-12}$ 30. $\lim_{x\to1^+} \dfrac{2x^2-9x-5}{x^2+x-2}$

♠ 31. $\lim_{x\to9} \dfrac{x-8}{(x-9)^2}$ 32. $\lim_{x\to6^-} \dfrac{e^x}{(x-6)^3}$ ♠ 33. $\lim_{x\to6^+} \ln(x^2-36)$

♠ 34. $\lim_{x\to\pi^-} \csc(x)$ ♠ 35. $\lim_{x\to-\infty} \dfrac{x-3}{x^2+2}$ ♠ 36. $\lim_{x\to\infty} \dfrac{x^4-7x^2+x}{x^3-x+2}$

37. $\lim_{x\to\infty} \dfrac{x+x^7}{10x^2-x^7}$ ♠ 38. $\lim_{x\to\infty} \dfrac{1+x-2x^2}{3x^2-4}$ 39. $\lim_{x\to-\infty} \dfrac{1+x-2x^2}{3x^2-4}$

40. $\lim_{x\to\infty} \dfrac{1-x+x^2-x^3}{4x^3-3x^2+2x-1}$ 41. $\lim_{x\to-\infty} \dfrac{1-x+x^2-x^3}{4x^3-3x^2+2x-1}$ 42. $\lim_{x\to\infty} \dfrac{4+x-2x^2}{-2-x+3x^2}$

43. $\lim_{x\to-\infty} \dfrac{4+x-2x^2}{-2-x+3x^2}$ 44. $\lim_{x\to\infty} \dfrac{2x^2-x^3}{1+3x^2-x}$

"To Infinity, and Beyond!" – Buzz from Toy Story 2

Lecture Videos

Vertical and Horizontal Asymptotes

Limits at Infinity and the Squeeze Theorem

Limits at Infinity Involving Radicals

Limits at Infinity Involving Exponentials

The End Behavior of Dampened Harmonic Motion

Limits at Infinity Involving Arctangent

2.8 Limits at Infinity II

Example 2.8.1. Find the vertical and horizontal asymptotes of the following.

(a) $y = \dfrac{2x+1}{x-2}$.

We see that y has a vertical asymptote at $\boxed{x=2}$ since $\lim\limits_{x\to 2^+} \dfrac{2x+1}{x-2} = \infty$, and y has a horizontal asymptote at $\boxed{y=2}$ since $\lim\limits_{x\to\pm\infty} \dfrac{2x+1}{x-2} = \dfrac{2}{1} = 2$.

(b) $y = \dfrac{x^2-4}{2x^2-3x-2}$.

Note that $y = \dfrac{x^2-4}{2x^2-3x-2} = \dfrac{(x-2)(x+2)}{(x-2)(2x+1)}$. Therefore, y has a vertical asymptote at $\boxed{x = -\dfrac{1}{2}}$, since $\lim\limits_{x\to -1/2^+} \dfrac{(x-2)(x+2)}{(x-2)(2x+1)} = \infty$, but not at $x = 2$ since $\lim\limits_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)(2x+1)} = \lim\limits_{x\to 2} \dfrac{x+2}{2x+1} = \dfrac{2+2}{2(2)+1} = \dfrac{4}{5}$. On the other hand, y has a horizontal asymptote at $\boxed{y = \dfrac{1}{2}}$ since $\lim\limits_{x\to\pm\infty} \dfrac{x^2-4}{2x^2-3x-2} = \dfrac{1}{2}$.

Example 2.8.2. Evaluate the limit $\lim\limits_{x\to\infty} e^{1/x}$.

Let $\exp(x) = e^x$. Then exp is a continuous function and

$$\lim_{x\to\infty} e^{1/x} = \lim_{x\to\infty} \exp\left(\frac{1}{x}\right) = \exp\left(\lim_{x\to\infty} \frac{1}{x}\right) = \exp(0) = e^0 = \boxed{1}.$$

Example 2.8.3. Evaluate $\lim\limits_{x\to\infty} \dfrac{\sin^2(x)}{x^2+1}$.

Notice that $\lim\limits_{x\to\infty} \sin^2(x)$ does not exist, so we cannot use the quotient limit property. Instead, we notice that for all values of x, $\dfrac{\sin^2(x)}{x^2+1} \geq 0$, since $\sin^2(x)$ and x^2+1 are always positive (except when $\sin^2(x) = 0$). Also, $\sin(x) \leq 1 \Longrightarrow \sin^2(x) \leq 1$. Hence,

$$0 \leq \frac{\sin^2(x)}{x^2+1} \leq \frac{1}{x^2+1}, \quad \text{for all } x.$$

Since $\lim\limits_{x\to\infty} \dfrac{1}{x^2+1} = 0$, the Squeeze Lemma gives us that $\lim\limits_{x\to\infty} \dfrac{\sin^2(x)}{x^2+1} = \boxed{0}$.

Example 2.8.4. Find the following limits.

(a) $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \left(\frac{1/x}{1/\sqrt{x^2}}\right) \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+0}} = \boxed{1}
\end{aligned}$$

(b) $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to\infty} \frac{\sqrt{x^2(2+1/x^2)}}{x(3-5/x)} = \lim_{x\to\infty} \frac{x\sqrt{2+1/x^2}}{x(3-5/x)} \\
&= \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{(3-5/x)} = \frac{\sqrt{2+0}}{3-5(0)} = \boxed{\frac{\sqrt{2}}{3}}
\end{aligned}$$

(c) $\lim\limits_{x\to\infty} (\sqrt{x^2-1} - x)$.

This limit has the indeterminant form $\infty - \infty$. To avoid this, we will rationalize the numerator.

$$\begin{aligned}
\lim_{x\to\infty} (\sqrt{x^2-1} - x) &= \lim_{x\to\infty} (\sqrt{x^2-1} - x) \left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}+x}\right) = \lim_{x\to\infty} \frac{(x^2-1)-x^2}{\sqrt{x^2-1}+x} \\
&= \lim_{x\to\infty} \frac{-1}{\sqrt{x^2-1}+x} = \boxed{0}.
\end{aligned}$$

(d) $\lim\limits_{x\to-\infty} (\sqrt{x^2-1} - x)$.

This limit has the form $\infty + \infty = \infty$. Therefore, $\lim\limits_{x\to-\infty} (\sqrt{x^2-1} - x) = \boxed{\infty}$.

It should be mentioned that it is not always the case that $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x)$, for example, $\lim\limits_{x\to-\infty} e^x = 0$ but $\lim\limits_{x\to\infty} e^x = \infty$.

Proposition 2.8.5 (Horizontal Asymptotes of Exponentials)**.** *For any $a > 1$,*

$$\lim_{x\to-\infty} a^x = 0.$$

Example 2.8.6. Evaluate the following limits.

(a) $\lim_{x\to 0^-} e^{1/x}$.

Let $y = 1/x$. Then $\lim_{x\to 0^-} y = \lim_{x\to 0^-} 1/x = -\infty$. Thus,

$$\lim_{x\to 0^-} e^{1/x} = \lim_{y\to -\infty} e^y = \boxed{0}.$$

(b) $\lim_{x\to\infty} \frac{2e^x}{e^x - 5}$.

As the limit is of the from $\frac{\infty}{\infty}$, we work to avoid this indeterminant form. Instead, we compute

$$\lim_{x\to\infty} \frac{2e^x}{e^x - 5} = \lim_{x\to\infty} \frac{2e^x}{e^x(1-5e^{-x})} = \lim_{x\to\infty} \frac{2}{1-5e^{-x}} = \frac{2}{1-5\cdot 0} = \boxed{2}.$$

(c) $\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{4e^{3x} + e^{-3x}}$.

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{4e^{3x} + e^{-3x}} = \lim_{x\to\infty} \frac{e^{3x}(1-e^{-6x})}{e^{3x}(4+e^{-6x})} = \lim_{x\to\infty} \frac{1-e^{-6x}}{4+e^{-6x}} = \frac{1-0}{4+0} = \boxed{\tfrac{1}{4}}.$$

Example 2.8.7. Evaluate $\lim_{x\to\infty} (e^{-x}\sin(x))$.

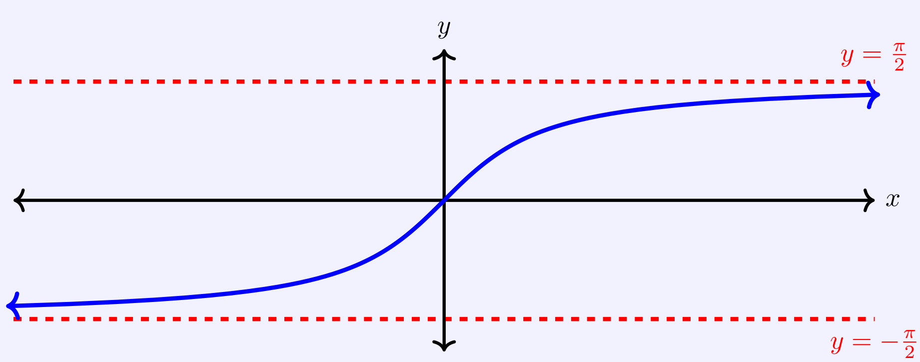
Notice that $\lim_{x\to\infty} \sin(x)$ does not exist, so we cannot use the quotient limit property. Instead, we notice that for all values of x, $-1 \le \sin(x) \le 1$. Since $e^{-x} > 0$ for all x, we have

$$-e^{-x} \le e^{-x}\sin(x) \le e^{-x}, \quad \text{for all } x.$$

Since $\lim_{x\to\infty} \pm e^{-x} = \pm \lim_{x\to\infty} e^{-x} = 0$, the Squeeze Lemma gives us that $\lim_{x\to\infty} e^{-x}\sin(x) = \boxed{0}$.

Proposition 2.8.8 (Horizontal Asymptotes of Arctangent)**.**

$$\lim_{x\to\pm\infty} \arctan(x) = \pm\frac{\pi}{2}$$



Example 2.8.9. Evaluate following limits.

(a) $\lim_{x\to\infty} \sin(\tan^{-1}(x))$.

Since sine is continuous, we have

$$\lim_{x\to\infty} \sin(\tan^{-1}(x)) = \sin\left(\lim_{x\to\infty} \tan^{-1}(x)\right) = \sin\left(\frac{\pi}{2}\right) = \boxed{1}.$$

Alternatively, if we denote $\tan^{-1}(x) = \theta$, then $\tan\theta = \frac{x}{1}$. Then $\sin(\tan^{-1}(x)) = \sin\theta = \frac{x}{\sqrt{x^2+1}}$. Therefore,

$$\lim_{x\to\infty} \sin(\tan^{-1}(x)) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1,$$

as we saw earlier.

(b) $\lim_{x\to-\infty} \arctan(e^{-x})$.

As $x \to -\infty$, we have that $e^{-x} \to \infty$ also. Let $t = e^{-x}$. Since arctan is continuous, we then have that

$$\lim_{x\to-\infty} \arctan(e^{-x}) = \lim_{t\to\infty} \arctan(t) = \boxed{\frac{\pi}{2}}.$$

(c) $\lim_{x\to2^-} \arctan\left(\frac{1}{x-2}\right)$.

Let $t = \frac{1}{x-2}$. So, $\lim_{x\to2^-} t = \lim_{x\to2^-} \frac{1}{x-2} = -\infty$. Therefore,

$$\lim_{x\to2^-} \arctan\left(\frac{1}{x-2}\right) = \lim_{t\to-\infty} \arctan(t) = \boxed{-\frac{\pi}{2}}.$$

[i]See §4.6 Limits at Infinity and Asymptotes in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–27, compute the given limit.

1. $\displaystyle\lim_{x\to\infty} \frac{x^2}{\sqrt{4x^4+1}}$

2. $\displaystyle\lim_{x\to-\infty} \frac{x^2}{\sqrt{4x^4+1}}$

3. $\displaystyle\lim_{x\to\infty} (\sqrt{9x^2+x}-3x)$

♠ 4. $\displaystyle\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{9t-t^2}$

♠ 5. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{9x^6-x}}{x^3+5}$

♠ 6. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{9x^6-x}}{x^3+5}$

♠ 7. $\displaystyle\lim_{x\to\infty} (\sqrt{9x^2+x}-3x)$

♠ 8. $\displaystyle\lim_{x\to-\infty} (x+\sqrt{x^2+4x})$

♠ 9. $\displaystyle\lim_{x\to\infty} (\sqrt{x^2+1}$

♠ 10. $\displaystyle\lim_{x\to\infty} \frac{x+2}{\sqrt{64x^2+1}}$

♠ 11. $\displaystyle\lim_{x\to-\infty} \frac{x+2}{\sqrt{64x^2+1}}$

12. $\displaystyle\lim_{x\to\infty} \frac{3e^x+4}{e^x-1}$

13. $\displaystyle\lim_{x\to-\infty} \frac{3e^x+4}{e^x-1}$

14. $\displaystyle\lim_{x\to\infty} \frac{1+3e^x}{2-4e^x}$

♠ 15. $\displaystyle\lim_{x\to\infty} \frac{5-e^x}{5+2e^x}$

♠ 16. $\displaystyle\lim_{x\to\pi/2^+} 9e^{\tan(x)}$

♠ 17. $\displaystyle\lim_{x\to\infty} \tan^{-1}(x^3-x^5)$

♠ 18. $\displaystyle\lim_{x\to\infty} \tan^{-1}(e^x)$

19. $\displaystyle\lim_{x\to\infty} \tan^{-1}(\ln x)$

20. $\displaystyle\lim_{x\to\infty} \cos(\tan^{-1}(x))$

21. $\displaystyle\lim_{x\to\infty} \sin(\tan^{-1}(x))$

22. $\displaystyle\lim_{x\to\infty} \sin(2\tan^{-1}(x))$

23. $\displaystyle\lim_{x\to\infty} \cos(2\tan^{-1}(x))$

♠ 24. $\displaystyle\lim_{x\to\infty} e^{-5x}\cos x$

25. $\displaystyle\lim_{x\to0^+} \tan^{-1}(\ln x)$

26. $\displaystyle\lim_{x\to\infty} \cos\left(\frac{1}{2}\tan^{-1}(x)\right)$

27. $\displaystyle\lim_{x\to\infty} \sin\left(\frac{1}{2}\tan^{-1}(x)\right)$

♠ 28. If

$$\frac{9e^x-24}{3e^x} < f(x) < \frac{3\sqrt{x}}{\sqrt{x-1}}$$

for all $x>1$, then find $\displaystyle\lim_{x\to\infty} f(x)$.

For Exercises 29–33, find the horizontal and vertical asymptotes of the function f, if any.

29. $f(x) = \dfrac{3x^2+18x-81}{2x^2+7x+3}$

♠ 30. $y = \dfrac{x^2-x}{x^2-6x+5}$

31. $f(x) = \dfrac{x+x^7}{10x^2-x^7}$.

♠ 32. $f(x) = \dfrac{3e^x}{e^x-6}$

33. $f(x) = \dfrac{1+x-2x^2}{3x^2-4}$

"Don't limit yourself. Many people limit themselves to what they think they can do. You can go as far as your mind lets you. What you believe, remember, you can achieve" – Mary Kay Ash

Lecture Videos

Tangent Lines

Instantaneous Rate of Change and Velocity

2.9 Tangent Lines

For a circle, a **tangent line** is a line which intersects the circle at a unique point and a **secant line** is a line which intersects a circle in two points. In this section we want to talk about the tangent line of a curve, aka graph of a function. Intuitively, the tangent line to an arbitrary curve at a point P on the curve should touch the curve at P, but not at any points nearby, and should indicate the direction of the curve.

Let $f(x)$ be a function and let P be a point on the graph of $f(x)$. Then we wish to construct the tangent line to f through P. A line is determined by its slope and a point on the line. Certainly, P will be on its tangent line, so we need to be able to find the slope of these tangent lines. Let R be any other point of the same curve. Then we can calculate the slope of the secant line connecting P and R. If $P = (a, f(a))$ and $R = (b, f(b))$, then the slope of the secant line through P and R is just the average rate of change of the function f on the interval $[a, b]$. If we move R closer and closer to P, then the corresponding secant lines will better approximate the tangent line at P. So, at $R \to P$, we have that $b \to a$ and the slope of the line approaches the instantaneous rate of change of the function f at $x = a$.

Definition 2.9.1. The **tangent line** of the graph $y = f(x)$ at the point $(a, f(a))$ is the line through this point having slope

$$m = \lim_{b\to a} \frac{f(b) - f(a)}{b - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists. If this limit does not exists, then this is no tangent at the point. The slope of the tangent line at a point is also called the **slope of the curve** at the point. It indicates the direction of the curve at that point.

Example 2.9.2. Consider the graph of $f(x) = x^2 + 2$.

(a) Find the slope and equation of the secant line through the points where $x = -1$ and $x = 2$.

We first find the points: $f(-1) = (-1)^2 + 2 = 3$ and $f(2) = (2)^2 + 2 = 6$. So the secant is the line through the points $(-1, 3)$ and $(2, 6)$. The slope of the secant line is the average rate of change of f on $[-1, 2]$:

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{6 - 3}{3} = \frac{3}{3} = 1.$$

Remember if m is the slope of a line and (x_1, y_1) is a point on the line, then the equation for the line can be given as $(y - y_1) = m(x - x_1)$, which is know as the **point-slope form** of a line. Because the representation can change depending on our choice of point, we prefer a unique representation of a line. Any equation of a line can be written uniquely in **slope-intercept form**, $y = mx+b$, where m is the slope of the line and b is the y-intercept of the line.

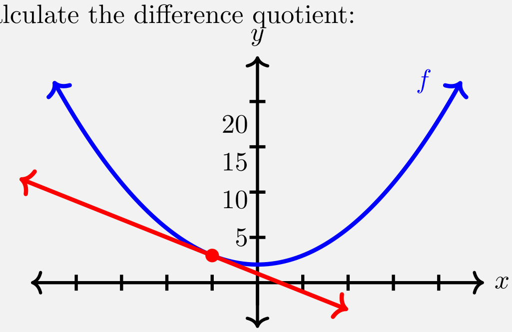
Using the point-slope form, our line can be represented at $y - 3 = 1(x - (-1))$. We proceed to put the line into slope-intercept form.

$$y - 3 = 1(x+1) \Rightarrow y = x + 4.$$

(b) Find the slope and equation of the tangent line at $x = -1$.

This example is just like the previous part, but instead the slope of the tangent line is the instantaneous rate of change at $x = -1$. We calculate the difference quotient:

$$\begin{aligned}
\lim_{b\to -1} \frac{f(b) - f(-1)}{b - (-1)} &= \lim_{b\to -1} \frac{(b^2+2) - 3}{b+1} \\
&= \lim_{b\to -1} \frac{b^2 - 1}{b+1} \\
&= \lim_{b\to -1} \frac{(b+1)(b-1)}{b+1} \\
&= \lim_{b\to -1} (b-1) \\
&= -1 - 1 = -2
\end{aligned}$$



So the slope of the tangent line is -2. Next, we find the equation of the line like before:

$$(y-3) = -2(x+1) \Rightarrow y - 3 = -2x - 2 \Rightarrow \boxed{y = -2x + 1}.$$

One of the main applications of Calculus is determining how one variable changes in relation to another. A marketing manager wants to know how profit changes with respect to the amount spent on advertising, while a physician wants to know how a patient's reaction to a drug changes with respect to the dose.

Definition 2.9.3. Let $f(x)$ be a function defined on the interval $[a, b]$. Then the **average rate of change** of $f(x)$ with respect to x for a function f as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a} \tag{2.9.1}$$

This quotient is often referred to as the **difference quotient**.

Now, the difference quotient is none other than the slope formula!

Example 2.9.4. Suppose a car is stopped at a traffic light. When the light turns green, the car begins to move along a straight road. Assume that the distance traveled by the car is given by the function $s(t) = 3t^2$, for $0 \le t \le 15$, where t is the time in seconds and $s(t)$ is the distance in feet. What is the exact speed of the car at $t = 10$?

Let us determine the exact speed of the car at $t = 10$. One idea is to use the average speed formula but choose the interval $[10, 10]$. Then we have

$$\frac{s(10) - s(10)}{10 - 10} = \frac{0}{0}.$$

Well, that didn't work... the problem is that the difference quotient measures a change of two points. In order to find the *instantaneous* change at an exact point in time, we will approximate the exact speed using the average speed formula but choose smaller and smaller interval starting at $t = 10$. For example,

$$\begin{array}{ll} t=10 \text{ to } t=10.1 & \dfrac{s(10.1)-s(10)}{10.1-10}=\dfrac{306.03-300}{0.1}=60.3 \\ t=10 \text{ to } t=10.01 & \dfrac{s(10.01)-s(10)}{10.01-10}=\dfrac{300.6003-300}{0.01}=60.03 \\ t=10 \text{ to } t=10.001 & \dfrac{s(10.001)-s(10)}{10.001-10}=\dfrac{300.060003-300}{0.001}=60.003 \end{array}$$

The results from this table tells us that the closer of a point to 10 that we choose, the better approximation of the instantaneous speed at $t = 10$, which looks like it is about 60 mph. Is there a best approximation? Of course, that is what a limit is!

Before we can start calculating limits, we need to play around a little bit more. We are approximating the exact speed at $t = 10$ by choosing a small interval starting at 10 and end a little bit bigger than 10. Let the length of this interval be denoted as h, so the smaller we make h, the better our approximation is. Thus, the average speed of the car on the interval $[10, 10 + h]$ is

$$\frac{s(10+h)-s(10)}{(10+h)-10}=\frac{s(10+h)-s(10)}{h}.$$

Thus, the instantaneous speed of the car at time 10 is

$$\lim_{h\to 0}\frac{s(10+h)-s(10)}{h}.$$

Now, the denominator goes to zero as $h \to 0$. Also, $s(10 + h) - s(10) \to 0$ as $h \to 0$. Thus, we will need to simplify the expression if we hope to calculate the limit.

So,

$$\begin{aligned} \lim_{h\to 0}\frac{s(10+h)-s(10)}{h} &= \lim_{h\to 0}\frac{3(10+h)^2-3(10)^2}{h}=\lim_{h\to 0}\frac{3(100+20h+h^2)-3(100)}{h} \\ &= \lim_{h\to 0}\frac{(300+60h+3h^2)-300}{h}=\lim_{h\to 0}\frac{60h+3h^2}{h} \\ &= \lim_{h\to 0}\frac{h(60+3h)}{h}=\lim_{h\to 0}60+3h=60 \end{aligned}$$

Therefore, the instantaneous speed of the car at $t = 10$ is $\boxed{\text{60 mph}}$, like we had guessed.

Definition 2.9.5. The **instantaneous rate of change** for a function f when $x = a$ is

$$\lim_{x\to a}\frac{f(a+h)-f(a)}{h}, \tag{2.9.2}$$

provided the limit exists.

Alternatively, we can define the instantaneous rate of change to more closely resemble the average rate of change formula, that is, the instantaneous rate of change of f at $x = a$ is

$$\lim_{b\to a}\frac{f(b)-f(a)}{b-a}.$$

Both definitions are equivalent and the student is free to use either of the two alternate forms that they choose.

In the example just discussed, we saw that the instantaneous rate of change gave the speed of the car.

But speed is always positive, while instantaneous rate of change can be positive or negative. We therefore will refer to **velocity** when we want to consider not only how fast something is moving but also in what direction it is moving. We say the forward direction is the positive direction and the backward direction is the negative direction.

Example 2.9.6. The distance in feet of an object from a starting point is given by $s(t) = 2t^2 - 5t + 40$, where t is the time in seconds.

(a) Find the average velocity of the object from 2 seconds to 4 seconds.

The average velocity is

$$\frac{s(4)-s(2)}{4-2} = \frac{(2(4)^2-5(4)+40)-(2(2)^2-5(2)+40)}{2} = \frac{52-38}{2} = \frac{14}{2} = \boxed{\text{7 feet per second}}.$$

(b) Find the instantaneous velocity at 4 seconds.

For $t = 4$, the instantaneous velocity is

$$\begin{aligned}
\lim_{h\to 0}\frac{s(4+h)-s(4)}{h} &= \lim_{h\to 0}\frac{(2(4+h)^2-5(4+h)+40)-52}{h} \\
&= \lim_{h\to 0}\frac{(2(16+8h+h^2)-20-5h+40)-52}{h} \\
&= \lim_{h\to 0}\frac{(32+16h+2h^2+20-5h)-52}{h} \\
&= \lim_{h\to 0}\frac{2h^2+11h+52-52}{h} \\
&= \lim_{h\to 0}\frac{h(2h+11)}{h} \\
&= \lim_{h\to 0}(2h+11) = 11
\end{aligned}$$

So, the instantaneous velocity at $t = 4$ is $\boxed{\text{11 feet per second}}$.

[i] See §2.1 A Preview of Calculus in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–5, a water tank's volume at given times, as it drains, are listed in the table below. Find the average rate of which the tank is draining on the given time interval.

Volume (gallon)	5000	3445	2210	1225	505	135	0
Time (min)	0	5	10	15	20	25	30

1. $[5, 15]$
♠ 2. $[10, 15]$
3. $[15, 20]$
♠ 4. $[15, 25]$
5. $[15, 30]$

For Exercises 6–9, a cardiac monitor is used to measure the heart rate of a patient after surgery, and the number of heartbeats after t minutes is recorded in the table below. Find the average rate of heartbeats on the given time interval.

Heartbeats	2523	2659	2795	2934	3066
Time (min)	36	38	40	42	44

6. $[36, 42]$
♠ 7. $[38, 42]$
8. $[40, 42]$
9. $[42, 44]$

For Exercises 10–19, a particle's position at given times are listed in the table below. Find the average velocity on the given time interval.

Position (ft)	0	20	50	75	80
Time (min)	0	5	10	15	20

10. $[0, 5]$
11. $[0, 10]$
12. $[0, 15]$
13. $[0, 20]$
14. $[5, 10]$
15. $[5, 15]$
16. $[5, 20]$
17. $[10, 15]$
18. $[10, 20]$
19. $[15, 20]$

♠ 20. The point $P = (9, -4)$ lies on the curve

$$f(x) = \frac{4}{8 - x}.$$

If $Q = (a, f(a))$ is likewise a point on the graph of f, compute the slope of the secant line passing through P and Q for:

$$a = 8.9, 8.99, 8.999, 8.9999, 9.1, 9.01, 9.001, 9.0001.$$

Use these secant slope to estimate the slope of the tangent line to f at P. Find the equation of this tangent line.

The following Desmos calculator may prove helpful

♠ 21. The point $P = (6, 1)$ lies on the curve

$$g(x) = \sqrt{x - 5}.$$

If $Q = (a, f(a))$ is likewise a point on the graph of g, compute the slope of the secant line passing through P and Q for:

$$a = 5.5, 5.9, 5.99, 5.999, 6.5, 6.1, 6.01, 6.001.$$

Use these secant slope to estimate the slope of the tangent line to g at P. Find the equation of this tangent line.

The following Desmos calculator may prove helpful

For Exercises 22–25, if a ball is thrown into the air with an initial velocity of 46 $\frac{\text{ft}}{\text{s}}$, its height in feet at time t seconds after launch is given by $s(t) = 46t - 16t^2$. Find the average velocity on the given time interval.

♠ 22. $[2, 2.5]$ ♠ 23. $[2, 2.1]$ ♠ 24. $[2, 2.05]$ ♠ 25. $[2, 2.01]$

♠ 26. Use the average velocities computed in Exercises 22-25 to estimate the instantaneous velocity of the ball at $t = 2$.

For Exercises 27–31, if a ball is thrown into the air of an alien planet with an initial velocity of 68 $\frac{\text{m}}{\text{s}}$, its height in feet at time t seconds after launch is given by $s(t) = 68t - 1.86t^2$. Find the average velocity on the given time interval.

♠ 27. $[1, 2]$ ♠ 28. $[1, 1.5]$ ♠ 29. $[1, 1.1]$ ♠ 30. $[1, 1.01]$ ♠ 31. $[1, 1.001]$

♠ 32. Use the average velocities computed in Exercises 27-31 to estimate the instantaneous velocity of the ball at $t = 1$.

For Exercises 33–36, a particle's position in centimeters, at given times in seconds, is given by the function $s(t) = 4\sin(\pi t) + 3\cos(\pi t)$. Find the average velocity on the given time interval.

♠ 33. $[1, 2]$ ♠ 34. $[1, 1.1]$ ♠ 35. $[1, 1.01]$ ♠ 36. $[1, 1.001]$

♠ 37. Use the average velocities computed in Exercises 33-36 to estimate the instantaneous velocity of the particle at $t = 1$.

'You must be the change you wish to see in the world." – Mahatma Gandhi

Lecture Videos

The Derivative of a Function

Computing Derivatives from the Definition (Tangent Lines)

Computing Derivatives from the Definition (Polynomial)

Computing Derivatives from the Definition (Rational)

Computing Derivatives from the Definition (Velocity)

Derivatives Tutorial[i]

2.10 Derivatives and Rates of Change

We discussed before the notion of rate of change. Let (x_1, y_1) and (x_2, y_2) be two points. Then the change of x is given by

$$\Delta x = x_2 - x_1,$$

the change of y is given by

$$\Delta y = y_2 - y_1,$$

and the (average) rate of change is given by

$$\left.\frac{\Delta y}{\Delta x}\right|_{[x_1,x_2]} = \frac{y_2 - y_1}{x_2 - x_1},$$

which gives us the difference quotient. If $y = f(x)$ for some function f and $x_1 + h = x_2$, then

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

The average rate of change geometrically measures the slope of the secant line connecting (x_1, y_1) and (x_2, y_2).

On the other hand, the instantaneous rate of change measures the slope of the tangent line at (x_1, y_1) and is given by the formula

$$\left.\frac{dy}{dx}\right|_{x=x_1} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}.$$

The instantaneous rate of change is useful in calculating the slope of the tangent line, the instantaneous velocity of an object, and many more applications.

Definition 2.10.1. The **derivative** of the function f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{b \to x} \frac{f(x) - f(b)}{x - b}, \tag{2.10.1}$$

provided this limit exists.

The function $f'(x)$ is called the derivative of f with respect to x. If x is a value in the domain of f and if $f'(x)$ exists, then f is **differentiable** at x. The process that produces f' is called **differentiation**.

The derivative is a *function* of x, since $f'(x)$ varies as x varies. This differs from both the slope of the tangent line and the instantaneous rate of change, either of which is represented by the number $f'(a)$ that corresponds to a number a. Otherwise, the formula for the derivative is identical to the formula for the slope of the tangent line and the formula for instantaneous rate of change given earlier.

With this perspective the derivative has two very important interpretations. First, the function $f'(x)$ represents the instantaneous rate of change of $y = f(x)$ with respect to x. This (instantaneous) rate of change could be interpreted as marginal cost if the original function f is a cost function or velocity if the original function f is a displacement function. Second, the function $f'(x)$ represents the slope of the graph $f(x)$ at any point x. If the derivative is evaluated at the point $x = a$, then $f'(a)$ is the slope of the tangent line of f at $x = a$. In particular, the formula for the tangent line to f at $x = a$ is

$$y - f(a) = f'(a)(x - a). \tag{2.10.2}$$

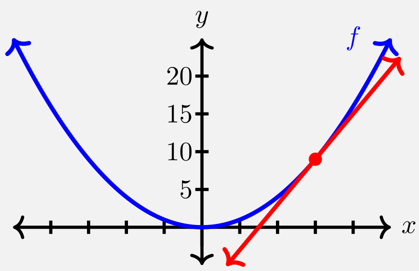
Example 2.10.2. Let $f(x) = x^2$.

(a) Find the derivative.

We will calculate the derivative using the alternate form for instantaneous rate of change.

$$\begin{aligned} f'(x) &= \lim_{b\to x} \frac{f(b) - f(x)}{b - x} = \lim_{b\to x} \frac{b^2 - x^2}{b - x} \\ &= \lim_{b\to x} \frac{(b - x)(b + x)}{b - x} = \lim_{b\to x}(b + x) \\ &= x + x = \boxed{2x} \end{aligned}$$

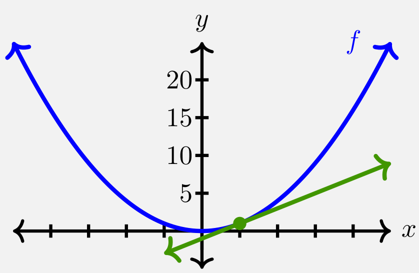
(b) Calculate $f'(3)$ and find the tangent line of f at $x = 3$.



First, $f'(3) = 2(3) = 6$. Second, the tangent line of f at $x = 3$ is given by

$$\begin{aligned} (y - f(3)) = f'(3)(x - 3) \quad &\Rightarrow \quad y - 9 = 6(x - 3) \\ &\Rightarrow \quad y - 9 = 6x - 18 \\ &\Rightarrow \quad \boxed{y = 6x - 9}. \end{aligned}$$

(c) Find the tangent line of f at $x = 1$.



The slope of the tangent line is $m = f'(1) = 2(1) = 2$. The point of tangency is $(1, f(1)) = (1, 1)$. Therefore, the equation of the tangent is

$$\begin{aligned} y - f(1) = f'(1)(x - 1) \quad &\Rightarrow \quad y - 1 = 2(x - 1) \\ &\Rightarrow \quad y - 1 = 2x - 2 \\ &\Rightarrow \quad \boxed{y = 2x - 1}. \end{aligned}$$

Example 2.10.3. Let $f(x) = 2x^3 + 4x$. Find $f'(x)$, $f'(2)$, and $f'(-1)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{(2(x + h)^3 + 4(x + h)) - (2x^3 + 4x)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(2(x^3+3hx^2+3h^2x+h^3)+4x+4h)-(2x^3+4x)}{h} \\
&= \lim_{h\to 0} \frac{((2x^3+6hx^2+6h^2x+2h^3)+4x+4h)-(2x^3+4x)}{h} \\
&= \lim_{h\to 0} \frac{6hx^2+6h^2x+2h^3+4h}{h} = \lim_{h\to 0} \frac{h(6x^2+6hx+2h^2+4)}{h} \\
&= \operatorname*{im}_{h\to 0}(6x^2+4+6hx+2h^2) = \boxed{6x^2+4}.
\end{aligned}
$$

So, $f'(2) = 6(2)^2+4 = 6(4)+4 = 24+4 = \boxed{28}$ and $f'(-1) = 6(-1)^2+4 = 6+4 = \boxed{10}$.

Example 2.10.4. Let $f(x) = \frac{4}{x}$. Find $f'(x)$.

$$
\begin{aligned}
f'(x) &= \lim_{b\to x} \frac{f(b)-f(x)}{b-x} = \lim_{b\to x} \frac{\frac{4}{b}-\frac{4}{x}}{b-x} \\
&= \lim_{b\to x} \left(\frac{\frac{4}{b}-\frac{4}{x}}{b-x}\right)\frac{bx}{bx} = \lim_{b\to x} \frac{4x-4b}{bx(b-x)} \\
&= \lim_{b\to x} \frac{-4(b-x)}{bx(b-x)} = \lim_{b\to x} \frac{-4}{bx} = \boxed{-\frac{4}{x^2}}.
\end{aligned}
$$

Example 2.10.5. Suppose that a ball is dropped from the upper observation deck of a 250 m tall tower. The distance in meters fallen after t seconds is $s(t) = 4.9t^2$.

(a) What is the velocity of the ball after 5 seconds?

Velocity is the rate at which position is changing over time, that is, velocity is the derivative of the position. We begin by computing the derivative of s.

$$
\begin{aligned}
v(t) &= \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} = \lim_{h\to 0} \frac{4.9(t+h)^2-4.9t^2}{h} \\
&= 4.9 \lim_{h\to 0} \frac{(t+h)^2-t^2}{h} = 4.9 \lim_{h\to 0} \frac{t^2+2th+h^2-t^2}{h} \\
&= 4.9 \lim_{h\to 0} \frac{2th+h^2}{h} = 4.9 \lim_{h\to 0} \frac{h(2t+h)}{h} \\
&= 4.9 \lim_{h\to 0}(2t+h) = 4.9(2t+0) = \boxed{9.8t}
\end{aligned}
$$

Therefore, the velocity at time $t=5$ is $v(5) = 9.8(5) = \boxed{49 \text{ m/s}}$.

(b) How fast is the ball traveling when it hits the ground?

We first need to use the position function s to determine at which time t does the ball travel 250 m.

$$
\begin{aligned}
s(t) &= 250 \\
4.9t^2 &= 250 \\
t^2 &= \frac{250}{4.9} = \frac{2500}{49}
\end{aligned}
$$

$$t = \sqrt{\frac{2500}{49}} = \frac{50}{7} \approx 7.14\ s$$

Therefore, the speed of the ball at the moment of impact was $v\left(\frac{50}{7}\right) = 9.8\left(\frac{50}{7}\right) = \frac{98(5)}{7} = 14(5) = \boxed{70 \text{ m/s}}$.

[i]A former student created the linked video that explains the basics of derivatives with examples. As some of the topics go beyond what we have learned so far, do not feel like you should watch the whole video. Instead, revisit it when we have learned the next topic.

[ii]See §3.1 Defining the Derivative in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–23, given the function g, find the equation of the tangent line of g at the given point of tangency. In some cases, g' is given for convenience.

1. $g(x) = \sin(x)$, $g'(x) = \cos(x)$ @ $x = 0$
2. $g(x) = \sin(x)$, $g'(x) = \cos(x)$ @ $x = \pi$
3. $g(x) = \sin(x)$, $g'(x) = \cos(x)$ @ $x = \frac{\pi}{2}$
4. $g(x) = \sin(x)$, $g'(x) = \cos(x)$ @ $x = \frac{\pi}{3}$
5. $g(x) = e^x$, $g'(x) = e^x$ @ $x = 1$
6. $g(x) = 4\ln x$, $g'(x) = \frac{4}{x}$ @ $x = e$
7. $g(x) = x - x^2$ @ $x = 1$
8. $g(x) = x^2 + 1$ @ $x = 1$
9. $g(x) = 2x^2 - x + 7$ @ $x = 1$

♠ 10. $g(x) = 5x - x^2$ @ $x = 1$

♠ 11. $g(x) = x - x^3$ @ $x = 1$

12. $g(x) = x^3 - 3x$ @ $x = -1$

♠ 13. $g(x) = x^3 - 2x + 2$ @ $x = 3$

14. $g(x) = x^2 - 3x + 5$ @ $x = 1$

♠ 15. $g(x) = 9 + 5x^2 - 2x^3$ @ $x = 1$

♠ 16. $g(x) = 9 + 5x^2 - 2x^3$ @ $x = 2$

♠ 17. $g(x) = \sqrt{x}$ @ $x = 1$

18. $g(x) = \sqrt{x}$ @ $x = 4$
19. $g(x) = 2\sqrt{x}$ @ $x = 9$
20. $g(x) = x^{3/2}$ @ $x = 4$
21. $g(x) = \dfrac{1}{x}$ @ $x = \frac{1}{2}$

♠ 22. $g(3) = -5$, $g'(3) = 2$ @ $x = 3$

23. $g(4) = 3$, $g'(4) = \frac{1}{4}$ @ $x = 4$

For Exercises 24–27, compute the derivative to verify the estimate of instantenous rate of change made in Section 2.9.

24. Exercise 2.9.20

♠ 25. Exercise 2.9.21

♠ 26. Exercise 2.9.26

27. Exercise 2.9.32

♠ 28. If the tangent line to $y = f(x)$ at $(6, 2)$ passes through the point $(0, 1)$, then find $f(6)$ and $f'(6)$.

29. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, then find $f(4)$ and $f'(4)$.

For Exercises 30–44, the limit represents the derivative of some function f at some number a. State such an f and a.

30. $\displaystyle\lim_{h\to 0} \frac{(1+h)^{10} - 1}{h}$

♠ 31. $\displaystyle\lim_{h\to 0} \frac{(1+h)^3 - 1}{h}$

32. $\displaystyle\lim_{h\to 0} \frac{\sqrt[4]{16+h} - 2}{h}$

♠ 33. $\displaystyle\lim_{h\to 0} \frac{\sqrt[4]{81+h} - 3}{h}$

34. $\displaystyle\lim_{x\to 1} \frac{x^3 - x}{x - 1}$
35. $\displaystyle\lim_{x\to 1} \frac{x^4 + x - 2}{x - 1}$
36. $\displaystyle\lim_{x\to 1} \frac{x^{356} - 1}{x - 1}$
37. $\displaystyle\lim_{x\to 5} \frac{2^x - 32}{x - 5}$

♠ 38. $\displaystyle\lim_{x\to 2} \frac{4^x - 16}{x - 2}$

39. $\displaystyle\lim_{h\to 0} \frac{(2+h)^{2+h} - 4}{h}$

♠ 40. $\displaystyle\lim_{x\to \pi/4} \frac{\tan(x) - 1}{x - \pi/4}$

41. $\displaystyle\lim_{x\to \pi/4} \frac{\tan x - 1}{x - \pi/4}$
42. $\displaystyle\lim_{x\to \pi/6} \frac{\sin x - 1/2}{x - \pi/6}$
43. $\displaystyle\lim_{x\to \pi/3} \frac{\sin x - \sqrt{3}/2}{x - \pi/3}$
44. $\displaystyle\lim_{h\to 0} \frac{\cos(\pi+h) + 1}{h}$

"All other virtues are derivatives and reflections of love!" — Neal A. Maxwell

Lecture Videos

Graphing the Derivative of a Function from Its Graph

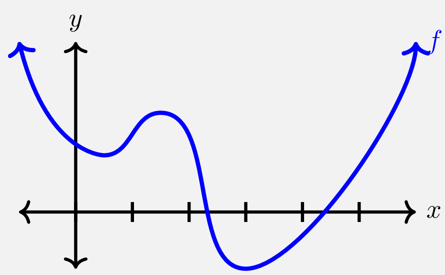
Criteria for a Function Being Differentiable

Graphing the Derivative of a Function from Its (Sometimes Non-Differentiable) Graph

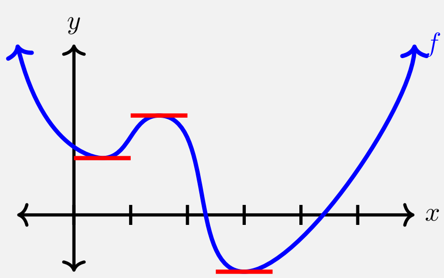
2.11 Derivatives and Rates of Change II

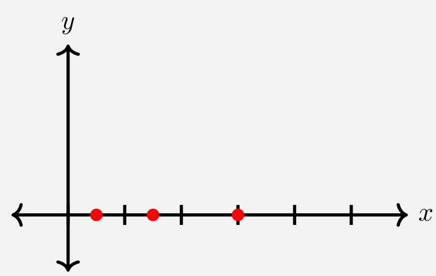
Since the derivative measures the rate of change of f, we can determine the graph of f' from the graph of f as illustrated in the below examples.

Example 2.11.1. The graph of a function f is given below. Use it to sketch the graph of its derivative f'.

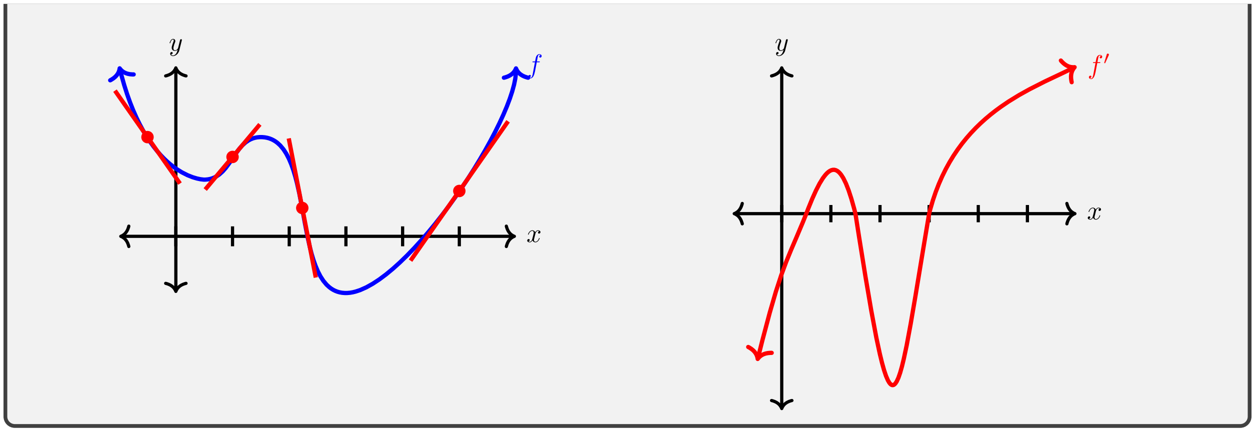


We begin by recognizing where the horizontal tangent lines on f are. These will correspond to x-intercepts on the derivative's graph.





Next, the derivative will be positive whenever the tangent slopes are positive. This happens exactly when the graph is increasing. Likewise, the derivative will be negative whenever the tangent slopes are negative. This happens exactly when the graph is decreasing.



We will now consider the phrase "provided the limit exists" in the definition of the derivative. The following theorem provides necessary and sufficient conditions for the derivative to exist at a point on a curve.

Theorem 2.11.2. *The derivative exists when a function f satisfies* ***all*** *of the following conditions at a point.*

(a) f is continuous,

(b) f is smooth, and

(c) f does not have a vertical tangent line.

The derivative does ***not*** *exist when* ***any*** *of the following conditions are true for a function at a point.*

(a) f is discontinuous,

(b) f has a sharp corner, or

(c) f has a vertical tangent line.

Proof. We will prove that statement that "if f is differentiable, then f is continuous."

Suppose that f is differentiable at $x = a$. Then the limit

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

exists. Of course, $\lim_{x\to a}(x-a) = 0$. Therefore,

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)-f(a)}{x-a} &= f'(a) \\
\left(\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right) &= f'(a)\left(\lim_{x\to a}(x-a)\right) \\
\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right) &= f'(a)(0) \\
\lim_{x\to a}(f(x)-f(a)) &= 0,
\end{aligned}$$

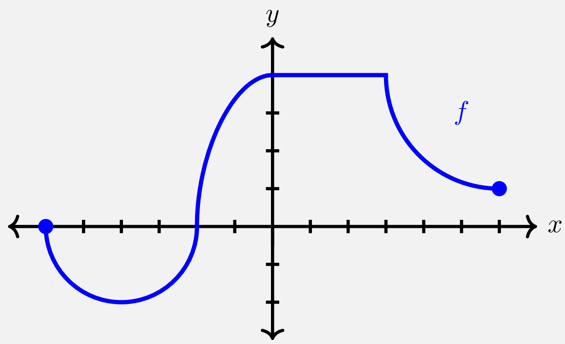
which shows that the limit $\lim_{x\to a} f((x)-f(a))$ exists. Of course, $\lim_{x\to a} f(a) = f(a)$. Hence,

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= 0 \\
\lim_{x\to a}(f(x)-f(a)) + \lim_{x\to a} f(a) &= \lim_{x\to a} f(a)
\end{aligned}$$

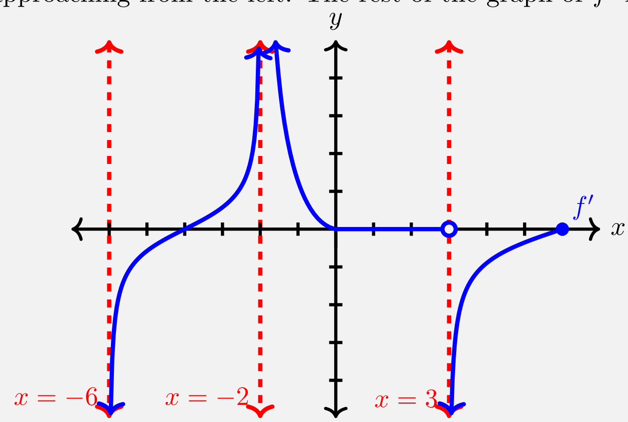
$$\lim_{x\to a} f(x) \quad = \quad f(a).$$

Therefore, f is continuous at $x = a$. □

Example 2.11.3. Identify the locations on the graph of f, provided below, where the derivative is undefined.



The derivative is undefined when $\boxed{x = -6,\ -2,\ 3}$ because there are vertical tangent lines at $x = -6,\ -2$ and a corner at $x = 3$. The derivative is defined at $x = 0$ because the approach from the left agrees with the approach from the right. Both sides say the tangent line is horizontal. The derivative also exists at $x = 6$. Because the approach from the right does not exist, the tangent slope is the limit of secant slopes approaching from the left. The rest of the graph of f' is provided below.



It should be mentioned that even though f is not differentiable at $x = -6,\ -2,\ 3$, f is continuous on its domain including $x = -6,\ -2,\ 3$. Although, these points are corners or have vertical tangent lines, there are no discontinuities at these points. This shows that a function f' may be discontinuous at a point where f is continuous.

We mention here that there are several commonly used notations for the derivative of a function f. So far we have used $f'(x)$ to denote the derivative of f. Also used are $\dfrac{dy}{dx}$, $\dfrac{d}{dx}[f(x)]$, and $D_x[f(x)]$. In order for the student also to become familiar with these different notations, the textbook will start rotating through them in subsequent examples.

Next, x isn't the only big variable on campus. For example, if $y = f(t)$ gives the population growth as a function of time, then the derivative of y with respect to t could be written as

$$f'(t), \quad \frac{dy}{dt}, \quad \frac{d}{dt}[f(t)], \quad \text{or} \quad D_t[f(t)].$$

Let $f(x)$ be a differentiable function. So, we can compute $f'(x)$. If $f'(x)$ is differentiable, we can calculate the second derivative $f''(x)$. Similarly, if $f''(x)$ is differentiable, we can calculate $f'''(x)$, the **third**

derivative. Even further, if $f'''(x)$ is differentiable, then we can calculate $f^{(4)}(x)$, the **fourth derivative**, etc.

Definition 2.11.4. The **second derivative** of $y = f(x)$ can be written using any of the following

$$f''(x),\ \ \frac{d^2y}{dx^2},\ \ \text{or}\ \ D_x^2\big[f(x)\big].$$

The third derivative can be written in a similar way. For $n \geq 4$, the nth derivative is written $f^{(n)}(x)$, but the other two notations follow the given pattern.

Earlier, we saw that the first derivative of a function represents the rate of change of the function. The second derivative, then, represents the rate of change of the first derivative. If a function describes the position of an object at time t, then the first derivative (*velocity*) gives the rate of change of position with respect to time. So, the second derivative of the position function is the rate at which velocity is changing over time. This is known as **acceleration**. For example, if velocity is positive and acceleration is positive, the velocity is increasing, so the object is speeding up. If the velocity is positive and the acceleration is negative, the object is slowing down. Continuing in this fashion, if velocity is negative and acceleration is positive, then the object is slowing down, since it is accelerating in the opposite direction of velocity. Lastly, if velocity and acceleration are negative, then the object is speeding up.

We can also interpret the third derivative of position physically as the derivative or rate of change for acceleration. This is known as **jerk**. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

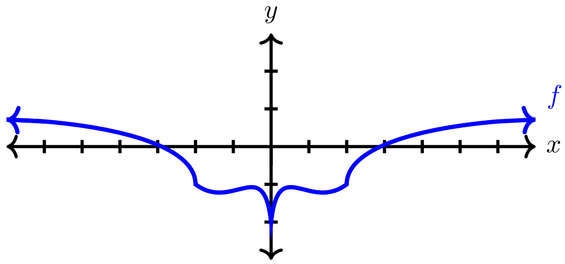
[i]See §3.2 The Derivative as a Function in OpenStax to find the corresponding section.

Exercises

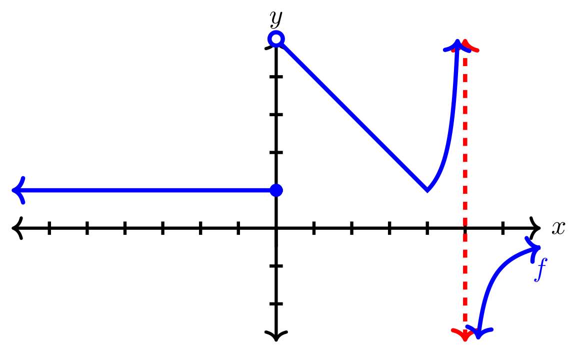
(Go to Solutions)

For Exercises 1–6, the function f is illustrated below. Determine all values of x where f is not differentiable.

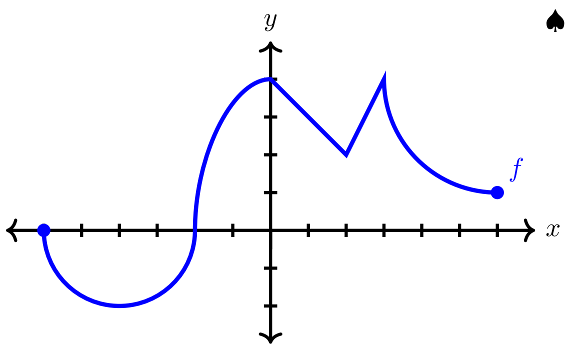
1.



♠ 2.



♠ 3.



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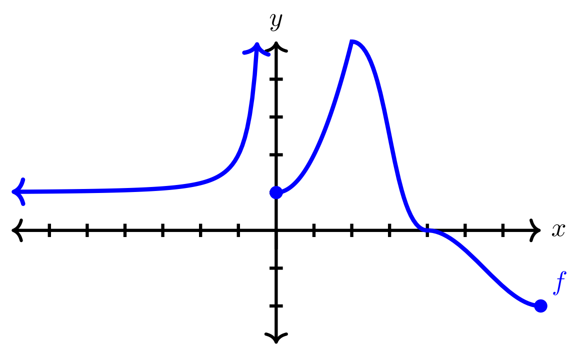

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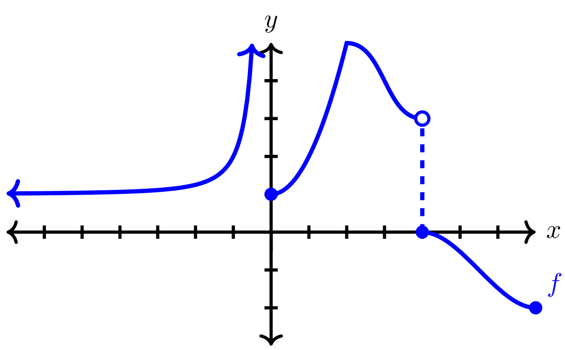

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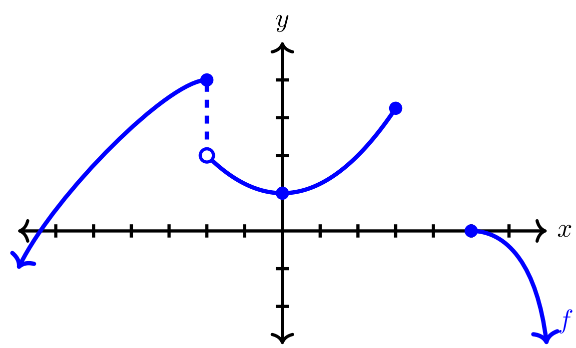
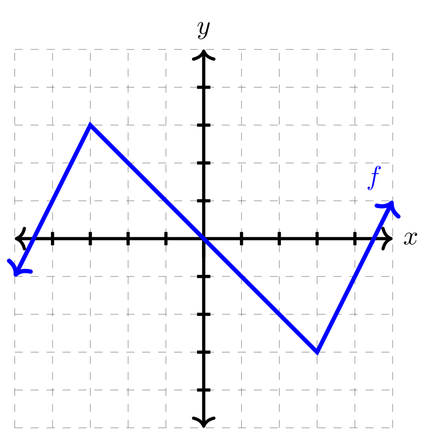
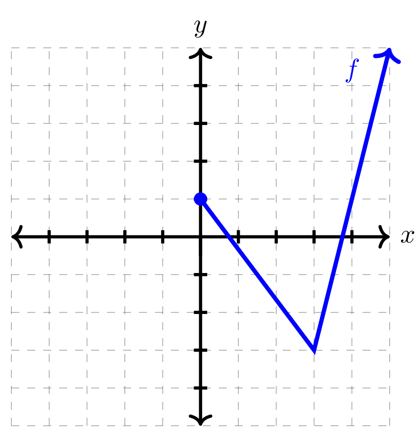


For Exercises 7–14, for the given graph of the function f, draw the graph of its derivative f'.

7.



♠ 8.



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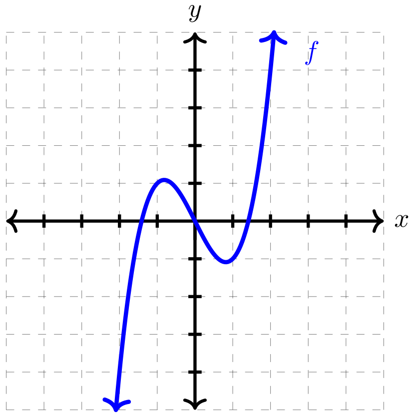

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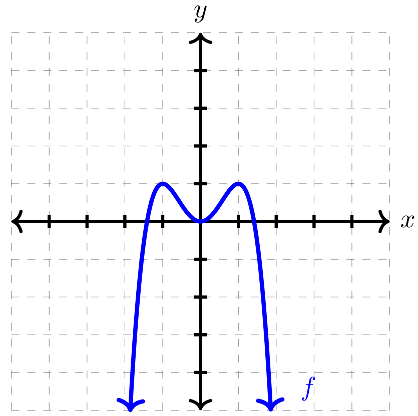
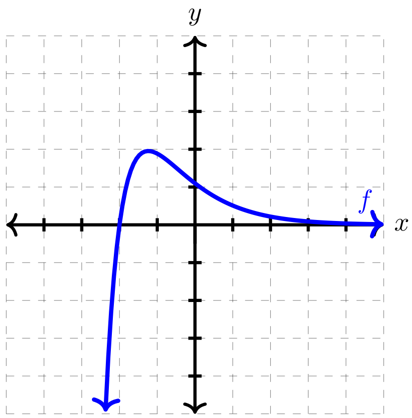
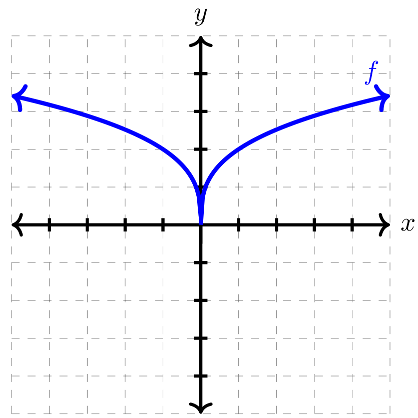


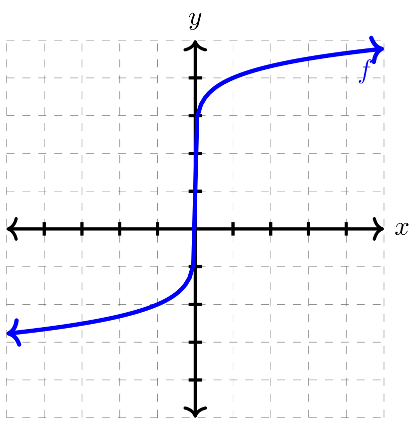
♠ 11.



♠ 12.



♠ 13.



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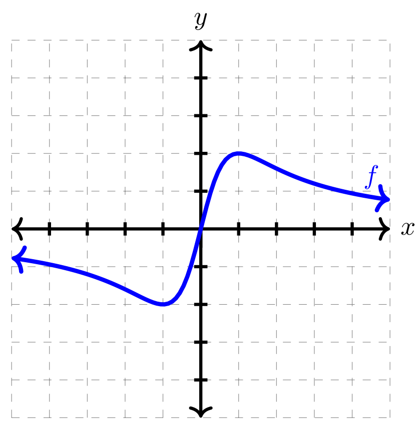

For Exercises 15–29, find the derivative of the function f using the definition of the derivative.

15. $f(x) = \frac{1}{4}x - \frac{1}{10}$

16. $f(x) = x^2$

♠ 17. $f(x) = 5x - 8x^2$

18. $f(x) = 2x^2 + 1$

19. $f(x) = 3x^2 + 1$

20. $f(x) = 2x^2 + x + 1$

21. $f(x) = 3x^2 - x + 1$

22. $f(x) = x^3$

23. $f(x) = \sqrt{x}$

♠ 24. $f(x) = \sqrt{8 - x}$

25. $f(x) = \frac{1}{x}$

26. $f(x) = x + \frac{1}{x}$

♠ 27. $f(x) = \dfrac{1-4x}{3+x}$ ♠ 28. $f(x) = \dfrac{1}{\sqrt{x}}$ 29. $f(x) = \dfrac{1}{x\sqrt{x}}$

Deeper Dive

Definition 2.11.5. The **left-handed** and **right-handed derivatives** of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \qquad \text{and} \qquad f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h},$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

For Exercises 30–33, let $f(x) = \begin{cases} 1, & x \leq 0 \\ 5 - x, & 0 < x < 4 \\ \dfrac{1}{5-x}, & x \geq 4 \end{cases}$.

30. Sketch the graph of f.

31. Where is f discontinuous?

32. Evaluate $f'_-(4)$ and $f'_+(4)$.

33. Where is f not differentiable?

Chapter 3

Derivatives

"Power is of two kinds. One is obtained by the fear of punishment and the other by acts of love. Power based on love is a thousand times more effective and permanent than the one derived from fear of punishment." – Mahatma Gandhi

Lecture Videos

The Power Rule

The Linearity of the Derivative

Finding Acceleration of a Motion Function

The Derivative of e^x

Derivatives Tutorial[i]

3.1 The Power Rule

Surely you have realized by now that the derivative can be very cumbersome to calculate. Fortunately, in this chapter, we will develop some techniques, aka shortcuts, for calculating the derivative of differentiable functions. It is our goal in this section to compute the derivatives of polynomial and exponential functions.

Theorem 3.1.1. *Let $f(x) = x^n$ for any positive integer n. Then*

$$\frac{d}{dx}f(x) = \frac{d}{dx}x^n = nx^{n-1}.$$

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0}\frac{(x^n + nx^{n-1}h + (n(n-1)/2)x^{n-2}h^2 + \ldots + h^n) - x^n}{h} \\ &= \lim_{h\to 0}\frac{nx^{n-1}h + (n(n-1)/2)x^{n-2}h^2 + \ldots + h^n}{h} \\ &= \lim_{h\to 0}(nx^{n-1} + (n(n-1)/2)x^{n-2}h + \ldots + h^{n-1}) = nx^{n-1}. \end{aligned}$$ □

Theorem 3.1.2 (The Power Rule)**.** *If n is any real number, then*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Example 3.1.3. Use the Power Rule to calculate the following derivatives.

(a) $y = x^6$

$$D_x y = 6x^{6-1} = \boxed{6x^5}.$$

(b) $y = t = t^1$

$$\frac{dy}{dt} = 1t^{1-1} = 1t^0 = \boxed{1}.$$

(c) $y = 1 = x^0$

$$\frac{dy}{dx} = 0x^{0-1} = \boxed{0}.$$

(d) $y = 1/x^3$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d(x^{-3})}{dx} = -3x^{-3-1} = -3x^{-4} = \boxed{\frac{-3}{x^4}}.$$

(e) $y = x^{4/3}$

$$D_x(x^{4/3}) = \frac{4}{3}x^{4/3-1} = \boxed{\frac{4}{3}x^{1/3}}.$$

(f) $y = \sqrt{z}$

$$\frac{dy}{dz} = \frac{d}{dz}(\sqrt{z}) = \frac{d}{dz}(z^{1/2}) = \frac{1}{2}z^{1/2-1} = \frac{1}{2}z^{-1/2} = \boxed{\frac{1}{2\sqrt{z}}}.$$

Theorem 3.1.4. *If f is differentiable and c is a constant, then*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

Proof.

$$(cf)'(x) = \lim_{h\to 0}\frac{(cf)(x+h)-(cf)(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x). \qquad \square$$

Example 3.1.5. Find the derivatives of the following functions.

(a) $y = 8x^4$

$$\frac{dy}{dx} = 8(4x^3) = 8(4x^3) = \boxed{32x^3}.$$

(b) $y = \dfrac{6}{x}$

$$y' = \left(\frac{6}{x}\right)' = 6\left(\frac{1}{x}\right)' = 6\left(\frac{-1}{x^2}\right) = \boxed{-\frac{6}{x^2}}.$$

Theorem 3.1.6. *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x)\pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

Proof.

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h} = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x).
\end{aligned}$$

The case for $f-g$ is handled similarly. $\square$

The collection of all these rules tell us that we can easily take the derivative of a polynomial without an appeal to the difference quotient.

Example 3.1.7. Find the derivative of each function.

(a) $y = 6x^3 + 15x^2$.

$$\begin{aligned} y' &= (6x^3+15x^2)' = (6x^3)' + (15x^2)' = 6(x^3)' + 15(x^2)' \\ &= 6(3x^2) + 15(2x) = \boxed{18x^2 + 30x}. \end{aligned}$$

(b) $f(x) = \dfrac{x^3 + 3\sqrt{x}}{x}$.

$$\begin{aligned} f'(x) &= \left(\frac{x^3+3\sqrt{x}}{x}\right)' = \left(\frac{x^3}{x} + \frac{3\sqrt{x}}{x}\right)' = \left(x^2 + 3\frac{x^{1/2}}{x}\right)' \\ &= \left(x^2 + 3x^{-1/2}\right)' = 2x - \frac{3}{2}x^{-3/2} = \boxed{2x - \frac{3}{2\sqrt{x^3}}} \end{aligned}$$

(c) $f(x) = (4x^2 - 3x)^2$.

$$f'(x) = \left((4x^2-3x)^2\right)' = \left(16x^4 - 24x^3 + 9x^2\right)' = \boxed{64x^3 - 72x^2 + 18x}.$$

Example 3.1.8. The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where s is measured in centimeters and t in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Acceleration is the second derivative of s. Note

$$\begin{aligned} s(t) &= 2t^3 - 5t^2 + 3t + 4 \\ v(t) = s'(t) &= (2t^3 - 5t^2 + 3t + 4)' = 6t^2 - 10t + 3 \\ a(t) = v'(t) = s''(t) &= (6t^2 - 10t + 3)' = \boxed{12t - 10} \end{aligned}$$

The acceleration after 2 s is $a(2) = 12(2) - 10 = 24 - 10 = \boxed{14\ \dfrac{\text{cm}}{\text{s}^2}}$.

Have you ever wondered why mathematicians like the number $e \approx 2.718281828\ldots$. Why should $y = e^x$ be the natural exponential? Why is $y = \ln(x) = \log_e(x)$ the natural logarithm? What makes e so natural?

Theorem 3.1.9 (Derivative of e^x)**.**

$$\frac{d}{dx}(e^x) = e^x.$$

Proof.

$$\begin{aligned} (e^x)' &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} \\ &= \lim_{h\to 0}\frac{e^x(e^h - 1)}{h} = e^x \cdot \lim_{h\to 0}\frac{e^h-1}{h} \end{aligned}$$

It turns out that $\displaystyle\lim_{h\to 0}\frac{e^h-1}{h} = 1$, which can be proven using the Squeeze Theorem. Thus,

$$f'(x) = e^x \cdot \lim_{h\to 0}\frac{e^h - 1}{h} = e^x \cdot (1) = e^x.$$

□

$(e^x)' = e^x$! This is incredible! You will love this because it makes calculating its derivative quite easy. Math geeks like me love this because this property will be very valuable in the process of solving differential equations, which we will do later in this course. In a nutshell, this is a very awesome property. In fact, the reason mathematicians first started considering the number e is because they wanted a function which exhibited such nice differential properties. This is the very reason interest, population growth, and radioactive decay can be modeled using the exponential e.

Example 3.1.10. If $f(x) = e^x - x$, find f' and f''.

$$f'(x) = (e^x - x)' = (e^x)' - (x)' = \boxed{e^x - 1}.$$

$$f''(x) = \frac{d}{dx}f'(x) = (e^x - 1)' = \boxed{e^x}.$$

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[i] A former student created the linked video that explains the basics of derivatives with examples. As some of the topics go beyond what we have learned so far, do not feel like you should watch the whole video. Instead, revisit it when we have learned the next topic.

[ii] See §3.3 Differentiation Rules and §3.9 Derivatives of Exponential and Logarithmic Functions in OpenStax to find the corresponding sections.

Exercises

(Go to Solutions)

For Exercises 1–28, differentiate the function.

1. $f(x) = x^{1/2}$
2. $f(x) = x^{-1}$
3. $f(x) = \sqrt[3]{x}$
4. $f(x) = \dfrac{1}{\sqrt{x}}$

♠ 5. $f(x) = \dfrac{3}{4}x^{12}$

♠ 6. $g(t) = 2t^{-3/4}$

♠ 7. $f(x) = 2^{80}$

♠ 8. $y = -\dfrac{14}{x^5}$

♠ 9. $y = \sqrt[6]{x}$

♠ 10. $f(x) = \dfrac{\sqrt{2}}{x^9}$

♠ 11. $f(t) = 5 - \dfrac{1}{3}t$

12. $f(x) = 3x^2 - 5x + 2$

♠ 13. $f(x) = x^5 - 9x + 1$

14. $y = 7x^3 + 2x + 6$

15. $y = -4x^3 + 3x - 5$

16. $y = 3x^3 - 7x^2 - 3$

17. $y = 5x^3 - x^2 + 7x + 2$

18. $y = \dfrac{3}{x^3} + x^{-4}$

♠ 19. $g(x) = x^2(1 - 3x)$

♠ 20. $f(t) = \sqrt{t}(t - 10)$

21. $y = x^{3/2}(x^3 + 2x)$

22. $f(x) = x\sqrt{x} + \sqrt{3x}$

♠ 23. $y = 8e^x + \dfrac{8}{\sqrt[3]{x}}$

24. $f(x) = e^x - x^3$

♠ 25. $f(x) = \dfrac{6x^2 + 6x + 8}{\sqrt{x}}$

♠ 26. $u = \sqrt[7]{x} + 4\sqrt{x^7}$

27. $f(x) = e^x + x^5$

28. $f(x) = x^{19} - \dfrac{\sqrt{x}}{3} + e^x$

For Exercises 30–29, find the equation of the tangent line to the curve at the given point.

♠ 29. $y = 5x - 4\sqrt{x}$ @ $(1, 1)$

♠ 30. $y = x^3 - 3x^2 + 3x + 18$ @ $(1, 19)$

For Exercises 31–36, find the second derivative of the function.

31. $f(x) = x^{4/5}$

32. $y = \sqrt{x}$

33. $f(x) = e^x + 2x$

34. $f(x) = e^x - x^3$

35. $y = \dfrac{3}{x^3} + x^{-4}$

36. $y = x^{3/2}(x^3 + 2x)$

"Friendships multiply joys and divide griefs." – Thomas Fuller

Lecture Videos

The Product Rule

The Quotient Rule

Combining the Quotient and Product Rules

We Don't Always Need the Quotient Rule

3.2 The Product Rule

Example 3.2.1. We learned previously that the derivative of a sum is equal to a sum of derivative. A natural question is whether the derivative of a product is equal to the product of derivatives. Let us experiment.

Let $f(x) = 2x+3$ and $g(x) = 3x^2$. Then $f'(x) = 2$ and $g'(x) = 6x$. So, $f'(x)g'(x) = 12x$. Next we compute $f(x)g(x) = (2x+3)(3x^2) = 6x^3+9x^2$. So, $(f(x)g(x))' = (6x^3+9x^2)' = 18x^2+18x \neq 12x = f'(x)g'(x)$. Alright, the derivative of a product is NOT equal to the product of derivatives. Although the derivative of a product of function is not this simple, there is a simple formula for computing the derivative of a product.

Theorem 3.2.2 (Product Rule)**.** *If $f(x)$ and $g(x)$ are differentiable functions, then*

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x).$$

Proof. First notice that $f(t)[g(t)-g(x)]+g(x)[f(t)-f(x)] = f(t)g(t)-f(t)g(x)+g(x)f(t)-f(x)g(x) = f(t)g(t)-f(x)g(x) = fg(t)-fg(x)$. Then

$$\begin{aligned}
\frac{d}{dx}(fg(x)) &= \lim_{t\to x}\frac{fg(t)-fg(x)}{t-x} \\
&= \lim_{t\to x}\frac{f(t)[g(t)-g(x)]+g(x)[f(t)-f(x)]}{t-x} \\
&= \lim_{t\to x}\frac{f(t)[g(t)-g(x)]}{t-x}+\frac{g(x)[f(t)-f(x)]}{t-x} \\
&= \lim_{t\to x}f(t)\lim_{t\to x}\frac{g(t)-g(x)}{t-x}+\lim_{t\to x}g(x)\lim_{t\to x}\frac{f(t)-f(x)}{t-x} \\
&= f(x)g'(x)+g(x)f'(x).
\end{aligned}$$

Notice that $\lim_{t\to x} f(t) = f(x)$ since f is continuous, which followed from f being differentiable. □

Example 3.2.3. Use the product rule to compute $\left((2x+3)(3x^2)\right)'$.

$$\begin{aligned}
\left((2x+3)(3x^2)\right)' &= (2x+3)(3x^2)'+(3x^2)(2x+3)' \\
&= (2x+3)(6x)+(3x^2)(2) \\
&= (12x^2+18x)+(6x^2) \\
&= \boxed{18x^2+18x}
\end{aligned}$$

We note that this agrees with our result from Example 3.2.1.

Example 3.2.4. Find the derivative of $y = (\sqrt{x}+3)(x^2-5x)$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[(\sqrt{x}+3)(x^2-5x)\right] \\
&= (\sqrt{x}+3)\frac{d(x^2-5x)}{dx} + \frac{d(\sqrt{x}+3)}{dx}(x^2-5x) \\
&= (x^{1/2}+3)(2x-5) + \left(\frac{1}{2}x^{-1/2}\right)(x^2-5x) \\
&= (2x^{3/2} - 5x^{1/2} + 6x - 15) + (\frac{1}{2}x^{3/2} - \frac{5}{2}x^{1/2}) \\
&= \boxed{\frac{5}{2}x^{3/2} + 6x - \frac{15}{2}x^{1/2} - 15}
\end{aligned}$$

Example 3.2.5. Find the nth derivative for $f(x) = xe^x$.

$$\begin{aligned}
f'(x) &= (xe^x)' = x'e^x + x(e^x)' = e^x + xe^x = \boxed{(1+x)e^x} = e^x + f(x). \\
f''(x) &= (e^x + f(x))' = (e^x)' + f'(x) = e^x + (1+x)e^x = \boxed{(2+x)e^x} = 2e^x + f(x). \\
f'''(x) &= (2e^x + f(x))' = \boxed{(3+x)e^x}. \\
f^{(n)}(x) &= \boxed{(n+x)e^x}.
\end{aligned}$$

The derivative of a quotient of function is a little bit more complicated.

Theorem 3.2.6 (The Quotient Rule)**.** *If $f(x)$ and $g(x)$ are differentiable functions and if $g(x) \neq 0$, then*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Proof. Although we can prove the Quotient Rule using a technique similar to the proof of the product rule, we will be able to prove the Quotient Rule quite easily after we develop the chain rule. □

The formula for the Quotient Rule is a little more complicated. So I present a mnemonic that I learned a while ago, which I honestly still use, in order to remember the Quotient Rule. First, let us rewrite the Quotient Rule using the D_x notation:

$$\frac{g(x)D_x f(x) - f(x)D_x g(x)}{g(x)^2}.$$

Hopefully, the follows rhyme will be useful:

"Low-Dee-High minus High-Dee-Low,
Square the bottom and here we go!"

"Low" refers to $g(x)$, the function in the denominator and "High" refers to $f(x)$, the function in the numerator. "Dee" of course means D_x, that is, take the derivative. Therefore, the Quotient Rule says

$$\frac{\text{Low}D_x(\text{High}) - \text{High}D_x(\text{Low})}{(\text{Low})^2}.$$

When using this rhyme, remember that it is supposed to rhyme. If it doesn't rhyme, then it means you have the numerator switched around.

Example 3.2.7. Find $f'(x)$ if $f(x) = \dfrac{2x-1}{4x+3}$.

$$\begin{aligned} f'(x) &= \frac{(4x+3)(2x-1)' - (2x-1)(4x+3)'}{(4x+3)^2} = \frac{(4x+3)(2) - (2x-1)(4)}{(4x+3)^2} \\ &= \frac{(8x+6)-(8x-4)}{(4x+3)^2} = \boxed{\frac{10}{(4x+3)^2}} \end{aligned}$$

It is rarely ever beneficial to expand the denominator of a derivative when using the quotient rule. So, we will not expect students to do so.

In Example 3.2.7, we had the expression $\dfrac{(4x+3)(2)-(2x-1)(4)}{(4x+3)^2}$. Students often incorrectly "cancel" the $4x+3$ in the numerator with one factor of the denominator. Because the numerator is a *difference* of two products, you can only cancel this factor if both terms in the numerator have $4x+3$ as a common factor. As we saw in Example 3.2.7, this is not always the case.

Example 3.2.8. Find $D_x\left[\dfrac{x^2e^x}{7x-9}\right]$.

$$\begin{aligned} D_x\left[\frac{x^2e^x}{7x-9}\right] &= \frac{(7x-9)D_x\left[x^2e^x\right] - x^2e^xD_x\left[7x-9\right]}{(7x-9)^2} \\ &= \frac{(7x-9)\left[(x^2)D_x[e^x] + D_x[x^2](e^x)\right] - x^2e^x(7)}{(7x-9)^2} \\ &= \frac{(7x-9)\left[x^2e^x + 2xe^x\right] - 7x^2e^x}{(7x-9)^2} \\ &= \frac{7x^3e^x + 14x^2e^x - 9x^2e^x - 18xe^x - 7x^2e^x}{(7x-9)^2} \\ &= \frac{7x^3e^x - 2x^2e^x - 18xe^x}{(7x-9)^2} \\ &= \frac{xe^x(7x^2-2x-18)}{(7x-9)^2} \end{aligned}$$

Example 3.2.9. Don't use the Quotient rule *every* time you see a quotient. Sometimes it's easier to rewrite the quotient first to put it in a form that is simpler for the purpose of differentiation. For example,

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}.$$

Although, we *could* use the quotient rule, it is much easier to first recognize that

$$F(x) = \frac{3x^2}{x} + \frac{2x^{1/2}}{x} = 3x + 2x^{-1/2}.$$

Thus,

$$F'(x) = \boxed{3 - 1x^{-3/2}}.$$

Example 3.2.10. Find an equation of the tangent line to the curve $y = e^x/(1+x^2)$ at the point $(1, e/2)$.

We first compute y'.

$$\begin{aligned} y &= \frac{e^x}{1+x^2} \\ y' &= \frac{(1+x^2)(e^x)' - (e^x)(1+x^2)'}{(1+x^2)^2} = \frac{(1+x^2)e^x - (e^x)(2x)}{(1+x^2)^2} \\ &= \frac{e^x[(1+x^2)-(2x)]}{(1+x^2)^2} = \frac{e^x(1-2x+x^2)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}. \end{aligned}$$

Thus, $\left.\frac{dy}{dx}\right|_{x=1} = \frac{e^1(1-1)^2}{(1+1^2)^2} = 0$. Therefore, the tangent line is the horizontal line $\boxed{y = e/2}$.

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[i]See §3.3 Differentiation Rules in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–26, differentiate the function.

1. $f(x) = x^4 e^x$
♠ 2. $f(x) = (1 + 4x^2)(x - x^2)$
3. $f(x) = \sqrt{x}e^x$
4. $f(x) = x^{5/2}e^x$
♠ 5. $g(x) = 7e^x\sqrt{x}$
6. $f(x) = (1 - e^x)(x + e^x)$
♠ 7. $y = (x - \sqrt{x})(x + \sqrt{x})$
♠ 8. $f(x) = \left(\frac{1}{x^2} - \frac{9}{x^4}\right)(x + 7x^3)$
♠ 9. $y = \dfrac{x^5}{3 - x^4}$
10. $f(x) = \dfrac{3x}{x^2 + 1}$
11. $f(x) = \dfrac{x^2 - 2}{1 + 2x}$
12. $f(x) = \dfrac{x^2 + 2x + 1}{x + 2}$
♠ 13. $y = \dfrac{x}{(x - 6)^2}$
14. $f(x) = \dfrac{x + 10}{x - 10}$
♠ 15. $f(x) = \dfrac{x + 4}{x^3 + x - 7}$
♠ 16. $f(t) = \dfrac{t^2 + 5}{t^4 - 4t^2 + 2}$
♠ 17. $f(t) = \dfrac{6t}{6 + \sqrt{t}}$
♠ 18. $h(x) = \dfrac{x - \sqrt{x}}{x^{1/7}}$
19. $f(x) = \dfrac{e^x}{x}$
♠ 20. $f(x) = \dfrac{6 - xe^x}{x + e^x}$
21. $f(x) = \dfrac{e^x}{1 - e^x}$
22. $f(x) = \dfrac{e^x + 1}{e^x}$
23. $y = \dfrac{e^x}{3 + x^2}$
24. $f(x) = \dfrac{x^2 + 1}{e^x}$
25. $f(x) = \dfrac{xe^x}{x^2 + 1}$
26. $h(x) = \dfrac{g(x)}{1 + f(x)}$

For Exercises 27–28, find the equation of the tangent line to the curve at the given point.

27. $y = \dfrac{3x}{x^2 + 1}$ @ $\left(2, \frac{6}{5}\right)$
28. $y = \dfrac{e^x}{x + e^x}$ @ $(0, 1)$

For Exercises 29–30, find the second derivative of the function.

29. $f(x) = x^8 e^x$
30. $f(x) = \dfrac{x^2}{e^x}$

Deeper Dive

Theorem 3.2.11 (Reciprocal Rule)**.** *If g is differentiable, then*

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-g'(x)}{g(x)^2}.$$

31. Use the Quotient Rule to prove the Reciprocal Rule.

32. Use the Reciprocal Rule to compute $\frac{d}{dx}\left(\frac{1}{x+ke^x}\right)$, where k is constant.

33. Use the Reciprocal Rule to verify that the Power Rule is valid for negative integers, that is,

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

for all positive integers n.

"Until he extends his circle of compassion to include all living things, man will not himself find peace."
– Albert Schweitzer

Lecture Videos

Trigonometric Limits	The Derivatives of Sine and Cosine	The Derivatives of Tangent and Other Trigonometric Functions	Higher Derivatives of Sine

3.3 Trigonometric Derivatives

In order to calculate the derivatives of trigonometric functions, we first need to talk about limits of trigonometric functions. The next proposition is an important result in this direction, which was proven previously using the Squeeze Theorem.

Proposition 3.3.1.

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

We can use this result to prove another important trigonometric identity concerning limits.

Proposition 3.3.2.

$$\lim_{x\to 0}\frac{\cos x-1}{x}=0.$$

Proof.

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x-1}{x} &= \lim_{x\to 0}\frac{(\cos x-1)}{x}\frac{(\cos x+1)}{(\cos x+1)}=\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}\\
&= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)},\ \text{ using the trig identity "}\sin^2 x+\cos^2 x=1\text{"}\\
&= \lim_{x\to 0}(-\sin x)\left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x+1}\right)\\
&= \lim_{x\to 0}(-\sin x)\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\cdot\lim_{x\to 0}\left(\frac{1}{\cos x+1}\right)\\
&= (-\sin(0))(1)\left(\frac{1}{1+1}\right)=(0)(1)(1/2)=0.
\end{aligned}$$

□

Example 3.3.3. Calculate the limit $\lim_{x\to 0} x\cot x$.

$$\lim_{x\to 0}x\cot x=\lim_{x\to 0}x\left(\frac{\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{\cos x}{\sin x/x}=\frac{\cos 0}{1}=\boxed{1}.$$

Example 3.3.4. Find $\lim_{x\to 0}\dfrac{\sin 7x}{4x}$.

Let us first note that $\dfrac{\sin 7x}{4x} = \dfrac{\sin 7x}{7x}\left(\dfrac{7}{4}\right)$. If we let $\theta = 7x$, then

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \lim_{x\to 0}\frac{7}{4}\cdot\frac{\sin 7x}{7x} = \lim_{\theta\to 0}\frac{7}{4}\cdot\frac{\sin\theta}{\theta} = \boxed{\frac{7}{4}}.$$

With these very important limits out of the way, we are ready to calculate the derivatives of $\sin x$ and $\cos x$.

Theorem 3.3.5 (Derivatives of $\sin x$ and $\cos x$)**.**

$$\frac{d}{dx}(\sin x) = \cos x \qquad \textit{and} \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

Proof. Let $f(x) = \sin x$. Then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h)-\sin(x)}{h}+\frac{\sin(h)\cos(x)}{h}\right) \\
&= \lim_{h\to 0}\left(\sin(x)\left(\frac{\cos(h)-1}{h}\right)+\cos(x)\left(\frac{\sin(h)}{h}\right)\right) \\
&= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 = \cos(x).
\end{aligned}$$

The proof that $\dfrac{d}{dx}(\cos x) = -\sin x$ is similar. □

Example 3.3.6. Find the derivative of each function.

(a) $f(x) = 3x\cos x$.

$$f'(x) = (3x)'\cos x + 3x(\cos x)' = \boxed{3\cos x - 3x\sin x}.$$

(b) $g(x) = x^2\sin x$.

$$g'(x) = (x^2)'\sin x + x^2(\sin x)' = \boxed{2x\sin x + x^2\cos(x)}.$$

Theorem 3.3.7 (Derivatives of Tangent of the Other Trigonometry Functions)**.**

$$\frac{d}{dx}(\tan x) = \sec^2 x, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x, \qquad \frac{d}{dx}(\sec x) = \sec x\tan x, \qquad \frac{d}{dx}(\csc x) = -\csc x\cot x.$$

Proof. Each of these statements can be proven using the quotient rule. We will demonstrate how using tangent.

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

□

Example 3.3.8. Find the derivative of $f(x) = x^2 \sec x$.

$$D_x(x^2 \sec x) = D_x(x^2)\sec x + x^2 D_x(\sec x) = \boxed{2x\sec x + x^2 \sec x \tan x}.$$

Example 3.3.9. Differentiate $f(x) = \dfrac{\sec x}{1+\tan x}$. For what value of x does the graph of f have a horizontal tangent line?

First

$$f'(x) = \frac{(1+\tan x)\sec x \tan x - \sec x(\sec^2 x)}{(1+\tan x)^2} = \frac{\sec x[(\tan x + \tan^2 x) - \sec^2 x]}{(1+\tan x)^2} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}.$$

Now,

$$\begin{aligned}
f'(x) &= 0 \\
\frac{\sec x(\tan x - 1)}{(1+\tan x)^2} &= 0 \\
\sec x(\tan x - 1) &= 0 \\
\tan x - 1 &= 0 \\
\tan x &= 1 \\
x &= \boxed{\frac{\pi}{4} + n\pi}
\end{aligned}$$

Example 3.3.10. Find the nth derivative of $f(x) = \sin x$.

$$\begin{aligned}
f'(x) &= \cos x \\
f''(x) &= -\sin x \\
f'''(x) &= -\cos x \\
f^{(4)}(x) &= \sin(x) = f(x) \\
f^{(5)}(x) &= \cos(x) \\
f^{(27)}(x) &= f^{(4\cdot 6+3)}(x) = f'''(x) = -\cos(x) \\
f^{(42)}(x) &= f''(x) = -\sin(x)
\end{aligned}$$

$$f^{(n)} = \boxed{\begin{cases} \sin(x), & n = 4k \\ \cos(x), & n = 4k+1 \\ -\sin(x), & n = 4k+2 \\ -\cos(x), & n = 4k+3. \end{cases}}$$

[i]See §3.5 Derivatives of Trigonometric Functions in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–17, compute the given limit.

1. $\lim_{x\to 0} \frac{2x}{\sin(x)}$

♠ 2. $\lim_{x\to 0} \frac{\sin(3x)}{x}$

3. $\lim_{\theta\to 0} \frac{1}{\theta\cot\theta}$

4. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$

♠ 5. $\lim_{x\to 0} \frac{\sin(2x)}{3x}$

6. $\lim_{x\to 0} \frac{\sin(x^5)}{x}$

♠ 7. $\lim_{x\to 0} \frac{\sin(5x)}{\sin(9x)}$

8. $\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)}$

9. $\lim_{\theta\to 0} \frac{\sin(5\theta)}{\sin(7\theta)}$

10. $\lim_{\theta\to 0} \frac{\sin(3\theta)\sin(5\theta)}{\theta^2}$

11. $\lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta}$

12. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}$

♠ 13. $\lim_{\theta\to 0} \frac{\cos(9\theta) - 1}{\sin(2\theta)}$

14. $\lim_{\theta\to 0} \frac{\sin^2(3\theta)}{(2\theta)^2}$

15. $\lim_{\theta\to 0} \frac{\tan(6\theta)}{\sin(2\theta)}$

16. $\lim_{\theta\to 0} \frac{\sin\theta}{5\theta^2 - 4\theta}$

17. $\lim_{\theta\to 1} \frac{\sin(\theta - 1)}{\theta^2 + \theta - 1}$

For Exercises 18–27, differentiate the function.

18. $f(x) = \tan(x)\sec(x)$

♠ 19. $f(x) = \sin x + \frac{7}{8}\cot x$

♠ 20. $y = 6x^5 - 3\cos(x)$

♠ 21. $f(\theta) = 8\sqrt{\theta}\sin\theta$

22. $y = e^x\cos(x)$

♠ 23. $f(x) = 9xe^x\csc x$

♠ 24. $y = 7x^2\cos x\cot x$

♠ 25. $y = \frac{6 - \sec x}{\tan x}$

♠ 26. $y = \frac{4x}{7 - \cot x}$

27. $f(\theta) = \frac{\sin\theta - 1}{\sin\theta + 1}$

For Exercises 28–30, find the equation of the tangent line to the curve at the given point.

28. $f(x) = \sin x$ @ $(\pi, 0)$

♠ 29. $f(x) = \sec x$ @ $\left(\frac{\pi}{3}, 2\right)$

30. $f(x) = \sec x$ @ $\left(\frac{\pi}{4}, \sqrt{2}\right)$

For Exercises 31–32, find the second derivative of the function.

31. $y = \sec(x)$

32. $y = e^x\cos(x)$

33. Prove that $\frac{d}{dx}(\cos x) = -\sin x$.

34. Prove that $\frac{d}{dx}(\sec x) = \tan(x)\sec(x)$.

♠ 35. Prove that $\frac{d}{dx}(\csc x) = -\csc x\cot x$

36. Prove that $\frac{d}{dx}(\cot x) = -\csc^2(x)$.

"A positive attitude causes a chain reaction of positive thoughts, events and outcomes. It is a catalyst and it sparks extraordinary results." – Wade Boggs

Lecture Videos

The Chain Rule

Trigonometric Derivatives and the Chain Rule

Exponential Derivatives and the Chain Rule

Combining the Product Rule and the Chain Rule

The Chain Rule and the Quotient Rule

3.4 The Chain Rule

Now, a natural question to ask, "Is there an effective way to take the derivative of a composition of functions?" The answer is yes!

Theorem 3.4.1 (The Chain Rule)**.** *If $g(x)$ and $f(x)$ are differentiable functions such that the range of g is contained in the domain of f, then*

$$\big[f \circ g\big]'(x) = \big[f(g(x))\big]' = f'\big[g(x)\big] \cdot g'(x).$$

Using the dy/dx notation, if $u = g(x)$ and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

or

$$[f(u)]' = f'(u) \cdot u'.$$

Proof. Let Δu be the change in u corresponding to a change of in x, denoted Δx, that is,

$$\Delta u = g(x + \Delta x) - g(x).$$

Let Δy denote the change in y corresponding to a change in u, that is,

$$\Delta y = f(u + \Delta u) - f(u).$$

Therefore,

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\left(\frac{\Delta u}{\Delta u}\right) \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\left(\frac{\Delta u}{\Delta x}\right) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}, \qquad \text{(Note that } \Delta u \to 0 \text{ as } \Delta x \to 0.\text{)} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}
\end{aligned}$$

□

Example 3.4.2. Let $h(x) = (5x^3 + 2)^2$. To find $h'(x)$, you could expand the polynomial to discover $h(x) = 25x^6 + 20x^3 + 4$. So, $h'(x) = 150x^5 + 60x^2$. But we can also recognize that if $f(x) = x^2$ and $g(x) = 5x^3 + 2$, then $h(x) = (f \circ g)(x)$. Using the chain rule, we see that

$$h'(x) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 2(5x^3 + 2) \cdot (15x^2) = \boxed{150x^5 + 60x^2}.$$

To visualize the real strength that the chain rule has, let $h(x) = (5x^3+2)^{100}$. Even using the Binomial Theorem, you would not want to expand this function and calculate the derivative using the Power Rule. However, using the Chain rule here is really no more difficult then above. Note,

$$h'(x) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 100(5x^3 + 2)^{99} \cdot (15x^2) = \boxed{1500x^2(5x^3 + 2)^{99}}.$$

Example 3.4.3. Find dy/dx if $y = (3x^2 - 5x)^{1/2}$.

Let $u = 3x^2 - 5x$. Then $y = u^{1/2}$. Using the chain rule, we have

$$y' = (u^{1/2})' = \frac{1}{2}u^{-1/2} \cdot u'.$$

If we substitute out the u, then we have

$$\begin{aligned} y' &= \frac{1}{2}u^{-1/2} \cdot u' = \frac{1}{2}(3x^2 - 5x)^{-1/2} \cdot (3x^2 - 5x)' \\ &= \frac{1}{2}(3x^2 - 5x)^{-1/2} \cdot (6x - 5) = \boxed{\frac{6x - 5}{2(3x^2 - 5x)^{1/2}}}. \end{aligned}$$

Example 3.4.4. Find the derivative of each trigonometric function.

(a) $y = \sin 6x$.

$$y' = (\sin 6x)' = \cos(6x) \cdot (6x)' = \boxed{6\cos 6x}.$$

(b) $f(x) = \sin(x^2)$.

$$f'(x) = \cos(x^2) \cdot (2x) = \boxed{2x\cos(x^2)}$$

(c) $g(x) = \sin^2(x)$.

$$g'(x) = 2\sin(x) \cdot \cos(x) = \boxed{\sin(2x)}$$

(d) $y = 5\sin(9x^2 + 2) + \cos\left(\frac{\pi}{7}\right)$.

$$\begin{aligned} y' &= \left(5\sin(9x^2 + 2) + \cos\left(\frac{\pi}{7}\right)\right)' \\ &= (5\sin(9x^2 + 2))' + \cos\left(\frac{\pi}{7}\right)' \\ &= 5\cos(9x^2 + 2)(18x) + 0 \end{aligned}$$

$$= \boxed{90x\cos(9x^2+2)}$$

(e) $f(x) = \cos^4 x$.

$$f'(x) = 4\cos^3(x)(\cos x)' = \boxed{-4\cos^3 x \sin x}.$$

(f) $y = \cot^6(x)$.

$$D_x(\cot^6 x) = 6\cot^5 x(-\csc^2 x) = \boxed{-6\cot^5 x \csc^2 x}.$$

Example 3.4.5. Find the derivative of each exponential function.

(a) $y = e^{5x}$

$$y' = (e^{5x})' = e^{5x}(5x)' = \boxed{5e^{5x}}.$$

(b) $y = 10e^{3x^2}$

$$y' = \left(10e^{3x^2}\right)' = \left(10e^{3x^2}\right)\cdot(3x^2)' = \left(10e^{3x^2}\right)(6x) = \boxed{60xe^{3x^2}}.$$

(c) $y = e^{\sin x}$.

$$\frac{dy}{dx} = \boxed{e^{\sin(x)}\cos(x)}.$$

Example 3.4.6. Find the derivative of $y = 4x(3x+5)^5$.

We will need to use the product rule and the chain rule for this example.

$$\begin{aligned}
y' &= (4x(3x+5)^5)' \\
&= (4x)'(3x+5)^5 + (4x)\left((3x+5)^5\right)' \quad \text{by the product rule} \\
&= 4(3x+5)^5 + (4x)\left[5(3x+5)^4\cdot(3)\right] \quad \text{by the chain rule} \\
&= 4(3x+5)^4\left[(3x+5) + x(5)(3)\right] \quad \text{taking out common factor} \\
&= \boxed{4(3x+5)^4(18x+5)}
\end{aligned}$$

Theorem 3.4.7 (The Quotient Rule)**.** *If $f(x)$ and $g(x)$ are differentiable functions and if $g(x) \neq 0$, then*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Proof.

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= (f(x)g(x)^{-1})' = f'(x)g(x)^{-1} + f(x)[g(x)^{-1}]' \\
&= f'(x)g(x)^{-1} + f(x)[-g(x)^{-2}\cdot g'(x)]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f'(x)}{g(x)}\left(\frac{g(x)}{g(x)}\right) - \frac{f(x)g'(x)}{g(x)^2} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}
$$

□

Example 3.4.8. Find the derivative of $f(x) = \dfrac{x^2+1}{3x+2}$.

Mimicking the proof of the previous theorem, we get that $f(x) = (x^2+1)(3x+2)^{-1}$. Thus,

$$
\begin{aligned}
f'(x) &= [(x^2+1)(3x+2)^{-1}]' = (x^2+1)'(3x+2)^{-1} + (x^2+1)[(3x+2)^{-1}]' \\
&= (2x)(3x+2)^{-1}(x^2+1)[-(3x+2)^{-2}\cdot(3)] = \frac{2x}{3x+2} - \frac{(x^2+1)(3)}{(3x+2)^2} \\
&= \frac{2x(3x+2)-(x^2+1)(3)}{(3x+2)^2}, \qquad \text{(This line is just the Quotient Rule).} \\
&= \frac{6x^2+4x-3x^2-3}{(3x+2)^2} = \boxed{\frac{3x^2+4x-3}{(3x+2)^2}}
\end{aligned}
$$

[i]See §3.6 The Chain Rule in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–31, differentiate the function.

1. $y = x^2 - e^{-x}$
2. $f(x) = (x^2 + e^x)^4$
3. ♠ $y = (x^4 + 5x^2 - 3)^3$
4. $f(x) = (x+5)^{231}$
5. $g(x) = (3x^2 - 5)^{100}$
6. $h(x) = (x^3 + x^2)^{125}$
7. $f(x) = \sqrt{x^2+1}$
8. $f(x) = \sqrt{x^2+x}$
9. $y = \sqrt{x^4 + 4x^2 + 25}$
10. ♠ $y = \sqrt[3]{1+8x}$
11. $y = \sqrt[3]{8x^3 + 27}$
12. $f(x) = \sqrt[4]{x^4 + 16}$
13. ♠ $y = \sqrt{5 + 8e^{4x}}$
14. $y = \tan(2x)$
15. $y = \sec(2x)$
16. $f(x) = \cot(2x)$
17. $g(x) = \sin(x^3 + 5x)$
18. ♠ $f(x) = a^7 + \cos^6(x)$
19. ♠ $f(x) = \cos(a^8 + x^8)$
20. $h(x) = \sin(3x + \cos x)$
21. $f(x) = e^{x^2}$
22. ♠ $y = e^{3\sqrt{x}}$
23. $f(x) = e^{\cos(x)} + \cos(e^x)$
24. $y = 4e^{-t} - 3(\sin t)^4$
25. ♠ $f(x) = 9xe^{-kx}$
26. $g(x) = e^{e^x}$
27. ♠ $f(x) = (2x-5)^4(x^2+x+1)^5$
28. $f(x) = (xe^x)^{10}$
29. $y = e^x \sin(x^3 - 2x)$
30. ♠ $y = \left(\dfrac{x^2+4}{x^2-4}\right)^3$
31. $f(x) = \left(\dfrac{x}{x+1}\right)^9$

"No man can put a chain about the ankle of his fellow man without at last finding the other end fastened about his own neck." – Frederick Douglass

Lecture Videos

Examples of the Chain Rule

Using the Chain Rule on the Composition of Three Functions

Finding the Equation of a Tangent Line using the Chain Rule

Using the Chain Rule Graphically

Derivatives of Exponential Functions

The Chain Rule and a Story Problem

3.5 The Chain Rule II

Example 3.5.1. Find the derivative.

(a) $y = (x^3 - 1)^{100}$.

$$y' = 100(x^3-1)^{99} \cdot (3x^2) = \boxed{300x^2(x^3-1)^{99}}.$$

(b) $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$.

$$f'(x) = -\frac{1}{3}(x^2+x+1)^{-4/3}(2x+1) = \boxed{-\frac{2x+1}{3\sqrt[3]{(x^2+x+1)^4}}}.$$

(c) $g(t) = \left(\dfrac{t-2}{2t+1}\right)^9$.

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8\left(\frac{(2t+1)(1)-(t-2)(2)}{(2t+1)^2}\right) = \boxed{\frac{45(t-2)^8}{(2t+1)^{10}}}.$$

(d) $y = (2x+1)^5(x^3-x+1)^4$.

$$\begin{aligned} y' &= 5(2x+1)^4(2)(x^3-x+1)^4 + (2x+1)^5(4)(x^3-x+1)^3)(3x^2-1) \\ &= \boxed{2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3)}. \end{aligned}$$

Example 3.5.2. Let $y = e^{x^2+1}\sqrt{5x+2}$. Find y'.

$$\begin{aligned} y' &= \left(e^{x^2+1}\right)(\sqrt{5x+2})' + \left(e^{x^2+1}\right)'(\sqrt{5x+2}) \\ &= \left(e^{x^2+1}\right)\left(\frac{1}{2}(5x+2)^{-1/2}(5)\right) + \left(e^{x^2+1}\right)(2x)(5x+2)^{(1/2)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{5e^{x^2+1}}{2\sqrt{5x+2}} + 2xe^{x^2+1}\sqrt{5x+2}\cdot\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} = \frac{5e^{x^2+1}+4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} \\
&= \frac{e^{x^2+1}\big[5+4x(5x+2)\big]}{2\sqrt{5x+2}} = \boxed{\frac{e^{x^2+1}(20x^2+8x+5)}{2\sqrt{5x+2}}}
\end{aligned}
$$

The chain rule applying no matter how many links are in the chain. For example,

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\
&= \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} \\
&= \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dt}\cdot\frac{dt}{dx} \\
&= \cdots
\end{aligned}
$$

Example 3.5.3. Find the derivative.

(a) $f(x) = \sin(\cos(\tan x))$.

$$
\begin{aligned}
f'(x) &= \cos(\cos(\tan(x)))\cdot[\cos(\tan(x))]' = \cos(\cos(\tan(x)))\cdot(-\sin(\tan(x))\cdot\tan(x)' \\
&= -\cos(\cos(\tan(x)))\sin(\tan(x))\cdot\sec^2(x) = \boxed{-\cos(\cos(\tan(x)))\sin(\tan(x))\sec^2(x)}.
\end{aligned}
$$

(b) $g(x) = \sqrt{1+\sqrt{1+\sqrt{x}}}$.

$$
\begin{aligned}
g'(x) &= \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\cdot\left[1+\sqrt{1+\sqrt{x}}\right]' \\
&= \frac{1}{2\sqrt{1+\sqrt{1+\sqrt{x}}}}\cdot\frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\cdot\left[1+\sqrt{x}\right]' \\
&= \frac{1}{4\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}\cdot\frac{1}{2}x^{-1/2} \\
&= \boxed{\frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}}
\end{aligned}
$$

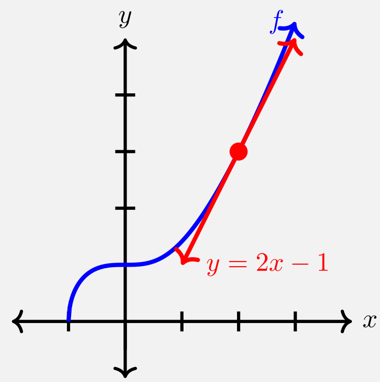
(c) $y = e^{\sec 3\theta}$.

$$
y' = e^{\sec(3\theta)}\cdot(\sec(3\theta)\tan(3\theta))\cdot(3) = \boxed{3e^{\sec(3\theta)}\sec(3\theta)\tan(3\theta)}.
$$

Example 3.5.4. Find the equation of the tangent of $f(x) = \sqrt{1+x^3}$ at the point $(2, 3)$.

Note that $f'(x) = \frac{1}{2}(1+x^3)^{-1/2} \cdot (3x^2) = \frac{3x^2}{2\sqrt{1+x^3}}$. Thus, $f'(2) = \frac{3(2)^2}{2\sqrt{1+(2)^3}} = \frac{3(2)^{\cancel{2}}}{\cancel{2}\sqrt{9}} = \frac{\cancel{3}(2)}{\cancel{3}} = 2$. Therefore, the tangent line is given as

$$y - 3 = 2(x-2) \Rightarrow \boxed{y = 2x - 1}.$$



Example 3.5.5. The graphs of functions f and g are given.

(a) Let $u = (f \circ g)(x)$. Compute $u'(1)$.

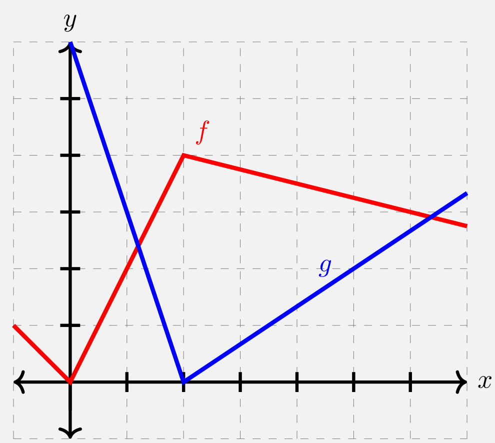
By the Chain Rule, $u'(1) = (f \circ g)'(1) = f'(g(1)) \cdot g'(1) = f'(3) \cdot g'(1) = \left(-\frac{1}{4}\right)(-3) = \boxed{\frac{3}{4}}$

(b) Let $v = (g \circ f)(x)$. Compute $v'(1)$.

By the Chain Rule, $v'(1) = (g \circ f)'(1) = g'(f(1)) \cdot f'(1) = g'(2) \cdot f'(1)$. But $g'(2)$ is undefined, which implies that $v'(1)$ is $\boxed{\text{undefined}}$.

(c) Let $w = (g \circ g)(x)$. Compute $w'(1)$.

By the Chain Rule, $w'(1) = (g \circ g)'(1) = g'(g(1)) \cdot g'(1) = g'(3) \cdot g'(1) = \left(\frac{2}{3}\right)(-3) = \boxed{-2}$



Theorem 3.5.6 (Derivatives of Exponential Functions)**.**

$$\frac{d}{dx}(a^x) = a^x(\ln a).$$

Proof. Note that $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$. Thus,

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x\ln(a)}\right) = e^{x\ln(a)} \cdot \ln(a) = a^x(\ln a).$$

□

Example 3.5.7. Find the derivative of each exponential function.

(a) $s = 3^t$

$$s' = (3^t)' = \boxed{(\ln 3)3^t}.$$

(b) $s = 8 \cdot 10^{1/t}$

$$\begin{aligned} s' &= \left(8 \cdot 10^{1/t}\right)' = 8\left(10^{1/t}\right)' = 8(\ln 10)\left(10^{1/t}\right) \cdot (1/t)' \\ &= 8(\ln 10)\left(10^{1/t}\right) \cdot (-1/t^2) = \boxed{\frac{-8(\ln 10)10^{1/t}}{t^2}}. \end{aligned}$$

Example 3.5.8. The revenue realized by a small city from the collection of fines from parking tickets is given by

$$R(n) = \frac{8000n}{n+2},$$

when n is the number of work-hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work-hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work-hours per day. How fast is revenue from parking fines decreasing at the outbreak of the epidemic?

We want to find dR/dt, the change in revenue with respect to time. By the chain rule,

$$\frac{dR}{dt} = \frac{dR}{dn} \cdot \frac{dn}{dt}.$$

First we find dR/dn, using the quotient rule.

$$\frac{dR}{dn} = \frac{(n+2)(8000n)' - (8000n)(n+2)'}{(n+2)^2} = \frac{8000(n+2) - 8000n}{(n+2)^2} = \frac{16,000}{(n+2)^2}.$$

Since 30 work-hours were used at the outbreak of the epidemic, $n = 30$, so $\left.\frac{dR}{dn}\right|_{n=30} = \frac{16,000}{(30+2)^2} = \frac{16000}{1024} = 15.625$. Also, $\frac{dn}{dt} = -6$. Thus,

$$\frac{dR}{dt} = \frac{dR}{dn} \cdot \frac{dn}{dt} = (15.625)(-6) = -93.75.$$

Revenue is being lost at the rate of about $\boxed{\$94 \text{ per day}}$ at the outbreak of the epidemic.

One lesson to learn from this section is that a derivative is always with respect to some variable. You need to pay close attention to which variable we are allowing to change.

[i]See §3.6 The Chain Rule in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–13, differentiate the function.

1. $y = e^{\tan(\sqrt{x})}$

♠ 2. $y = \sin(\tan(7x))$

♠ 3. $y = \cos\left(\dfrac{1-e^{2x}}{1+e^{2x}}\right)$

♠ 4. $y = \cot^2(\sin\theta)$

♠ 5. $y = \cos(\cos(\cos x))$

6. $f(x) = x^3 - 3^x$

7. $f(x) = x^\pi - \pi^x$

8. $y = 3^{x^2}$

9. $f(x) = 2^{x^2-x+1}$

♠ 10. $y = 3^{\sqrt{x}}$

♠ 11. $f(t) = 6^{t^3}$

♠ 12. $f(x) = 9^{C/x}$

13. $f(x) = 2^{\sec(x)}$

For Exercises 14–17, find the equation of the tangent line to the curve at the given point.

14. $f(x) = \sin(\sin(x))$ @ $(\pi, 0)$

♠ 15. $y = (1+2x)^8$ @ $(0,1)$

♠ 16. $y = \sin(3x) + \sin^2(3x)$ @ $(0,0)$

17. $f(x) = 4e^{-x} - 3(\sin x)^4$ @ $\left(\frac{\pi}{4}, \frac{16-3e^{\pi/4}}{4e^{\pi/4}}\right)$

♠ 18. Find all horizontal tangent lines to the graph of the function $f(x) = 2\cos x + \cos^2 x$.

For Exercises 19–22, use the table below to evaluate the expression.

19. $h'(3)$ if $h = f \circ g$

♠ 20. $h'(0)$ if $h = f \circ g$

♠ 21. $h'(3)$ if $h(x) = \sqrt{3 + 2f(x)}$

22. $h'(3)$ if $h = g \circ f$

x	0	1	2	3	4	5
$f(x)$	5	2	0	1	3	4
$g(x)$	2	1	3	5	4	0
$f'(x)$	0	1	2	3	4	5
$g'(x)$	3	1	2	0	4	5

For Exercises 23–26, find the second derivative of the function.

23. $f(x) = e^{5x}$

♠ 24. $y = e^{6e^x}$

♠ 25. $y = \sqrt{\sin x}$

26. $f(x) = x^3 - 3^x$

27. Prove that the derivative of an even function is odd.

28. Prove that the derivative of an even function is even.

"A piece of advice always contains an implicit threat, just as a threat always contains an implicit piece of advice." – Jose Bergamin

Lecture Videos

Implicit Differentiation

Implicit Differentiation vs. Explicit Differentiation

Implicit Differentiation (Polynomial Relation)

Implicit Differentiation (Radical Relation)

Implicit Differentiation (Trigonometric Relation)

3.6 Implicit Differentiation

In almost all of the examples and applications we have done so far, all equations have been of the form

$$y = f(x),$$

where y is given **explicitly** in terms of x, that is, y is equal to some algebraic expression in terms of x. This is typical for functions. For example,

$$y = 3x - 2, \;\; y = x^2 + x + 6, \;\; \text{and} \;\; y = -\frac{x^3+2}{x^4-1}$$

are all **explicit functions** of x. On the other hand, some equations of x and y do not have y explicitly solved, for example,

$$4xy - 3x = 6, \;\; y^2 + 2yx + 4x^2 = 0, \;\; \text{and} \;\; y^5 + 8y^3 + 6y^2x^2 + 2yx^3 + 6 = 0.$$

In such equations, y is said to be given **implicitly** in terms of x. Now, **implicit equations**[i] can sometimes be solved for a particular variable. For example,

$$4xy - 3x = 6 \quad \Rightarrow \quad 4xy = 3x + 6 \quad \Rightarrow \quad y = \frac{3x+6}{4x}$$

Similarly, $y^2 + 2yx + 4x^2 = 0$ can be solved for y by completing the square or by the quadratic equation, but this can be tedious and will still not be an explicit function in general. Even worse, there is no hope of solving $y^5 + 8y^3 + 6y^2x^2 + 2yx^3 + 6 = 0$ in terms of y.

On the other hand, even though implicit equations usually do not correspond to explicit functions or even explicit equations, they are of great importance to mathematics and her applications (physics, economics, etc.). For example, suppose that we have the equation $y^2 = x^3 + 17$ and we want to determine the tangent line of this graph at the point $(-2, 3)$, how would we be able to do this? We used the derivative of a function to find the slope of the tangent lines previously. $\dfrac{dy}{dx}$ will still represent the slope of the tangent line, and, as we will see, this can still be found even if the equation is implicit. This is known as **implicit differentiation**.

It is often useful to use dy/dx here rather than y' to make it clear which variable is independent and which is dependent. On the other hand, if it is clear from the context which variable we are differentiating with respect to, then the prime notation may still be used.

Example 3.6.1. If $x^2 + y^2 = 25$, then find $\dfrac{dy}{dx}$. Use this to then find the tangent line equation at the point $(3, 4)$.
We will find this derivative and equation in two ways.[ii] First, we try implicit differentiation:

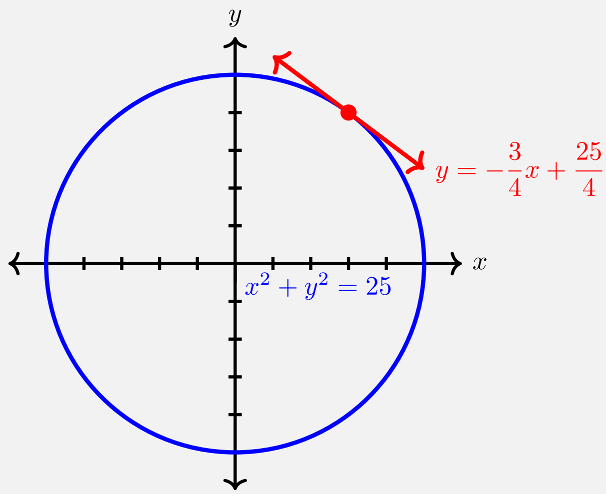
$$\begin{aligned}
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(25) \\
\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \\
2x + 2y\frac{dy}{dx} &= 0 \\
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= \boxed{-\frac{x}{y}}
\end{aligned}$$

Thus, the slope of the tangent line at $(3, 4)$ is

$$\left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4}.$$

Therefore, the tangent line is given as

$$y - 4 = -\frac{3}{4}(x - 3) \qquad \Rightarrow \qquad \boxed{y = -\frac{3}{4}x + \frac{25}{4}}.$$



Alternatively, we can solve for y explicitly and find the tangent line as we have in the past. Note that $y = \pm\sqrt{25 - x^2}$. Since $(3, 4)$ lies on the upper semicircle, we choose our function to be $y = \sqrt{25 - x^2}$. Then

$$\begin{aligned}
\frac{dy}{dx} &= \left[\sqrt{25 - x^2}\right]' = \frac{1}{2}\left[25 - x^2\right]^{-1/2} \cdot (-2x) \\
&= \frac{-2x}{2\sqrt{25 - x^2}} = -\frac{x}{\sqrt{25 - x^2}}
\end{aligned}$$

Thus,

$$\left.\frac{dy}{dx}\right|_{x=3} = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{\sqrt{25 - 9}} = -\frac{3}{\sqrt{16}} = \boxed{-\frac{3}{4}},$$

which is the same slope as earlier.

Example 3.6.2. Find dy/dx if $3xy + 4y^2 = 10$.

When using differentiation implicitly, we will differentiate both sides of the equation with respect to

x and will pay close attentition to the chain rule when necessary. Thus,

$$\begin{aligned}
\frac{d}{dx}(10) &= \frac{d}{dx}(3xy + 4y^2) \\
0 &= 3\frac{d}{dx}(xy) + 4\frac{d}{dx}(y^2) \\
0 &= 3\left(y\frac{d}{dx}(x) + x\frac{d}{dx}(y)\right) + 4\left(2y\frac{d}{dx}(y)\right) \\
0 &= 3\left(y + x\frac{dy}{dx}\right) + 8y\frac{dy}{dx} \\
0 &= 3y + 3x\frac{dy}{dx} + 8y\frac{dy}{dx} \\
-3y &= \frac{dy}{dx}(3x + 8y) \\
\frac{dy}{dx} &= \boxed{\frac{-3y}{3x + 8y}}
\end{aligned}$$

Example 3.6.3. Find dy/dx for $x + \sqrt{x}\sqrt{y} = y^2$.

Take the derivative on both sides with respect to x.

$$\begin{aligned}
(x + \sqrt{x}\sqrt{y})' &= (y^2)' \\
x' + (\sqrt{x}\sqrt{y})' &= 2yy' \\
1 + (\sqrt{x}\sqrt{y}' + \sqrt{x}'\sqrt{y}) &= 2yy' \\
1 + \sqrt{x}\frac{y'}{2\sqrt{y}} + \sqrt{y}\frac{1}{2\sqrt{x}} &= 2yy' \\
1 + \frac{\sqrt{y}}{2\sqrt{x}} &= 2yy' - \frac{\sqrt{x}}{2\sqrt{y}}y' \\
1 + \frac{\sqrt{y}}{2\sqrt{x}} &= \left(2y - \frac{\sqrt{x}}{2\sqrt{y}}\right)y' \\
\frac{1 + \frac{\sqrt{y}}{2\sqrt{x}}}{2y - \frac{\sqrt{x}}{2\sqrt{y}}} &= y' \\
\left(\frac{1 + \frac{\sqrt{y}}{2\sqrt{x}}}{2y - \frac{\sqrt{x}}{2\sqrt{y}}}\right)\left(\frac{2\sqrt{x}\sqrt{y}}{2\sqrt{x}\sqrt{y}}\right) &= y' \\
y' &= \boxed{\frac{2\sqrt{x}\sqrt{y} + y}{4y\sqrt{x}\sqrt{y} - x}}
\end{aligned}$$

Example 3.6.4. Find dy/dx for $\sin(x + y) = y^2 \cos x$.

$$\begin{aligned}
(\sin(x + y))' &= (y^2 \cos x)' \cos(x + y)(1 + y') \\
&= (y^2)' \cos x + (y^2)(\cos x)'
\end{aligned}$$

$$\begin{aligned}
\cos(x+y) + y'\cos(x+y) &= 2yy'\cos x - y^2\sin x \\
\cos(x+y) + y^2\sin x &= 2yy'\cos x - y'\cos(x+y) \\
\cos(x+y) + y^2\sin x &= y'(2y\cos x - \cos(x+y)) \\
y' &= \boxed{\frac{\cos(x+y) + y^2\sin x}{2y\cos x - \cos(x+y)}}.
\end{aligned}$$

[i] The term "equation" here is used instead of function because more often than not the graph of an implicit equation fails the vertical line test and hence is not a function at all.

[ii] A third way can be done using Geometric and Trigonometric methods, which we will omit.

[iii] See §3.8 Implicit Differentiation in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–14, find $\dfrac{dy}{dx}$.

1. $y^3 + x^2 = 3y$

♠ 2. $4x^2 - y^2 = 7$

♠ 3. $\dfrac{3}{x} + \dfrac{3}{y} = 7$

♠ 4. $x^8 + y^3 = 1$

♠ 5. $4\sqrt{x} + \sqrt{y} = 2$

♠ 6. $9x^2 + 7xy - y^2 = 8$

♠ 7. $2x^3 + x^2y - xy^3 = 7$

♠ 8. $x^3(x + y) = y^2(4x - y)$

♠ 9. $y \cos x = 4x^2 + 2y^2$

♠ 10. $7 \cos x \sin y = 5$

♠ 11. $e^{x/y} = 5x - y$

♠ 12. $e^y \cos x = 6 + \sin(xy)$

13. $x^2y + \sin y + x + y^2$

14. $x \sin y + y \cos x$

"The opposite of love is not hate, it's indifference." – Elie Wiesel

Lecture Videos

Implicit Differentiation (Folium of Descartes)

Second Derivatives with Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

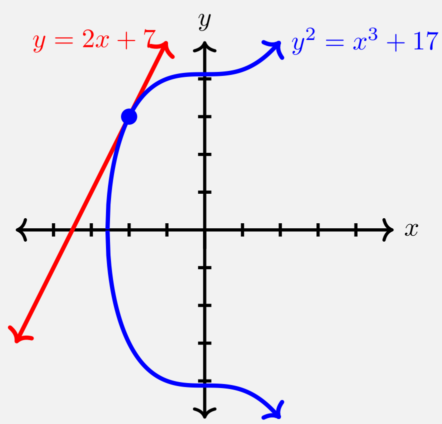
3.7 Implicit Differentiation II

Example 3.7.1. Find the tangent line of the curve $y^2 = x^3 + 17$ at the point $(-2, 3)$.

We use implicit differentiation to find the tangent slope. Note that

$$\begin{aligned} \frac{d}{dx}(y^2) &= \frac{d}{dx}(x^3 + 17) \\ 2y\frac{dy}{dx} &= 3x^2 \\ \frac{dy}{dx} &= \frac{3x^2}{2y}. \end{aligned}$$

Thus, $\left.\frac{dy}{dx}\right|_{(-2,3)} = \frac{3(-2)^2}{2(3)} = \frac{12}{6} = 2.$

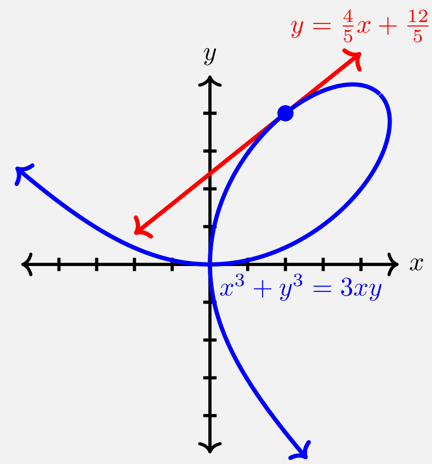


We then calculate the tangent equation.

$$(y - 3) = 2(x - (-2)) \quad \Rightarrow \quad y - 3 = 2x + 4 \quad \Rightarrow \quad \boxed{y = 2x + 7}.$$

Example 3.7.2. Find the equation of the tangent line at point $(2, 4)$ of the equation $x^3 + y^3 = 9xy$.

We find the equation of the tangent line before but need to use Implicit Differentiation to find the slope.



$$\begin{aligned}
(x^3+y^3)' &= (9xy)' \\
(x^3)' + (y^3)' &= 9(xy)' \\
3x^2 + 3y^2y' &= 9(xy' + x'y) = 9xy' + 9y \\
3y^2y' - 9xy' &= 9y - 3x^2 \\
y'(3y^2 - 9x) &= 9y - 3x^2 \\
y' &= \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}
\end{aligned}$$

Thus, at $(2,4)$, we have the slope

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \frac{3(4) - (2)^2}{(4)^2 - 3(2)} = \frac{12-4}{16-6} = \frac{8}{10} = \frac{4}{5}.$$

Then the equation of the tangent is given as

$$y - 4 = \frac{4}{5}(x-2) \quad \Rightarrow \quad y = \frac{4}{5}x - \frac{8}{5} + 4 \quad \Rightarrow \quad \boxed{y = \frac{4}{5}x + \frac{12}{5}}$$

Example 3.7.3. Find y'' if $x^4 + y^4 = 16$.

We first find y'.

$$\begin{aligned}
(x^4 + y^4)' &= 16' \\
4x^3 + 4y^3y' &= 0 \\
4y^3y' &= -4x^3 \\
y' &= \frac{-4x^3}{4y^3} = -\frac{x^3}{y^3}.
\end{aligned}$$

We next compute y''.

$$\begin{aligned}
y'' = (y')' &= \left(-\frac{x^3}{y^3}\right)' = -\frac{y^3(x^3)' - x^3(y^3)'}{(y^3)^2} = -\frac{y^3(3x^2) - x^3(3y^2y')}{y^6} \\
&= -\frac{3x^2y^3 - 3x^3y^2(-x^3/y^3)}{y^6} = -\frac{3x^2y^3 + 3x^6y^{-1}}{y^6} = -\frac{3x^2y^{-1}(y^4+x^4)}{y^6} \\
&= -\frac{3x^2(16)}{y^7} = \boxed{-\frac{48x^2}{y^7}}.
\end{aligned}$$

In general, for a curve of the form $ax^n + by^n = c$, for constants a, b, c, this type of substitution will always occur in the second derivative.

Theorem 3.7.4 (The Derivative of Inverse Sine)**.**

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

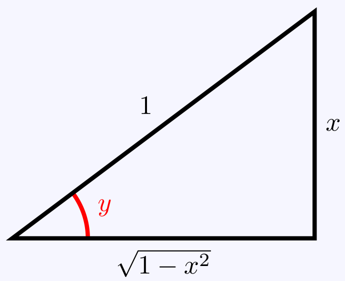
Proof. We will calculate the derivative using implicit differentiation. Notice that

$$y = \arcsin(x) \iff \sin(y) = x.$$

Thus, we calculate dy/dx.

$$\begin{aligned} \sin(y)' &= x' \\ \cos(y)y' &= 1 \\ y' &= \frac{1}{\cos(y)}. \end{aligned}$$

Now, we hope to be able to find a formula using only x. To do this, we will use that fact that $y = \arcsin(x)$.



Thus, $y' = \dfrac{1}{\cos y} = \dfrac{1}{\cos(\sin^{-1} x)}$. Thus, we can construct a right triangle for y. Since $\cos(y) = \dfrac{\text{adj}}{\text{hyp}}$, we conclude that $\cos(\arcsin(x)) = \sqrt{1-x^2}$ and

$$y' = \boxed{\frac{1}{\sqrt{1-x^2}}}.$$

□

Theorem 3.7.5.

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Proof. Again, we will calculate the derivative using implicit differentiation. Notice that

$$y = \arctan(x) \quad \iff \quad \tan(y) = x.$$

Thus, we calculate dy/dx.

$$\begin{aligned} \tan(y)' &= x' \\ \sec^2(y)y' &= 1 \\ y' &= \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}. \end{aligned}$$

□

Theorem 3.7.6 (Derivatives of Inverse Trigonometric Functions)**.**

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Example 3.7.7. Differentiate

(a) $y = \dfrac{1}{\arcsin x}$

$$y' = [(\arcsin x)^{-1}]' = (-1)(\arcsin x)^{-2}(1/\sqrt{1-x^2}) = \boxed{-\frac{1}{(\arcsin x)^2\sqrt{1-x^2}}}.$$

(b) $y = x \arctan\sqrt{x}$

$$\begin{aligned} y' &= (x)' \arctan\sqrt{x} + x(\arctan\sqrt{x})' = \arctan\sqrt{x} + x(1/(1+\sqrt{x}^2)) \cdot (1/2)x^{-1/2} \\ &= \arctan\sqrt{x} + \frac{x}{2\sqrt{x}(1+x)} = \boxed{\arctan\sqrt{x} + \frac{\sqrt{x}}{2(1+x)}}. \end{aligned}$$

[i]See §3.7 Derivatives of Inverse Functions and §3.8 Implicit Differentiation in OpenStax to find the corresponding sections.

Exercises

(Go to Solutions)

For Exercises 1–6, differentiate the function.

1. $f(x) = \sin^{-1}(x^2)$
♠ 2. $y = \sin^{-1}(2x+1)$
♠ 3. $y = (\tan^{-1}(7x))^2$
♠ 4. $f(x) = \sqrt{1-9x^2}\cos^{-1}(3x)$
♠ 5. $g(x) = \tan^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)$
6. $f(x) = x\tan^{-1}(x)$

♠ 7. Find $f'(1)$ if $f(x) + x^2 f(x)^4 = 18$ and $f(1) = 2$

For Exercises 8–11, find the equation of the tangent line to the curve at the given point.

8. $x^2 + y^2 = 25$ @ $(3, 4)$
♠ 9. $x^2 + y^2 = (3x^2 + 4y^2 - x)^2$ @ $\left(0, \frac{1}{4}\right)$
♠ 10. $y\sin(8x) = x\cos(2y)$ @ $\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$
11. $x^2 - y^2 = 4$ @ $(3, \sqrt{5})$

For Exercises 12–15, find $\dfrac{d^2y}{dx^2}$.

12. $xy = 1$
♠ 13. $2x^3 + 5y^3 = 4$
♠ 14. $xy + 8e^y = 8e$
15. $x^2 + y^2 = 25$

♠ 16. Prove that $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$.
17. Prove that $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$.
18. Prove that $\dfrac{d}{dx}(\csc^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$.
19. Prove that $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{1}{1+x^2}$.

Deeper Dive

For Exercises 20–22, suppose f is a one-to-one, differentiable function and its inverse function f^{-1} is also differentiable.

20. Use implicit differentiation to show that $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.
21. If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, find $(f^{-1})'(5)$.
22. Let $f(x) = x + e^x$. Since the function is always increasing, f is one-to-one. Find $f^{-1}(1)$ and $(f^{-1})'(1)$.

"People reveal so much of their mental processes online, simply because the psychological effect of anonymity just means that a whole raft of inhibitions are left alone when people log on." – Joanne Harris

Lecture Videos

Derivatives of Logarithms

Derivatives of Logarithms with Absolution Value

Logarithmic Differentiation

The Proof of the Power Rule by Logarithmic Differentiation

Taking Derivatives of Functions involving Exponents and Bases

Taking Derivatives of Functions involving Absolute Values

3.8 Logarithmic Differentiation

Theorem 3.8.1 (Derivative of the Natural Logarithm)**.**

$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

Proof. We can calculate the derivative implicitly. If $y = \ln x$, then $e^y = x$. The implicit derivative is then

$$\begin{aligned} e^y y' &= 1 \\ y' &= \frac{1}{e^y} = \frac{1}{x}. \end{aligned}$$

□

Theorem 3.8.2 (Derivative of $\log_a x$)**.**

$$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}.$$

Proof.

$$(\log_a(x))' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

□

Example 3.8.3. Find the derivative of each function.

(a) $f(x) = \ln 6x$.
By the chain rule we have

$$f'(x) = (\ln 6x)' = \frac{1}{6x}(6x)' = \frac{1}{6x}(6) = \frac{1}{x}.$$

(b) $f(x) = \ln(x^2+1)$.

$$f'(x) = \ln(x^2+1)' = \frac{1}{(x^2+1)}(x^2+1)' = \frac{1}{x^2+1}(2x) = \boxed{\frac{2x}{x^2+1}}.$$

(c) $y = -\ln(\cos x)$.

$$y' = -\frac{\cos x'}{\cos x} = -\frac{-\sin x}{\cos x} = \boxed{\tan x}.$$

(d) $g(x) = \sqrt{\ln x}$.

$$g'(x) = (1/2)(\ln x)^{-1/2}(1/x) = \boxed{\frac{1}{2x\sqrt{\ln x}}}.$$

(e) $y = \log_2(3x^2-4x)$.

$$y' = \log_2(3x^2-4x)' = \frac{1}{(\ln 2)(3x^2-4x)}(3x^2-4x)' = \boxed{\frac{6x-4}{(\ln 2)(3x^2-4x)}}.$$

(f) $y = \log_{10}(2+\sin x)$

$$y' = \frac{(2+\sin x)'}{(\ln 10)(2+\sin x)} = \boxed{\frac{\cos x}{(\ln 10)(2+\sin x)}}.$$

(g) $y = \ln(-x)$.

$$y' = (\ln(-x))' = \frac{1}{-x}(-1) = \boxed{\frac{1}{x}}.$$

By the Chain Rule we see that

$$(\ln g(x))' = \frac{g'(x)}{g(x)}.$$

Remember that $\log_a 0$ is not defined and $\lim_{x\to 0^+} \log_a x = -\infty$. Also, the logarithm of a negative number is not a real number[i]. Thus, the domain of $\log_a(x)$ is $x > 0$, or $(0,\infty)$ in interval notation. An easy way to extend a logarithm to a function with a larger domain is to consider the composite function $\log_a |x|$. Although, $\log_a |x|$ is not defined at $x = 0$, $\log_a |x|$ is a real number even if x is negative. In fact, Example 3.8.3 (g) shows us that the derivative is the same when x is negative.

Theorem 3.8.4.

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad \textit{and} \qquad \frac{d}{dx}(\log_a |x|) = \frac{1}{(\ln a)x}.$$

Example 3.8.5. Find the derivative of each function.

(a) $y = \ln|5x|$.

If $g(x) = 5x$, then

$$y' = (\ln|5x|)' = \frac{g'(x)}{g(x)} = \frac{5}{5x} = \boxed{\frac{1}{x}}.$$

(b) $f(x) = 3x\ln|x^2|$.

$$f'(x) = (3x)(\ln|x^2|)' + (3x)'(\ln|x^2|) = (3x)\left(\frac{2x}{x^2}\right) + (3)(\ln|x^2|)$$

$$= \frac{6x^2}{x^2} + 3\ln|x^2| = \frac{6x^2}{x^2} + 3\ln x^2 = \boxed{6 + \ln x^6}$$

Example 3.8.6. Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.

One can calculate the derivative of this function like before, using the derivative of the natural log, chain rule, and quotient rule. Alternatively, one could also expand the expression into simpler terms using properties of logarithms. This actually leads to a simpler calculation.

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}[\ln(x+1) - \ln\sqrt{x-2}] = \frac{d}{dx}[\ln(x+1) - \frac{1}{2}\ln(x-2)] = \boxed{\frac{1}{x+1} - \frac{1}{2(x-2)}}.$$

This last example shows us that the logarithmic properties used before the use of Calculus can dramatically simplify a calculation. If we use these log properties in conjugation with implicit differentiation we have a very effective derivative technique known as **logarithmic differentiation.**

Example 3.8.7. Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Using logarithmic differentiation, we have

$$\begin{aligned}
y &= \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} \\
\ln y &= \ln\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2) \\
(\ln y)' &= \left(\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right)' \\
\frac{y'}{y} &= \frac{3}{4x} + \frac{\cancel{2}x}{\cancel{2}(x^2+1)} - \frac{5\cdot 3}{3x+2} \\
y' &= y\cdot\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) = \boxed{\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)}.
\end{aligned}$$

Steps in Logarithmic Differentiation

(i) Take natural logarithms of both sides of an equation $y = f(x)$ and use Laws of Logarithms to simplify.

(ii) Differentiate implicitly with respect to x.

(iii) Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can write $|y| = |f(x)|$ and recognize that $|y|' = \dfrac{y'}{y}$. Because of this fact, we might think we were somewhat careless on Example 3.8.7. Actually, the domain of the function y was $x \geq 0$, since other wise $x^{3/4}$ would not be real. If $x \geq 0$, then $3x+2 > 0$ and $x^2+1 > 0$. The only possible problem was $x^{3/4} = 0$ if $x = 0$, but inspecting the derivative, we see that y' is undefined at $x = 0$.

Theorem 3.8.8 (The Power Rule)**.** *If n is any real number and $f(x) = x^n$, then*

$$f'(x) = nx^{n-1}.$$

Proof. Let $y = x^n$ and we will calculate the derivative logarithmically. If $x \neq 0$,

$$\begin{aligned} \ln|y| &= \ln|x^n| = n \ln|x| \\ \frac{y'}{y} &= \frac{n}{x} \\ y' &= \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}. \end{aligned}$$

If $x = 0$, then $f'(0) = \lim_{h\to 0} \frac{(0+h)^n - 0^n}{h} = \lim_{h\to 0} \frac{h^n}{h} = \lim_{h\to 0} h^{n-1} = 0 = n(0)^{n-1}$. □

Example 3.8.9. Differentiate $y = x^{\sin(x)}$.

Such a function can be easily differentiated by logarithmic differentiation. Notice that the domain of y is $x \geq 0$, otherwise the outputs for negative x's could be non-real.

$$\begin{aligned} \ln y &= \ln x^{\sin x} = \sin x \ln x \\ \frac{y'}{y} &= \cos x \ln x + \sin x \left(\frac{1}{x}\right) = \cos x \ln x + \frac{\sin x}{x} \\ y' &= \boxed{x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)}. \end{aligned}$$

[i] It is actually an imaginary number, e.g. $\ln(-1) = \pi i$

[ii] See §3.9 Derivatives of Exponential and Logarithmic Functions in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–54, differentiate the function.

1. $f(x) = x^2 + \ln(x)$
2. $f(x) = 5x + 2\ln(x) + 3$
3. $f(x) = e^x + \ln(x)$
4. $f(x) = 10e^x + 5\ln(x)$
5. $y = \ln(x) - e^{4-5x}$
6. $f(x) = \dfrac{e^x}{5} + \dfrac{\ln(x)}{3}$
7. $f(x) = \ln(5x - 7)$
8. $y = x^4 + x^3 - \ln(8 - 2x)$
9. $y = \ln(x^2 + 1)$
10. $f(x) = (\ln(x))^5$
11. $f(x) = \sin(7\ln x)$
12. $y = x\ln(x)$
13. $g(x) = x^2\ln(x)$
14. $f(t) = e^t\ln(t)$
15. $g(t) = t^3\ln(t)$
16. ♠ $f(x) = 4x\ln(3x) - 4x$
17. $f(x) = x^2\ln(x)e^x$
18. $f(x) = e^{x^2}\ln(x)$
19. $g(x) = \cos x\ln x$
20. $f(x) = e^{2x-1}\ln(2x - 1)$
21. $f(x) = x^8 e^{2-\ln(2x)}$
22. ♠ $f(x) = \ln(16\sin^2 x)$
23. ♠ $g(x) = \ln(x\sqrt{x^2 - 1})$
24. ♠ $h(x) = \ln\left(\dfrac{(4x+1)^3}{\sqrt{x^2+1}}\right)$
25. ♠ $f(x) = \dfrac{15}{\ln x}$
26. $f(x) = \dfrac{\ln x}{x^2}$
27. $y = \dfrac{e^x}{\ln(x^2 + 1)}$
28. $f(x) = \dfrac{\ln(x)}{x^3}$
29. $f(x) = \dfrac{x^2}{\ln(x)}$
30. $f(x) = \dfrac{\ln(x)}{x}$
31. $f(x) = e^{x^3\ln(x)}$
32. $f(x) = \ln|1 + x - x^2|$
33. $f(x) = \ln|\ln x|$
34. $f(x) = \log_5(xe^x)$
35. $f(x) = \log_2(x^3 + x + 1)$
36. ♠ $f(x) = \log_{10}(x^4 + 8)$
37. ♠ $f(x) = \log_{15}(xe^x)$
38. $f(x) = \log_2|1 + x - x^2|$
39. $g(x) = \log_5(1 + \cos(\sqrt{x}))$ *Remember that* $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{\sin\theta}{1+\cos\theta}$
40. ♠ $y = (x^3 + 2)^2(x^5 + 4)^4$
41. $y = \dfrac{x\sqrt{x^3 - 2}}{(5x + 4)^6}$
42. $y = \sqrt{\dfrac{e^{2x}\cos(x)}{(x + 4)^3}}$
43. ♠ $y = \sqrt{\dfrac{x - 2}{x^4 + 2}}$
44. $y = \left|\dfrac{(x + 2)^4}{\cos(x)}\right|$
45. $y = x^x$
46. ♠ $y = x^{4x}$
47. $y = x^{6x}$
48. $y = (1 + x)^{2x}$
49. ♠ $y = x^{8\cos x}$
50. $y = (\cos(x))^x$
51. $y = (\sqrt{x + 1})^x$
52. $y = |\sin(x)|$
53. ♠ $y = |\cos(2x)|$
54. $x^y = y^x$

For Exercises 55–57, find the equation of the tangent line to the curve at the given point.

55. $f(x) = \ln(1 + e^{2x})$ @ $(0, \ln(2))$
56. $y = \ln(x^2 + 4)$ @ $(-2, \ln(8))$
57. $f(x) = \log_5(x)$ @ $(1, 0)$

For Exercises 58–60, find $\dfrac{d^2y}{dx^2}$.

58. $f(x) = \ln(2x)$

59. $f(x) = x^3 + \ln(x)$

60. $f(x) = x\ln(x)$

"Most people say that it is the intellect which makes a great scientist. They are wrong: it is character."
– Albert Einstein

Lecture Videos

Rates of Change in Science

Derivatives and Linear Density

Derivatives and Isothermal Compressibility

Derivatives and Population Growth

Derivatives and Economics

3.9 Rates of Changes within the Sciences

We know that if $y = f(x)$, then the derivative $\dfrac{dy}{dx}$ can be interpreted as the rate of change of y with respect to x. In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Remember that if x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1).$$

The difference quotient

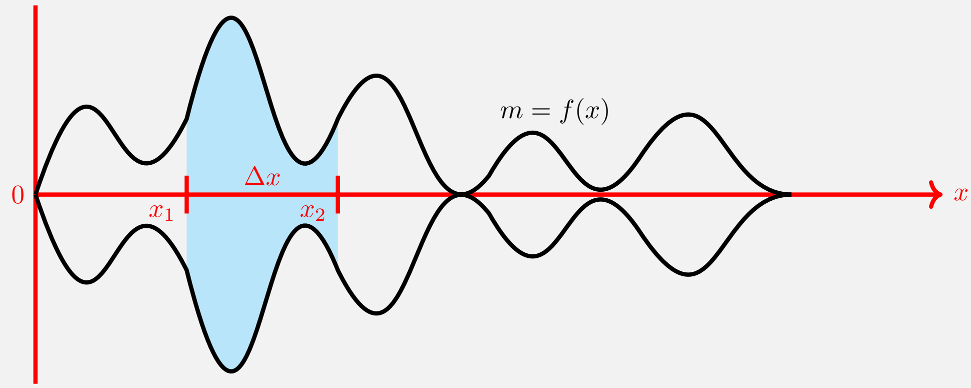
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$. Its limit as $\Delta x \to 0$ is the derivative $f'(x)$, which can be interpreted as the **instantaneous rate of change of y with respect to x**. In particular,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change. For example, in physics we have already studied this rate of change problem with respect to the position function $s = f(t)$. The derivative, the velocity function, $v = f'(t)$, represents the change of position with respect to time. Continuing on, the derivative of velocity, the acceleration function, $a = f''(t)$, represents the change of velocity with respect to time. In this section, we will explore further interpretations of derivatives in the sciences. We will begin with an alternative physics example.

Example 3.9.1. If a rod or piece of wire is homogeneous (that is, its mass and shape are uniform across the entire object), then its linear density is uniform and is defined as the mass per unit length ($\rho = \frac{m}{\ell}$) and measured in kilograms per meter. Suppose, however, that the rod is not homogeneous but that its mass measured from its left end to a point x is given as $m = f(x)$



The mass of the part of the rod that lies between x_1 and x_2 is given by $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

If we now let $\Delta x \to 0$, we are computing the average density over smaller and smaller intervals. The **linear density** ρ at $x = x_1$ is the limit of these average densities as $\Delta x \to 0$. Thus, the linear density is the rate of change of mass with respect to length:

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}.$$

Thus the linear density of the rod is the derivative of mass with respect to length.

Let $m = \sqrt{x}$, where x is measured in meters and m in kilograms. The average density of the portion of the rod given by $1 \leq x \leq 1.2$ is

$$\frac{\Delta m}{\Delta x} = \frac{m_2 - m_1}{x_2 - x_1} = \frac{\sqrt{1.2} - \sqrt{1}}{1.2 - 1} = \frac{\sqrt{1.2} - 1}{0.2} \approx \boxed{0.48 \text{ kg/m}}.$$

On the other hand, the linear density at $x = 1$ is

$$\rho = \left.\frac{dm}{dx}\right|_{x=1} = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = \frac{1}{2} = \boxed{0.5 \text{ kg/m}}.$$

Velocity and density are not the only rates of change that are important in physics. In fact, derivatives occur all over in physics. Others include current (the rate of change of electrical charge over time, see Exercise 3.9.15), power (the rate at which work is done), the rate of heat flow, temperature gradient (the rate of change of temperature with respect to position), and the rate of decay of a radioactive substance in nuclear physics.

Derivatives are also useful in the study of chemistry, for example, the chemical idea of reaction and concentration which introduces the idea of instantaneous rate of reaction. Our next example illustrates the idea of isothermal compressibility.

Example 3.9.2. One of the quantities of interest in thermodynamics is compressibility. If a given substance is kept at a constant temperature, then its volume V depends on its pressure P. We can consider the rate of change of volume with respect to pressure— namely, the derivative dV/dP. As P increases, V decreases, so $dV/dP < 0$. The **compressibility** is defined by introducing a minus sign and dividing this derivative by the volume V:

$$\text{isothermal compressibility} = \beta = -\frac{1}{V}\frac{dV}{dP}.$$

Thus β measures how fast, per unit volume, the volume of a substance decreases as the pressure on it increases at constant temperature.

For instance, the volume V in cubic meters of a sample of air at 25°C was found to be related to the pressure P in kilopascals by the equation

$$V = \frac{5.3}{P}.$$

The dV/dP when $P = 50$kPa is

$$\left.\frac{dV}{dP}\right|_{P=50} = \left.\left(\frac{5.3}{P}\right)'\right|_{P=50} = -\left.\frac{5.3}{P^2}\right|_{P=50} = -\frac{5.3}{2500} = -0.00212 \text{ m}^3/\text{kPa}.$$

The compressibility at that pressure is

$$\beta = -\left.\frac{1}{V}\frac{dV}{dP}\right|_{P=50} = -\frac{50}{5.3}\left(-\frac{5.3}{2500}\right) = \frac{50}{2500} = \frac{1}{50} = \boxed{0.02}.$$

In Biology, population growth is important, as is the rate at which the population is growing.

Example 3.9.3. Let $n = f(t)$ be the number of organisms in an animal/plant population at time t. The change in the population between the times t_1 and t_2 is $\Delta n = f(t_2) - f(t_1)$, and so the average rate of growth during the time period $t_1 \leq t \leq t_2$ is

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

The **instantaneous growth rate** is obtained from the limit of the average growth rates as $\Delta t \to 0$, that is,

$$\textbf{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}.$$

To be more specific, consider a population of bacteria in a homogeneous nutrient medium (that is, every organism has equal access to vital nutrients). Suppose that by sampling the population at certain intervals it is determined that the population doubles every hour. If the initial population is n_0 and the time t is measured in hours, then

$$n = f(t) = n_0(2^t).$$

So the growth rate of the bacteria population at time t is

$$\frac{dn}{dt} = \frac{d}{dt}(n_0 2^t) = n_0 \frac{d}{dt}(2^t) = n_0(\ln 2)2^t.$$

For example, suppose that we start with an initial population of $n_0 = 100$ bacteria. Then the growth rate after 4 hours is

$$\left.\frac{dn}{dt}\right|_{t=4} = 100 \cdot 2^4(\ln 2) = 1600 \ln 2 \approx 1109.$$

This means that, after 4 hours, the bacteria population is growing at a rate of about $\boxed{\text{1109 bacteria per hour}}$.

In biology, rates of change can appear as blood flow, such as law of laminar flow and velocity gradient.

In economics, the derivative is useful for studying cost, revenue, and profit.

Example 3.9.4. Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, then the **additional cost** is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}.$$

The limit of this quantity as $\Delta x \to 0$, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}.$$

Taking $\Delta x = 1$ and n large so that Δx is smalled compared to n, we have

$$C'(n) \approx C(n+1) - C(n).$$

Thus the marginal cost of producing n units is approximating equal to the cost of producing one more unit.

For instance, suppose a company has estimated that the cost in dollars of producing x items is

$$C(x) = 10000 + 5x + 0.01x^2.$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x.$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500) = 5 + 10 = \boxed{\$15/\text{item}}.$$

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$C(501) - C(500) = [10000 + 5(501) + 0.01(501)^2] - [10000 + 5(500) + 0.01(500)^2] = \boxed{\$15.01}.$$

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows into or out of a reservoir. An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height.

In psychology, those interested in learning theory study the so-called learning curve which graphs the performance $P(t)$ of someone learning a skill as a function of the training time t. Of particular interest is the rate at which performance improves as times passes, that is, dP/dt.

In sociology, differential calculus is used in analyzing the spread of rumors (or fads or fashions). If $p(t)$ denotes the proportion of a population that knows a rumor by time t, then the derivative dp/dt represents the rate of spread of the rumor.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept such as the derivative can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more effective than developing properties of special concepts in each separate science.

[i]See §3.4 Derivatives as Rates of Change in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–6, the position of a particle along a straight line is given by $s(t)$ where s is in meters, t is in seconds, and $0 \leq t \leq 10$.

1. $s(t) = t^3 - 9t^2 + 24t + 4$
   (a) What is acceleration of the velocity?
   (b) What is acceleration of the particle?
   (c) When is the particle moving forward?

2. $s(t) = t^3 - 3t$
   (a) What is acceleration of the velocity?
   (b) Find the velocity at $t = 2$.
   (c) What is acceleration of the particle?

♠ 3. $s(t) = t^3 - 15t^2 + 72t$
   (a) What is velocity of the particle?
   (b) What is the velocity at $t = 5$?
   (c) When is the particle at rest?
   (d) When is the particle moving forward?
   (e) Find the total distance traveled during the first 7 s.
   (f) What is acceleration of the particle?
   (g) Find the acceleration at $t = 5$.
   (h) When is the particle speeding up?
   (i) When is the particle slowing down?

♠ 4. $s(t) = 0.01t^4 - 0.03t^3$
   (a) What is velocity of the velocity?
   (b) What is the velocity at $t = 2$?
   (c) When is the particle at rest?
   (d) When is the particle moving forward?
   (e) Find the total distance traveled during the first 9 s.
   (f) What is acceleration of the particle?
   (g) Find the acceleration at $t = 2$.
   (h) When is the particle speeding up?
   (i) When is the particle slowing down?

♠ 5. $s(t) = \cos\left(\dfrac{\pi t}{4}\right)$
   (a) What is velocity of the particle?
   (b) What is the velocity at $t = 1$?
   (c) When is the particle at rest?
   (d) When is the particle moving forward?
   (e) Find the total distance traveled during the first 8 s.
   (f) What is acceleration of the particle?
   (g) Find the acceleration at $t = 1$.
   (h) When is the particle speeding up?
   (i) When is the particle slowing down?

6. $s(t) = t\sin(\pi t)$
   (a) What is acceleration of the velocity?
   (b) Determine the velocity at $t = 3$.
   (c) What is acceleration of the particle?

7. A rock is thrown from the top of a bridge. Its height s above the water, measured in feet, is given by

$$s(t) = 96 + 80t - 16t^2,$$

   where t is measured in seconds.
   (a) What is the velocity of the rock when $t = 2$ seconds?
   (b) When will the rock hit the water?
   (c) What will the velocity of the rock be when it hits the water?

♠ 8. The height of a projectile shot vertically upward from a point 2 m above ground level with an initial velocity of 22.5 $\dfrac{\text{m}}{\text{s}}$ is

$$s(t) = 2 + 22.5t - 4.9t^2$$

   after t seconds.
   (a) Find the velocity at $t = 2$.
   (b) Find the velocity at $t = 4$.
   (c) When does the projectile reach its maximum height?
   (d) What is the maximum height?
   (e) When does it hit the ground?
   (f) With what velocity does it hit the ground?

For Exercises 9–12, suppose a tank of water holds 4500 gallons and drains from the bottom of the tank in 50 minutes. According to Toricelli's Law, the volume V of the water in the tank after t minutes is given as

$$V = 4500\left(1 - \frac{1}{50}t\right)^2$$

for $0 \le t \le 50$. Find the rate at which water is draining from the tank after the given time.

♠ 9. 5 min ♠ 10. 10 min ♠ 11. 20 min ♠ 12. 50 min

♠ 13. Using the volume function from Exercises 9-12, at what time is the water flowing out the fastest?

♠ 14. Using the volume function from Exercises 9-12, at what time is the water flowing out the slowest?

A current I exists whenever an electric charge Q moves, that is, current is the change of charge over time. For Exercises 15–17, suppose the quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t in seconds is given by

$$Q = t^3 - 2t^2 + 4t + 2.$$

Note that amperes (or amps), denoted by A, is a coulombs per second, that is, $1\text{ A} = 1\ \dfrac{\text{C}}{\text{s}}$.

♠ 15. Find the current at $t = 0.1$ s.

♠ 16. Find the current at $t = 1$ s.

♠ 17. At what time is the current the lowest?

♠ 18. Newton's law of gravitation is given as

$$F = \frac{GmM}{r^2},$$

where G is the gravitational constant, m and M are the masses of two bodies, r is the distance between the bodies, and F is the gravitational force exerted between the bodies. If the earth attracts an object with a force that decreases at the rate of $4\ \dfrac{\text{N}}{\text{km}}$ when r =30,000 km. How fast does this force change when r =15,000 km?

♠ 19. According to ideal gas law,

$$PV = nRT,$$

where P is pressure (in atmospheres), V is volumne (in liters), n is the number of moles of the gas, R is the gas constant, and T is temperature (in kelvin). Suppose that for a certain gas with $R = 0.0821$, at a certain instant, $P = 7.0$ atm and is increasing at a rate of $0.13\ \dfrac{\text{atm}}{\text{min}}$, and $V = 13$ L and is decreasing at a rate of $0.17\ \dfrac{\text{L}}{\text{min}}$. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

For Exercises 20–22, the total number of bacteria (in millions) present in a culture is given by $N(t)$ where t represents time (in hours) after the beginning of an experiment.

20. $N(t) = 2t\sqrt{5t+9} + 12$
    (a) How many millions of bacteria will be present at 8 hours?
    (b) How fast is the culture growing 1 hour and 24 minutes ($t = 7/5$) into the experiment?

♠ 21. $N(t) = 100(3^t)$; How fast is the culture growing at 3.5 hours.

22. $N(t) = 1000e^{0.5t}$
    (a) What is the initial population of the bacteria?
    (b) At what rate is the population increasing at the beginning of the experiment?

23. WidgetCo specializes in manufacturing widgets. The cost function for widgets is

$$C(x) = 400 + 0.5x + 0.01x^2,$$

where x is the number of units produced and $C(x)$ is dollars.

(a) How much does it cost to produce 100 widgets?

(b) What is the marginal cost at the production level of 100 units?

24. Suppose that the total cost in hundreds of dollars to produce x thousand barrels of root beer is given by $C(x) = 4x^2 + 100x + 500$.

(a) Find the marginal cost function (the rate at which cost is changing per unit).

(b) Calculate the marginal cost when 5,000 barrels ($x = 5$) are produced.

♠ 25. The cost, in dollars, of producing x yards of a certain fabric is

$$C(x) = 1400 + 11x - 0.1x^2 + 0.0005x^3.$$

(a) Find the marginal cost function.

(b) Find $C'(400)$.

(c) Find the additional cost of the 401st yard of fabric.

♠ 26. The cost, in dollars, of producing x commodities is

$$C(x) = 335 + 24x - 0.08x^2 + 0.0001x^3.$$

(a) Find $C'(100)$.

(b) Find the additional cost of the 101st commodity.

"Every particular in nature, a leaf, a drop, a crystal, a moment of time is related to the whole, and partakes of the perfection of the whole." – Ralph Waldo Emerson

Lecture Videos

Related Rates

Related Rates and a Falling Ladder

Related Rates and an Inverted Conical Tank of Water

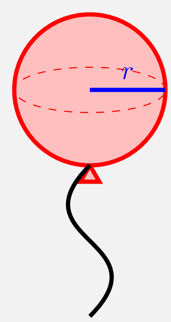
Related Rates and Two Approaching Cars

3.10 Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change (derivative) of one quantity in terms of the rate of change (derivative) of another quantity, which may be more easily measured. The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Example 3.10.1. Air is being pumped into a spherical balloon so that is volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?



We start by identifying two things: the *given information*"

the rate of increase of the volume of air is 100cm^3/s

and the *unknown information*:

the rate of increase of the radius when the diameter is 50cm.

In order to express these quantities mathematically, we search for a mathematical formula/equation/function which relates the two quantities. For volume of a sphere, we have

$$V = \frac{4}{3}\pi r^3,$$

where V denotes the volume of the sphere in cm^3 and r denotes the radius of the sphere in cm. The key thing to remember is that rates of change are derivatives. Thus,

$$\frac{dV}{dt} = \text{the rate of change of Volume with respect to time measured in cm}^3\text{/s}$$

and

$$\frac{dr}{dt} = \text{ the rate of change in Radius with respect to time measured in cm/s.}$$

Therefore, our known information is that $\frac{dV}{dt} = 100\text{cm}^3/\text{s}$ and our unknown information is $\frac{dr}{dt}$ when $r = 25$ cm.

To solve for $\frac{dr}{dt}$, we will implicitly differentiate the volume formula **with respect to time**, which will require the Chain Rule.

$$\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
\frac{dV}{dt} &= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\frac{d}{dt}\left(r^3\right) \\
&= \frac{4\pi}{3}(3r^2)\frac{dr}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right) \\
\frac{dr}{dt} &= \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right).
\end{aligned}$$

Now, $\frac{dV}{dt} = 100$. So, if $r = 25$, we can solve for $\frac{dr}{dt}$:

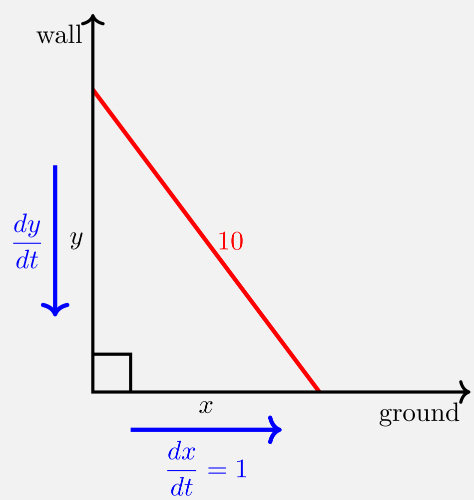
$$\left.\frac{dr}{dt}\right|_{r=25} = \frac{1}{4\pi(25)^2}(100) = \boxed{\frac{1}{25\pi}}.$$

Therefore, the radius of the balloon is increasing at the rate of $1/25\pi \approx 0.0127$ cm/s.

As these related rate problem nearly always measure the rate of change of different quantities *with respect to time*, we will use the prime notation to denote the derivative of a quantity with respect to time t, unless otherwise stated.

Example 3.10.2. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Often times it is resourceful to draw a diagram relating the known and unknown information. We then construct a right triangle with the horizontal leg having length x ft, the vertical leg having length y ft, and the hypotenuse having length 10 ft. So, we know $x' = \frac{dx}{dt} = 1$ ft/s and we would like to first know y' when $x = 6$. The Pythagorean Equation relates these two quantities. Thus,



$$\begin{aligned}
100 &= x^2 + y^2 \\
0 &= (x^2 + y^2)' = 2xx' + 2yy' \\
2yy' &= -2xx' \\
y' &= \frac{-2xx'}{2y} = \frac{-xx'}{y}.
\end{aligned}$$

It appears that we will also need to know y when $x = 6$. By the Pythagorean Equation, $100 = 6^2 + y^2 \Rightarrow y^2 = 64 \Rightarrow y = 8$. Thus,

$$\left.\frac{dy}{dt}\right|_{x=6} = \frac{-(6)(1)}{8} = \frac{-6}{8} = \boxed{-\frac{3}{4}}.$$

The fact that y' is negative means that the distance from the top to the ground is decreasing. Therefore, the ladder is sliding down at a rate of $\frac{3}{4}$ ft/s.

Example 3.10.3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Let V denote the volume of the water in the tank in m^3 and let h denote the height of the water in m. Thus, we need to compute $\frac{dh}{dt}$. We know that $V' = 2\text{m}^3/\text{min}$ and the volume of the cone will relate the two quantities:

$$V = \frac{1}{3}\pi r^2 h.$$

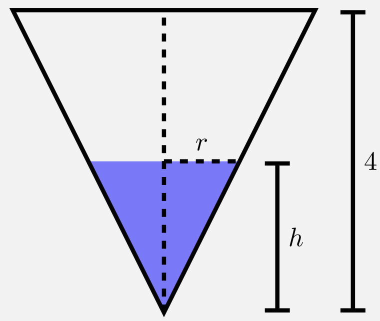
Now this equation has 3 variables: volume, radius, and height. Since we don't know nor care to know the rate of change of the radius, we will replace r with a function of h using similar triangles.

Note that the right triangle with legs 4 and 2 is similar to the triangle with corresponding legs h and r. Thus,

$$\frac{r}{h} = \frac{1}{2} \Rightarrow r = \frac{1}{2}h.$$

Hence, we will use the formula

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12}h^3.$$

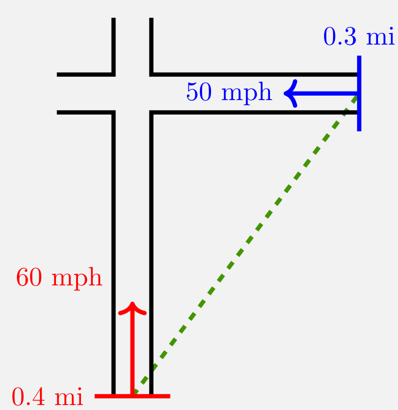


$$\begin{aligned} V' &= \frac{\pi}{12}(h^3)' = \frac{\pi}{12}(3h^2h') \\ h' &= \frac{12V'}{3\pi h^2} = \frac{4V'}{\pi h^2} \\ &= \frac{4(2)}{\pi(3)^2} = \frac{8}{9\pi}. \end{aligned}$$

Thus, the water level is rising at a rate of $8/9\pi \approx \boxed{0.28 \text{ m/min}}$.

Example 3.10.4. Car A is traveling west at 50 mph and car B is traveling north at 60 mph. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when Car A is 0.3 mi and car B is 0.4 mi from the intersection?

Let x denote the distance in miles that car A is from the intersection and let y denote the distance in miles that car B is from the same intersection. Then the distance between them, z, is related to x and y by the Pythagorean Equation. We want to find $\frac{dz}{dt}$. Notice first that when car A is 0.3 miles and car B is 0.5 miles from the intersection, the distance between them is $z = \sqrt{0.3^2 + 0.4^2} = \sqrt{0.9 + 1.6} = \sqrt{2.5} = 0.5$ miles. Also, $x' = -50$ and $y' = -60$. These are negative since x and y are decreasing. Thus,



$$z^2 = x^2 + y^2$$

$$\begin{aligned} 2zz' &= 2xx' + 2yy' \\ z' &= \frac{2xx' + 2yy'}{2z} = \frac{xx' + yy'}{z} \\ &= \frac{(0.3)(-50) + (0.4)(-60)}{0.5} = \frac{-15 - 24}{.5} = \frac{-39}{.5} = \frac{-390}{5} = -78. \end{aligned}$$

Therefore, the cars are approaching each other at a rate of $\boxed{\text{78 mph}}$.

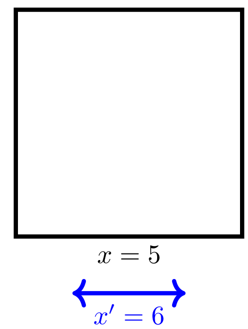
[i]See §4.1 Related Rates in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

1. Consider a square where each side is increasing at a constant rate of 6 $\dfrac{\text{cm}}{\text{s}}$. Find the rate the area of this square is increasing at the moment its side is 5 cm.

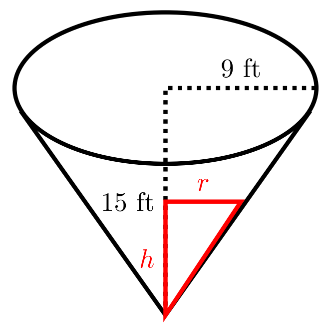
♠ 2. Consider a square where each side is increasing at a constant rate of 2 $\dfrac{\text{cm}}{\text{s}}$. Find the rate the area of this square is increasing at the moment its area is 25 cm^2.



♠ 3. Consider a rectangle whose length is increasing at a constant rate of 4 $\dfrac{\text{cm}}{\text{s}}$ and its width is increasing at a constant rate of 8 $\dfrac{\text{cm}}{\text{s}}$. Find the rate the area of the rectangle is increasing when the length is 9 cm and the width is 4 cm.

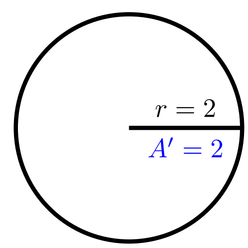
4. Consider an inverted conical water tank which is 15 ft high and has a radius of 9 ft. If the water is drained from the tank at a constant rate of 9 $\dfrac{\text{ft}^3}{\text{min}}$, find how fast the water level is changing when the water is 10 ft deep.

♠ 5. Consider a cylindrical water tank with radius 3 m, being filled with water at a constant rate of 3 $\dfrac{\text{m}^3}{\text{min}}$. Find how fast the height of the water is increasing.



6. Consider the area of a circle, which is increasing at a constant rate of 2 $\dfrac{\text{cm}^2}{\text{s}}$. Find the rate the radius is increasing when the diameter is 4 cm.

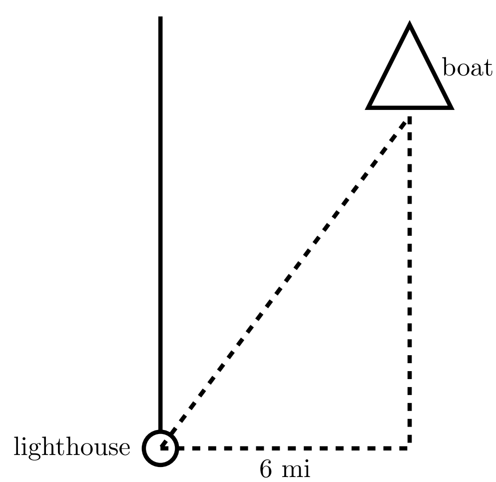
♠ 7. Consider the radius of a sphere, which is increasing at a constant rate of 5 $\dfrac{\text{mm}}{\text{s}}$. Find how fast the volume is increasing when the diameter is 40 mm.



8. Consider a ship traveling due north 6 miles east of the Maryland shore at a constant speed of 15 miles per hour. On the shore is a lighthouse. Find the rate the distance between the ship and the lighthouse is increasing when the ship is 8 miles north of the lighthouse.

♠ 9. Consider a child watching a plane flying horizontally over her at an altitude of 2 mi and at a speed of 570 mph. Find the rate at which the distance between the plane and the child is increasing when the plane is a total distance of 3 mi away.

♠ 10. Consider a ship which is 130 km west of a barge. The ship is sailing east at 25 kmph, and the barge is sailing north at 15 kmph. Find how fast is the distance between the ship and the barge is changing four hours later.



♠ 11. Consider two cars start moving from the same point. The first travels due south at 48 mph, and the second travels due west at 20 mph. Find the rate the distance between the cars is increasing three hours later.

♠ 12. Consider a man walking north at 6 $\frac{\text{ft}}{\text{s}}$ from a busy intersection. Five minutes later a woman starts walking south at 7 $\frac{\text{ft}}{\text{s}}$ from a different intersection 500 ft due east. Find the speed these people are moving apart 15 min after the woman starts walking.

♠ 13. Consider a triangle $\triangle ABC$ such that $m\angle A$ is increasing at a rate of 0.06 $\frac{\text{rad}}{\text{s}}$. If $b = 6$ m and $c = 7$ m are fixed, find the rate at which the area of $\triangle ABC$ is increasing whe $m\angle A = \frac{\pi}{6}$.

14. Consider a 13 ft long ladder, whose bottom is sliding away from the wall at a rate of $\frac{1}{4}$ feet per second. When the bottom of the ladder is 12 feet from the wall, find the rate at which the top of the ladder is falling.

♠ 15. Consider a ladder, whose top is sliding down a wall at a constant rate of 0.675 $\frac{\text{m}}{\text{s}}$. When the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.9 $\frac{\text{m}}{\text{s}}$. Find the length of the the ladder.

"The length of a film should be directly related to the endurance of the human bladder." – Alfred Hitchcock

Lecture Videos

Strategies for Solving Related Rates Problems	Related Rates and a Trapezoidal Trough	Related Rates and Expanding Gases	Related Rates and Rotating Searchlight

3.11 Related Rates II

Related Rates Problem Solving Strategy

(i) Read the problem carefully.

(ii) Draw a diagram if possible.

(iii) Introduce notation. Assign symbols to all quantities that are functions of time.

(iv) Express the given information and the required rate in terms of derivatives.

(v) Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitutuion.

(vi) Use the Chain Rule to differentiate both sides of the equation with respect to t.

(vii) Substitute the given information into the resulting equation and solve for the unknown rate.

Example 3.11.1. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

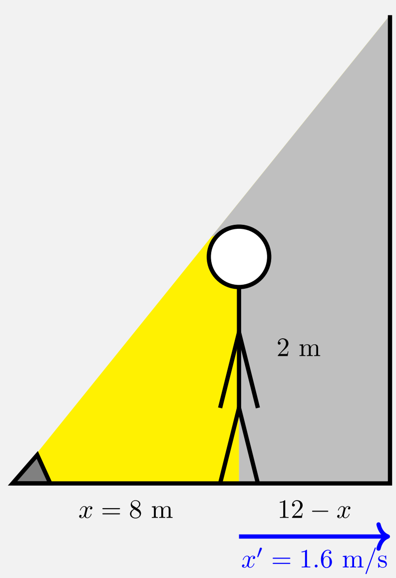
Let x denote the distance the man is from the spotlight. Let y denote the height of the shadow. Then we are given $x = 12 - 4 = 8$ and $x' = 1.6$, but we need y'. The two variable are related to each other by similar triangles. Note that

$$\frac{2}{x} = \frac{y}{12} \qquad \Rightarrow \qquad y = \frac{24}{x}.$$

Therefore,

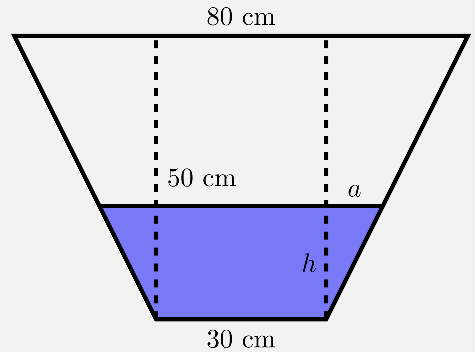
$$\begin{aligned} y' &= \left(\frac{24}{x}\right)' = 24\left(\frac{-(x)'}{x^2}\right) = -\frac{24x'}{x^2} \\ &= -\frac{24(1.6)}{8^2} = -\frac{38.4}{64} = \boxed{-0.6 \text{ m/s}} \end{aligned}$$

That is, the height of the shadow is dropping at a rate of 0.6 m/s.



Example 3.11.2. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

Let h denote the height of the water and let a denote the distance in the figure. For an isosceles trapezoid, the area of the cross-section is given as



$$\begin{aligned} \text{Area} &= \frac{1}{2}(\text{base}_1 + \text{base}_2) \cdot (\text{height}) \\ &= \frac{1}{2}(0.30 + (0.30 + 2a))h \\ &= \frac{1}{2}(0.60 + 2a)h = (0.30 + a)h. \end{aligned}$$

The volume of the water is then

$$V = 10(0.30 + a)h.$$

Using similar triangles, we note that

$$\begin{aligned} \frac{a}{h} &= \frac{0.25}{0.50} = \frac{1}{2} \\ a &= \frac{h}{2} \end{aligned}$$

Therefore,

$$V = 3h + 5h^2.$$

Taking the derivative, we have

$$\begin{aligned} V' &= 3h' + 10hh' = (3 + 10h)h' \\ h' &= \frac{V'}{3 + 10h} = \frac{0.2}{3 + 10(0.30)} = \frac{0.2}{6} = \boxed{\frac{1}{30} \text{ m/min}}, \end{aligned}$$

that is, the height is increasing at a rate of 0.033 m/min, or 3.3 cm/min.

Example 3.11.3. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min. At what rate is the volume increasing at this time?

We are considering the case when $V = 400\ \text{cm}^3$, $P = 80$ kPa, and $P' = -10$ kPa/min. We need to know V'. Using the above relation, we take the derivative, we get that

$$\begin{aligned} (PV^{1.4})' &= C' \\ P'V^{1.4} + 1.4PV^{0.4}V' &= 0 \\ 1.4PV^{0.4}V' &= -P'V^{1.4} \\ V' &= \frac{-P'V^{1.4}}{1.4PV^{0.4}} = -\frac{P'V}{1.4P} \\ &= -\frac{(-10)(400)}{1.4(80)} = \frac{100}{1.4(2)} = \frac{100}{2.8} = \boxed{\frac{250}{7} \text{ cm}^3/\text{min}}, \end{aligned}$$

that is, the volume is increasing at a rate of 35.714 cm^3/min.

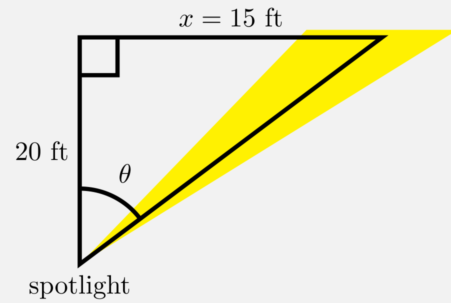
Example 3.11.4. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closet to the searchlight?

Let θ be the angle made by the searchlight and the line perpendicular to the path the man is traveling. Let x denote the distance in ft between the man and this perpendicular line. This then makes a right triangle with angle θ, adjacent leg having length 20 ft, and opposite leg having length x. We know the rate the man is walking, $\frac{dx}{dt}$, and we want to know the rate the angle is changing, $\frac{d\theta}{dt}$, when x is 15. We can use the tangent ratio of this triangle to relate the rates, that is,

$$\frac{x}{20} = \tan\theta.$$

Thus,

$$\begin{aligned} \left(\frac{x}{20}\right)' &= (\tan\theta)' \\ \frac{x'}{20} &= \sec^2(\theta)\theta' \\ \theta' &= \frac{x'}{20\sec^2\theta} = \frac{x'\cos^2\theta}{20} \\ &= \frac{(4)\cos^2\theta}{20} = \frac{1}{5}\cos^2\theta. \end{aligned}$$



To finish the problem, we need to know θ when $x = 15$. Actually, we only need to know what $\cos^2\theta$ is equal to. Now the hypotenuse of this triangle has length $\sqrt{20^2+15^2} = \sqrt{400+225} = \sqrt{625} = 25$. Hence, $\cos\theta = \frac{20}{25} = \frac{4}{5}$. Therefore,

$$\frac{d\theta}{dt} = \frac{1}{5}\cos^2\theta = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125}.$$

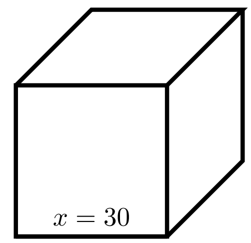
Therefore, the searchlight is rotating at a rate of $16/125 = \boxed{0.128 \text{ rad/s}}$.

[i]See §4.1 Related Rates in OpenStax to find the corresponding section.

Exercises

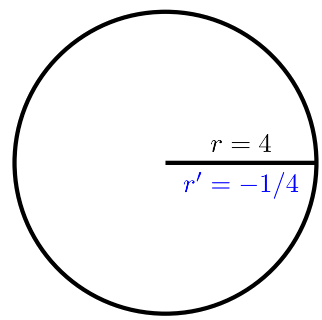
(Go to Solutions)

1. Consider a cube whose volume is increasing at a constant rate of $10\ \text{cm}^3/\text{min}$. Find how fast its surface area is increasing when the length of an edge is 30 cm.



♠ 2. Consder a street light mounted at the top of a 15-ft-tall pole. If a man, 6 ft tall, walks away from the pole with a constant speed of $7\ \frac{\text{ft}}{\text{s}}$ along a straight path, find how fast his shadow is lengthening when he is 45 ft from the pole.

3. Consider the radius r of a melting snowball which decreases at a constant rate of $\frac{1}{4}$ inches per minute. Find the rate its volume is decreasing when the radius is $r = 4$ inches.

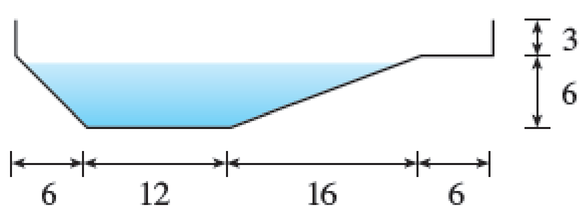


♠ 4. Consider the surface area of a melting snowball which decreases at a constant rate of $3\ \frac{\text{cm}^2}{\text{min}}$. Find the rate its diameter is decreasing when the diameter is 8 cm.

♠ 5. Consider a pulley attached 1 m above a dock which is pulling in a boat by a rope. Supposing the the rope is reeled in at a constant rate of $1\ \frac{\text{m}}{\text{s}}$, find how fast the boat is approaching the dock at the moment it is 9 m from the dock.

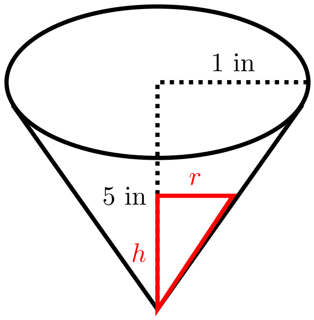
♠ 6. Consider a 14 ft long trough whose ends each are isosceles triangles 4 ft across at the top and 1 ft tall at the middle. Supposing the trough is filled with water at a constant rate of $15\ \frac{\text{ft}^3}{\text{min}}$, find the rate at which the water level is rising when the water is 9 inches deep.

♠ 7. A swimming pool is 20 ft wide, 40 ft long, 3 ft deep at the shallow end, and 9 ft deep at is deepest point. A cross-section is shown in the figure. If the pool is being filled at a rate of $0.7\ \frac{\text{ft}^3}{\text{min}}$, how fast is the water level rising when the depth at the deepest point is 5 ft?



8. Consider a creamery where soft-serve ice cream is being poured into a cone with a height of 5 inches and radius 1 inch at its widest part. If the cone is being filled at a constant rate of $1\ \text{in}^3/\text{s}$, find the rate at which the ice cream level is rising when the ice cream is 2 inches deep.

♠ 9. Consider sand, spilling from above, forms a pile shaped like a cone whose height and base diameter remain in one-to-one proportion. Supposing the sand spills at a constant rate of $15\ \frac{\text{ft}^3}{\text{min}}$, find how fast the height of the sand pile is increasing at the moment the pile is 13 ft high.

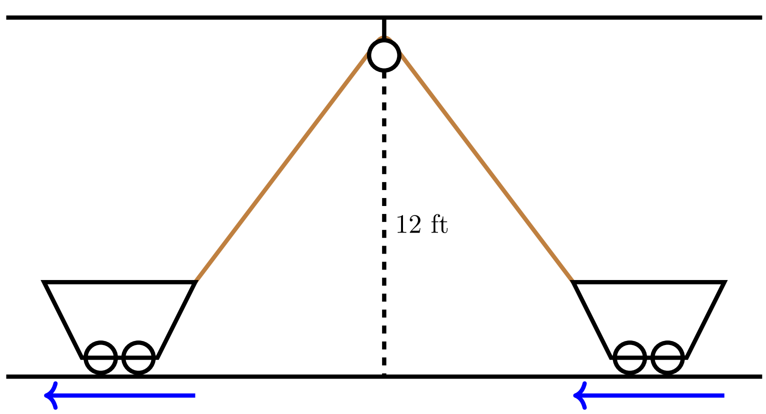


10. Consider a balloon rising at a constant speed of $5\dfrac{\text{ft}}{\text{s}}$. A boy is cycling along a straight road at a speed of $15\dfrac{\text{ft}}{\text{s}}$. When he passes under the balloon, it is 45 ft above him. Find how fast the distance between the boy and the balloon is increasing 3 s later.

♠ 11. Consider a child flying a kite 100 ft above the ground. Because of the wind, the kites moves horizontally in the sky at a constant speed of $8\ \dfrac{\text{ft}}{\text{s}}$. As the kite blows and the child is letting out more string, the angle between the string and the ground is decreasing. At what rate, in radians, is this angle decreasing when the string is 200 ft long.

♠ 12. Consider a 10 ft long ladder, whose bottom is sliding away from a wall[ii] at a constant rate of $1.3\ \dfrac{\text{ft}}{\text{s}}$. When the bottom of the ladder is 8 ft from the wall, find the rate at which the angle between the ladder and the ground is decreasing.

13. Consider two carts, see below, which are connected together by a 33 ft long rope that passes through a pulley which is 12 ft above the ground. If the left cart is pulled to the left at a speed of 2 ft/s, then how fast is the right cart moving left at the moment when the left cart is 5 ft from the pulley (along the ground, not the length of the rope between the cart and the pulley).



[ii]This appears to be a very common problem for this brand of ladders. It is recommended that the climbers secure their ladders better in the future.

"Inch by inch, life's a [sinh]. Yard by yard, life's hard." – John Bytheway

Lecture Videos

The Hyperbolic Functions | Derivatives of the Hyperbolic Functions | The Inverse Hyperbolic Functions | Derivatives of the Inverse Hyperbolic Functions

3.12 Hyperbolic Functions

Mimicking the definitions of the six trigonometric functions, we get the six **hyperbolic functions**.

Definition 3.12.1 (Definition of the Hyperbolic Functions)**.**

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

The hyperbolic functions get their name because hyperbolic cosine and hyperbolic sine are related to each by a hyperbolic equation, similar to the Pythagorean (circular) relationship between cosine and sine.

Proposition 3.12.2.

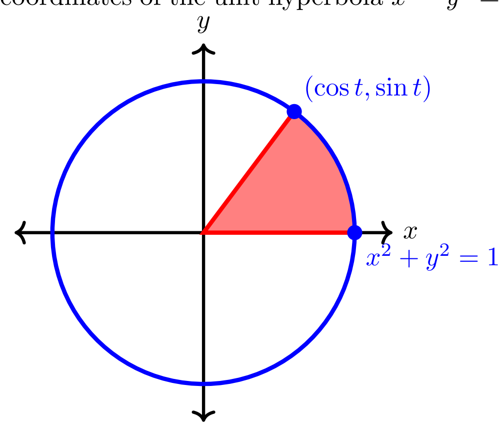
$$\cosh^2 x - \sinh^2 x = 1$$

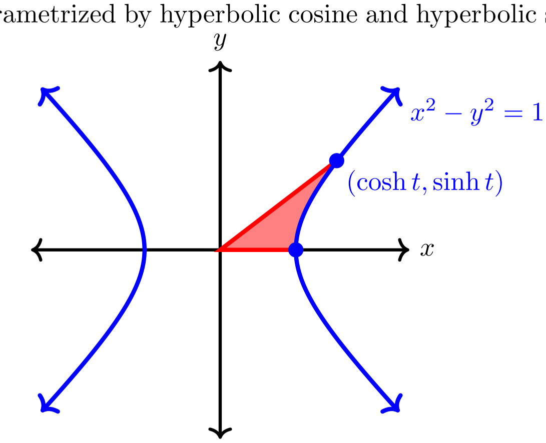
Proof.

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}[(e^x + e^{-x})^2 - (e^x - e^{-x})^2] \\ &= \frac{1}{4}[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})] = \frac{4}{4} = 1\end{aligned}$$

□

In fact, despite the defintion of hyperbolic sine and hyperbolic cosine, the definitions of the hyperbolic functions are analogous to the trigonometric definitions. The only real difference is for trigonometric functions the coordinates of the unit circle $x^2 + y^2 = 1$ are parametrized by cosine and sine and for hyperbolic functions the coordinates of the unit hyperbola $x^2 - y^2 = 1$ are parametrized by hyperbolic cosine and hyperbolic sine.





As a consequence of these similarities, the hyperbolic functions exhibit similar identities as the trigonometric functions.

Proposition 3.12.3 (Hyperbolic Identities)**.**

$$\begin{aligned} \sinh(-x) &= -\sinh(x) & \cosh(-x) &= \cosh(x) \\ \cosh^2(x) - \sinh^2(x) &= 1 & 1 - \tanh^2(x) &= \operatorname{sech}^2(x) \end{aligned}$$

$$\begin{aligned} \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \end{aligned}$$

In many more ways, as we will see below, hyperbolic functions act like trigonometric functions. Because of this, many applications of hyperbolic functions can also be accomplished by trigonometry. So it would appear that the hyperbolic functions offer nothing new. This is certainly not the case. Applications of hyperbolic functions to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguised, for the decay can be represented by hyperbolic functions. For example, hyperbolic functions can be used to describe the shape of a hanging wire suspended between two points of the same height or the velocity of waves over the ocean surface. In these cases, hyperbolic functions are needed.

Proposition 3.12.4 (Derivatives of Hyperbolic Functions)**.**

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \cosh(x) & \frac{d}{dx}(\tanh x) &= \operatorname{sech}^2(x) & \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech}(x)\tanh(x) \\ \frac{d}{dx}(\cosh x) &= \sinh(x) & \frac{d}{dx}(\coth x) &= -\operatorname{csch}^2(x) & \frac{d}{dx}(\operatorname{csch} x) &= -\operatorname{csch}(x)\coth(x) \end{aligned}$$

Proof.

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2} \cdot \left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right] \\ &= \frac{1}{2}\left[e^x + e^{-x}\right] = \cosh(x) \end{aligned}$$

Using the Quotient Rule when necessary, the remaining derivative calculations are similar. □

Example 3.12.5. Compute $\frac{d}{dx}(\cosh\sqrt{x})$.

$$\frac{d}{dx}(\cosh\sqrt{x}) = \sinh\sqrt{x} \cdot \frac{1}{2\sqrt{x}} = \boxed{\frac{\sinh\sqrt{x}}{2\sqrt{x}}}$$

The six hyperbolic functions also have inverse functions (after restricting domains).

Proposition 3.12.6 (Inverse Hyperbolic Functions)**.**

$$\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right) \qquad \cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right) \quad (\textit{for } x \geq 1)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (\textit{for } -1<x<1) \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \quad (\textit{for } |x|>1)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \quad (\textit{for } 0<x\leq 1) \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) \quad (\textit{for } x \neq 0)$$

Proof.

$$\begin{aligned}
\sinh^{-1}(x) &= y \\
x &= \sinh(y) = \frac{e^y - e^{-y}}{2} \\
2x &= e^y - e^{-y} \\
e^y - 2x - e^{-y} &= 0 \\
e^y(e^y - 2x - e^{-y}) &= e^y \cdot 0 \\
e^{2y} - 2xe^y - 1 &= 0 \\
e^y &= \frac{2x + \sqrt{(-2x)^2+4}}{2} = \frac{2x+2\sqrt{x^2+1}}{2} = x + \sqrt{x^2+1} \\
y &= \ln\left(x+\sqrt{x^2+1}\right)
\end{aligned}$$

The other statements are proven similarly. □

Proposition 3.12.7 (Derivatives of Inverse Hyperbolic Functions)**.**

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

Proof. Since $\cosh^2 t - \sinh^2 t = 1$, it is also true that $\cosh t = \sqrt{1+\sinh^2 t}$. Then

$$\begin{aligned}
\sinh^{-1}(x) &= y \\
x &= \sinh(y) \\
1 &= \cosh(y)y' \\
y' &= \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2(y)}} = \frac{1}{\sqrt{1+x^2}}
\end{aligned}$$

The remaining derivatives are handled similarly. □

Notice that the formulas for the derivatives of $\tanh^{-1}$ and $\coth^{-1}$ appear to be identical. But the domains of these functions have no numbers in common, since the domains of $\tanh^{-1}$ and $\coth^{-1}$ have common elements in their domains.

Example 3.12.8. Find $\dfrac{d}{dx}(\tanh^{-1}(\sin x))$.

$$\frac{d}{dx}(\tanh^{-1}(\sin x)) = \frac{1}{1-(\sin x)^2} \cdot \cos x = \frac{\cos x}{1-\sin^2 x} = \frac{\cos x}{\cos^2 x} = \boxed{\sec x}.$$

On the other hand, it can be shown that

$$\tanh^{-1}(\sin x) = \ln(\sec x + \tan x),$$

in which case the derivative can be calculated using methods that we have already seen.

[i]See §6.9 Calculus of the Hyperbolic Functions in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–7, compute the hyperbolic expression.

1. $\tanh(0)$
♠ 2. $\sinh(0)$
♠ 3. $\cosh(0)$
♠ 4. $\cosh(6)$
5. $\cosh(\ln 2)$
♠ 6. $\cosh(\ln 6)$
7. $\sinh\left(\ln\left(\frac{1}{3}\right)\right)$

For Exercises 8–9, given a hyperbolic ratio, find the values of the other hyperbolic functions at x.

♠ 8. $\tanh(x) = \dfrac{5}{13}$
♠ 9. $\cosh(x) = \dfrac{25}{24}$

♠ 10. Prove $\sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \sinh(x+y)$

For Exercises 11–22, compute the limit.

11. $\lim\limits_{x\to\infty} \cosh x$
12. $\lim\limits_{x\to-\infty} \cosh x$
♠ 13. $\lim\limits_{x\to\infty} \sinh x$
♠ 14. $\lim\limits_{x\to-\infty} \sinh x$
♠ 15. $\lim\limits_{x\to\infty} \tanh x$
♠ 16. $\lim\limits_{x\to-\infty} \tanh x$
♠ 17. $\lim\limits_{x\to\infty} \coth x$
♠ 18. $\lim\limits_{x\to-\infty} \coth x$
♠ 19. $\lim\limits_{x\to0^+} \coth x$
♠ 20. $\lim\limits_{x\to0^-} \coth x$
21. $\lim\limits_{x\to\infty} \operatorname{sech} x$
22. $\lim\limits_{x\to-\infty} \operatorname{sech} x$

For Exercises 23–33, differentiate the function.

23. $f(x) = 3\tanh(x)$
♠ 24. $y = x\sinh x - 4\cosh x$
♠ 25. $y = \tanh(2 + e^{4x})$
♠ 26. $y = \cosh(\ln x)$
♠ 27. $y = e^{\cosh(6x)}$
♠ 28. $y = \sinh(\cosh x)$
29. $f(x) = x\coth(1 + x^2)$
30. $f(x) = \arctan(\tanh x)$
31. $f(x) = \ln(\cosh x)$
32. $f(x) = \operatorname{sech}^2(e^x)$
33. $f(x) = \ln(\sinh(x^2+1))$

34. Prove that $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2(x)$.
35. Prove that $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2(x)$.
♠ 36. Prove that $\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech}(x)\tanh(x)$.
♠ 37. Prove that $\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch}(x)\coth(x)$.
♠ 38. Prove that $\dfrac{d}{dx}(\cosh^{-1}(x)) = \dfrac{1}{\sqrt{x^2-1}}$.
♠ 39. Prove that $\dfrac{d}{dx}(\tanh^{-1}(x)) = \dfrac{1}{1-x^2}$.
40. Prove that $\dfrac{d}{dx}(\coth^{-1}(x)) = \dfrac{1}{1-x^2}$.
41. Prove that $\dfrac{d}{dx}(\operatorname{sech}^{-1}(x)) = -\dfrac{1}{x\sqrt{1-x^2}}$.

Chapter 4

Applications of Derivatives

"It is said that any virtue when taken to an extreme can become a vice." – Dieter F. Uchtdorf

Lecture Videos

Local Extrema

Critical Numbers

Absolute Extrema

The Extreme Value Theorem

The Extreme Value Problem

4.1 Extrema

Definition 4.1.1. Let c be a number in the domain of a function f. Then $f(c)$ is a **relative** (or **local**) **maximum** for f if there exists an open interval (a, b) containing c such that

$$f(x) \leq f(c)$$

for all x in (a, b), and $f(c)$ is a **relative** (or **local**) **minimum** for f if there exists an open interval (a, b) containing c such that

$$f(x) \geq f(c)$$

for all x in (a, b).

A function has a **relative** (or **local**) **extremum** (plural: **extrema**) at c if c is a relative maximum or minimum.

If c is an endpoint of the domain of f, we only consider x in the half-open interval that is in the domain. In other words, an endpoint of a graph can be an extremum if it is extreme from the left or right, depending which side is in the domain of the function.

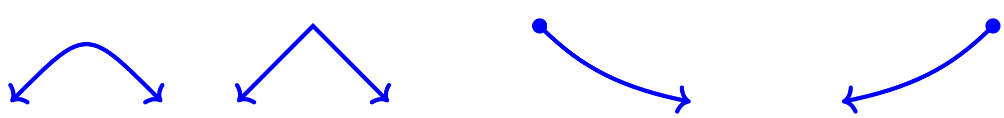
Example 4.1.2. The parabola $y = x^2$ has a unique local minimum at its vertex $(0, 0)$ and has no local maxima.

Similarly, $y = |x|$ has a unique local minimum at the origin but no local maxima.

A sine wave has infinitely many local maxima and minima, corresponding to the x-values which output 1 and -1, $x = \pi/2 + 2\pi n$ and $x = -\pi/2 + 2\pi n$ respectively.

The function $y = x^3$ has no local extrema whatsoever.

Extreme points come in one of a few ways. Examples of maxima are illustrated below. The examples of minima are the reflection of those below.



$f'(x) = 0$ $\quad$ $f'(x)$ DNE $\quad$ Left Domain Boundary $\quad$ Right Domain Boundary

Definition 4.1.3. A **critical point** of a function f is a point $(c, f(c))$ on the graph such that $f'(c) = 0$ or $f'(c)$ is undefined.

Theorem 4.1.4 (Fermat's Theorem)**.** *If a function f has a relative extremum at c, then c is a critical point of f or c is an endpoint of the domain.*

The converse to this statement is not true. Not every critical point has to be a local extremum. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and $x = 0$, is a critical value. But $f(x) = x^3$ has no local max or min at $x = 0$.

Definition 4.1.5. Let f be a function defined on some interval. Let c be a number in this interval. Then $f(c)$ is the **absolute maximum** of f on the interval if

$$f(x) \leq f(c)$$

for every x in the interval, and $f(c)$ is the **absolute minimum** of f on the interval if

$$f(x) \geq f(c)$$

for every x in the interval.

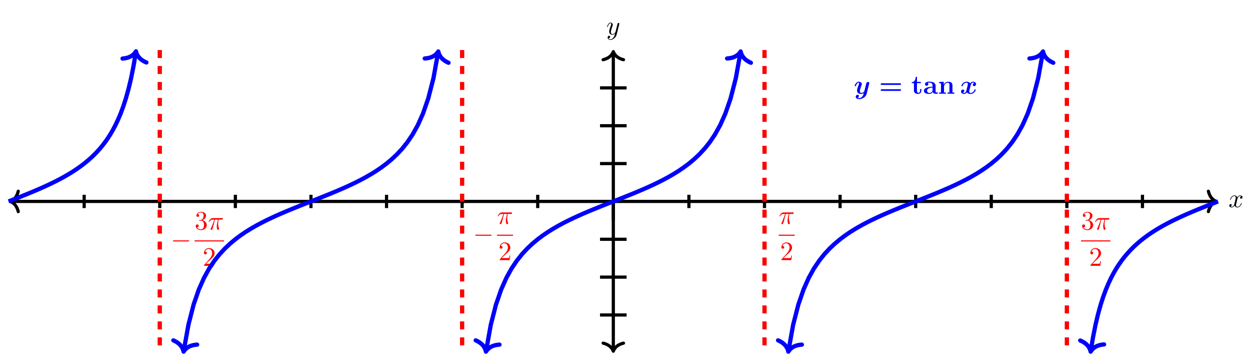
A function has an **absolute extremum** at c if $f(c)$ is the absolute maximum or minimum of f.

We now compare a few differences between relative extrema and absolute extrema. First, for a function f, the absolute maximum (if it exists) is unique, while f can have several relative maximal values. With this said, it might be true that more than one x-value obtains this absolute maximal y-value. Also, the absolute maximum will be among the relative maxima. If the interval being considered is closed, then the endpoints can also be relative maxima and should be considered when searching for the absolute maximum. Analogous statements also hold for minima.

Does the absolute maximum or minimum always exist on an interval in the domain of a function f? If we are picky about our choice of intervals, then yes.

Theorem 4.1.6 (Extreme Value Theorem)**.** *A function f that is continuous on a closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum on the interval.*

The assumptions that the interval be closed and that f be continuous on the interval are necessary. For example, on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan(x)$ has no absolute maximum, because the graph approaches its vertical asymptotes at $x = \pm\frac{\pi}{2}$. Likewise, tangent has no absolute maximum on the interval $[0, \pi]$ since it has a vertical asymptote at $x = \frac{\pi}{2}$.



Finding Absolute Extrema — To find absolute extrema for a function f continuous on a closed interval $[a, b]$:

(a) Find all critical numbers for f in (a, b).

(b) Evaluate f for all critical numbers in (a, b).

(c) Evaluate f for the endpoints a and b, that is, determine $f(a)$ and $f(b)$.

(d) The largest value found is the absolute maximum for f on $[a, b]$ and the smallest value found is the absolute minimum for f on $[a, b]$.

Example 4.1.7. Find the absolute extrema of the function

$$f(x) = x^{8/3} - 16x^{2/3}$$

on the interval $[-1, 8]$.

We first find the critical numbers by taking the first derivative:

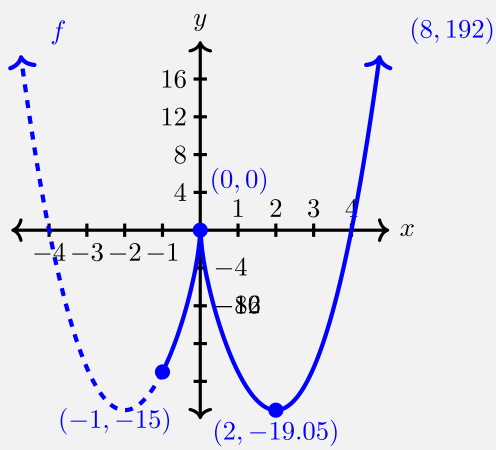
$$\begin{aligned} f'(x) &= \frac{8}{3}x^{5/3} - 16\left(\frac{2}{3}\right)x^{-1/3} = \frac{8}{3}\left(x^{5/3} - \frac{4}{x^{1/3}}\right) \\ &= \frac{8}{3}\left(x^{5/3} \cdot \frac{x^{1/3}}{x^{1/3}} - \frac{4}{x^{1/3}}\right) = \frac{8}{3}\left(\frac{x^2 - 4}{x^{1/3}}\right) \end{aligned}$$

Thus, $f'(x) = 0 \Rightarrow (x^2 - 4) = 0 \Rightarrow x = \pm 2$. Also, $f'(x)$ is undefined at $x = 0$. So, the critical values which are in the interval $[-1, 8]$ are 0 and 2. Including the endpoints $x = -1$ and $x = 8$, we now list the local extrema to the right.

x-value	y-value
-1	-15
0	0
2	-19.05
8	192

Therefore, the absolute maximum of f on $[-1, 8]$ is $\boxed{\text{192 and is obtained when } x = 8}$.

Likewise, the absolute minimum of f on $[-1, 8]$ is about $\boxed{-19.05 \text{ and is obtained when } x = 2}$.



Example 4.1.8.

(a) Find the locations of the absolute extrema, if they exist, for the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 2.$$

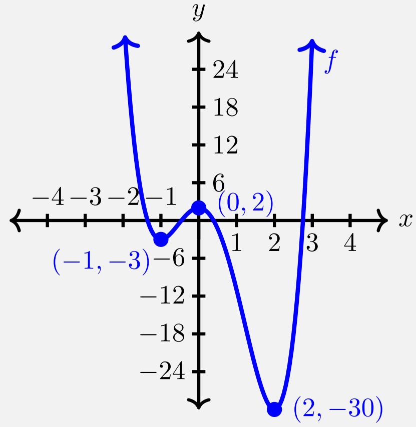
Now f is defined for all real numbers, that is, the domain of f is $(-\infty, \infty)$. In particular, the Extreme Value Theorem does not apply here because $(-\infty, \infty)$ is not a closed, finite interval. In order to determine if the absolute extrema exist, we first determine the end behavior of f. Note that

$$\lim_{x \to \pm\infty} f(x) = \infty.$$

Therefore, f has no absolute maximum because we can always find a larger y-value by choosing a larger x-value. But because both ends go off toward ∞, f *does* have an absolute minimum, and this absolute minimum will be among the relative minima.

Note that $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$. Thus, the critical numbers of f are $x = 0, -1, 2$. Now, each local minimum point is a critical point. So, the absolute minimum value of f comes from one of these critical x-values. Note that:

x-value	y-value
-1	-3
0	2
2	-30

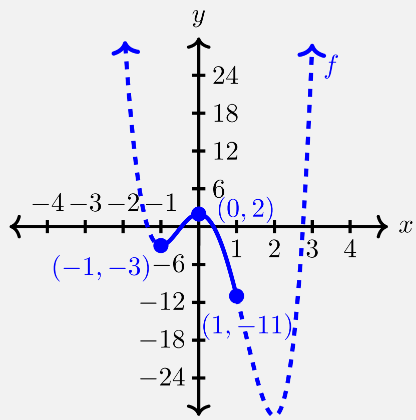


Therefore, $\boxed{(2, -30)}$ is the absolute minimum of f.

(b) Find the absolute extrema of f on the interval $[-1, 1]$.

By our previous work, we see that critical numbers are $x = -1, 0, 2$, only two of which sit in our interval. Thus, the Extreme Value Theorem guarantees the existence of an absolute maximum and minimum, and Fermat's Theorem guarantees it must be at $-1, 0$, or 1.

x-value	y-value
-1	-3
0	2
1	-11



Therefore, on the interval $[-1, 1]$, f has an absolute maximum at $\boxed{x = 0 \text{ with value } y = 2}$ and has an absolute minimum at $\boxed{x = 1 \text{ with value } y = -11}$.

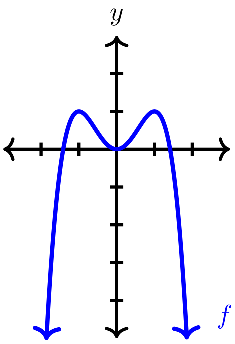
[i]See §4.3 Maxima and Minima in OpenStax to find the corresponding section.

Exercises

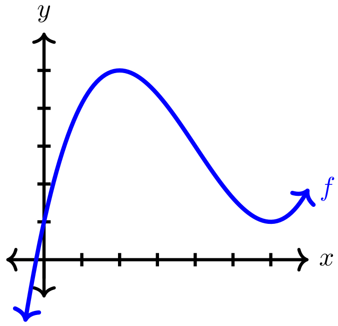
(Go to Solutions)

For Exercises 1–12, the function f is illustrated below. Find all local maxima and minima. Find all absolute maxima and minima.

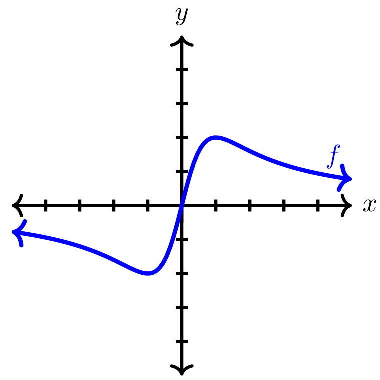
1.



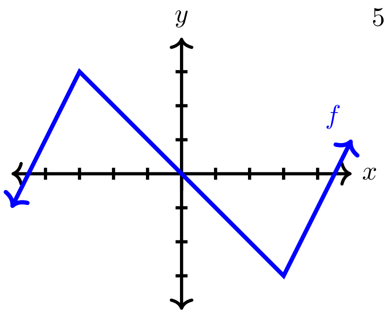
2.



♠ 3.



4.



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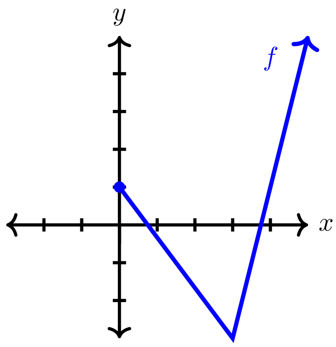

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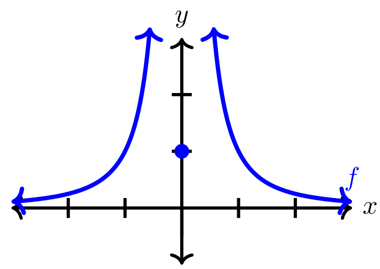

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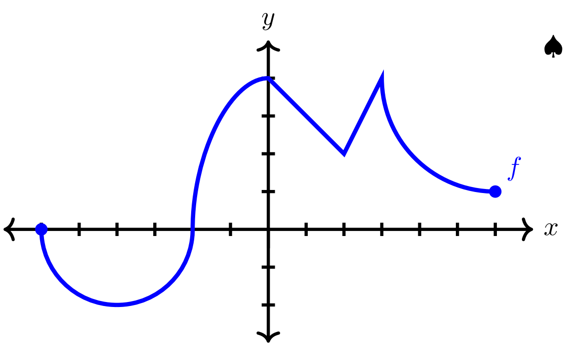

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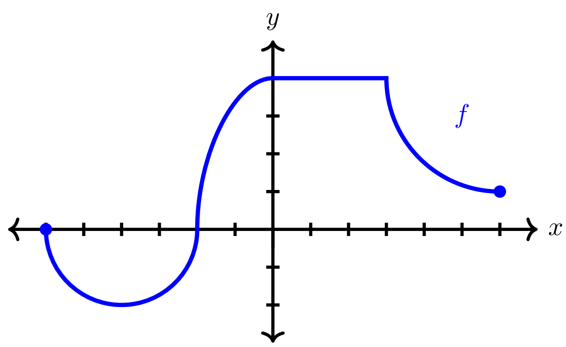

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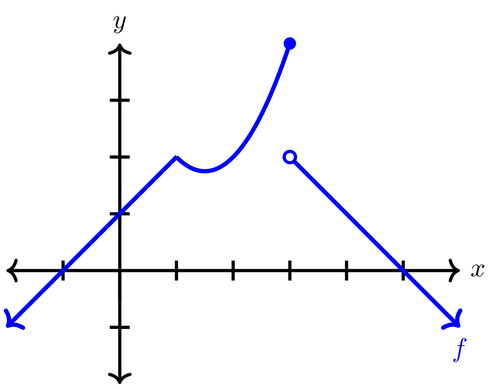

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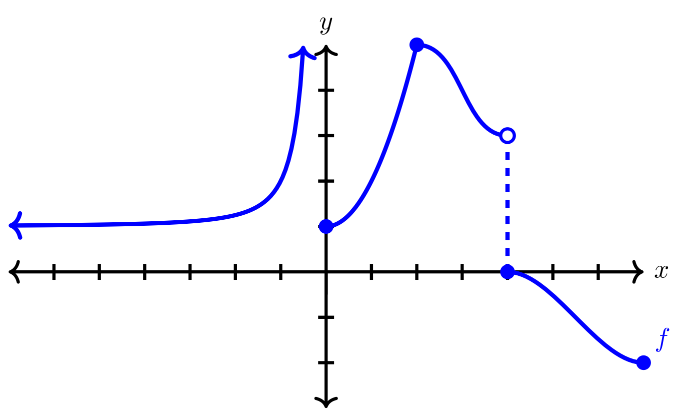

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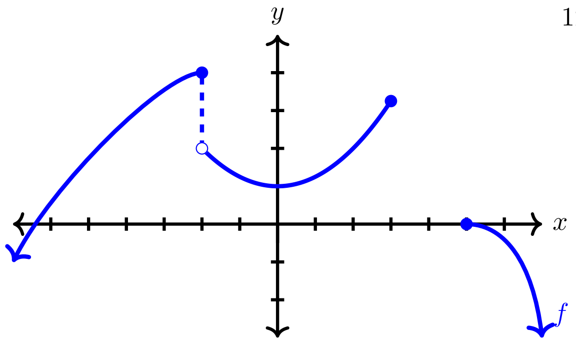

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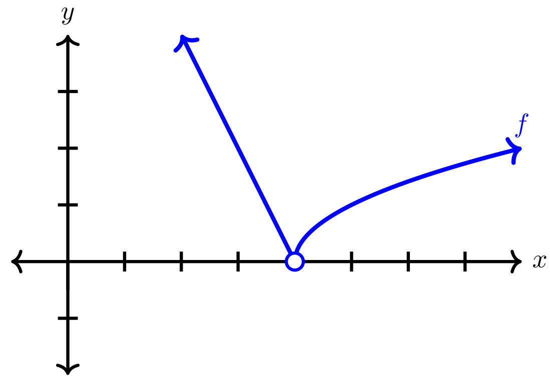


For Exercises 13–24, find the critical numbers of the function.

13. $f(x) = x^3 - 2x^2 - 7x + 3$

♠ 14. $f(x) = x^3 + 6x^2 - 15x$

15. $f(x) = 2x^3 - 3x^2 - 12x + 1$

16. $f(x) = x^3 + 6x^2 - 15x$

17. $f(x) = 3x^{4/3} - 9x^{1/3}$

18. $f(t) = t^{1/3} - t^{-2/3}$

♠ 19. $g(t) = t^{3/4} - 6t^{1/4}$

20. $f(x) = 3xe^x$

21. $f(x) = x \ln x$

22. $f(x) = x - \tan x$

23. $f(x) = 4x - 13 \tan^{-1} x$

24. $f(x) = \cosh(x)$

For Exercises 25–41, find the absolute minimum and maximum of the function on the given domain.

25. $f(x) = x^2 + 5x + 4,\ [-5, 5]$

26. $f(x) = x^3 - 27x + 1,\ [-2, 4]$

27. $f(x) = x^3 - 6x^2 + 5,\ [-3, 5]$

♠ 28. $f(x) = 4x^3 - 18x^2 - 48x + 7,\ [-2, 5]$

29. $f(x) = 2x^3 - 3x^2 - 12x + 1,\ [-2, 4]$

30. $f(x) = 2x^3 + 3x^2 + 1,\ [-1, 1]$

31. $f(x) = 3x^4 + 4x^3 - 12x^2 + 7,\ [0, 2]$

32. $f(x) = 4x^3 + 36x^2 + 11,\ [-1, 1]$

33. $f(x) = -3x^4 + 4x^3 + 36x^2 + 11,\ [-3, 1]$

34. $f(x) = \dfrac{x-1}{x^2+3},\ [0, 2]$

♠ 35. $f(x) = \dfrac{x}{x^2 - x + 1},\ [0, 3]$

♠ 36. $f(x) = x\sqrt{64 - x^2},\ [-1, 8]$

♠ 37. $f(x) = 16 \cos x + 8 \sin(2x),\ \left[0, \dfrac{\pi}{2}\right]$

♠ 38. $g(x) = 8x + 8 \cot\left(\dfrac{x}{2}\right),\ \left[\dfrac{\pi}{4}, \dfrac{7\pi}{4}\right]$

♠ 39. $f(x) = \ln(x^2 + 7x + 15),\ [-4, 1]$

40. $f(x) = \dfrac{1}{2} \sin(2x),\ \left[0, \dfrac{2\pi}{3}\right]$

41. $f(x) = x - 2 \tan^{-1}(x),\ [0, 4]$

"Just because something doesn't do what you planned it to do doesn't mean it's useless."
– Thomas A. Edison

Lecture Videos

Rolle's Theorem

The Mean Value Theorem

Proving that an Equation has Exactly One Solution

The Assumptions of the Mean Value Theorem

Inferences of the Mean Value Theorem

Two Functions with the Same Derivative Differ by a Constant

4.2 The Mean Value Theorem

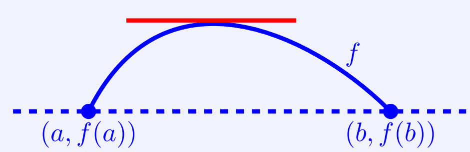
Theorem 4.2.1 (Rolle's Theorem)**.** *Let f be a function that satisfies the following:*

(i) f is continuous on the closed interval $[a, b]$.

(ii) f is differentiable on the open interval (a, b).

(iii) $f(a) = f(b)$.

Then there exists some c such that $a < c < b$ and $f'(c) = 0$.



Example 4.2.2. Let's apply Rolle's Theorem to the position function $s = f(t)$ of a moving object. If the object is in the same place at two different instants $t = a$ and $t = b$, then $f(a) = f(b)$. Rolle's Theorem says that there is some instant of time $t = c$ between a and b when $f'(c) = 0$, that is, the velocity is 0. In particular, you can see that this is true when a ball is thrown directly upward.

Example 4.2.3. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

Proof. We first will prove that it has a solution. Let $f(x) = x^3 + x - 1$, which is a continuous function. Since $\lim_{x\to\infty} f(x) = \infty$, there exists some $x = t$ such that $f(t) > 0$. In particular, $f(1) = 1+1-1 = 1 > 0$. Similarly, $\lim_{x\to-\infty} f(x) = -\infty$, which implies that there exists some $x = s$ such that $f(s) < 0$, and we may assume that $s < t$. In particular, $f(0) = 0+0-1 = -1 < 0$. The Intermediate Value Theorem applies and guarantees that there exists some r such that $f(r) = 0$. Note that this argument applies to any odd degree polynomial. Thus, every odd degree polynomial has an x-intercept. For our particular polynomial, we know there is a solution between 0 and 1.

In order to prove that r is the unique solution to the equation, let us suppose that there are at least two solutions, a and b, that is, $f(a) = f(b) = 0$. Since f is a polynomial, f is continuous on $[a, b]$ and differentiable on (a, b). Thus, Rolle's Theorem applies and we have some c such that $a < c < b$ such that $f'(c) = 0$. But

$$f'(x) = 3x^2 + 1 \geq 1 \qquad \text{for all } x,$$

which is a contradiction. Therefore, f has a unique x-intercept, which implies that the equation has a unique solution. □

Theorem 4.2.4 (The Mean Value Theorem)**.** *Let f be a function that satisfies the following:*

(i) f is continuous on the closed interval $[a, b]$.

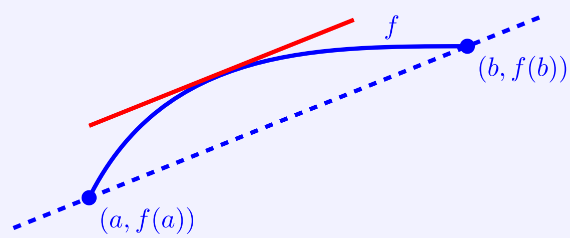
(ii) f is differentiable on the open interval (a, b).

Then there exists some c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or, equivalently

$$f(b) - f(a) = f'(c)(b - a).$$



Example 4.2.5. Let $f(x) = x^3 - x$, $a = 0$, and $b = 2$. Since f is a polynomial, it is continuous and differentiable for all x including the intervals $[0, 2]$ and $(0, 2)$. Therefore, by the Mean Value Theorem, there exists some number $0 < c < 2$ such that

$$f(2) - f(0) = f'(c)(2 - 0).$$

Now, $f(2) = 2^3 - 2 = 6$, $f(0) = 0^0 - 0 = 0$, and $f'(x) = 3x^2 - 1$.

Let

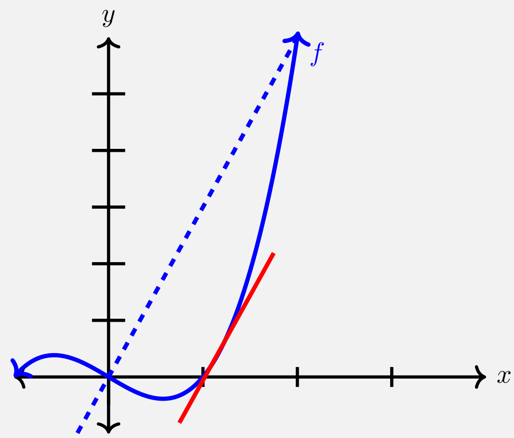
$$m = \frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2 - 0} = \frac{6}{2} = 3.$$

Thus, the Mean Value Theorem guarantees that there is some tangent line between 0 and 2 with slope equal to 3, that is, is parallel to the secant line connecting $(0, 0)$ and $(2, 6)$. We can find this value of c explicitly. Now,

$$f'(c) = 3c^2 - 1 = 3 \quad \Rightarrow \quad 0 = 3c^2 - 4$$

$$\Rightarrow \quad 4 = 3c^2 \quad \Rightarrow \quad \frac{4}{3} = c^2 \quad \Rightarrow \quad c = \pm\frac{2}{\sqrt{3}}$$

Now, $0 < 2/\sqrt{3} < 2$. Therefore, $\boxed{c = 2/\sqrt{3}}$.



Example 4.2.6. Let $f(x) = |x|$. Then

$$f'(x) = \frac{x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Now $\dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{1 - 1}{2} = 0$. So, there is no c such that $f'(c) = 0$. Therefore, the Mean Value Theorem may fail if the differentiability assumption is removed.

Example 4.2.7. Suppose now that $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Consider the interval $[0, 1]$. First, $\dfrac{f(1) - f(0)}{1 - 0} = \dfrac{-1}{1} = -1$, but $f'(x) = 0$ for all $0 < x < 1$. Therefore, there exists no c such that $0 < c < 1$ and $f'(c) = -1$. This shows us that the continuity at the endpoints is also required for the Mean Value Theorem.

Example 4.2.8. If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ if $f'(c)$. Thus the Mean Value Theorem tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity. For instance, if a car traveled 180 miles in 2 hours, then the speedometer must have read 90 mph at least once. Highway Patrol sometimes use this trick to catch speeders.

In general, the Mean Value Theorem can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change.

Example 4.2.9. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

We first notice that, by assumption, f is differentiable (and hence continuous) everywhere. Therefore, the Mean Value Theorem guarantees that

$$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$

for some $0 < c < 2$. By assumption, $f'(c) \leq 5$. Hence,

$$f'(c) \leq 5 \quad \Rightarrow \quad \frac{f(2) - f(0)}{2 - 0} \leq 5 \quad \Rightarrow \quad \frac{f(2) + 3}{2} \leq 5$$

$$\Rightarrow \quad f(2) + 3 \leq 10 \quad \Rightarrow \quad f(x) \leq 7.$$

Therefore, the largest possible value of $f(2)$ is $\boxed{7}$.

Example 4.2.10. Suppose that the derivative of a function satisfies $2 \leq f'(x) \leq 4$ for all values of x. Find the best values A, B such that

$$A \leq f(10) - f(5) \leq B$$

for every such function.

We will determine A and B using the Mean Value Theorem. Notice that $\dfrac{f(10) - f(5)}{10 - 5} = f'(c)$ for some $c \in (5, 10)$ by the Mean Value Theorem. By our assumption, $2 \leq f'(c) \leq 4$. Thus,

$$2 \leq f'(c) \leq 4 \quad \Rightarrow \quad 2 \leq \frac{f(10) - f(5)}{10 - 5} \leq 4$$

$$\Rightarrow \quad 2 \leq \frac{f(10) - f(5)}{5} \leq 4 \quad \Rightarrow \quad 10 \leq f(10) - f(5) \leq 20$$

Therefore, $\boxed{A = 10 \text{ and } B = 20}$.

The Mean Value Theorem is a very important result to prove several other results for differential calculus.

Theorem 4.2.11. *If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).*

Proof. Choose x_1 and x_2 such that $a < x_1 < x_2 < b$. Since f is differentiable on (a, b), it is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. Applying the Mean Value Theorem gives the existence of some number $x_1 < c < x_2$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0.$$

Thus, $f(x_2) = f(x_1)$. Therefore, f has the same value at *any* two numbers $x_1, x_2 \in (a, b)$. This means f is constant on (a, b). □

Corollary 4.2.12. *If $f'(x) = g'(x)$ for all x on (a, b), then $f - g$ is constant on (a, b), that is, $f(x) = g(x) + C$ where C is a constant number.*

Proof. Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x on (a, b) by assumption. Theorem 4.2.11 implies that F is constant on (a, b), that is, $F(x) = C$ for all x on (a, b). Therefore, $f(x) - g(x) = C \Longrightarrow f(x) = g(x) + C$ for all $x \in (a, b)$. □

[i] See §4.4 The Mean Value Theorem in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–10, the function f is given. Explain why f sastifies or does not satisfy the hypotheses of the Mean Value Theorem on the given closed interval $[a, b]$. Find any numbers c that satisfy the conclusion of the Mean Value Theorem for f on $[a, b]$. If there are no such values, explain why not.

1. $f(x) = x^2$, $[0, 2]$
2. ♠ $f(x) = 2x^2 - 3x + 1$, $[0, 2]$
3. ♠ $f(x) = 4 - 32x + 4x^2$, $[3, 5]$
4. $f(x) = -4x^3 + 7x - 3$, $[0, 1]$
5. ♠ $f(x) = \sqrt{x}$, $[0, 25]$
6. ♠ $f(x) = \sqrt{x} - \dfrac{1}{6}x$, $[0, 36]$
7. ♠ $f(x) = 1 - x^{2/3}$, $[-1, 1]$
8. ♠ $f(x) = \cos(3x)$, $\left[\dfrac{\pi}{12}, \dfrac{7\pi}{12}\right]$
9. ♠ $f(x) = (x - 3)^{-2}$, $[2, 5]$
10. $f(x) = \dfrac{1}{x}$, $[1, 2]$
11. ♠ If $f(3) = 1$ and $f'(x) \geq 3$ for $3 \leq x \leq 6$, how small can $f(6)$ possibly be?
12. ♠ If $2 \leq f'(x) \leq 4$ for all x, what are the minimum and maximum possible values of $f(5) - f(1)$?
13. ♠ Does there exist a function f such that $f(0) = -8$, $f(2) = 9$, and $f'(x) \leq 6$ for all x? Why or why not?

For Exercises 14–19, show that the equation has exactly one real root.

14. $x^3 + 3x - 15 = 0$
15. $2x + \sin x = 0$
16. $3x + \sin(2x) = 0$
17. ♠ $7x + \cos x = 0$
18. $e^x + x = 0$
19. $\ln(x) + x = 0$

Deeper Dive

Definition 4.2.13. A number a is called a **fixed point** of a function f if $f(a) = a$.

20. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

"Every noble work is at first impossible." – Thomas Carlyle

Lecture Videos

The First Derivative Test

Determining Local Extrema using the First Derivative Test

A Remark about Critical Numbers

The Test for Concavity

4.3 The First Derivative Test

Proposition 4.3.1. *Suppose a function f is differentiable on an open interval; then*

(a) *if $f'(x) > 0$ for each x in the interval, then f is increasing and the slope of the tangent line is positive at each point;*

(b) *if $f'(x) < 0$ for each x in the interval, then f is decreasing and the slope of the tangent line is negative at each point;*

(c) *if $f'(x) = 0$ for each x in the interval, then f is constant and the slope of the tangent line is zero at each point, that is, the tangent line is horizontal.*

By the Intermediate Value Theorem, if the critical numbers of a function are used as the end points of an open interval, then the sign of the derivative at any point in the interval will be the same as the sign of any other point. Therefore, the process of finding the increasing and decreasing intervals can be outlined as simply solving the inequality $f'(x) > 0$.

Theorem 4.3.2 (The First Derivative Test)**.** *Let c be a critical number for a function f. Suppose that f is continuous on (a, b) and differentiable on (a, b) except possibly at c, and that c is the only critical number for f in (a, b).*

(a) *$f(c)$ is a relative maximum of f if the derivative f' is positive in the interval (a, c) and negative in the interval (c, b).*

(b) *$f(c)$ is a relative minimum of f if the derivative f' is negative in the interval (a, c) and positive in the interval (c, b).*

(c) *If the derivative f' has the same sign on the intervals (a, c) and (c, b), then $f(c)$ is not a relative extreme.*

Example 4.3.3. Find all relative extrema for the following functions, as well as where each function is increasing and decreasing.

(a) $f(x) = 2x^3 - 3x^2 - 72x + 15$

First, we have $f'(x) = 6x^2 - 6x - 72 = 6(x^2 - x - 12) = 6(x - 4)(x + 3)$. Thus, the critical numbers of f are $x = 4, -3$. We next construct the appropriate sign chart.

	$(-\infty, -3)$	$(-3, 4)$	$(4, \infty)$
$6(x+3)(x-4)$	$+$	$-$	$+$

Therefore, f is increasing on the intervals $(-\infty, -3)$ and $(4, \infty)$ and decreasing on the interval $(-3, 4)$. By the First Derivative Test, $f(-3) = 150$ is a local maximum and $f(4) = -193$ is a local minimum.

(b) $f(x) = 6x^{2/3} - 4x$

First, we have $f'(x) = \dfrac{2}{3}\left(6x^{-1/3}\right) - 4 = \dfrac{12}{3x^{1/3}} - 4 = \dfrac{4}{x^{1/3}} - 4$. So, if $f'(x) = 0$, then

$$\begin{aligned} f'(x) &= 0 \\ \frac{4}{x^{1/3}} - 4 &= 0 \\ \frac{4}{x^{1/3}} &= 4 \\ 4 &= 4x^{1/3} \\ 1 &= x^{1/3} \\ 1 &= x \end{aligned}$$

Thus, $x = 1$ is a critical value of f. Also, $x = 0$ is a critical value since $f'(0)$ is undefined. We next construct the appropriate sign chart.

	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$-$	$+$	$-$

Therefore, f is decreasing on the intervals $(-\infty, 0)$ and $(1, \infty)$ and increasing on the interval $(0, 1)$. By the First Derivative Test, $f(0) = 0$ is a local minimum and $f(1) = 2$ is a local maximum.

(c) $f(x) = xe^{2-x^2}$

First, $f'(x) = (1)e^{2-x^2} + x(e^{2-x^2}(-2x)) = e^{2-x^2}\left(1 - 2x^2\right)$. To find the critical points, we solve the equation $f'(x) = 0$. Now, $e^x > 0$ for all real numbers x. Furthermore, if $g(x)$ is any real-valued function, $e^{g(x)} > 0$. So, $e^{2-x^2} > 0$. So, $f'(x) = 0$ if and only if

$$\begin{aligned} 1 - 2x^2 &= 0 \\ 2x^2 &= 1 \\ x^2 &= \frac{1}{2} \\ x &= \pm\sqrt{\frac{1}{2}} \\ x &\approx \pm 0.707. \end{aligned}$$

Therefore,

	$\left(-\infty, -\sqrt{\frac{1}{2}}\right)$	$\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$	$\left(\sqrt{\frac{1}{2}}, \infty\right)$
$f'(x)$	$-$	$+$	$-$

Hence, $f\left(-\sqrt{\frac{1}{2}}\right) \approx -3.17$ is a local minimum and $f\left(\sqrt{\frac{1}{2}}\right) \approx 3.17$ is a local maximum.

Table 4.3.1: Monotonicity

f	Increasing	Decreasing
f'	$+$	$-$

A critical point must be in the domain of the function. For example, the derivative of $f(x) = x/(x-4)$ is $f'(x) = -4/(x-4)^2$, which is undefined at $x = 4$. But $f(4)$ is also undefined, so 4 is not a critical point of f, and the function has no relative extrema. In fact, $x = 4$ corresponds to a vertical asymptote of f.

The first derivative has been used to show where a function is increasing or decreasing and where the extrema occur. The second derivative gives the rate of change of the first derivative; it indicates *how fast* the function is increasing or decreasing. The rate of change of the derivative affects the *shape* of the graph.

Definition 4.3.4. A function is **concave upward** on an interval (a, b) if the graph of the function lies above its tangent line at each point of (a, b). A function is **concave downward** on (a, b) if the graph of the function lies below its tangent line at each point of (a, b). A point where the graph changes **concavity** is called an **inflection point**.

Intuitively, we say that a graph is concave upward on an interval if it "holds water" and concave downward if it "spills water."

If $f''(x) > 0$ on an interval (a, b), then the tangent lines on this interval are getting steeper and steeper, that is, the graph is curving upward. If $f''(x) < 0$ on an interval (a, b), then the tangent lines on this interval are falling lower and lower, that is, the graph is curving downward.

Proposition 4.3.5 (Test for Concavity)**.** *Let f be a function with derivatives f' and f'' existing at all points in an interval (a, b). Then f is concave upward on (a, b) if $f''(x) > 0$ for all x in (a, b), and concave downward on (a, b) if $f''(x) < 0$ for all x in (a, b).*

Corollary 4.3.6. *At an inflection point for a function f, the second derivative is 0 or does not exist.*

Example 4.3.7. Find all the intervals where $f(x) = x^4 - 8x^3 + 18x^2$ is concave upward and downward, and find all the inflection points.

We first calculate the first and second derivatives:

$$f'(x) = 4x^3 - 24x^2 + 36x$$

$$f''(x) = 12x^2 - 48x + 36$$

$$
\begin{aligned}
&= 12(x^2 - 4x + 3) \\
&= 12(x-1)(x-3)
\end{aligned}
$$

So, the roots of $f''(x)$ are $x = 1, 3$. We then construct a sign chart.

	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
$12(x-1)(x-3)$	$+$	$-$	$+$

Therefore, f is concave upward on the intervals $(-\infty, 1)$ and $(3, \infty)$ and concave downward on the interval $(1, 3)$. The inflection points of f are $x = 1$ and 3.

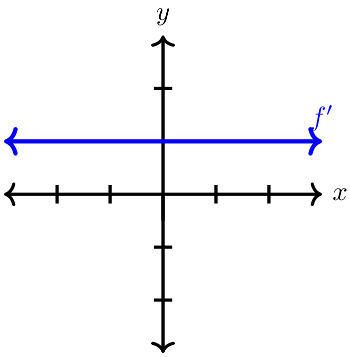
[i]See §4.5 Derivatives and the Shape of a Graph in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–16, the derivative f' is illustrated below. On which intervals is the function f increasing or decreasing? Where are the local maxima and minima of the function f? On which intervals is the function f concave upward or downward? Where are points of inflection of the function f?

1.



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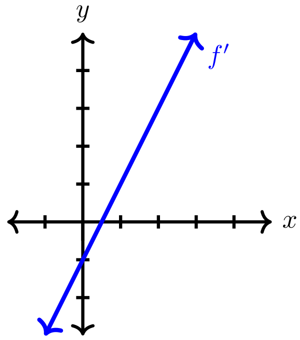
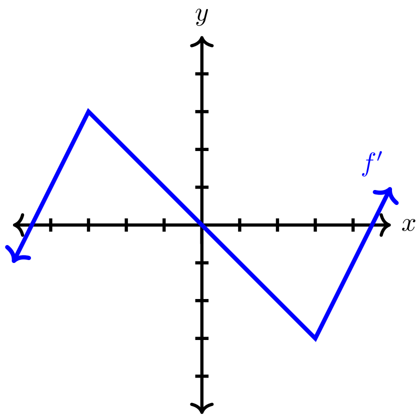
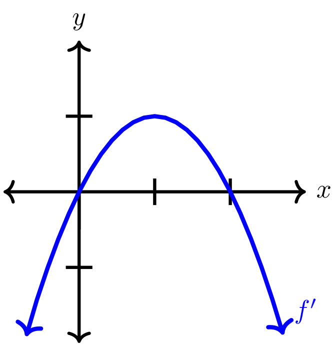


♠ 3.



♠ 4.



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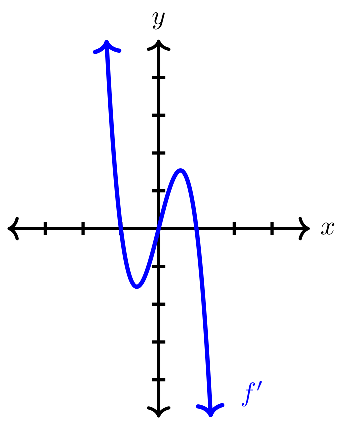

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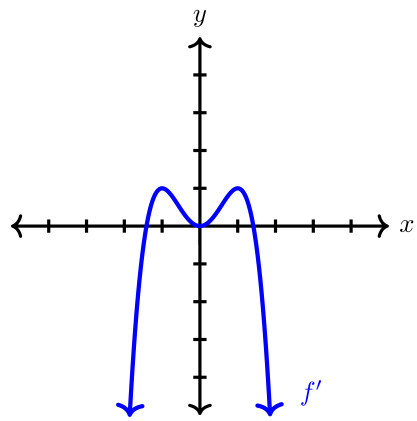

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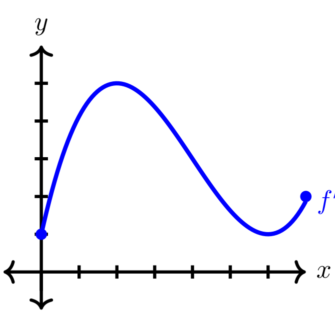

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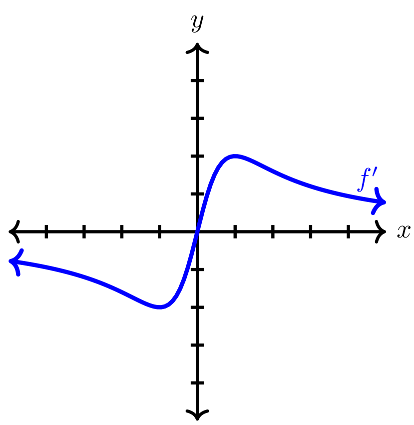
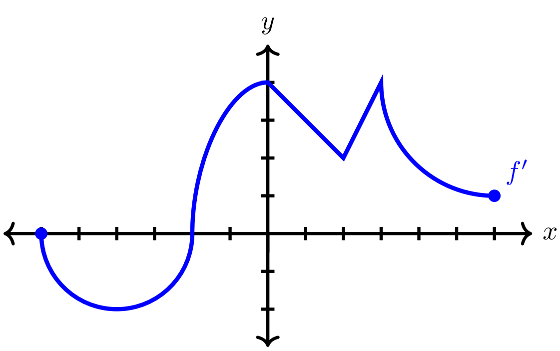


♠ 9.



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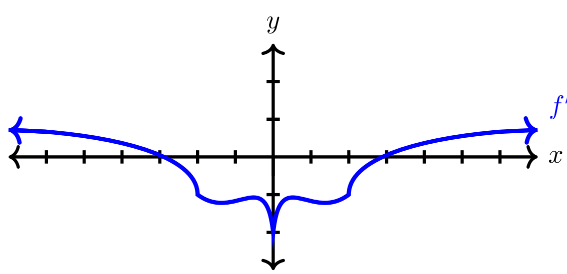

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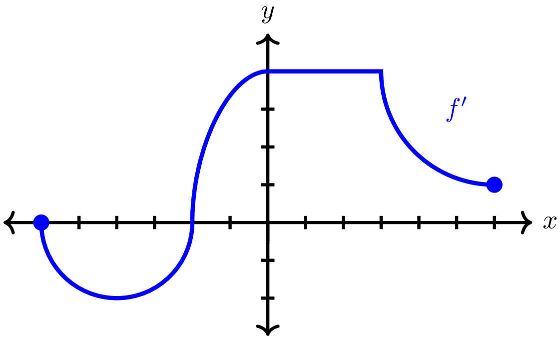

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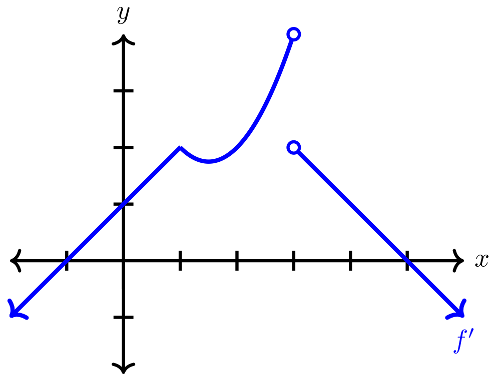

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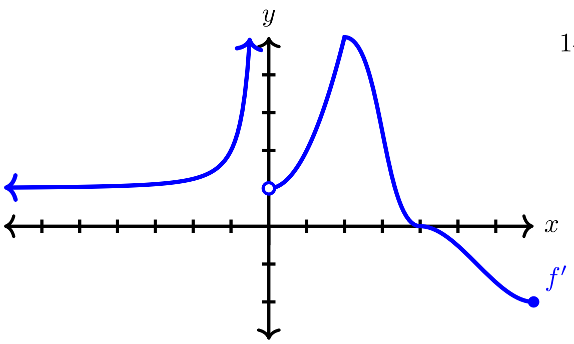

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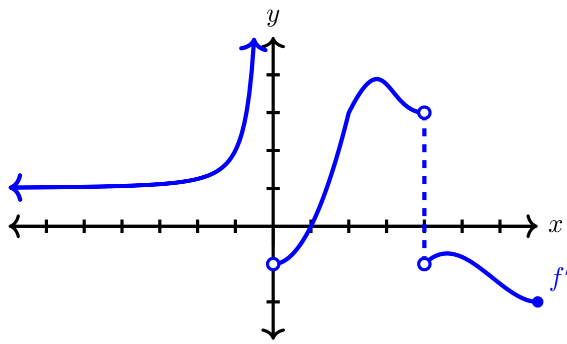
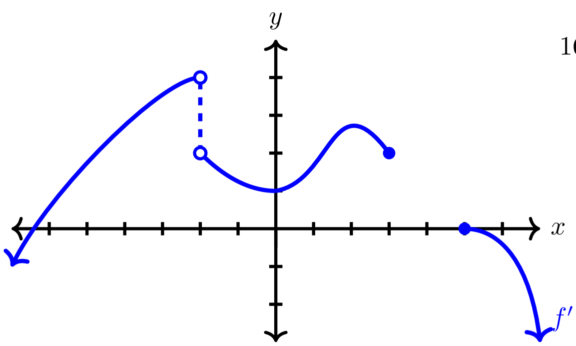


♠ 15.



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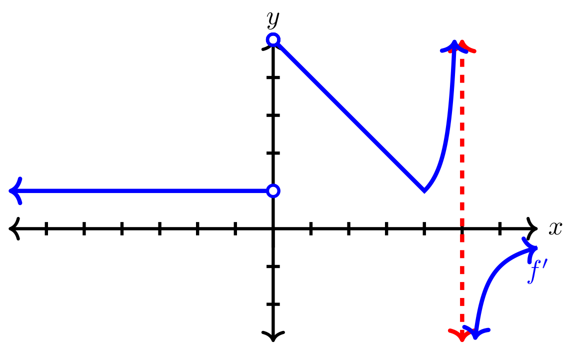


For Exercises 17–30, on which intervals is the function f increasing or decreasing? Where are the local maxima and minima of the function f? On which intervals is the function f concave upward or downward? Where are points of inflection of the function f?

17. $f(x) = x^3 - 3x$

18. $f(x) = 2x^3 - 3x^2 - 12x + 1$

19. $f(x) = \dfrac{x^3}{3} + x^2 - 3x$

20. $f(x) = \dfrac{x^4}{4} - 2x^2$

21. $f(x) = \dfrac{5}{3}x^3 + \dfrac{19}{2}x^2 - 4x + 11$

22. $f(x) = \dfrac{1}{4}x^4 - \dfrac{7}{2}x^2 + \dfrac{11}{4}$

♠ 23. $f(x) = 2x^3 + 3x^2 - 72x$

♠ 24. $f(x) = 4x^3 + 6x^2 - 24x + 8$

25. $f(x) = -3x^3 + x^2 + 3x - 1$

♠ 26. $f(x) = x^4 - 32x^2 + 6$

♠ 27. $f(x) = 9\sin x + 9\cos x$ on $[0, 2\pi]$

♠ 28. $f(x) = 9\cos^2 x - 18\sin x$ on $[0, 2\pi]$

♠ 29. $f(x) = e^{2x} + e^{-x}$

30. $f(x) = x^{1/3} - x^{4/3}$

Deeper Dive

31. Show that $f(x) = \tan x - x$ is increasing on the interval $\left(0, \dfrac{\pi}{2}\right)$.

"People do make mistakes and I think they should be punished. But they should be forgiven and given the opportunity for a second chance. We are human beings." – David Millar

Lecture Videos

The Second Derivative Test

l'Hospital's Rule

4.4 The Second Derivative Test

Concavity can also be used to determine whether a critical value is a local extremum. First, a curve with a horizontal tangent at a point c and which concaves downward on an open interval containing c also has a local maximum at c. A local minimum occurs when a graph has a horizontal tangent at a point d and is concave upward on an open interval containing d.

Proposition 4.4.1 (The Second Derivative Test)**.** *Let f'' exist on some open interval containing c, and let $f'(c) = 0$.*

(a) If $f''(c) > 0$, then $f(c)$ is a relative minimum.

(b) If $f''(c) < 0$, then $f(c)$ is a relative maximum.

(c) If $f''(c) = 0$ or $f''(c)$ is undefined, then the test gives no information about extrema, so use the First Derivative Test.

Example 4.4.2. Find all the relative extrema for $f(x) = 4x^3 + 7x^2 - 10x + 8$.

We will find them using the Second Derivative test. First, note

$$f'(x) = 12x^2 + 14x - 10 = 2(6x^2 + 7x - 5) = 2(3x + 5)(2x - 1)$$

Hence, the critical values are $x = -\frac{5}{3}$ and $\frac{1}{2}$. Next,

$$f''(x) = 24x + 14.$$

Thus,
$f''\left(-\frac{5}{3}\right) = 24\left(-\frac{5}{3}\right) + 14 = -40 + 14 = -26 < 0$. So, $\boxed{x = -\frac{5}{3} \text{ is a local maximum}}$, by the Second Derivative test. Since $f''\left(\frac{1}{2}\right) = 26 > 0$, $\boxed{x = \frac{1}{2} \text{ is a local minimum}}$.

We began the course talking about limits, and we used limits to define the derivative of a function. So, the study of derivatives will be eternally grateful to the study of limits. l'Hospital's Rule is a way for derivatives to pay back their debt to limits.

Suppose we need to evaluate the limit

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

Table 4.4.1: Concavity

f	Concave Up	Concave Down
f'	Increasing	Decreasing
f''	$+$	$-$

Although, $\ln x$ and $x-1$ are nice, continuous functions, we can't simply evaluate the functions at $x=1$ to determine the limit since if we did we would obtain the **indeterminant form** $\frac{0}{0}$. Our previous limit rules don't apply since $x-1=0$ when $x=1$. Similarly, if we need to determine

$$\lim_{x\to\infty}\frac{\ln x}{x-1}$$

and we blindly substitute in "$x=\infty$" we get another **indeterminant form** $\frac{\infty}{\infty}$.

Similar to the last example, consider the end behavior of a rational function

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1}=\frac{1}{2}.$$

Although, this limit is an indeterminate form $\frac{\infty}{\infty}$, the limit still exists. l'Hospital's Rule is a method which can alleviate us of these problems of indeterminate forms for more general functions.

Theorem 4.4.3 (l'Hospital's Rule)**.** *Let f and g be functions and let a be a real number or let $a=\pm\infty$ such that*

$$\lim_{x\to a} f(x)=0 \text{ and } \lim_{x\to a} g(x)=0,$$

or

$$\lim_{x\to a} f(x)=\pm\infty \text{ and } \lim_{x\to a} g(x)=\pm\infty.$$

Let f and g be differentiable at each point in some open interval containing a[i]

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$, *then* $\lim_{x\to a}\frac{f(x)}{g(x)}=L$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *does not exist, then* $\lim_{x\to a}\frac{f(x)}{g(x)}$ *does not exist.*

Remark 4.4.4. In particular, l'Hospital's Rule can be used to find the limit of indeterminate forms, that is, the evaluation of $\frac{f(x)}{g(x)}$ at $x=a$ leads to $\frac{0}{0}$ or $\frac{\infty}{\infty}$. ▼

Example 4.4.5. Find the limits using l'Hospital's Rule.

(a) $\lim_{x\to 1}\frac{\ln x}{x-1}$.

As seen before, this limit has the indeterminant form $\frac{0}{0}$. Thus, by l'Hospital's Rule,

$$\lim_{x\to 1}\frac{\ln x}{x-1}\overset{\text{l'H}}{=}\lim_{x\to 1}\frac{1/x}{1}=\lim_{x\to 1}\frac{1}{x}=\boxed{1}$$

(b) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x-1}$.

As seen before, this limit has the indeterminant form $\dfrac{\infty}{\infty}$. Thus, by l'Hospital's Rule,

$$\lim_{x\to\infty}\frac{\ln x}{x-1}\overset{\text{l'H}}{=}\lim_{x\to\infty}\frac{1/x}{1}=\lim_{x\to\infty}\frac{1}{x}=\boxed{0}$$

(c) $\displaystyle\lim_{x\to 2}\frac{3x-6}{\sqrt{2+x}-2}$.

By mere evaluation, we see that $\dfrac{3(2)-6}{\sqrt{2+(2)}-2}=\dfrac{6-6}{\sqrt{4}-2}=\dfrac{0}{0}$, which is an indeterminate form. So, we use l'Hospital's Rule. Note that $(3x-6)'=3$ and $\left[\sqrt{2+x}-2\right]'=\dfrac{1}{2\sqrt{2+x}}$. Therefore,

$$\lim_{x\to 2}\frac{3x-6}{\sqrt{2+x}-2}\overset{\text{l'H}}{=}\lim_{x\to 2}\frac{3}{\left(\dfrac{1}{2\sqrt{2+x}}\right)}=\lim_{x\to 2}3\cdot 2\sqrt{2+x}=6\sqrt{2+2}=6\cdot 2=\boxed{12}.$$

(d) $\displaystyle\lim_{x\to 1}\frac{\ln x}{(x-1)^2}$.

Since $\displaystyle\lim_{x\to 1}\ln x=\ln 1=0$ and $\displaystyle\lim_{x\to 1}(x-1)^2=(1-1)^2=0$, we have the indeterminate form $\dfrac{0}{0}$. By l'Hospital's Rule,

$$\lim_{x\to 1}\frac{\ln x}{(x-1)^2}\overset{\text{l'H}}{=}\lim_{x\to 1}\frac{(\ln x)'}{\left[(x-1)^2\right]'}=\lim_{x\to 1}\frac{1/x}{2(x-1)}=\lim_{x\to 1}\frac{1}{2x(x-1)},$$

which implies that the original limit $\boxed{\text{does not exist}}$ since $x=1$ corresponds to a vertical asymptote of $\dfrac{1}{2x(x-1)}$.

(e) $\displaystyle\lim_{x\to\infty}\frac{x^2}{e^x}$.

Since $\displaystyle\lim_{x\to\infty}x^2=\infty$ and $\displaystyle\lim_{x\to\infty}e^x=\infty$, the limit is an indeterminate form $\dfrac{\infty}{\infty}$. By l'Hospital's Rule,

$$\lim_{x\to\infty}\frac{x^2}{e^x}\overset{\text{l'H}}{=}\lim_{x\to\infty}\frac{(x^2)'}{(e^x)'}=\lim_{x\to\infty}\frac{2x}{e^x}.$$

Now, $\displaystyle\lim_{x\to\infty}2x=\infty$ and $\displaystyle\lim_{x\to\infty}e^x=\infty$, the limit is still the indeterminate form $\dfrac{\infty}{\infty}$. We can then repeat this process:

$$\lim_{x\to\infty}\frac{x^2}{e^x}\overset{\text{l'H}}{=}\lim_{x\to\infty}\frac{2x}{e^x}\overset{\text{l'H}}{=}\lim_{x\to\infty}\frac{(2x)'}{(e^x)'}=\lim_{x\to\infty}\frac{2}{e^x}=\boxed{0}$$

(f) $\displaystyle\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}$.

If we blindly attempted to use l'Hospital's Rule, we would get

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\lim_{x\to\pi^-}\frac{\cos x}{\sin x}=\frac{-1}{0^+}=-\infty.$$

This is **wrong!** Although the numerator $\sin x \to 0$ as $x \to \pi^-$, notice that the denominator $(1 - \cos x)$ does not approach 0, so l'Hospital's rule can't apply. Instead, note that

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi^-}{1 - \cos \pi^-} = \frac{0}{1 - (-1)} = \boxed{0}.$$

Remember, it is important to check that the limit is first an indeterminate form.

[i] If $a = \infty$, then this means that f and g are differentiable on an open interval (b, ∞) for some real number b. Similarly, this means the interval $(-\infty, b)$ if $a = -\infty$.

[ii] See §4.5 Derivatives and the Shape of a Graph and §4.8 L'Hôspital's Rule in OpenStax to find the corresponding sections.

Exercises

(Go to Solutions)

For Exercises 1–4, find the local maxima and minima of f using both the First and Second Derivative Tests.

1. $f(x) = -x^3 + 4x^2 + 16x + 15$

♠ 2. $f(x) = x^5 - 5x + 8$

♠ 3. $f(x) = x^6(x-2)^5$

4. $f(x) = \dfrac{x^4}{4} - \dfrac{5x^3}{3} + 3x^2 + \dfrac{23}{2}$

For Exercises 5–26, compute the limit.

5. $\lim\limits_{x\to 4} \dfrac{x^2 - 10x + 24}{x-4}$

♠ 6. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-2x}$

♠ 7. $\lim\limits_{x\to 1} \dfrac{x^2+2x-3}{x-1}$

♠ 8. $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{\tan(3x)}$

♠ 9. $\lim\limits_{x\to \pi/2^+} \dfrac{\cos x}{1-\sin x}$

10. $\lim\limits_{x\to \pi^-} \dfrac{\sin x}{1-\cos x}$

11. $\lim\limits_{x\to 0^+} \dfrac{x^2}{x-\sin x}$

12. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{4x^3}$

13. $\lim\limits_{x\to 0} \dfrac{e^x-1}{x}$

14. $\lim\limits_{x\to 0} \dfrac{x^2}{e^x-x-1}$

15. $\lim\limits_{x\to 0} \dfrac{e^x-1}{x^2+1}$

16. $\lim\limits_{x\to 0} \dfrac{x^2-1}{e^x+1}$

17. $\lim\limits_{x\to 0} \dfrac{e^x-e^{-x}}{2x}$

18. $\lim\limits_{x\to 0} \dfrac{2-2e^x}{e^{-x}-1}$

19. $\lim\limits_{x\to \infty} \dfrac{x^2}{e^{3x^2}}$

20. $\lim\limits_{x\to 0} \dfrac{e^{3x}-1-3x}{x^2}$

21. $\lim\limits_{x\to 0} \dfrac{xe^x}{e^{2x}-1}$

22. $\lim\limits_{x\to \infty} \dfrac{e^{x^2}}{x}$

23. $\lim\limits_{x\to 0} \dfrac{e^x-1}{\sin x}$

24. $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{\sin x}$

25. $\lim\limits_{x\to 1} \dfrac{\ln(x)}{x-1}$

26. $\lim\limits_{x\to 1} \dfrac{\ln x}{\cos \frac{\pi}{2} x}$

"The very first requirement in a hospital is that it should do the sick no harm." – Florence Nightingale

Lecture Videos

More Practice on l'Hospital's Rule

l'Hospital's Rule an Product Indeterminants

l'Hospital's Rule and Exponential Indeterminants

l'Hospital's Rule and Difference Indeterminants

4.5 l'Hospital's Rule

We first introduced l'Hospital's rule in the previous section. In this section, we continue to present examples.

Example 4.5.1. Compute the following limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$

The limit has the form $\dfrac{\infty}{\infty}$, so by l'Hospital's Rule,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{(x^{-2/3})/3} = \lim_{x\to\infty} \frac{3x^{2/3}}{x} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = \boxed{0}$$

(b) $\displaystyle\lim_{x\to0} \frac{\tan x - x}{x^3}$

The limit has the form $\dfrac{0}{0}$, so by l'Hospital's Rule,

$$\lim_{x\to0} \frac{\tan x - x}{x^3} \stackrel{\text{l'H}}{=} \lim_{x\to0} \frac{\sec^2 x - 1}{3x^2}.$$

This new limit also has the indeterminant form $\dfrac{0}{0}$, so we apply l'Hospital's Rule again, which gives

$$\lim_{x\to0} \frac{\sec^2 x - 1}{3x^2} \stackrel{\text{l'H}}{=} \lim_{x\to0} \frac{2\sec x(\tan x \sec x)}{6x} = \lim_{x\to0} \frac{\sec^2 x \tan x}{3x}.$$

This new limit still has the indeterminant form $\dfrac{0}{0}$, so we can apply l'Hospital's Rule one more time, which gives

$$\begin{aligned}\lim_{x\to0} \frac{\sec^2 x \tan x}{3x} &\stackrel{\text{l'H}}{=} \lim_{x\to0} \frac{2\sec x(\sec x \tan x) + \sec^2 x(\sec^2 x)}{3} \\ &= \lim_{x\to0} \frac{\sec^2 x(2\tan x + \sec^2 x)}{3} = \frac{1^2(2(0)+1^2)}{3} = \boxed{\frac{1}{3}}.\end{aligned}$$

Alternatively,

$$\lim_{x\to0} \frac{\sec^2 x \tan x}{3x} = \frac{1}{3}\left(\lim_{x\to0} \frac{\sin x}{x\cos^3 x}\right) = \frac{1}{3}\left(\lim_{x\to0} \frac{\sin x}{x}\right)\left(\lim_{x\to0} \frac{1}{\cos^3 x}\right) = \frac{1}{3}(1)(1) = \frac{1}{3}.$$

Remember that l'Hosptal's Rule is not always necessary. When a more elementary method exists, it is often more simple than taking derivatives.

Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$. Then their product limit is an **indeterminate form** $0 \cdot \infty$. L'Hospital's Rule can be used to find the limit if we convert to compounded fractions, that is, use the fact

$$fg = \frac{f}{1/g}.$$

Example 4.5.2. Evaluate $\lim_{x\to 0^+} x\ln x$.

Note that this limit has the indeterminate form $0 \cdot \infty$. Notice that $x \ln x = \frac{\ln x}{x^{-1}} = \frac{\ln x}{1/x}$. Then by L'Hospital's Rule, we have

$$\begin{aligned} \lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0^+} \frac{1}{-1/x} = \lim_{x\to 0^+} -x = \boxed{0} \end{aligned}$$

Several **indeterminate forms** arise from the limit

$$\lim_{x\to a} [f(x)]^{g(x)}$$

(i) **Type** 0^0 : $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$,

(ii) **Type** ∞^0 : $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$,

(iii) **Type** 1^∞ : $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

In these cases we use the identity

$$[f(x)]^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}.$$

Example 4.5.3. Find $\lim_{x\to 0^+} x^x$.

Note that

$$x^x = e^{\ln(x^x)} = e^{x\ln x}.$$

If $\exp(x) = e^x$, then

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = \lim_{x\to 0^+} \exp(x\ln x) = \exp\left(\lim_{x\to 0^+} x\ln x\right).$$

We then need to compute $\lim_{x\to 0^+} x\ln x$, which has the indeterminant form $0 \cdot \infty$. As we computed earlier, $\lim_{x\to 0^+} x\ln x = 0$. Therefore,

$$\lim_{x\to 0^+} x^x = \exp\left(\lim_{x\to 0^+} x\ln x\right) = \exp(0) = e^0 = \boxed{1}.$$

Example 4.5.4. Calculate $\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x}$.

We first check if the limit is an indeterminate form, which is verified as 1^∞. Thus,

$$\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x\to 0^+} e^{\cot(x)\ln(1+\sin 4x)} = \exp\left(\lim_{x\to 0^+} \cot(x)\ln(1 + \sin 4x)\right).$$

Thus, we need to compute $\lim_{x\to 0^+} \cot(x)\ln(1+\sin 4x)$.

$$\lim_{x\to 0^+} \cot(x)\ln(1+\sin 4x) = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \stackrel{\text{l'H}}{=} \lim_{x\to 0^+} \frac{(4\cos 4x)/(1+\sin 4x)}{\sec^2 x}$$
$$= \lim_{x\to 0^+} \frac{4\cos^2 x\cos 4x}{1+\sin 4x} = 4$$

Therefore,

$$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = \boxed{e^4}.$$

Example 4.5.5. Calculate $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x$.

We first check if the limit is an indeterminate form, which is verified as 1^∞. Thus,

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+3/x)} = \exp\left(\lim_{x\to\infty} x\ln(1+3/x)\right).$$

Thus, we need to compute $\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right)$.

$$\lim_{x\to\infty} x\ln(1+3/x) = \lim_{x\to\infty} \frac{\ln(1+3/x)}{1/x} \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{(-3/x^2)/(1+3/x)}{-1/x^2} = \lim_{x\to\infty} \frac{3}{1+3/x} = 3$$

Therefore,

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \boxed{e^3}.$$

The last **indeterminate form** to be wary of is $\infty-\infty$, that is, suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then their difference

$$\lim_{x\to a} f(x) - g(x)$$

is "$\infty - \infty$". Now, L'Hospital's rule doesn't apply directly here, but sometimes we can turn the difference into a quotient and then apply L'Hospital's Rule, like in the following example.

Example 4.5.6. Compute $\lim_{x\to\pi/2^-} (\sec x - \tan x)$.

First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2^-)$. Here we use a common denominator:

$$\begin{aligned}
\lim_{x\to\pi/2^-} (\sec x - \tan x) &= \lim_{x\to\pi/2^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) \\
&= \lim_{x\to\pi/2^-} \frac{1-\sin x}{\cos x}, \qquad \text{which has the indeterminate form } 0/0 \\
&= \lim_{x\to\pi/2^-} \frac{-\cos x}{-\sin x} = \boxed{0}.
\end{aligned}$$

[i]See §4.8 L'Hôspital's Rule in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–26, compute the limit.

1. $\displaystyle\lim_{x\to-\infty}\frac{x-3}{x^2+2}$

♠ 2. $\displaystyle\lim_{x\to\infty}\frac{x+x^2}{9-2x^2}$

♠ 3. $\displaystyle\lim_{x\to\infty}\frac{\ln(3x)}{\sqrt{3x}}$

4. $\displaystyle\lim_{x\to\infty}x^3e^{-x^2}$

♠ 5. $\displaystyle\lim_{x\to0}\frac{\sqrt{1+10x}-\sqrt{1-9x}}{x}$

♠ 6. $\displaystyle\lim_{x\to0}\frac{e^{7x}-1-7x}{x^2}$

♠ 7. $\displaystyle\lim_{x\to0}\frac{\tanh(x)}{\tan(x)}$

♠ 8. $\displaystyle\lim_{x\to0}\frac{\sin^{-1}(x)}{7x}$

9. $\displaystyle\lim_{x\to\infty}5x\sin\left(\frac{3}{x}\right)$

♠ 10. $\displaystyle\lim_{x\to\infty}x\sin\left(\frac{8\pi}{x}\right)$

♠ 11. $\displaystyle\lim_{x\to0^+}\cot(2x)\sin(4x)$

♠ 12. $\displaystyle\lim_{x\to\infty}x^7e^{-x^6}$

13. $\displaystyle\lim_{x\to\infty}x\sin\left(\frac{\pi}{x}\right)$

14. $\displaystyle\lim_{x\to\pi^-}(x-\pi)\csc(x)$

15. $\displaystyle\lim_{x\to0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

♠ 16. $\displaystyle\lim_{x\to1}\left(\frac{9x}{x-1}-\frac{9}{\ln x}\right)$

♠ 17. $\displaystyle\lim_{x\to0^+}(\tan(8x))^x$

♠ 18. $\displaystyle\lim_{x\to0}(1-8x)^{1/x}$

19. $\displaystyle\lim_{x\to0^+}(1-2x)^{1/x}$

20. $\displaystyle\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{3x}$

21. $\displaystyle\lim_{x\to\pi/2^-}(\tan x)^{\cos x}$

22. $\displaystyle\lim_{x\to0}(\cos x)^{1/x}$

23. $\displaystyle\lim_{x\to0^+}x^{\tan x}$

24. $\displaystyle\lim_{x\to\infty}(e^x+x)^{1/x}$

"A smile is a curve that sets everything straight." – Phyllis Diller

Lecture Videos

Curve Sketching

Curve Sketching (Polynomial Function)

Curve Sketching (Rational Function with Oblique Asymptote)

Curve Sketching (Rational Function with Horizontal Asymptote)

4.6 Curve Sketching

We are ready to put together all of our techniques for sketching the graph of a function. We list all of these concepts together.

Curve Sketching — To sketch the graph of a function f:

(i) **Domain** : Consider the domain of the function, and note any restrictions. (That is, avoid dividing by 0, taking a square root of a negative number, taking the logarithm of 0 or a negative number, etc.)

(ii) **Intercepts** : Find the y-intercept (if it exists) by substituting $x = 0$ into $f(x)$. Find any x-intercepts by solving $f(x) = 0$. If $f(x)$ is a polynomial or rational function, this will involve polynomial factoring.

(iii) **Symmetry** : Determine whether the function is even, odd, or neither. Furthermore, a function f is **periodic** if $f(x + p) = f(x)$ for all $x \in \operatorname{dom} f$, where p is a positive constant, called the **period** of f. Trigonometric functions are examples of periodic functions.

(iv) **Discontinuities** : Find any removable and jump discontinuities. This might require analyzing the switching points of a piece-wise function. Find any vertical asymptotes of the graph. For a rational function, this corresponds to when the denominator is zero but the numerator is nonzero. For logarithmic functions, there is a vertical asymptote when the parameter of the logarithm is equal to zero.

(v) **End Behavior** : Determine the end behavior of function, that is, determine the limit as $x \to \pm\infty$. For example, does the function have a horizontal asymptote, oblique asymptote, or any other asymptotic behavior? L'Hospital's Rule may be helpful here.

(vi) **First Derivative** : Find $f'(x)$. Locate any critical points by solving the equation $f'(x) = 0$ and determining where $f'(x)$ does not exist. Find any local extrema and determine where f is increasing or decreasing.

(vii) **Second Derivative** : Find $f''(x)$. Locate any potential inflection points by solving the equation $f''(x) = 0$ and determining where $f''(x)$ is undefined. Determine where f is concave upward and concave downward.

(viii) **Plot** : Plot the intercepts, the discontinuities, local extrema, the inflection points, and other points as needed.

(ix) **Connect** : Connect the points with a smooth curve using the correct concavity, being careful not to connect points where the function is not defined.

Example 4.6.1. Graph $f(x) = x^3 + x^2 - 5x + 3$.

Since $f(x)$ is a polynomial, its domain is all real numbers, $(-\infty, \infty)$. In particular, $f(x)$ will have no vertical asymptotes.

Since $f(0) = 3$, the y-intercept is $(0, 3)$. We search next for the x-intercepts. By the Rational Roots Theorem,[i] the only possible rational roots of $f(x)$ are $\pm 1, \pm 3$. By experimentation, we can discover that $f(-3) = 0$. Thus, -3 is a root of $f(x)$. By polynomial division, $f(x) = (x+3)(x^2 - 2x + 1) = (x+3)(x-1)^2$. The two x-intercepts of $f(x)$ are then $x = -3, 1$, where intercept $x = 1$ has multiplicity 2.

The end behavior of $f(x)$ can be determined as $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Also, $f(x)$ exhibits no symmetry since it has monomials of even and odd powers.

The first derivative of $f(x)$ is

$$f'(x) = 3x^2 + 2x - 5 = (x-1)(3x+5).$$

Thus, the critical numbers are $x = 1$ and $x = -\dfrac{5}{3}$. Also, the second derivative of $f(x)$ is

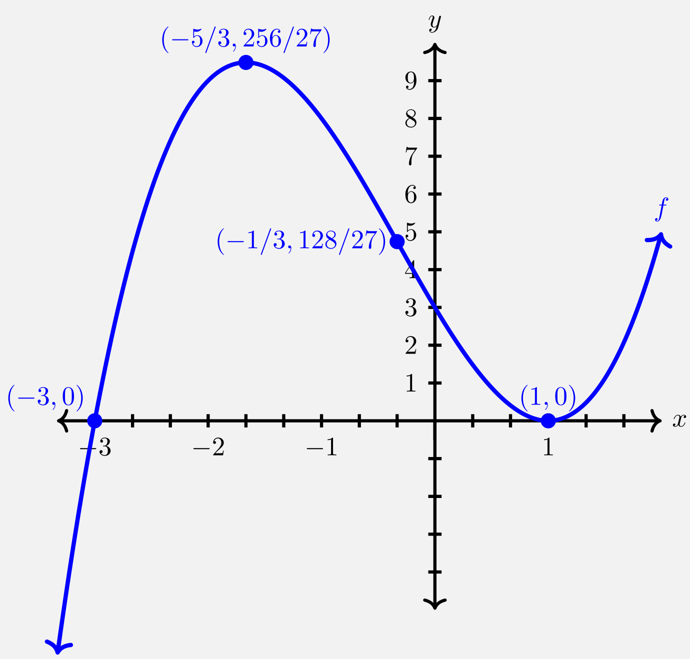
$$f''(x) = 6x + 2 = 2(3x+1).$$

Thus, $x = -\dfrac{1}{3}$ is a potential inflection point. Then by the (First) Second Derivative Test,

	$\left(-\infty, -\frac{5}{3}\right)$	$\left(-\frac{5}{3}, -\frac{1}{3}\right)$	$\left(-\frac{1}{3}, 1\right)$	$(1, \infty)$
$f'(x)$	$+$	$-$	$-$	$+$
$f''(x)$	$-$	$-$	$+$	$+$

Therefore, $\left(-\frac{5}{3}, f\left(-\frac{5}{3}\right)\right) = \left(-\frac{5}{3}, \frac{256}{27}\right)$ is a local maximum and $(1, f(1)) = (1, 0)$ is a local minimum. Also, $\left(-\frac{1}{3}, f\left(-\frac{1}{3}\right)\right) = \left(-\frac{1}{3}, \frac{128}{27}\right)$ is an inflection point.

Putting this all together gives the graph



Example 4.6.2. Graph $f(x) = x + \frac{1}{x}$.

We first notice that $x = 0$ is outside the domain of the function. So, the function does not have a y-intercept. By finding a common denominator, we recognize that $f(x) = \frac{x^2+1}{x}$. Therefore, $f(x) = 0$ if and only if $x^2 + 1 = 0$, which has no real roots. So, $f(x)$ has no x-intercepts. It does have a vertical asymptote at $x = 0$.

Calculating the derivative we have

$$f'(x) = (x + x^{-1})' = 1 - x^{-2} = \frac{x^2 - 1}{x^2}.$$

Thus, $f'(x) = 0$ when $x^2 - 1 = 0$, which means $f'(\pm 1) = 0$. Also, the derivative is undefined at $x = 0$. The second derivative is

$$f''(x) = (1 - x^{-2})' = 2x^{-3} = \frac{2}{x^3}.$$

Now, $f''(x) = 0$ has no solutions, but $f''(x)$ is undefined at $x = 0$. Hence,

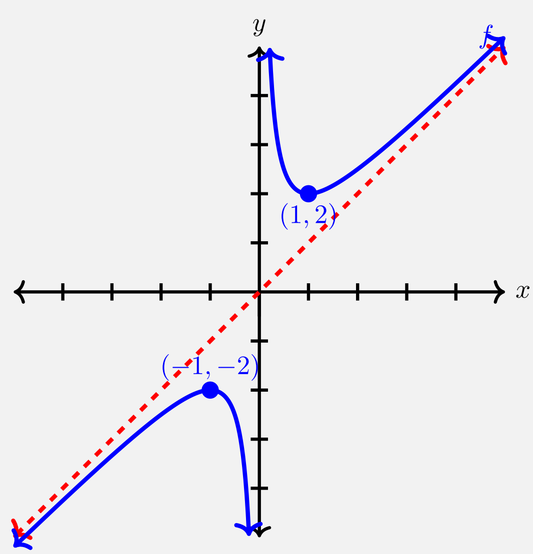
	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$+$	$-$	$-$	$+$
$f''(x)$	$-$	$-$	$+$	$+$

which gives the intervals that $f(x)$ is increasing, decreasing, concave upward, and concave downward. Also, there is a local minimum at $(-1, f(-1)) = (-1, -2)$ and a local maximum at $(1, f(1)) = (1, 2)$. Now, $x = 0$ is not an inflection point since $f(0)$ is undefined.

Lastly, we analyze end behavior and symmetry. We note that $f(-x) = (-x) + \frac{1}{-x} = -\left(x + \frac{1}{x}\right) = -f(x)$. Thus, $f(x)$ is symmetric about the origin.

Also, $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x$ since $\lim_{x\to\pm\infty} \frac{1}{x} = 0$. In particular, the end behavior of $f(x)$ is the same as the end behavior of the line $y = x$ and, as $x \to \pm\infty$, $f(x) \to x$. So, $f(x)$ is asymptotic to this line, which is called an **oblique asymptote**. Oblique asymptotes can also be found using polynomial division.

Putting this all together gives the graph



Example 4.6.3. Graph $f(x) = \frac{3x^2}{x^2+5}$.

Since $x^2 + 5$ has no real roots, the domain of $f(x)$ is all real numbers. The y-intercept is $f(0) = \frac{0}{5} = 0$ and the x-intercept is $x = 0$ since $\frac{3x^2}{x^2+5} = 0 \Rightarrow x = 0$. $f(x)$ is an even function, because $f(-x) = \dfrac{3(-x)^2}{(-x)^2+5} = \dfrac{3x^2}{x^2+5} = f(x)$. So, $f(x)$ is symmetric about the y-axis. Also, $\lim\limits_{x\to\pm\infty} \dfrac{3x^2}{x^2+5} = \dfrac{3}{1} = 3$. Thus, there is a horizontal asymptote at $y=3$.

Using the quotient rule, we have

$$f'(x) = \frac{(x^2+5)(6x)-(3x^2)(2x)}{(x^2+5)^2} = \frac{6x\left[(x^2+5)-x^2\right]}{(x^2+5)^2} = \frac{30x}{(x^2+5)^2}.$$

So, $f'(0)=0$ and the domain of $f'(x)$ is all real numbers like before. So, the only critical number is $x=0$. Continuing on,

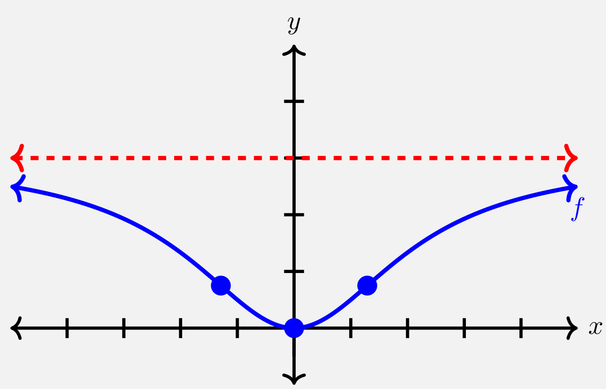
$$f''(x) = \frac{(x^2+5)^2(30)-(30x)2(x^2+5)(2x)}{(x^2+5)^4} = \frac{30(x^2+5)\left[(x^2+5)-4x^2\right]}{(x^2+5)^4} = \frac{30(5-3x^2)}{(x^2+5)^3}.$$

Since $30(5-3x^2)=0 \Rightarrow (5-3x^2)=0 \Rightarrow 3x^2=5 \Rightarrow x^2=\dfrac{5}{3} \Rightarrow x=\pm\sqrt{\dfrac{5}{3}}$, $x=\pm\sqrt{\dfrac{5}{3}} \approx \pm 1.29$ are potential inflection points. We construct a sign chart:

	$(-\infty,-1.29)$	$(-1.29,0)$	$(0,1.29)$	$(1.29,\infty)$
f'(x)	$-$	$-$	$+$	$+$
f"(x)	$-$	$+$	$+$	$-$

Therefore, $(0,f(0))=(0,0)$ is a local minimum of f and $(\pm1.29,\pm0.75)$ are inflection points since $f(\pm1.29)\approx 0.75$.

Putting this all together gives the graph



[i]**Rational Roots Theorem** – Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a + 0$ with a_i all integers, then every rational root of $f(x)$ is of the form $\dfrac{p}{q}$ where p is a divisor of a_0 and q is a divisor of a_n.

[ii]See §4.6 Limits at Infinity and Asymptotes in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–14, produce the graph of the function f that reveals all the important aspects of the curve. In particular, you should indicate the domain of f and label its intercepts, symmetry, discontinuities, end behavior, local extrema, and points of inflections, if present. You should use the first and second derivative to determine the appropriate parts of this curve sketch.

1. $f(x) = 2x^2 - 7x - 4$

♠ 2. $f(x) = x^3 - 6x^2 + 9x$

♠ 3. $f(x) = \dfrac{x}{x-1}$

♠ 4. $f(x) = \dfrac{1}{x^2 - 16}$

5. $f(x) = \dfrac{2x}{x^2 - 1}$

♠ 6. $f(x) = \dfrac{x}{x^2 - 25}$

7. $f(x) = \dfrac{x-1}{x^2}$

♠ 8. $f(x) = \dfrac{x}{x^2 + 36}$

♠ 9. $f(x) = \dfrac{x^2 - 4}{x^2 - 2x}$

♠ 10. $f(x) = \dfrac{x - x^2}{4 - 5x + x^2}$

♠ 11. $f(x) = \dfrac{x-4}{x^2}$

♠ 12. $f(x) = \dfrac{x}{x^3 - 1}$

♠ 13. $f(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

14. $f(x) = \dfrac{x^2}{x^2 + 9}$

"You can't do sketches enough. Sketch everything and keep your curiosity fresh." – John Singer Sargent

Lecture Videos

Curve Sketching (Radical Ratio)	Curve Sketching (Logarithmic Ratio)	Curve Sketching (Trigonometric Ratio)

4.7 Curve Sketching II

Example 4.7.1. Graph $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

The y-intercept of f is $y = 0$. Likewise, the only x-intercept is $x = 0$. The domain of f is $(-1, \infty)$ and, in fact,

$$\lim_{x\to -1^+} \frac{x^2}{\sqrt{x+1}} = \infty,$$

so f has a vertical asymptote at $x = -1$. This represents the end behavior on the left. For the right end behavior, consider the limit $\displaystyle\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}}$. This limit has the indeterminant form $\dfrac{\infty}{\infty}$. By l'Hospital's Rule,

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{2x}{1/2\sqrt{x+1}} = \lim_{x\to\infty} 4x\sqrt{x+1} = \infty.$$

The first derivative can be computed:

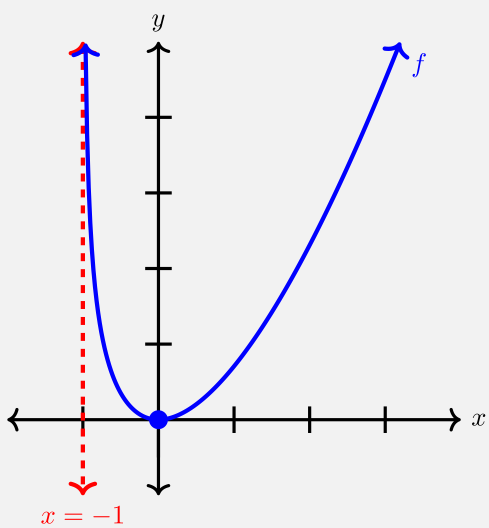
$$\begin{aligned} f'(x) &= \left(\frac{x^2}{\sqrt{x+1}}\right)' = \frac{\sqrt{x+1}(2x) - x^2(1/2\sqrt{x+1})}{x+1} \\ &= \frac{(x+1)(4x) - x^2}{2(x+1)\sqrt{x+1}} = \frac{4x^2 + 4x - x^2}{2(x+1)^{3/2}} \\ &= \frac{3x^2 + 4x}{2(x+1)^{3/2}} = \frac{x(3x+4)}{2(x+1)^{3/2}} \end{aligned}$$

Therefore, the only critical numbers of f is $x = 0$. We might be tempted to also include $x = -4/3$ and -1 as critical numbers, but remember that these numbers are outside the domain of f. Next, we compute the second derivative.

$$\begin{aligned} f''(x) &= \left(\frac{3x^2 + 4x}{2(x+1)^{3/2}}\right)' = \frac{2(x+1)^{3/2}(6x+4) - (3x^2+4x)(3(x+1)^{1/2})}{4(x+1)^3} \\ &= \frac{(x+1)^{1/2}[2(x+1)(6x+4) - 3(3x^2+4x)]}{4(x+1)^3} = \frac{(12x^2 + 20x + 8) - (9x^2 + 12x)}{4(x+1)^{5/2}} \\ &= \frac{3x^2 + 8x + 8}{4(x+1)^{5/2}} \end{aligned}$$

Since $\sqrt{x} > 0$ for all x in its domain, we have that $4(\sqrt{x+1})^5 > 0$. Likewise, $3x^2 + 8x + 8 > 0$ for all x since its discriminant is $-32 < 0$. Since both the numerator and denominator are always positive, the second derivative is always positive. This implies that f is always concave upward. By the Second Derivative Test, $x = 0$ is a local minimum.

Putting this all together gives the graph below:



Example 4.7.2. Graph $f(x) = \dfrac{\ln x}{x^2}$.

The domain is all positive real numbers, that is, $x > 0$. So, $f(x)$ has no y-intercept. Now, $\ln x = 0 \Rightarrow x = 1$. So, $f(1) = 0$. We don't have to worry about symmetry since $f(x)$ is not defined for $x < 0$. We will use L'Hospital's Rule to determine the end behavior of $f(x)$. Now,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x^2} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(\ln x)'}{(x^2)'} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

So, $f(x)$ has a horizontal asymptote at $y = 0$. What then is the end behavior on the left? It doesn't make sense to ask $\lim\limits_{x\to-\infty} f(x)$. Instead, to determine the end behavior on the left, we ask $\lim\limits_{x\to 0^+} f(x)$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln x}{x^2} = \lim_{x\to 0^+} (\ln x)\frac{1}{x^2} = (-\infty)(\infty) = -\infty$$

So, $f(x)$ has a vertical asymptote at $x = 0$.

The first and second derivative can be computed:

$$f'(x) = (x^{-2}\ln x)' = (-2x^{-3})\ln x + (x^{-2})(1/x) = \frac{-2\ln x}{x^3} + \frac{1}{x^3} = \frac{1 - 2\ln x}{x^3}$$

and

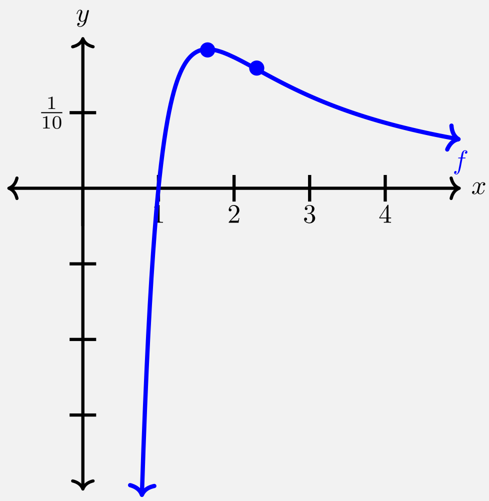
$$f''(x) = (x^{-3}(1-2\ln x))' = (-3x^{-4})(1-2\ln x) + (x^{-3})(-2/x) = \frac{-3+6\ln x}{x^4} + \frac{-2}{x^4} = \frac{6\ln x - 5}{x^4}.$$

Next, $f'(x) = 0 \Rightarrow 1 - 2\ln x = 0 \Rightarrow 2\ln x = 1 \Rightarrow \ln x = 1/2 \Rightarrow x = e^{1/2} \approx 1.65$. Likewise, $f''(x) = 0 \Rightarrow 6\ln x - 5 = 0 \Rightarrow 6\ln x = 5 \Rightarrow \ln x = 5/6 \Rightarrow x = e^{5/6} \approx 2.3$. We next construct the sign chart:

	$(0, 1.65)$	$(1.65, 2.3)$	$(2.3, \infty)$
$f'(x)$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$+$

Thus, $f(x)$ is increasing on the interval $(0, 1.65)$ and decreasing on the interval $(1.65, \infty)$. Likewise, $f(x)$ concaves downward on the interval $(0, 2.3)$ and concaves upward on $(2.3, \infty)$. So, $(1.65, f(1.65)) \approx (1.65, 0.18)$ is a maximum point and $(2.3, f(2.3)) \approx (2.3, 0.16)$ is a point of inflection.

Putting this all together gives the graph below:



Example 4.7.3. Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

To determine the domain, we solve the equation $2 + \sin x = 0 \Rightarrow \sin x = -2$, which has no solutions. Thus, $\operatorname{dom} f = (-\infty, \infty)$. Also, f has no discontinuities. We can also check that f is neither even nor odd. But f is periodic with period 2π. So it suffices to sketch the graph on the interval $[0, 2\pi]$. The periodicity of f also makes it silly to ask the end behavior of f.

Next, we compute

$$f(0) = \frac{\cos 0}{2 + \sin 0} = \frac{1}{2}.$$

Also,

$$f(x) = \frac{\cos x}{2 + \sin x} = 0 \Rightarrow \cos x = 0,$$

which implies that f has 2 x-intercepts $x = \pi/2, 3\pi/2$ on the interval $[0, 2\pi]$.

Next we compute

$$f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x(\cos x)}{(2 + \sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} = \frac{-1 - 2\sin x}{(2 + \sin x)^2}.$$

Solve for $-1 - 2\sin x = 0$, we get that the critical numbers are $x = 7\pi/6,\ 11\pi/6$.

Next we compute

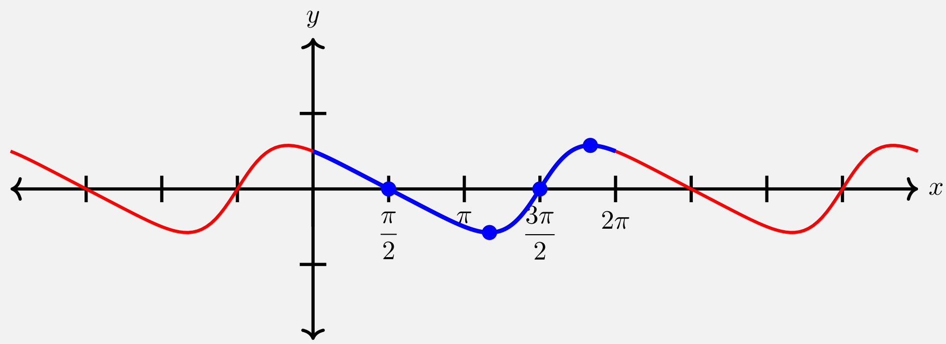
$$\begin{aligned} f''(x) &= \frac{(2 + \sin x)^2(-2\cos x) - (-1 - 2\sin x)2(2 + \sin x)(\cos x)}{(2 + \sin x)^4} \\ &= \frac{(2 + \sin x)[(2 + \sin x)(-2\cos x) + (1 + 2\sin x)(2\cos x)}{(2 + \sin x)^4} \\ &= \frac{-4\cos x - 2\sin x\cos x + 2\cos x + 4\sin x\cos x}{(2 + \sin x)^3} \end{aligned}$$

$$= \frac{-2\cos x + 2\sin x\cos x}{(2+\sin x)^3} = \frac{2\cos x(\sin x - 1)}{(2+\sin x)^3}.$$

Thus, $f''(x) = 0$ when $\cos x = 0 \Rightarrow x = \pi/2,\ 3\pi/2$ and $\sin x - 1 = 0 \Rightarrow x = \pi/2$. By the Second Derivative Test,

	$(0, \pi/2)$	$(\pi/2, 7\pi/6)$	$(7\pi/6, 3\pi/2)$	$(3\pi/2, 11\pi/6)$	$(11\pi/6, 2\pi)$
$f'(x)$	$-$	$-$	$+$	$+$	$-$
$f''(x)$	$-$	$+$	$+$	$-$	$-$

Putting this all together gives the graph below:



[i]See §4.6 Limits at Infinity and Asymptotes in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–14, produce the graph of the function f that reveals all the important aspects of the curve. In particular, you should indicate the domain of f and label its intercepts, symmetry, discontinuities, end behavior, local extrema, and points of inflections, if present. You should use the first and second derivative to determine the appropriate parts of this curve sketch.

1. $f(x) = \dfrac{x}{\sqrt{x^2-1}}$

♠ 2. $f(x) = 2\sqrt{x} - x$

♠ 3. $f(x) = \sqrt{x^2+4x} - x$

♠ 4. $f(x) = \dfrac{x}{\sqrt{x^2+5}}$

5. $f(x) = x\sqrt{3-x^2}$

♠ 6. $f(x) = x + \sin x$

7. $f(x) = \tan x$

♠ 8. $f(x) = \dfrac{\sin x}{1+\cos x}$

♠ 9. $f(x) = e^{-x}\cos x$

♠ 10. $f(x) = (1+e^x)^{-2}$

11. $f(x) = x^2 e^x$

12. $f(x) = e^{1/x}$

♠ 13. $f(x) = x - \ln x$

14. $f(x) = \dfrac{\ln x}{x^2}$

Deeper Dive

Definition 4.7.4. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occur in probability, where it is called the **normal density function**. The constant μ is called the **mean** and the positive constant σ is called the **standard deviation**.

For Exercises 15–20, for simplicity, let us scale the normal density function so as to remove the factor $1/\sigma\sqrt{2\pi}$ and let us analyze the special case where $\mu = 0$. So, we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}.$$

15. Find all x- and y-intercepts of f, if any.
16. Determine if f has any symmetry.
17. Compute the end behavior of f by evaluating the limits $\lim_{x\to\pm\infty} f(x)$.
18. Compute $f'(x)$ and all critical numbers of f. Determine the intervals for which f is increasing, the intervals for which f is decreasing, and which critical points are a local extrema.
19. Compute $f''(x)$. Determine the intervals for which f is concave upward, the intervals for which f is concave downward, and all points of inflection of f.
20. Sketch the graph of f on the gridlines below using the value $\sigma = 1$. This graph is called the **standard normal curve**.

"I always like to look on the optimistic side of life, but I am realistic enough to know that life is a complex matter." – Walt Disney

Lecture Videos

Optimization

Maximizing the Product of Two Points on a Line

Finding the Minimal Distance between a Point and a Parabola

Finding a Maximum Rectangle in a Semicircle

4.8 Optimization

The methods we have learned for finding extreme values have practical applications in many areas of life. A business–person wants to minimize cost and maximize profits. A traveler wants to minimize transportation time. Fermat's Principle in optics states that light follows the path that takes the least time. In this section we solve such problems as maximizing areas, volumes, and profits and minimizing distances, time, and costs.

Solving an Optimization Problem

(a) Read the problem carefully. Make sure you understand what is given and what is unknown.

(b) If possible, sketch a diagram. Label the various parts.

(c) Decide on the variable that must be maximized or minimized. Express that variable as a function of *one* variable.

(d) Find the domain of the function.

(e) Find the critical points for the function.

(f) If the domain is a closed interval, evaluate the function at the endpoints and at each critical number to see which yields the absolute extremum. If the domain is open, evaluate the function at the critical numbers and also find the limit as the endpoints of the interval are approached to determine *if* an absolute extremum exists at one of the critical points.

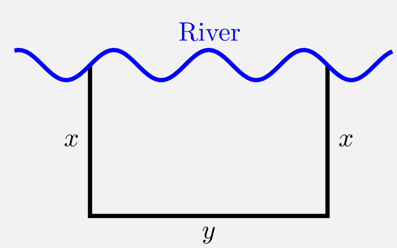
Example 4.8.1. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Let y be the dimension of the rectangle parallel to the river and let x denote the other dimension. Then the area of the rectangle is given by

$$A = xy.$$

But we are given a constraint of perimeter, in particular,

$$P = 2400 = 2x + y.$$



If we first experiment with different values of x and y, we get various possible areas:

x	y	A
100	2,200	220,000
700	1,000	700,000
1,000	400	400,000

Thus, finding the optimal values of x and y is not a trivial matter. We will next optimize the area by making $A = xy$ into a function of x. In particular, if $2400 = 2x + y$, then $y = 2400 - 2x$ and

$$A(x) = x(2400 - 2x) = 2400x - 2x^2.$$

We notice that for this problem, $x \in [0, 1200]$. Then we optimize:

$$\begin{aligned} A'(x) &= 2400 - 4x \\ 0 &= 2400 - 4x \\ 4x &= 2400 \\ x &= 600. \end{aligned}$$

So, $x = 600$ is the unique critical number. To solve the problem we investigate the function at the endpoints of the domain and at $x = 600$.

x	A
0	0
600	720,000
1,200	0

Therefore, $x = 600$ is optimal and the rectangle's largest area $720,000$ ft^2 is obtained when the field is $\boxed{\text{600 ft by 1200 ft}}$.

Example 4.8.2. Find two nonnegative numbers x and y for which $2x + y = 30$, such that xy^2 is maximized.

We wish to maximize the function $M = xy^2$. We need to remove one of the variables. For simplicity, we eliminate y using the substitution $2x + y = 30 \Rightarrow y = 30 - 2x$, given by the constraint. So, $M = x(30 - 2x)^2$. Since x and y are nonnegative, we have the inequalities $x \geq 0$ and $y = 30 - 2x \geq 0 \Rightarrow -2x \geq -30 \Rightarrow x \leq 15$.

Next, we search for the absolute maximum of M on the interval $[0, 15]$. Taking the derivative,

$$\begin{aligned} \frac{dM}{dx} &= (30 - 2x)^2 + x[2(30 - 2x)(-2)] \\ &= (30 - 2x)^2 - 4x(30 - 2x) \\ &= (30 - 2x)((30 - 2x) - 4x) \\ &= (30 - 2x)(30 - 6x) \\ &= 2(15 - x)6(5 - x) \\ &= 12(15 - x)(5 - x). \end{aligned}$$

Therefore, the critical numbers are $x = 5, 15$. We next evaluate M at the critical numbers and at the endpoints.

x	M
0	0
5	2,000
15	0

Therefore, the maximum value of M is 2000 and is obtained when $\boxed{x = 5 \text{ and } y = 20}$.

Example 4.8.3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

The distance between the point $(1, 4)$ and (x, y) is given by the distance formula

$$d = \sqrt{(x-1)^2 + (y-4)^2}.$$

In order to eliminate one of the variables we use the constraint given by the parabola $y^2 = 2x \Rightarrow x = \dfrac{y^2}{2}$. Then

$$d = \sqrt{(y^2/2-1)^2 + (y-4)^2}.$$

Since y can be any real number, we will use the first derivative test to determine the minimum. In order to simplify computations, we will minimize d^2, that is, if $f(y) = d^2$, we want to minimize the function

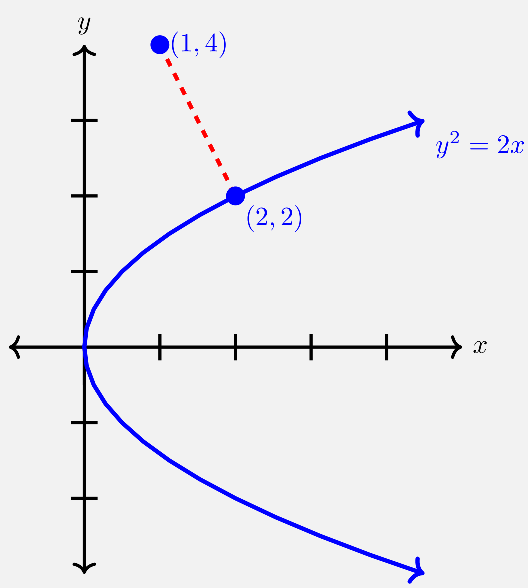
$$f(y) = (y^2/2-1)^2 + (y-4)^2.$$

Taking the derivative with respect to y, we have

$$\begin{aligned} f'(y) &= 2(y^2/2-2)(2y/2) + 2(y-4) \\ &= (y^2-2)(y) + (2y-8) \\ &= y^3 - 2y + 2y - 8 \\ &= y^3 - 8 \\ &= (y-2)(y^2+2y+4). \end{aligned}$$

Therefore, our critical number is $y = 2$.

	$(-\infty, 2)$	$(2, \infty)$
$f'(x)$	$-$	$+$



Therefore, $y = 2$ is the minimal number for $f(y)$. Then $x = (2)^2/2 = 2$. Therefore, $\boxed{(2,2)}$ is the closest point on the parabola $y^2 = 2x$ to the point $(1, 4)$ and the distance between them is $d = \sqrt{(2-1)^2 + (2-4)^2} = \sqrt{1+4} = \sqrt{5}$.

Example 4.8.4. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Let θ be the angle formed between the x-axis and the vertex of the rectangle in the first quadrant which rests on the circumference of the semicircle. Using trigonometry, the height of the rectangle is given by $r\sin\theta$ and width of the rectangle is given by $2r\cos\theta$. Therefore,

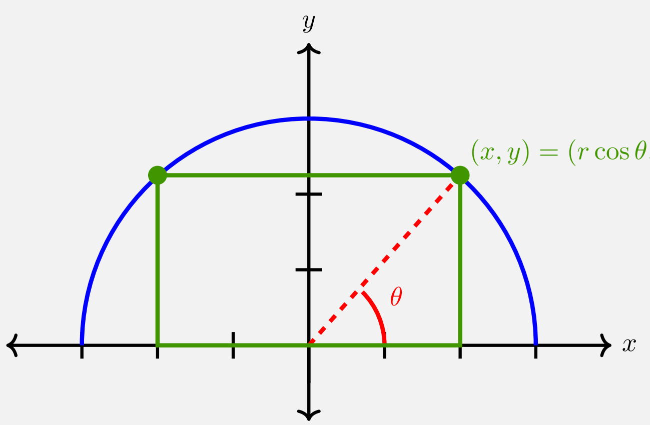
$$A(\theta) = (r\sin\theta)(2r\cos(\theta)) = r^2\sin(2\theta).$$

Notice that $0 \le \theta \le \pi/2$. Thus,

$$A'(\theta) = 2r^2\cos(2\theta)$$

and

$$\begin{aligned} 2r^2\cos(2\theta) &= 0 \\ \cos(2\theta) &= 0 \\ 2\theta &= \frac{\pi}{2} + \pi k \\ \theta &= \frac{\pi}{4} + \frac{\pi}{2}k = \frac{\pi}{4}, \frac{\pi}{2} \end{aligned}$$



Therefore, $\theta = \pi/4$ is optimal, that is, the rectangle should have dimensions $r\sin(\pi/4) = \boxed{\frac{r\sqrt{2}}{2}}$ and $2r\cos(\pi/4) = \frac{r\sqrt{2}}{2} = \boxed{r\sqrt{2}}$.

[i]See §4.7 Applied Optimization Problems in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–17, find the optimal value of the described problem.

1. The minimum product of two numbers whose difference is 50.
♠ 2. The maximum product of two numbers whose sum is 23.
♠ 3. The minimum product of two numbers whose difference is 48.
♠ 4. The minimum sum of two positive numbers whose product is 64.
5. The maximum area of a rectangle with perimeter of 12 cm.
6. The maximum area of a rectangle with perimeter of 16 cm.
♠ 7. The maximum area of a rectangle with perimeter 60 m.
♠ 8. The minimum perimeter of a rectangle with area 216 m^2.
9. The minimum perimeter of a rectangle with area of 25 cm^2.
10. The minimum perimeter of a rectangle with area of 100 cm^2.
♠ 11. The maximum vertical distance between the line $y = x + 56$ and the parabola $y = x^2$ if $-7 \leq x \leq 8$.
♠ 12. The closest point on the line $y = 2x + 4$ to the origin.
♠ 13. The farthest point on the ellipse $4x^2 + y^2 = 4$ from the point $(1, 0)$.
♠ 14. The maximum area of a rectangle inscribed in a circle of radius r.
♠ 15. The maximum area of a rectangle inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
♠ 16. The maximum area of a rectangle inscribed in a right triangle with legs of lengths 3 cm and 8 cm if two sides of the rectangle lie along the legs.
17. The largest possible product a^2b, where a and b are two positive integers whose sum is 30.

"If you optimize everything, you will always be unhappy." – Donald Knuth

Lecture Videos

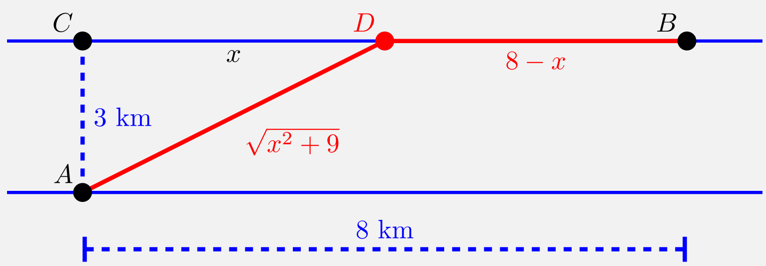
Finding Minimum Distance of a Path

Finding Minimum Distance of a Path Reprise

Finding the Maximum Volume of a Box

4.9 Optimization II

Example 4.9.1. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?



Let x denote the distance between C and D, then the distance the man must run will be

$$|CD| = 8 - x$$

and the distance he must row is

$$|AD| = \sqrt{x^2+9}.$$

Certainly, $0 \leq x \leq 8$. We know that distance = (rate)(time), which implies that

$$\text{time} = \frac{\text{distance}}{\text{rate}}.$$

Hence, we need to minimize the equation

$$T(x) = \frac{8-x}{8} + \frac{\sqrt{x^2+9}}{6}.$$

We take the derivative:

$$\begin{aligned}
T'(x) &= \frac{-1}{8} + \frac{1}{6}\left(\frac{1}{2}\right)(x^2+9)^{-1/2}(2x) = -\frac{1}{8} + \frac{x}{6\sqrt{x^2+9}} \\
0 &= -\frac{1}{8} + \frac{x}{6\sqrt{x^2+9}} \\
\frac{1}{8} &= \frac{x}{6\sqrt{x^2+9}} \\
6\sqrt{x^2+9} &= 8x \\
9(x^2+9) &= 16x^2 \\
81 &= 7x^2
\end{aligned}$$

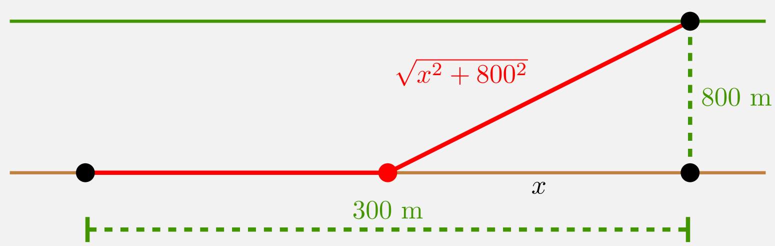
$$\begin{aligned} x^2 &= \frac{81}{7} \\ x &= \sqrt{\frac{81}{7}} = \frac{9}{\sqrt{7}} \approx 3.40168. \end{aligned}$$

Computing we have

x	T
0	$3/2 = 1.5$
$9/\sqrt{7}$	$1 + \sqrt{7}/8 \approx 1.33$
8	$\sqrt{73}/6 \approx 1.42$

Therefore, the minimal time is obtained when $\boxed{x = 9/\sqrt{7} \approx 3.4}$, that is, the man should row downstream 3.4 km and run the rest of the way.

Example 4.9.2. A Boy Scout participating in the sport of orienteering must get to a specific tree in the woods as fast as possible. He can get there by traveling east along the trail for 300 m and then north through the woods for 800 m. He can run 160 m per minute along the trail but only 70 m per minute through the woods. Running directly through the woods toward the tree minimizes the distance, but he will be going slowly the whole time. Find the path that will get him to the tree in the minimum time.



Let x denote the dimension illustrated in the diagram. By dimension considerations, $0 \le x \le 300$. Again,

$$\text{time} = \frac{\text{distance}}{\text{rate}}.$$

So, we can construct a function of time with respect to x:

$$T(x) = \frac{300 - x}{160} + \frac{\sqrt{x^2 + 800^2}}{70}$$

and

$$T'(x) = -\frac{1}{160} + \frac{1}{70}\left(\frac{1}{2\sqrt{x^2+800^2}}\right)(2x) = \frac{x}{70\sqrt{x^2+800^2}} - \frac{1}{160}.$$

To find the critical numbers, we check

$$\begin{aligned} T'(x) &= 0 \\ \frac{x}{70\sqrt{x^2+800^2}} - \frac{1}{160} &= 0 \\ \frac{x}{70\sqrt{x^2+800^2}} &= \frac{1}{160} \\ 160x &= 70\sqrt{x^2+800^2} \end{aligned}$$

$$16x = 7\sqrt{x^2+800^2}$$

$$\begin{aligned} 256x^2 &= 49(x^2 + 800^2) \\ 207x^2 &= 49 \cdot 800^2 \\ x^2 &= \frac{49 \cdot 800^2}{207} \\ x &= \frac{7 \cdot 800}{\sqrt{207}} \approx 389. \end{aligned}$$

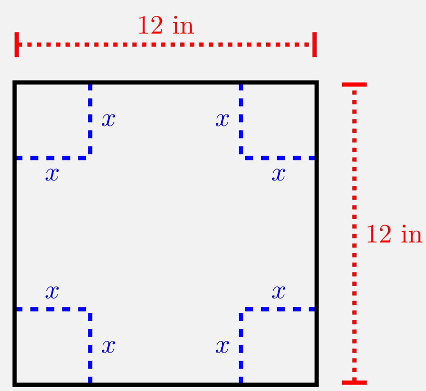
Now, 389 is not in the interval $[0, 300]$. So, one of the endpoints must be the absolute minimum.

x	T
0	13.3
300	12.21

We see from the table that the time is minimized when $\boxed{x = 300}$. Thus, the scout should race straight to the tree.

Example 4.9.3. An open box is made by cutting a square from each corner of a 12-in by 12-in piece of metal and then folding up the sides. What size square should be cut from each corner to produce a box of maximum volume?

Let x denote the dimension illustrated in the diagram. Certainly, $x \geq 0$. On the other hand, $12 - 2x \geq 0 \Rightarrow x \leq 6$. The volume of a rectangular prism is given by $V = \ell wh$. By the way we are folding the sheet, we see that $\ell = w = 12 - 2x$ and $h = x$. Thus,



$$V(x) = x(12 - 2x)^2 = 4x(6 - x)^2.$$

In order to find the critical values, we determine

$$\begin{aligned} V'(x) &= 4\big[(6-x)^2 - 2(6-x)x\big] \\ &= 4(6-x)(6-x-2x) \\ &= 4(6-x)(6-3x) \\ &= 12(6-x)(2-x) \end{aligned}$$

Thus, the critical numbers are $x = 6, 2$. Using the Extreme Value Theorem, we see that

x	V
0	0
2	128
6	0

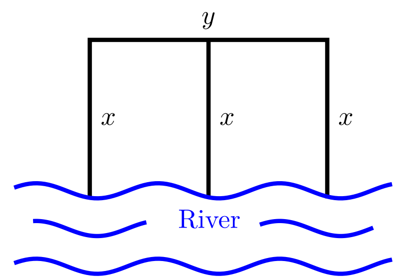
Therefore, the absolute maximum is $(2, 128)$. So, a $\boxed{2 \text{ in} \times 2 \text{ in}}$ square should be cut from each corner. This will create a box with dimensions 2 in $\times$ 8 in $\times$ 8 in with volume 128 in^3.

[i] See §4.7 Applied Optimization Problems in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

1. Consider a farmer who has 1200 ft of fencing and wants to fence off a rectangular field that borders a straight river. He also wants to divide the field into two pens with fencing perpendicular to the river, so that both pens have access to the river. He does not need a fence along the river (See the diagram to the right). Find the largest possible area of the field.



♠ 2. Consider a farmer who wants to fence off a rectangular field that that is 24 million square feet (about 550 acres). The field is divided in half by fencing down the middle, parallel to one side of the rectangular field. Find the dimensions of the field that has the smallest perimeter, thus minimizing the cost of fencing.

♠ 3. Consider an open box with a square base and a volume of 4000 cm^3. Find the minimum surface area of the box, thus minimizing cost of construction.

♠ 4. Consider an open box which has a volume of 10 m^3. The length of this base is twice the width. Material for the base costs 5/m^2, but material for the sides costs 3/m^2. Find the minimum cost of constructing such a box.

5. Consider a box with a square base and no top which needs to be constructed from 75 ft^2 of material. Find the largest possible volume of the box.

6. Consider a rectangular box with a square bottom and an open top which has a volume of 10 cubic inches. Material for the base costs 5 cents per square inch and material for the sides costs 2 cents per square inch. Find the cost of the cheapest such box.

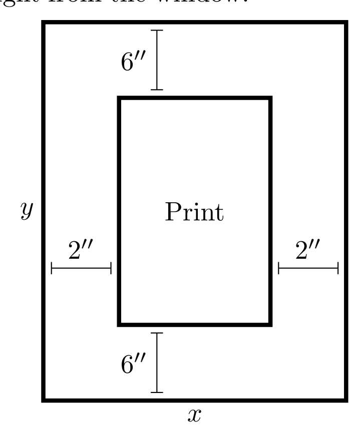
7. Consider a rectangular box with a square bottom and an open top which has a volume of 10 m^3. Material for the base costs \$10 per square meter and material for the sides costs \$4 per square meter. Find the cost of the cheapest such box.

8. Consider a can which is made to contain 1000π cm^3 of liquid. Also consider that it costs 5 cents per cm^2 for the metal used on the top and bottom of the can, but it costs 10 cents per cm^2 for the metal on the side. Find the exact cost of the cheapest such can.

♠ 9. Consider a cylinder which is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

♠ 10. Consider a Norman window, shaped as the union of a rectangle and semicircle such that the width of the rectangle is the diameter of the semicircle. If the perimeter of this window is 24 ft, find the width in order to maximize the area, thus maximizing the greatest possible amount of light from the window.

11. Consider the top and bottom margins of a poster are each 6 inches and the side margins are each 2 inches. If the area of the poster is fixed at 300 in^2, find the dimensions of the poster that will maximize the area of the printed material.



♠ 12. Consider a poster with printed area of 384 cm^2 and margins 6 cm on the top and bottom but 4 cm on the sides. Find the dimensions of the poster with smallest area.

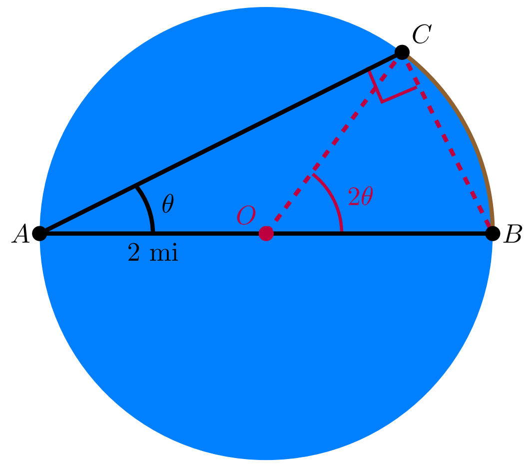
♠ 13. Consider a piece of wire, 7 m long, which is cut into two pieces: one piece is bent into a square and the other piece is bent into an equilateral triangle. Find the perimeter of this square so that the total area, square and triangle together, is maximum. Find the perimeter which is minimum.

♠ 14. Consider an 8-ft-tall fence 4 ft away from a tall building. Find the length of the shortest ladder that will reach from the ground to the side of the building, going over the fence.

♠ 15. Consider a conical paper drinking cup which can hold 33 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

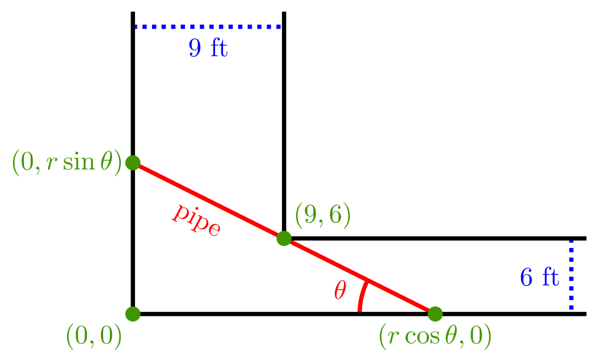
♠ 16. Consider again Example 4.9.1 such that the river is now 4 km wide and B is 3 km downstream from A. Where should he land to reach B as soon as possible?

♠ 17. Consider a woman at a point A on the shore of a circular lake with radius 2 mi. She wants to arrive at the point B diametrically opposite A on the other side of the lake. She could row her boat directly across the lake to point B, she could walk around the lake to B, or she could row to some point C on the other side of the lake and then walk the rest of the way to B. If she can row 2 mph and walk 4 mph, find the shortest time required for her to reach B.



18. Consider a steel pipe of length r ft to be carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. Find the length of the longest pipe that can be carried horizontallly around the corner.

Note that the longest pipe length that can pass through the corner is the minimal *length that the line segment illustrated can be.*



Definition 4.9.4. The **revenue** function is the price function times the number of units sold x, that is, $R(x) = xp(x)$, and the **profit** function is the total revenue minus total cost, that is, $P(x) = R(x) - C(x)$.

19. Consider a company that has a total cost function, in millions of dollars, of $C(x) = x^2 + x + 3$. The company's price function is also a function of how many ten thousand units sold, x, and is given by $p(x) = 25 - 2x$, also in millions of dollars. Find the maximum profit.

20. Consider a company which produces x items per month where $x \geq 0$. If $C(x) = 75000 + 60x + 0.1x^2$ is the cost of producing x items and $R(x) = 330x$ is the revenue generated from selling those items, find the value of x which obtains the maximum profit.

"A man only becomes wise when he begins to calculate the approximate depth of his ignorance."
– Gian Carlo Menotti

Lecture Videos

Tangent Line Approximation

Newton's Method

4.10 Newton's Method

Suppose f is a differentiable function at $x = a$. The tangent line at $x = a$ is given by

$$\begin{aligned} y - f(a) &= f'(a)(x-a) \\ L(x) &= f(a) + f'(a)(x-a). \end{aligned} \tag{4.10.1}$$

We will call L the **linearization** of f at $x = a$.

If $x \approx a$, then $f(x) \approx L(x)$. The process of approximating a quantity or function using the tangent is known as **linear approximation** or **tangent line approximation**.

Example 4.10.1. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the number $\sqrt{3.98}$ and $\sqrt{4.05}$.

We first will compute the derivative of f.

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}.$$

So, $f'(1) = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{2(2)} = \dfrac{1}{4}$. Likewise, $f(1) = \sqrt{1+3} = \sqrt{4} = 2$. Thus,

$$\begin{aligned} L(x) &= f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) \\ &= 2 + \frac{x}{4} - \frac{1}{4} = \frac{x}{4} + \frac{7}{4}. \end{aligned}$$

Therefore, for x close to 1, we have

$$\boxed{\sqrt{x+3} \approx \frac{x}{4} + \frac{7}{4}}$$

when x is close to 1. In order to approximate $\sqrt{3.98}$, notice that $\sqrt{3.98} = \sqrt{0.98+3} = f(0.98) \approx L(0.98)$. Thus,

$$\sqrt{3.98} \approx L(0.98) = \frac{0.98}{4} + \frac{7}{4} = \frac{7.98}{4} = \boxed{1.995}.$$

Similarly, $\sqrt{4.05} = \sqrt{1.05+3} = f(1.05) \approx L(1.05)$. Thus,

$$\sqrt{4.05} \approx L(1.05) = \frac{1.05}{4} + \frac{7}{4} = \frac{8.05}{4} = \boxed{2.0125}.$$

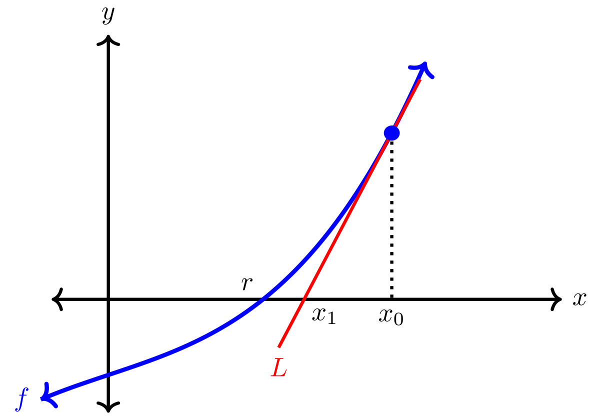
Since f concaves downward at $a = 1$, the tangent line is above the function. Therefore, these approximations are over-estimates. According to a calculator,

$$\sqrt{3.98} \approx 1.994993734$$

and

$$\sqrt{4.05} \approx 2.01246118.$$

We will now develop another approximation method used in calculating x-intercepts of functions. Consider a function f is given and f has an x-intercept at $x = r$, that is, $f(r) = 0$. Although we do not know what r is, we can make a guess. Suppose that x_0 is our first guess, which is probably incorrect. We will construct the tangent line of f at x_0. Notice that the tangent will intersect the x-axis somewhere closer to r (if x_0 was a good guess of r). Let this x-intercept of the tangent line be called x_1. Now x_1 is a better guess to the value r.



We know that the linearization of f at $x = x_0$ has the form

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

and is the equation for the tangent line of f at x_0. To find the x-intercept of L, we set $y = 0$ and solve for $x = x_1$:

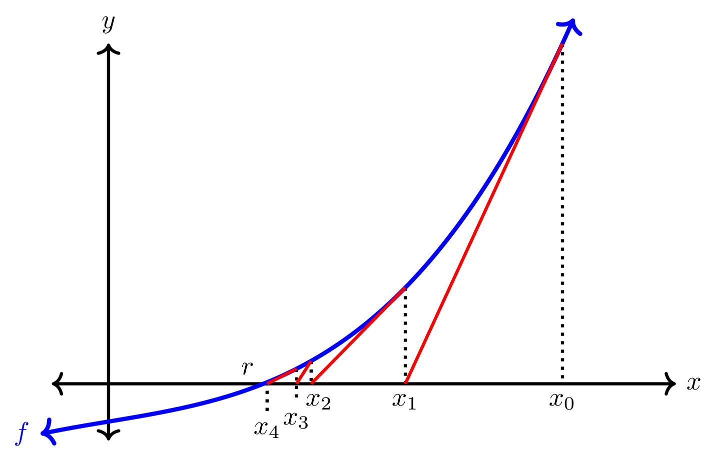
$$\begin{aligned} f(x_0) + f'(x_0)(x_1 - x_0) &= 0 \\ f'(x_0)(x_1 - x_0) &= -f(x_0) \\ x_1 - x_0 &= -\frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)}. \end{aligned}$$

To obtain a better approximation we can repeat this process, that is, construct the tangent of f at x_1, intersect it with the x-axis, define x_2 to be the x-intercept of the tangent line of x_1. By repeating this process, we get **Newton's Method**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{4.10.2}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n\to\infty} x_n = r.$$



Example 4.10.2. Starting with $x_0 = 2$, find the second approximation x_2 to the root of the equation $x^3 - 2x - 5 = 0$.

Let $f(x) = x^3 - 2x - 5$. To apply Newton's Method we need first the derivative. Note

$$f'(x) = 3x^2 - 2.$$

We next must compute $f(x_0) = f(2)$ and $f'(x_0) = f'(2)$.

$$\begin{aligned} f(2) &= (2)^3 - 2(2) - 5 = 8 - 4 - 5 = -1 \\ f'(2) &= 3(2)^2 - 2 = 12 - 2 = 10 \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

$$= \quad 2 - \frac{-1}{10} = \frac{21}{10} = \boxed{2.1}$$

Repeating we get,

$$\begin{aligned} f(2.1) &= (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 = 0.061 \\ f'(2.1) &= 3(2.1)^2 - 2 = 3(4.41) - 2 = 13.23 - 2 = 11.23 \\ x_2 &= 2.1 - \frac{0.061}{11.23} \approx 2.1 - 0.0054 = \boxed{2.0946} \end{aligned}$$

It turns out that this third approximation, $x_2 \approx 2.0946$, is accurate to four decimal places.

Example 4.10.3. Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

As a rule of thumb, we can stop when successive approximations x_n and x_{n+1} agree to on the appropriate number of digits, in our case, eight.

In order to approximate $\sqrt[6]{2}$, we note that $\sqrt[6]{2}$ is a root of the function $f(x) = x^6 - 2$. Since $f'(x) = 6x^5$, Newton's method provides the formula

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

for approximation. If we set $x_0 = 1$, then we obtain

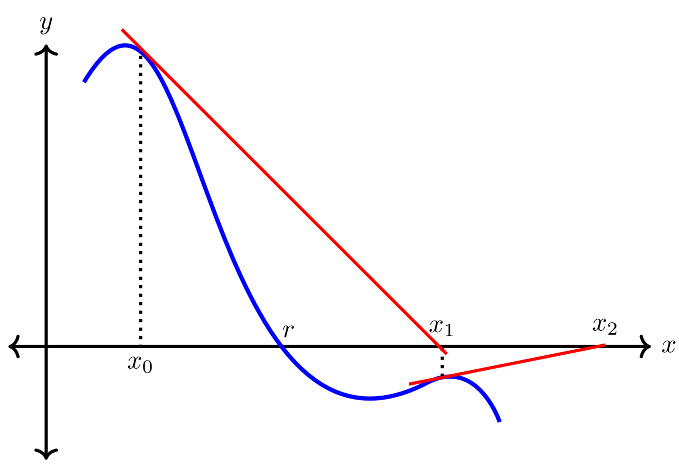
$$\begin{aligned} x_1 &\approx 1.16666667 \\ x_2 &\approx 1.12644368 \\ x_3 &\approx 1.12249707 \\ x_4 &\approx 1.12246205 \\ x_5 &\approx 1.12246205. \end{aligned}$$

Since x_4 and x_5 agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx \boxed{1.2246205},$$

accurate to eight decimal places.

We will not prove the convergence of $\lim\limits_{n\to\infty} x_n$ at this time, but let it be known that the convergence depends on our choice of x_0. That is, if x_0 is chosen too poorly, then the sequence $\{x_n\}$ will not converge to r, that is, Newton's Method will NOT approximate the root. For example, if x_0 is a local extremum, then x_1 does not exist. Another problem is illustrated to below.



[i]See §4.2 Linear Approximation and Differentials and §4.9 Newton's Method in OpenStax to find the corresponding sections.

Exercises

(Go to Solutions)

For Exercises 1–13, find the linearization $L(x)$ of the function f at $x = a$. The the linearization to approximate the given value(s).

1. $f(x) = \sqrt{x}$, $a = 9$; $\sqrt{10}$
2. $f(x) = \sqrt{x}$, $a = 64$; $\sqrt{65}$
3. ♠ $f(x) = x^4 + 6x^2$, $a = 1$; $f(1.1)$
4. $f(x) = \sqrt[3]{x}$, $a = 8$; $\sqrt[3]{7}$
5. $f(x) = \sqrt[4]{x}$, $a = 16$; $\sqrt[4]{17}$
6. ♠ $f(x) = x^{2/3}$, $a = 8$; $\sqrt[3]{65.61}$
7. ♠ $f(x) = \sqrt{4 - x}$, $a = 0$; $\sqrt{3.9}, \sqrt{3.99}$
8. ♠ $f(x) = \sqrt[3]{1 + x}$, $a = 0$; $\sqrt[3]{0.95}, \sqrt[3]{1.1}$
9. $f(x) = \sqrt{x^3 + 1}$, $a = 2$; $\sqrt{10.261}$
10. $f(x) = xe^{x-2}$, $a = 2$; $(2.1)e^{0.1}$
11. $f(x) = \sin x$, $a = 0$; $\sin(0.1)$
12. $f(x) = \cos x$, $a = \dfrac{\pi}{6}$; $\cos\left(\dfrac{1}{2}\right)$
13. $f(x) = \tan x$, $a = \dfrac{\pi}{4}$; $\tan\left(\dfrac{\pi}{5}\right)$

For Exercises 14–24, use Newton's method on the function f with the initial value x_0 to find x_2 (accurate to five decimal places).

The following Calculator Online calculator may prove helpful

The following Desmos calculator may prove helpful

14. $f(x) = x^2 - 2$, $x_0 = 2$
15. $f(x) = x^3 - 4$, $x_0 = 1$
16. ♠ $f(x) = x^3 + x + 3$, $x_0 = -1$
17. $f(x) = x^3 + x + 1$, $x_0 = 1$
18. $f(x) = x^3 - x^2 - 4x - 2$, $x_0 = 2$
19. ♠ $f(x) = x^4 - x - 3$, $x_0 = 1$
20. $f(x) = x^4 - x^3 - 2x^2 - 6x - 4$, $x_0 = 2$
21. $f(x) = x^5 - x - 1$, $x_0 = 1$
22. ♠ $f(x) = x^5 - x - 1$, $x_0 = 1$
23. ♠ $f(x) = \dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 + 10 = 0$, $x_0 = -3$
24. $f(x) = x^7 + 4$, $x_0 = 1$

For Exercises 25–28, use Newton's method to estimate the value (accurate to eight decimal places).

25. ♠ $\sqrt[5]{23}$
26. ♠ $\sqrt[101]{101}$
27. ♠ the root of $x^4 - 2x^3 + 6x^2 - 8$ on $[1, 2]$
28. ♠ the root of $2.2x^5 - 4.4x^3 + 1.3x^2 - 0.2x - 1.3$ on $[-2, -1]$

For Exercises 29–30, given $f(a)$, $f'(a)$, and x_0, using Newton's method find x_1?

29. $f(2) = 3$, $f'(2) = 1$, $x_0 = 2$
30. $f(5) = 1$, $f'(5) = 2$, $x_0 = 5$

For Exercises 31–32, if Newton's method is used to estimate a root of the function f, what is the relation between x_n and x_{n+1}?

31. $f(x) = x^2 - x - 10$

32. $f(x) = e^x - 7x$

"You can't undo the past... but you can certainly not repeat it." – Bruce Willis

Lecture Videos

What is an Antiderivative?

The Power Rule for Antiderivatives

Linearity Property of Antiderivatives

Basic Antiderivatives

Initial Value Problem for Antiderivatives

4.11 Antiderivatives

Functions used in applications in previous sections have provided information about a *total amount* of a quantity, such as cost, revenue, profit, or distance. Derivatives of these functions provided information about the rate of change of these quantities and allowed us to answer important questions about the extrema of the functions. It is not always possible to find ready-made functions that provide information about the total amount of a quantity, but it is often possible to collect enough data to come up with a function that gives the *rate of change* of a quantity. For example, if an asteroid is flying through space, initially we would not have a function which told us where the asteroid would be at any point in time, but we can measure its speed and trajectory (aka velocity) and use this to construct a position function for the asteroid. We know that derivatives give the rate of change when the total amount is known. The reverse of finding a derivative is known as **antidifferentiation**.

Definition 4.11.1. If $F'(x) = f(x)$, then $F(x)$ is an **antiderivative** of $f(x)$.

Example 4.11.2. If $F(x) = 10x$, then $F'(x) = 10$, so $\boxed{F(x) = 10x}$ is an antiderivative of $f(x) = 10$.

Likewise, if $F(x) = x^2$, then $F'(x) = 2x$, making $\boxed{F(x) = x^2}$ an antiderivative of $f(x) = 2x$.

Even more, if $\boxed{G(x) = x^2 + 2}$, then $G'(x) = 2x$ and $G(x)$ is also an antiderivative of $f(x)$. Also, $\boxed{H(x) = x^2 - 7}$ is another antiderivative of $f(x)$. So, antiderivatives need not be unique.

Even though a function can have multiple antiderivatives, they can only differ by their y-intercepts, that is, any two antiderivatives are vertical translates of each other.

Theorem 4.11.3. *If $F(x)$ and $G(x)$ are both antiderivatives of a function $f(x)$ on an interval, then there is a constant C such that*

$$F(x) - G(x) = C.$$

Definition 4.11.4. The family of all antiderivatives of the function f is indicated by

$$\int f(x)\,dx.$$

The symbol $\int$ is the **integral sign**, $f(x)$ is the **integrand**, and $\int f(x)\,dx$ is called the **indefinite integral**, the most general antiderivative of f.

Corollary 4.11.5. *If $F'(x) = f(x)$, then*

$$\int f(x)\,dx = F(x) + C,$$

for any real number C.

Example 4.11.6. $\int 2x\,dx = \boxed{x^2 + C}$.

Proposition 4.11.7 (Power Rule)**.** *For any real number $n \neq -1$,*

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Proof.

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{1}{n+1}\left(\frac{dx^{n+1}}{dx}\right) + \frac{dC}{dx} = \frac{n+1}{n+1}x^n + 0 = x^n.$$

□

Example 4.11.8. Use the power rule to find each indefinite integral.

(a) $\int t^3\,dt$

$$\int t^3\,dt = \frac{t^{3+1}}{3+1} + C = \boxed{\frac{t^4}{4} + C}.$$

(b) $\int \frac{1}{t^2}\,dt$

$$\int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = \frac{t^{-1}}{-1} + C = \boxed{-\frac{1}{t} + C}.$$

(c) $\int \sqrt{u}\,du$

$$\int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \boxed{\frac{2}{3}u^{3/2} + C}.$$

(d) $\int dx$

$$\int dx = \int 1\,dx = \int x^0\,dx = \frac{1}{1}x^1 + C = \boxed{x + C}.$$

Proposition 4.11.9 (Constant Multiple Rule and Sum or Difference Rule)**.** *If all indicated integral exist,*

$$\int k \cdot f(x)\,dx = k\int f(x)\,dx, \quad \textit{for any real number } k$$

and

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx,$$

In fact, every rule of differentiation can become a rule for antidifferentiation!

Example 4.11.10. Use the rules to find each integral.

(a) $\int 2v^3\,dv$.

$$\int 2v^3\,dv = 2\int v^3\,dv = 2\left(\frac{v^4}{4}\right) + C = \boxed{\frac{1}{2}v^4 + C}.$$

(b) $\int \frac{12}{z^5}\,dz$.

$$\int \frac{12}{z^5}\,dz = \int 12z^{-5}\,dz = 12\int z^{-5}\,dz = 12\left(\frac{z^{-4}}{-4}\right) + C = \boxed{\frac{-3}{z^4} + C}.$$

(c) $\int (3z^2 - 4z + 5)\,dz$.

$$\begin{aligned} \int ((3z^2 - 4z + 5)\,dz &= \int 3z^2\,dz - \int 4z\,dz + \int 5\,dz = 3\int z^2\,dz - 4\int z\,dz + 5\int dz \\ &= 3\left(\frac{1}{3}z^3\right) - 4\left(\frac{1}{2}z^2\right) + 5z + C = \boxed{z^3 - 2z^2 + 5z + C} \end{aligned}$$

(d) $\int \frac{x^2+1}{\sqrt{x}}\,dx = \int \left[\frac{x^2}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right]dx = \int \left[\frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}}\right]dx = \int \left[x^{3/2} + x^{-1/2}\right]dx = \boxed{\frac{2}{5}x^{5/2} + 2x^{1/2} + C}$

(e) $\int (x^2 - 1)^2\,dx = \int (x^4 - 2x^2 + 1)\,dx = \boxed{\frac{1}{5}x^5 - \frac{2}{3}x^3 + x + C}.$

(f) $\int 9e^t\,dt = 9\int e^t\,dt = \boxed{9e^t + C}.$

(g) $\int \frac{4}{x}\,dx = 4\int \frac{dx}{x} = \boxed{4\ln|x| + C}.$

(h)

$$\begin{aligned}
\int \left(4\sin x + \frac{2x^5 - \sqrt{x}}{x}\right) dx &= \int 4\sin x\, dx + \int \frac{2x^5 - \sqrt{x}}{x}\, dx \\
&= 4\int \sin x\, dx + 2\int \frac{x^5}{x}\, dx - \int \frac{\sqrt{x}}{x}\, dx \\
&= 4\int \sin x\, dx + 2\int x^4\, dx - \int x^{-1/2}\, dx \\
&= \boxed{-4\cos x + \frac{2x^5}{5} - 2x^{1/2} + C}.
\end{aligned}$$

You can always check your answer by differentiating your integral. If you get the original function again, then you found the correct antiderivative. This might be useful on your future exams.

Example 4.11.11. Find a function f whose graph has slope $f'(x) = 6x^2 + 4$ and goes through the point $(1, 1)$.

Since $f'(x) = 6x^2 + 4$,

$$f(x) = \int (6x^2 + 4)\, dx = 2x^3 + 4x + C.$$

The graph of f goes through $(1, 1)$, so C can be found by substituting 1 for x and 1 for $f(x)$.

$$\begin{aligned}
1 &= 2(1)^3 + 4(1) + C \\
1 &= 6 + C \\
C &= -5
\end{aligned}$$

Finally, $\boxed{f(x) = 2x^3 + 4x - 5.}$

Example 4.11.12. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Since $v(t) = a'(t) = 6t + 4$, we have that $v(t) = \int a(t)\, dt = \int (6t + 4)\, dt = \int 3t^2 + 4t + C$. Now, we have the initial value $v(0) = -6 = 3(0)^2 + 4(0) + C \Rightarrow C = -6$. Thus, $v(t) = 3t^2 + 4t - 6$.

Next, $s(t) = v'(t)$, which implies that $s(t) = \int v(t)\, dt = \int (3t^2 + 4t - 6)\, dt = t^3 + 2t^2 - 6t + D$. Now, we have the initial value $s(0) = 9 = (0)^3 + 2(0)^2 - 6(0) + D \Rightarrow D = 9$. Therefore, $\boxed{s(t) = t^3 + 2t^2 - 6t + 9}$.

[i]See §4.10 Antiderivatives in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–22, find each indefinite integral.

1. $\int (x-7)\,dx$

♠ 2. $\int \left(\frac{1}{2}x^2 - 2x + 8\right)\,dx$

♠ 3. $\int \left(\frac{1}{2} + \frac{2}{3}x^2 - \frac{6}{7}x^3\right)\,dx$

♠ 4. $\int (7x^9 - 4x^6 + 11x^3)\,dx$

♠ 5. $\int (5x^2 + 4x^{-2})\,dx$

♠ 6. $\int (x+3)(2x-3)\,dx$

♠ 7. $\int \left(\frac{4}{5} - \frac{7}{x}\right)\,dx$

8. $\int \left(\frac{5}{x} - 6\sqrt{x}\right)$

♠ 9. $\int \frac{2+x+x^2}{\sqrt{x}}\,dx$

10. $\int e^5\,dx$

11. $\int \left(\frac{1}{1+x^2} + 7\right)\,dx$

12. $\int \left(e^x + \frac{2}{\sqrt{1-x^2}}\right)\,dx$

♠ 13. $\int (9e^t - 6\cosh(t))\,dt$

♠ 14. $\int (3\csc^2\theta - 4e^\theta)\,d\theta$

♠ 15. $\int \left(4x^2 + 6 + \frac{5}{x^2+1}\right)\,dx$

16. $\int (6\sqrt{\theta} + 8\cos\theta)\,d\theta$

17. $\int (x^2 - \sin(x) + e^x)\,dx$

18. $\int (2\sqrt{x} + 6\cos x)\,dx$

19. $\int (3\sin x + x^2 - e^x)\,dx$

20. $\int (\csc\theta\cot\theta + 2e^{2\theta})\,d\theta$

21. $\int \frac{1+\cos^2 x}{\cos^2 x}\,dx$

22. $\int \left(\frac{1}{x} + \sinh(x)\right)\,dx$

For Exercises 23–33, find the position $s(t)$ of a moving particle given the following information about its velocity $v(t)$ and/or acceleration $a(t)$.

23. $v(t) = 2t+1,\ s(0) = 1$

24. $a(t) = 2t+1,\ s(0) = 3,\ v(0) = -2$

25. $v(t) = -9.8t - 25,\ s(0) = 1000$

♠ 26. $v(t) = 5t^4 - 2t^5,\ s(0) = 6$

♠ 27. $a(t) = 24t^3 - 15t^2 + 10t$

♠ 28. $a(t) = 2t+9,\ s(0) = 8,\ v(0) = -4$

29. $v(t) = \frac{3}{2}\sqrt{t},\ s(0) = 0$

30. $v(t) = 4e^{2t} + 6t^2,\ s(0) = 1$

31. $v(t) = 2t - 3\sin t,\ s(0) = 5$

32. $a(t) = \cos(t) + 2,\ s(0) = 4,\ v(0) = 1$

33. $a(t) = \sin t + 3\cos t,\ s(0) = 3,\ v(0) = 2$

Chapter 5

Integrals

"We have to forego [sum] good things in order to choose others that are better or best." – Dallin H. Oaks

Lecture Videos

Sigma Notation

Properties of Sigma

Examples of Sigma Notation

5.1 Sigma Notation

Definition 5.1.1. If $\{a_i\}$ is a sequence of real numbers, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n.$$

Example 5.1.2.

(a) $\displaystyle\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30.$

(b) $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots + (n-1) + n.$

(c) $\displaystyle\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63.$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$

Example 5.1.3. Write the sum $2^3 + 3^3 + \ldots + n^3$ in summation notation.

Note,

$$\sum_{i=2}^{n} i^3 = 2^3 + 3^3 + \ldots + n^3.$$

On the other hand,

$$\sum_{k=1}^{n-1} (k+1)^3 = (1+1)^3 + (2+1)^3 + \ldots + (n-1+1)^3 = 2^3 + 3^3 + \ldots + n^3.$$

Therefore, the same sum can be represented using sigma notation in multiple ways.

Theorem 5.1.4. *If c is any constant and $\{a_k\}$ and $\{b_k\}$ are sequences, then* *(a)* $\displaystyle\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$,

(b) $\displaystyle\sum_{i=m}^{n}(a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$,

(c) $\displaystyle\sum_{i=m}^{n}(a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$.

Proof. Note,

$$\sum_{i=m}^{n} ca_i = ca_m + ca_{m+1} + \ldots + ca_n = c(a_m + a_{m+1} + \ldots + a_n) = c\sum_{i=m}^{n} a_i.$$

Next,

$$\begin{aligned}\sum_{i=m}^{n}(a_i + b_i) &= (a_m + b_m) + (a_{m+1} + b_{m+1}) + \ldots + (a_n + b_n)\\ &= (a_m + a_{m+1} + \ldots + a_n) + (b_m + b_{m+1} + \ldots + b_n)\\ &= \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i.\end{aligned}$$

The last case is handled similarly. □

Theorem 5.1.5. *Let c be a constant and n a positive integer. Then*

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$,

(b) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$,

(c) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$,

(d) $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Proof.

(a)

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \ldots + 1}_{n \text{ terms}} = n.$$

(b) Let $S = \displaystyle\sum_{i=1}^{n} i$. Writing S twice, once in reverse order, we have:

$$\begin{array}{ccccccccccccc} S & = & 1 & + & 2 & + & 3 & + & \ldots & + & (n-1) & + & n \\ S & = & n & + & (n-1) & + & (n-2) & + & \ldots & + & 2 & + & 1. \end{array}$$

Adding each column together gives

$$\begin{array}{ccccccccccccc} 2S & = & (n+1) & + & (n+1) & + & (n+1) & + & \ldots & + & (n+1) & + & (n+1). \end{array}$$

Thus, $2S = n \cdot (n+1)$, which implies that $S = \dfrac{n(n+1)}{2}$.

(c) Let $S = \displaystyle\sum_{i=1}^{n} i^2$. Then

$$\begin{aligned}
\sum_{i=1}^{n}[(1+i)^3 - i^3] &= (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \ldots + ((n+1)^3 - n^3) \\
&= (\cancel{2^3} - 1^3) + (\cancel{3^3} - \cancel{2^3}) + (\cancel{4^3} - \cancel{3^3}) + \ldots + ((n+1)^3 - \cancel{n^3}) \\
&= (n+1)^3 - 1 = (n^3 + 3n^2 + 3n + 1) - 1 \\
&= n^3 + 3n^2 + 3n.
\end{aligned}$$

This method of cancelling terms is referred to as a **telescoping sum**.

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{n}[(1+i)^3 - i^3] &= \sum_{i=1}^{n}[(1 + 3i + 3i^2 + i^3) - i^3) \\
&= \sum_{i=1}^{n}[1 + 3i + 3i^2] = \sum_{i=1}^{n} 1 + 3\sum_{i=1}^{n} n + 3S \\
&= (n) + 3\left(\frac{n(n+1)}{2}\right) + 3S.
\end{aligned}$$

Therefore,

$$\begin{aligned}
3S + \frac{3n(n+1)}{2} + n &= n^3 + 3n^2 + 3n \\
3S &= n^3 + 3n^2 + 2n - \frac{3n(n+1)}{2} = \frac{2n^3 + 6n^2 + 4n - 3n(n+1)}{2} \\
&= \frac{2n^3 + 6n^2 + 4n - (3n^2 + 3n)}{2} = \frac{2n^3 + 3n^2 + n}{2} \\
S &= \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

(d) This last case is handled similar to the previous one.

□

Example 5.1.6. Evaluate $\displaystyle\sum_{i=1}^{n} i(4i^2 - 3)$.

$$\begin{aligned}
\sum_{i=1}^{n} i(4i^2 - 3) &= \sum_{i=1}^{n}(4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i = 4\left(\frac{n(n+1)}{2}\right)^2 - 3\left(\frac{n(n+1)}{2}\right) \\
&= \left(\frac{n(n+1)}{2}\right)(2n(n+1) - 3) = \boxed{\frac{n(n+1)(2n^2 + 2n - 3)}{2}}.
\end{aligned}$$

Example 5.1.7. Evaluate $\sum_{i=1}^{n}[i^4 - (i-1)^4]$.

$$\begin{aligned}\sum_{i=1}^{n}[i^4 - (i-1)^4] &= (1^4 - 0^4) + (2^4 - 1^4) + (3^4 - 2^4) + (4^4 - 3^4) + \ldots + (n^4 - (n-1)^4) \\ &= (\cancel{1^4} - 0^4) + (\cancel{2^4} - \cancel{1^4}) + (\cancel{3^4} - \cancel{2^4}) + (\cancel{4^4} - \cancel{3^4}) + \ldots + (n^4 - \cancel{(n-1)^4}) \\ &= n^4 - 0^4 = \boxed{n^4}.\end{aligned}$$

Example 5.1.8. Find $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty}\left(\frac{3}{n}\right)\sum_{i=1}^{n}\left[\frac{i^2}{n^2}+1\right] = \lim_{n\to\infty}\left[\left(\frac{3}{n^3}\right)\sum_{i=1}^{n} i^2 + \left(\frac{3}{n}\right)\sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty}\left[\left(\frac{3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{3}{n}\right)(n)\right] \\ &= \lim_{n\to\infty}\left[\left(\frac{(n+1)(2n+1)}{2n^2}\right) + 3\right] \\ &= \frac{2}{2} + 3 = \boxed{4}.\end{aligned}$$

[i]See §5.1 Approximating Areas in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–6, write condense or expand each sum.

♠ 1. $\displaystyle\sum_{i=1}^{6} \frac{7}{i+7}$

♠ 2. $\displaystyle\sum_{k=0}^{4} \frac{6k-1}{6k+1}$

♠ 3. $3+6+9+12+15+18+21+24+27+30$

♠ 4. $\dfrac{1}{1}+\dfrac{1}{4}+\dfrac{1}{9}+\dfrac{1}{16}+\dfrac{1}{25}+\dfrac{1}{36}$

♠ 5. $\dfrac{1}{10}+\dfrac{2}{11}+\dfrac{3}{12}+\ldots+\dfrac{19}{28}$

♠ 6. $2-2x+2x^2-2x^3+\ldots+(-1)^n 2x^n$

For Exercises 7–21, find the sum.

7. $\displaystyle\sum_{k=1}^{100} 4$

8. $\displaystyle\sum_{k=1}^{100} (2-5k)$

♠ 9. $\displaystyle\sum_{i=4}^{12} (5i-4)$

10. $\displaystyle\sum_{k=1}^{14} (k^2-4)$

11. $\displaystyle\sum_{k=3}^{6} i(i+2)$

♠ 12. $\displaystyle\sum_{i=1}^{24} i(i+9)$

13. $\displaystyle\sum_{k=1}^{20} (k+1)^2$

♠ 14. $\displaystyle\sum_{n=1}^{6} 3^{n+1}$

♠ 15. $\displaystyle\sum_{n=1}^{100} (-1)^n 4$

16. $\displaystyle\sum_{k=1}^{20} \left(k^4-(k-1)^4\right)$

17. $\displaystyle\sum_{k=1}^{99} \left(\frac{1}{k}-\frac{1}{k+1}\right)$

♠ 18. $\displaystyle\sum_{i=0}^{4} (3^i+i^3)$

19. $\displaystyle\sum_{k=1}^{6} \frac{1}{2^k}$

20. $\displaystyle\sum_{k=1}^{6} 5\left(-\frac{1}{3}\right)^k$

21. $\displaystyle\sum_{k=-2}^{4} 2^{3-k}$

"I believe any success in life is made by going into an area with a blind, furious optimism."
– Sylvester Stallone

Lecture Videos

Geometric Sums

Approximating π using Rectangles

5.2 Area Under the Curve

Definition 5.2.1. A sequence of real numbers $\{a_n\}$ is called **geometric** if it is of the form $a_n = ar^{n-1}$ for some *initial value* a and *constant ratio* r. In particular, geometric sequences are characterized by the property that $\dfrac{a_n}{a_{n-1}} = r$ for all n. Also, $a = a_1$, the initial value of the sequence.

Example 5.2.2. The sequence $2, 6, 18, 54, 162, \ldots$ is geometric since $\dfrac{162}{54} = \dfrac{54}{18} = \dfrac{18}{6} = 3$. Likewise, the sequence $\{s_n\} = \{2^{-n}\} = \left\{\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dfrac{1}{32}, \dfrac{1}{64}, \ldots\right\}$ is geometric. The sequence $\{t_n\} = \{4^n\} = \{4, 16, 64, 256, \ldots\}$ is also geometric.

Suppose we want to add together the first n-terms of a geometric sequence. Let S_n denote this sum. Then

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} r^{k-1}a = a + ar + ar^2 + \ldots + ar^{n-1} \\
rS_n &= r(a + ar + ar^2 + \ldots + ar^{n-1}) = ar + ar^2 + ar^3 + \ldots + ar^n \\
S_n - rS_n &= (a + ar + ar^2 + \ldots + ar^{n-1}) - (ar + ar^2 + ar^3 + \ldots + ar^n) \\
(1-r)S_n &= a - ar^n = a(1-r^n) \\
S_n &= a\left(\frac{1-r^n}{1-r}\right).
\end{aligned}$$

Thus, in summary,

$$S_n = a\left(\frac{1-r^n}{1-r}\right). \tag{5.2.1}$$

Example 5.2.3. Compute the geometric sum $\displaystyle\sum_{i=1}^{8} \frac{1}{2^i}$.

Notice that $a = r = \dfrac{1}{2}$. Thus,

$$\begin{aligned}
\sum_{i=1}^{8} \frac{1}{2^i} &= \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{256} = a\left(\frac{1-r^8}{1-r}\right) = \frac{1}{2}\left(\frac{1-1/2^8}{1-1/2}\right) \\
&= \frac{1}{2}\left(\frac{1-1/256}{1/2}\right) = 1 - \frac{1}{256} = \boxed{\frac{255}{256}}.
\end{aligned}$$

Calculating the area of a geometric figure is an important application of geometry. The easiest of all shapes to compute the area is a square or rectangle. The area of a parallelogram can be computed by dissecting the polygon and rearranging it into a rectangle. Likewise, we note that every triangle is half of a parallelogram, which gives the classic formula $\frac{1}{2}bh$. Lastly, since every polygon, such as a trapezoid or hexagon, can be dissected into triangles, we can calculate the area of any polygon.

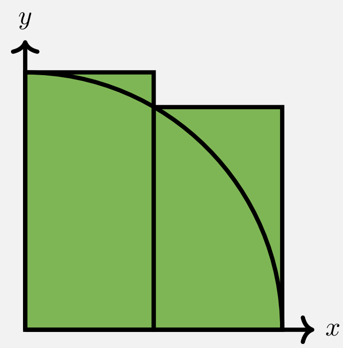
Now, there are plenty of interesting shapes other than polygons, e.g. circles, ellipses. How can we find the area of such a shape? For circles, we have the time-tested formula $A = \pi r^2$.[i] But where did such a formula come from? One way to calculate the area of a circle (and any other shape) is with Calculus.

Example 5.2.4. Consider the function $f(x) = \sqrt{4 - x^2}$, whose graph is a semicircle of radius 2 above the x-axis. Let us find the area of the region bounded by the curve, the x-axis, and the y-axis.

Notice that this region represents one quarter of the area of a circle with radius 2, so the area should be $A = \dfrac{\pi r^2}{4} = \dfrac{4\pi}{4} = \pi$. So, if we can approximate the area of this region, then we can actually compute π.

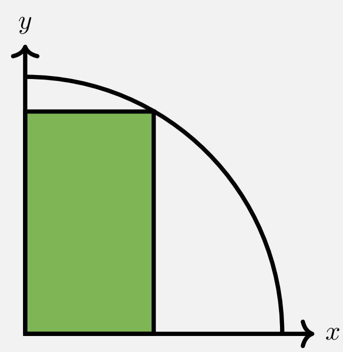
A very rough approximation of the area of this region can be found by using two rectangles. The height of the rectangle on the left is $f(0) = 2$ and the height of the rectangle on the right is $f(1) = \sqrt{3}$. The width of each rectangle is 1, making the total area of the two rectangles

$$1 \cdot f(0) + 1 \cdot f(1) = 2 + \sqrt{3} \approx 3.7321.$$



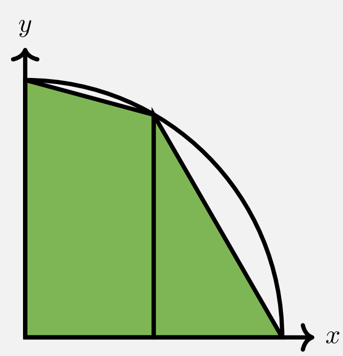
In this example, the function is decreasing, and we will overestimate the area when we evaluate the function at the left end points to determine the height of that interval. If we use the right endpoints, the answer will be too small. The area of the two rectangles is

$$1 \cdot f(1) + 1 \cdot f(2) = \sqrt{3} + 0 \approx 1.7321.$$



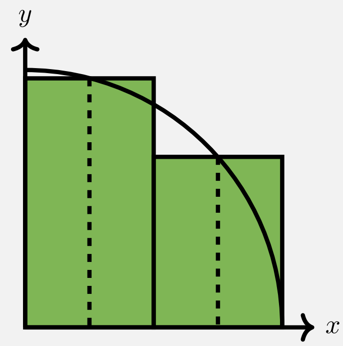
If the left endpoint gives an answer too big, and the right endpoint an answer too small, it seems reasonable to average the two answers. This produces the method called the *trapezoidal rule*. For this example, we get

$$\frac{3.7321 + 1.7321}{2} = 2.7321.$$



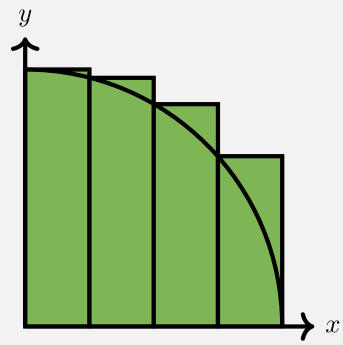
Another way to get an improved answer would be to use the midpoint of each interval, rather than the left endpoint or the right endpoint. This is called the *midpoint rule*. We compute the area as

$$1 \cdot f(0.5) + 1 \cdot f(1.5) = \sqrt{3.75} + \sqrt{1.75} \approx 3.2594.$$



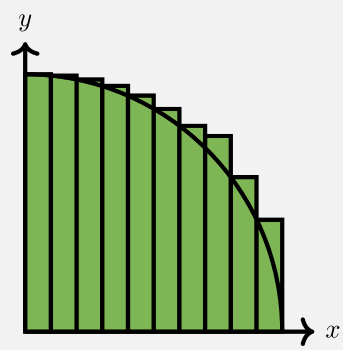
To improve accuracy on either of these methods, we could divide the interval $[0,2]$ into more parts. For example, the area can be approximated using 4 rectangles and left endpoints. This time, each rectangle has width $1/2$. Calculating the area gives

$$\begin{aligned} & \frac{1}{2} \cdot f(0) + \frac{1}{2} \cdot f\left(\frac{1}{2}\right) + \frac{1}{2} \cdot f(1) + \frac{1}{2} \cdot f\left(\frac{3}{2}\right) \\ = & \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \left(\frac{\sqrt{15}}{2}\right) + \frac{1}{2} \cdot \sqrt{3} + \frac{1}{2} \cdot \left(\frac{\sqrt{7}}{2}\right) \\ = & 1 + \frac{\sqrt{15}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{4} \approx 3.4957. \end{aligned}$$



This approximation looks better, but it is still greater than the actual area. To improve the approximation, divide the interval $[0,2]$ into more pieces. Figure to the right shows the approximation using 10 rectangles and left endpoints. In this case, each rectangle has width $1/5$ and the area approximates as 3.3045.

Table 5.2.1 shows the approximations for the four methods with different numbers of rectangles.



As the number of rectangles increases without bound, the sum of the areas of these rectangles gets closer and closer to the actual area of the region, π. This can be written as

$$\lim_{n\to\infty} (\text{sum of areas of } n \text{ rectangles}) = \pi^{\text{ii}}.$$

Definition 5.2.5. Let L_n denote the approximation of the area under the curve $y = f(x)$ using n rectangles and rectangle heights are determined by left-endpoints, called the **Left Sum**.

Let R_n denote the approximation of the area under the curve $y = f(x)$ using n rectangles and rectangle heights are determined by right-endpoints, called the **Right Sum**.

Let M_n denote the approximation of the area under the curve $y = f(x)$ using n rectangles and rectangle heights are determined by midpoints, called the **Midpoint Sum**.

Lastly, let T_n denote the approximation of the area under the curve $y = f(x)$ using n trapezoids connecting the left- and right-endpoints, called the **Trapezoidal Sum**. Note that $T_n = \dfrac{L_n + R_n}{2}$.

n	L_n	R_n	T_n	M_n
2	3.7321	1.7321	2.7321	3.2594
4	3.4957	2.4957	2.9957	3.1839
8	3.3398	2.8398	3.0898	3.1567
10	3.3045	2.9045	3.1045	3.1524
20	3.2285	3.0285	3.1285	3.1454
50	3.1783	3.0983	3.1383	3.1426
100	3.1604	3.1204	3.1404	3.1419
500	3.1455	3.1375	3.1415	3.1416

Table 5.2.1: Approximations of π, as outlined in Example 5.2.4

[i] "Pies are square? No! Pies are round! What are they teaching in school these days?"

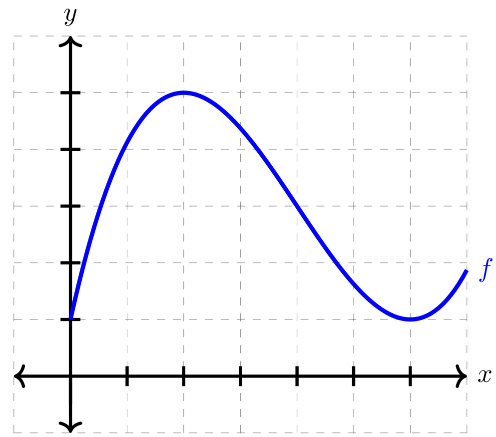
[ii] The value of π was originally found by a process similar to this. Many approximations have been used for π over the years. A passage in the Bible (1 Kings 7:23) indicates a value of 3. The Egyptians used the value 3.16, and Archimedes showed that its value must be between 22/7 and 223/71. A Hindu writer, Brahmaqupta, used $\sqrt{10} \approx 3.16228$ as its value in the seventh century. The search for the digits of π has continued into modern times. Yasumasa Kanada and his coworkers at the University of Tokyo recently computed the value to over 1.2 trillion places.

[iii] See §5.1 Approximating Areas in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–12, estimate the area under the graph of $y = f(x)$ from $x = 0$ to $x = 6$ using the prescribed approximation method. If the area under the curve is $\frac{39}{2}$, does the approximation overestimate or underestimate the area under the curve.



1. L_2 ♠ 2. R_2 3. T_2 ♠ 4. M_2

♠ 5. L_3 6. R_3 ♠ 7. T_3 ♠ 8. M_3

9. L_6 ♠ 10. R_6 11. T_6 12. M_6

For Exercises 13–24, estimate (accurate to six decimal places) the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using the prescribed approximation method. If the area under the curve is $\frac{16}{3}$, does the approximation overestimate or underestimate the area under the curve.

The following eMathHelp calculator may prove helpful

The following Desmos calculator may prove helpful

13. L_2 14. R_2 15. T_2 16. M_2

♠ 17. L_4 ♠ 18. R_4 ♠ 19. T_4 ♠ 20. M_4

21. L_8 22. R_8 23. T_8 24. M_8

For Exercises 25–36, estimate, accurate to six decimal places, the area under the graph of $f(x) = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ using the prescribed approximation method. If the area under the curve is 1, does the approximation overestimate or underestimate the area under the curve.

25. L_2 26. R_2 27. T_2 28. M_2

♠ 29. L_3 ♠ 30. R_3 ♠ 31. T_3 ♠ 32. M_3

33. L_4 34. R_4 35. T_4 36. M_4

For Exercises 37–48, estimate, accurate to six decimal places, the area under the graph of $f(x) = \sin x$ from $x = 0$ to $x = \frac{3\pi}{2}$ using the prescribed approximation method. If the area under the curve is 1, does the approximation overestimate or underestimate the area under the curve.

37. L_3 38. R_3 39. M_3 40. T_3

♠ 41. L_6 ♠ 42. R_6 ♠ 43. M_6 ♠ 44. T_6

45. L_9 46. R_9 47. M_9 48. T_9

For Exercises 49–60, estimate, accurate to six decimal places, the area under the graph of $f(x) = x^2$ from $x = -1$ to $x = 2$ using the prescribed approximation method. If the area under the curve is 3, does the approximation overestimate or underestimate the area under the curve.

49. L_3 50. R_3 51. T_3 52. M_3

♠ 53. L_5 ♠ 54. R_5 ♠ 55. T_5 ♠ 56. M_5

57. L_6 58. R_6 59. T_6 60. M_6

"One can never know for sure what a deserted area looks like." – George Carlin

Lecture Videos

Area under the Curve

Riemann Sum Calculators

Upper and Lower Sums

Velocity, Displacement, and Area under the Curve

The Definite Integral

5.3 Riemann Sums

To develop a process that would approximate the exact area under a curve, begin by subdividing the interval into n-many equal-width subintervals. Let Δx denote the width of each subinterval. Since the subintervals have equal width, we have that

$$\Delta x = \frac{b-a}{n}.$$

Let $x_i = a + i\Delta x$. Then x_i is the right endpoint of the ith interval $[x_{i-1},\ x_i]$. Let x_i^* be an arbitrary point in the ith interval, that is, x_i^* is some point that satisfies $x_{i-1} \leq x_i^* \leq x_i$. Then we will make the height of the ith rectangle be $f(x_i^*)$ for each subinterval. So,

$$\text{Area of the } i\text{th rectangle} = f(x_i^*) \cdot \Delta x.$$

The total area under the curve is approximated by the sum of the areas of all n of the rectangles. So,

$$\text{Area of all } n \text{ rectangles} = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Example 5.3.1. Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be the right endpoints and $a = 0$, $b = 3$, and $n = 6$.

We compute: $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{6} = \dfrac{1}{2}$, $x_i^* = x_i = a + i\Delta x = 0 + i\left(\dfrac{1}{2}\right) = \dfrac{i}{2}$, and:

i	x_i	$f(x_i)$
1	$1/2 = 0.5$	$-23/8 = -2.875$
2	1	-5
3	$3/2 = 1.5$	$-45/8 = -5.625$
4	2	-4
5	$5/2 = 2.5$	$5/8 = 0.625$
6	3	9

Thus,

$$\begin{aligned}
R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x \\
&= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\
&\qquad + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\
&= \frac{1}{2}\left(-\frac{23}{8} - 5 - \frac{45}{8} - 4 + \frac{5}{8} + 9\right) \\
&= \frac{1}{2}\left(-\frac{63}{8}\right) = -\frac{63}{16} = \boxed{-3.9375}.
\end{aligned}$$

Example 5.3.2. Evaluate the Riemann sum for $f(x) = x^2$, taking the sample points to be the midpoints and $a = 0$, $b = 1$, and $n = 4$.

We compute: $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{4} = \dfrac{1}{4}$, $x_i^* = \overline{x_i} = \dfrac{(a+(i-1)\Delta x)+(a+i\Delta x)}{2} = \dfrac{2a+2i\Delta x-\Delta x}{2} = 2a + i\Delta x - \dfrac{1}{2}\Delta x = 2(0) + \dfrac{i}{4} - \dfrac{1}{2(4)} = \dfrac{2i-1}{8}$, and:

i	$\overline{x_i}$	$f(\overline{x_i})$
1	$1/8 = 0.125$	$1/64 = 0.015625$
2	$3/8 = 0.375$	$9/64 = 0.140625$
3	$5/8 = 0.625$	$25/64 = 0.375$
4	$7/8 = 0.875$	$49/64 = 0.765625$

Thus,

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(\overline{x_i})\Delta x \\ &= f(\overline{x_1})\Delta x + f(\overline{x_2})\Delta x + f(\overline{x_3})\Delta x + f(\overline{x_4})\Delta x \\ &= \frac{1}{4}\left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right) = \frac{1}{4}\left(\frac{84}{64}\right) \\ &= \frac{84}{256} = \boxed{0.32815}. \end{aligned}$$

Definition 5.3.3. Let LS_n denote the approximation of the area under the curve $y = f(x)$ using n rectangles and rectangle heights are determined by the minimum value of f on the ith subinterval, called the **Lower Sum**.

Let US_n denote the approximation of the area under the curve $y = f(x)$ using n rectangles and rectangle heights are determined by the maximum value of f on the ith subinterval, called the **Upper Sum**.
Lastly, let A_n denote the approximation of the area under the curve $y = f(x)$ using n rectangles and rectangle heights are determined by the midpoint between the maximum and minimum values of f on the ith subinterval, called the **Average Sum**. Note that $A_n = \dfrac{LS_n + US_n}{2}$.

Example 5.3.4. Evaluate the upper and lower sums for $f(x) = 1 + x^2$ on the interval $[-1, 1]$ with $n = 4$.

First, note that

$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} = 0.5.$$

Since $f'(x) = 2x$, f is increasing on the interval $(0, \infty)$ and decreasing on the interval $(\infty, 0)$. When a function is increasing, the maximum value is obtained at the right endpoint and the minimum value is obtained at the left endpoint. For a decreasing function, the situation is reversed. Thus, $x_i^* = x_{i-1}$ or x_i, depending on the monotonicity.

i	x_i	$f(x_i)$
0	-1	2
1	$-1/2 = -0.5$	$5/4 = 1.25$
2	0	1
3	$1/2 = 0.5$	$5/4 = 1.25$
4	1	2

Thus,

$$US_4 = \Delta x(f(-1) + f(-1/2) + f(1/2) + f(1)) = \frac{1}{2}\left(2 + \frac{5}{4} + \frac{5}{4} + 2\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{13}{2}\right) = \frac{13}{4} = \boxed{3.25} \\
LS_4 &= \Delta x(f(-1/2) + f(0) + f(0) + f(1/2)) = \frac{1}{2}\left(\frac{5}{4} + 1 + 1 + \frac{5}{4}\right) \\
&= \frac{1}{2}\left(\frac{9}{2}\right) = \frac{9}{4} = \boxed{2.25}
\end{aligned}
$$

A better approximation can be obtained by the average of the upper and lower sums, which is $\boxed{A_2 = 2.75}$.

"Area" under the curve shows up in many applications from science.

Example 5.3.5. A driver traveling on a business trip checks the speedometer each hour. The table shows the driver's velocity at several times.

Time (hr.)	0	1	2	3
Velocity (mph)	0	52	58	60

Approximate the total distance traveled during the 3-hour period using the left endpoint of each interval, then the right endpoint.

Since velocity is the rate of change of position, the total distance traveled, that is, the total change of position, corresponds to the area under the velocity curve. Since each interval represents one hour, $\Delta x = 1$. Thus, using left endpoints, the total distance is

$$0 \cdot 1 + 52 \cdot 1 + 58 \cdot 1 = 110 \text{ miles.}$$

With right endpoints, we get

$$52 \cdot 1 + 58 \cdot 1 + 60 \cdot 1 = 170 \text{ miles.}$$

The left endpoints give a total that is too small, while right endpoints give a total that is too large. The average of these, that is, the Trapezoidal Method, is $T_3 = \boxed{140 \text{ miles}}$, is a better estimate of the total distance traveled.

The greater the number of rectangles used to approximate the area under the curve, the greater the accuracy of the approximation. In fact, the exact area is equal to the limit of the Riemann sum (if the limit exists) as the number of rectangles increases without bound:

$$\text{Exact Area under the Curve} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Definition 5.3.6. If f is defined on the interval $[a, b]$, the **definite integral**[i] of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists, where $\Delta x = (b-a)/n$ and x_i is *any* value of x in the ith subinterval.

The function $f(x)$ is called the **integrand** and the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann sum**.

If the limit exists and does not depend on the choice of x_i^*, then we say that f is **integrable** on $[a, b]$.

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$, although the converse need not be true. In fact, if f has only finitely many jump or removable discontinuities on $[a, b]$, then it is still integrable. This will go without proof.

Example 5.3.7. Find $\int_0^4 2x\,dx$, the area of the region under the graph of $f(x) = 2x$, above the x-axis, and between $x = 0$ and $x = 4$.

Since the region is a triangle, we may calculate $\int_0^4 2x\,dx$ using our formula from geometry. Thus,

$$\int_0^4 2x\,dx = \frac{1}{2}bh = \frac{1}{2}(4)(8) = \boxed{16}.$$

Example 5.3.8. Evaluate $\int_0^1 \sqrt{1 - x^2}\,dx$.

Let $f(x) = \sqrt{1 - x^2}$. The curve f is a semicircle with radius 1, that is, f is the upper semicircle of $x^2 + y^2 = 1$. Thus, the region under f between 0 and 1 is one quarter of a circle. The total area of the circle is $\pi r^2 = \pi(1)^2 = \pi$. Thus, $\int_0^1 \sqrt{1 - x^2}\,dx = \frac{1}{4}\text{Area of circle} = \boxed{\frac{\pi}{4}}$.

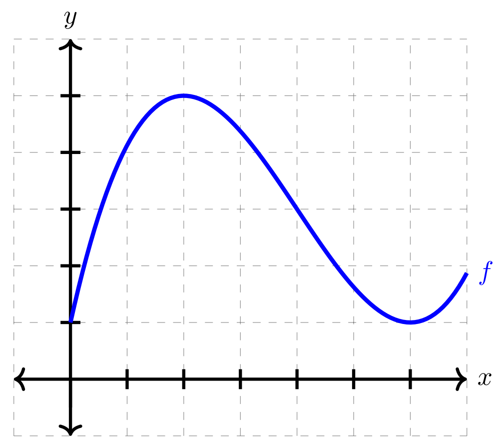
[i] We will see in the near future that there is a strong relationship between definite and indefinite integrals which is not yet apparent.

[ii] See §5.1 Approximating Areas and §5.2 The Definite Integral in OpenStax to find the corresponding sections.

Exercises

(Go to Solutions)

For Exercises 1–9, estimate the area under the graph of $y = f(x)$ from $x = 0$ to $x = 6$ using the prescribed approximation method. If the area under the curve is $\frac{39}{2}$, does the approximation overestimate or underestimate the area under the curve.



1. LS_2
2. US_2
3. A_2
4. LS_3
5. US_3
6. A_3
7. LS_6
8. US_6
9. A_6

For Exercises 10–30, estimate (accurate to six decimal places) the area under the graph of $f(x) = 1 + \sin x$ from $x = 0$ to $x = \pi$ using the prescribed approximation method. If the area under the curve is $2 + \pi$, does the approximation overestimate or underestimate the area under the curve.

The following GeoGebra calculator may prove helpful

10. L_2
11. R_2
12. T_2
13. M_2
♠ 14. LS_2
♠ 15. US_2
♠ 16. A_2
17. L_4
18. R_4
♠ 19. T_4
♠ 20. M_4
♠ 21. LS_4
♠ 22. US_4
♠ 23. A_4
24. L_6
25. R_6
26. T_6
27. M_6
♠ 28. LS_6
♠ 29. US_6
30. A_6

For Exercises 31–37, a table of values of an increasing function f is shown. Use the table to approximate $\int_0^{30} f(x)\,dx$ using the prescribed estimate.

x	0	5	10	15	20	25	30
$f(x)$	−12	−6	−2	1	3	8	12

31. L_3
♠ 32. R_3
♠ 33. T_3
♠ 34. M_3
♠ 35. L_6
36. R_6
37. T_6

For Exercises 38–46, expression each limit of a Riemann sum as a definite integral or vice versa.

38. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$

♠ 39. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{2}{n}\left(7 + \frac{2i}{n}\right)^{10}$

♠ 40. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{\pi}{5n}\tan\left(\frac{i\pi}{5n}\right)$

41. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{2}{n}\left(\left(3 + \frac{2i}{n}\right)^2 + \sqrt{1 + 2\left(3 + \frac{2i}{n}\right)}\right)$

♠ 42. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\cos\left(2\pi+\frac{\pi}{n}\right)}{\left(2\pi+\frac{\pi}{n}\right)}\left(\frac{\pi}{n}\right)$

43. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{5i}{n}\right)\ln\left(1+\left(2+\frac{5i}{n}\right)^2\right)\left(\frac{5i}{n}\right)$

♠ 44. $\displaystyle\int_3^6 \frac{x}{2+x^4}\,dx$

45. $\displaystyle\int_2^4 (x-5\ln x)\,dx$

46. $\displaystyle\int_0^{\ln 2} e^x\,dx$

For Exercises 47–52, evaluate the integral by interpreting it in terms of areas.

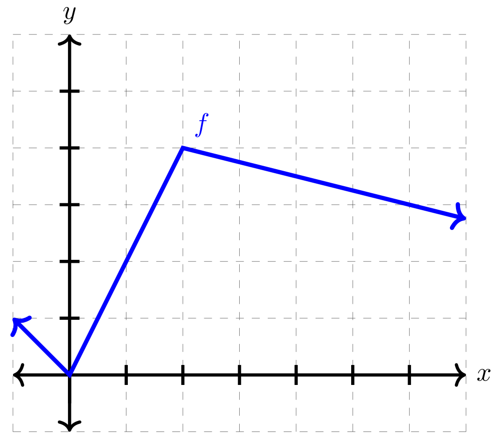
47. $\displaystyle\int_{-3}^{4} |x|\,dx$

♠ 48. $\displaystyle\int_0^{20} |x-10|\,dx$

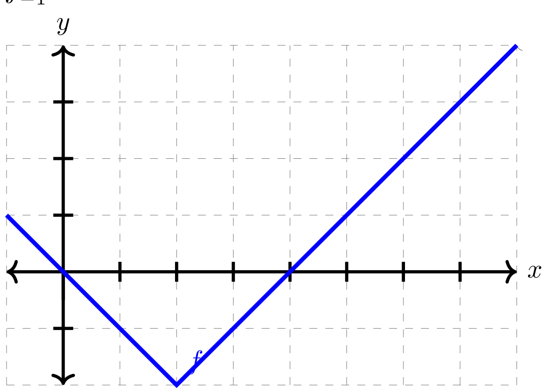
♠ 49. $\displaystyle\int_0^6 f(x)\,dx$ if:

$$f(x)=\begin{cases}4, & x<4\\ x, & x\geq 4\end{cases}$$

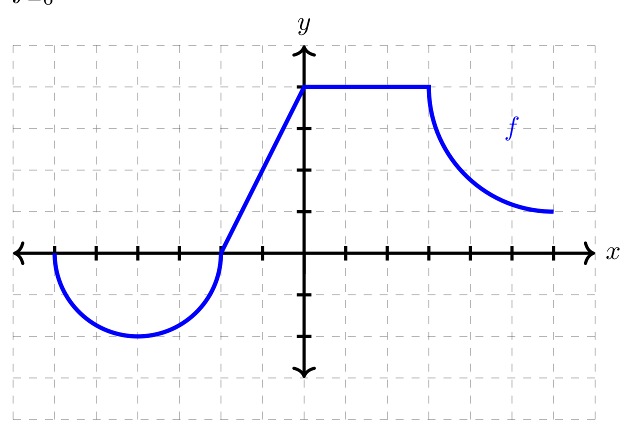
♠ 50. $\displaystyle\int_{-1}^{6} f(x)\,dx$ if:



♠ 51. $\displaystyle\int_{-1}^{8} f(x)\,dx$ if:



52. $\displaystyle\int_{-6}^{6} f(x)\,dx$ if:



"Joy can only be real if people look upon their life as a service and have a definite object in life outside themselves and their personal happiness." – Leo Tolstoy

Lecture Videos

Definition of Definite Integrals

Computing Definite Integrals by the Definition

Computing Definite Integrals by the Definition involving a Geometric Sum

Properties of Definite Integrals

Comparison Test of Definite Integrals

5.4 Definite Integrals

Suppose that f is an integrable function on $[a,b]$. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$ and x_i^* is any number such that $a+(i-1)\Delta x \le x_i^* \le a+i\Delta x$. For integrable functions, the choice of x_i^* is irrelevant. Thus, unless otherwise stated, we will set $x_i^* = x_i = a+i\Delta x$, that is, x_i is the right end point of the subinterval.

Example 5.4.1. Evaluate $\displaystyle\int_0^3 (x^3-6x)\,dx$.

$$\begin{aligned}
\int_0^3 (x^3-6x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^n f\left(a+i\Delta x\right)\left(\frac{b-a}{n}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^n f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty}\left(\frac{3}{n}\right)\sum_{i=1}^n\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^n\left[\frac{27i^3}{n^3}-\frac{18i}{n}\right] = \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^n i^3 - \frac{54}{n^2}\sum_{i=1}^n i\right]\\
&= \lim_{n\to\infty}\left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]\\
&= \lim_{n\to\infty}\left[\frac{81n^2(n+1)^2}{4n^4} - \frac{54n(n+1)}{2n^2}\right] = \lim_{n\to\infty}\left[\frac{81(n+1)^2}{4n^2} - \frac{27(n+1)}{n}\right]\\
&= \frac{81}{4} - 27 = \boxed{-\frac{27}{4} = -6.75}.
\end{aligned}$$

As a reminder, we computed $R_6 = -3.9375$.

Example 5.4.2. Evaluate $\displaystyle\int_0^1 x^2\,dx$.

$$\begin{aligned}
\int_0^1 x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a+i\Delta x\right)\left(\frac{b-a}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{i=1}^{n}\frac{i^2}{n^2}\\
&= \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty}\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\\
&= \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \boxed{\frac{1}{3}\approx 0.33333}.
\end{aligned}$$

As a reminder, we computed $M_4 = 0.32815$.

Example 5.4.3. Evaluate $\displaystyle\int_1^3 2^x\,dx$.

$$\begin{aligned}
\int_1^3 2^x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(1+i\Delta x)\left(\frac{2}{n}\right)\\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} 2^{1+2i/n}\\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} 2(4^{i/n}) = \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}(4^{1/n})^i,\ \text{this is a geometric sum,}\\
&= \lim_{n\to\infty}\frac{4}{n}(4^{1/n})\left(\frac{1-(4^{1/n})^n}{1-4^{1/n}}\right) = \lim_{n\to\infty}\frac{4^{1+1/n}}{n}\left(\frac{1-4}{1-4^{1/n}}\right)\\
&= \lim_{n\to\infty}\frac{(-3)4^{1+1/n}(1/n)}{1-4^{1/n}} = \lim_{n\to\infty}\frac{(-12)4^{1/n}(1/n)}{1-4^{1/n}},\ \text{this is the indeterminate form } 0/0,\\
&\overset{l'H}{=} \lim_{n\to\infty}\frac{-12[(\ln 4)4^{1/n}(-1/n^2)(1/n)+4^{1/n}(-1/n^2)]}{-(\ln 4)(4^{1/n})(-1/n^2)}\\
&= \lim_{n\to\infty}\frac{-12(4^{1/n})(-1/n^2)[(\ln 4/n+1]}{-(\ln 4)(4^{1/n})(-1/n^2)}\\
&= \lim_{n\to\infty}\frac{12(\ln 4/n+1)}{\ln 4} = \frac{12(0+1)}{\ln 4}\\
&= \frac{12}{\ln 4} = \frac{12}{2\ln 2} = \boxed{\frac{6}{\ln 2}\approx 8.65617}.
\end{aligned}$$

Key properties of definite integrals are listed below.

Proposition 5.4.4 (Properties of Definite Integrals)**.** *If all indicated definite integrals exist,*

(i) $\displaystyle\int_a^a f(x)\,dx = 0,$

(ii) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$

(iii) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$

(iv) $\displaystyle\int_a^b [f(x)\pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx,$

(v) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Example 5.4.5. Evaluate $\int_0^1 (4+3x^2)\,dx$. Remember that $\int_0^1 x^2\,dx = \frac{1}{3}$.

$$\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx = 4(1-0)+3(1/3) = 4+1 = \boxed{5}.$$

Example 5.4.6. If $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, then calculate $\int_8^{10} f(x)\,dx$.

Note that

$$\int_0^{10} f(x)\,dx = \int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx \Rightarrow 17 = 12 + \int_8^{10} f(x)\,dx,$$

which implies that $\int_8^{10} f(x)\,dx = \boxed{5}$.

Proposition 5.4.7 (Comparison Properties of Definite Integrals)**.** *If all indicated definite integrals exist,*

(i) If $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq 0$.

(ii) If $f(x) \geq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(iii) If $m \leq f(x) \leq M$ for all $x \in [a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Example 5.4.8. Estimate $\int_0^1 e^{-x^2}\,dx$ using 5.4.7.(ii).

Let $f(x) = e^{-x^2}$. On the interval $[0,1]$, f is a decreasing function. Thus, $f(1) \leq f(x) \leq f(0)$ for all $x \in [0,1]$. Since $f(1) = e^{-1}$ and $f(0) = e^0 = 1$, we have by 5.4.7.(ii) that

$$(0.367 \approx)\,1/e \leq \int_0^1 e^{-x^2}\,dx \leq 1.$$

[i]See §5.2 The Definite Integral in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–20, evaluate the definite integral $\int f(x)\,dx$ (without using the Fundamental Theorem of Calculus).

1. $\int_1^4 x^2\,dx$
2. $\int_2^5 x^2\,dx$
3. $\int_{-1}^2 x^2\,dx$
4. $\int_0^3 (x^2-4x+2)\,dx$
5. $\int_0^4 (x^2-4x+2)\,dx$
6. ♠ $\int_0^4 (x^2-6x)\,dx$
7. ♠ $\int_1^9 (x^2-4x+9)\,dx$
8. ♠ $\int_{-2}^0 (7x^2+7x)\,dx$
9. $\int_0^5 (x^2+2x+1)\,dx$
10. $\int_1^3 x^3\,dx$
11. $\int_0^2 (x^3-1)\,dx$
12. ♠ $\int_{-9}^3 (3+\sqrt{81-x^2})\,dx$
13. ♠ $\int_\pi^\pi \sin^3 x\cos^4 x\,dx$
14. ♠ $\int_1^0 12x\sqrt{x^2+9}\,dx$ if $\int_0^1 12x\sqrt{x^2+9}\,dx = 40\sqrt{10}-108$
15. ♠ $\int_0^1 (7-3x^2)\,dx$ if $\int_0^1 x^2\,dx = \dfrac{1}{3}$
16. ♠ $\int_1^3 (2e^x-2)\,dx$ if $\int_1^3 e^x\,dx = e^3-e$
17. ♠ $\int_1^3 5e^{x+4}\,dx$ if $\int_1^3 e^x\,dx = e^3-e$
18. ♠ $\int_1^6 f(x)\,dx$ if $\int_1^7 f(x)\,dx = 14$ and $\int_6^7 f(x)\,dx = 5.9$
19. ♠ $\int_0^{10} (2f(x)+4g(x))\,dx$ if $\int_0^{10} f(x)\,dx = 39$ and $\int_0^{10} g(x)\,dx = 14$
20. $\int_1^2 x^{-2}\,dx$

    Choose $x_i^* = \sqrt{x_{i-1}x_i}$, the geometric mean of x_{i-1} and x_i. Also, use the identity $\dfrac{1}{m(m+1)} = \dfrac{1}{m} - \dfrac{1}{m+1}$. This will lead to a telescoping sum.

"Most of the fundamental ideas of science are essentially simple, and may, as a rule, be expressed in a language comprehensible to everyone." – Albert Einstein

Lecture Videos

Integral Functions

The Fundamental Theorem of Calculus - Part 1

Computing Derivatives using the Fundamental Theorem of Calculus - Part 1

Computing Derivatives using the Fundamental Theorem of Calculus, where the limits are backward

Computing Derivatives using the Fundamental Theorem of Calculus, where the limits are functions

5.5 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. In particular, the Fundamental Theorem enables us to compute areas and integrals very easily without having to compute them as limits of sums as we did before.

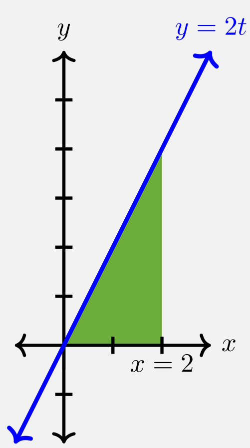
Before we present the Fundamental Theorem, we mention that definite integrals can be used to invent new functions. For example, suppose that f is continuous on $[a, b]$. Then we can define the function g on the same domain as f by the following

$$g(x) = \int_a^x f(t)\,dt.$$

Thus $g(x)$ is the value of the area under the curve f on the interval $[a, x]$.

Example 5.5.1. Let $g(x) = \displaystyle\int_0^x 2t\,dt$.

$$\begin{aligned}
g(0) &= \int_0^0 2t\,dt = 0 \\
g(1) &= \int_0^1 2t\,dt = \frac{1}{2}(1)(2\cdot 1) = 1 \\
g(2) &= \int_0^2 2t\,dt = \frac{1}{2}(2)(2\cdot 2) = 4 \\
g(3) &= \int_0^3 2t\,dt = \frac{1}{2}(3)(2\cdot 3) = 9 \\
g(x) &= \int_0^x 2t\,dt = \frac{1}{2}(x-0)(2x) = x^2.
\end{aligned}$$



One can't help but notice that $g'(x) = 2x$ from the previous example and $2x$ is exactly the integrand of that example. It turns out that this is not a mere coincidence and the result holds in more generality.

Theorem 5.5.2 (The Fundamental Theorem of Calculus, Part 1)**.** *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

f is continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) = f(x)$.

Proof. If x and $x + h$ are in (a, b), then

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

So for $h \neq 0$, we have

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

First, assume that $h > 0$. So, $x < x + h$ and f is continuous on the interval $[x, x + h]$. The Extreme Value Theorem then gives that there exists x-values u, v such that $f(u) = m$ is the absolute minimum and $f(v) = M$ is the absolute maximum values of f on $[x, x + h]$. Therefore, $m \leq f(t) \leq M$ on the interval $[x, x + h]$. Using properties of integrals, we have

$$\begin{aligned}
\int_x^{x+h} m\,dt &\leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} M\,dt \\
m(x + h - x) &\leq \int_x^{x+h} f(t)\,dt \leq M(x + h - x) \\
mh &\leq \int_x^{x+h} f(t)\,dt \leq Mh \\
f(u)h &\leq \int_x^{x+h} f(t)\,dt \leq f(v)h \\
f(u) &\leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(v) \qquad \text{since } h > 0.
\end{aligned}$$

If $h < 0$, a similar argument shows that

$$f(v) \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq f(u).$$

In either case, if $h \to 0$, then $u, v \to x$, since $u, v \in [x, x + h]$. Therefore,

$$\lim_{h\to 0} f(u) = \lim_{h\to 0} f(v) = f(x)$$

since f is continuous on $[a, b]$. The Squeeze Theorem now applies and gives that

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - (g)}{h} = f(x).$$

(If $x = a, b$, then the above limit should be interpreted as a one-sided limit). □

Using alternate notation, the Fundamental Theorem of Calculus says that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Example 5.5.3. Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\,dt$.

First, notice that $f(t) = \sqrt{1+t^2}$ is continuous on all real numbers. Therefore the FTC applies and we have that

$$g'(x) = \frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt = \boxed{\sqrt{1+x^2}}.$$

Example 5.5.4. Find $\dfrac{d}{dx}\displaystyle\int_x^1 t^2\,dt$.

Although, t^2 is continuous on its domain, FTC1 doesn't exactly apply here. It only applies to the context $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt$. Thus, we need to move the variable x to the upper limit of the integral instead of the lower limit. Therefore,

$$\frac{d}{dx}\int_x^1 t^2\,dt = \frac{d}{dx}\left(-\int_1^x t^2\,dt\right) = -\frac{d}{dx}\int_1^x t^2\,dt = \boxed{-x^2}.$$

Example 5.5.5. Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\,dt$.

Although, $\sec t$ is continuous on its domain, FTC1 again doesn't exactly apply here. In conjugation with the chain rule, we can calculate the derivative. First let $u = x^4$. Then FTC1 gives us that

$$\frac{d}{du}\int_1^u \sec t\,dt = \sec u.$$

The Chain Rule then gives

$$\frac{d}{dx}\int_1^{x^4} \sec t\,dt = \frac{d}{du}\left(\int_1^u \sec t\,dt\right)\frac{du}{dx} = (\sec u)\frac{du}{dx} = \sec(x^4)\frac{d(x^4)}{dx} = \boxed{4x^3\sec x^4}.$$

Example 5.5.6. Find $\dfrac{d}{dx}\displaystyle\int_{\cos x}^{\sin x} \ln(1+2t)\,dt$.

Note that both the limits of the integral are functions of x. Thus, we need to adjust things before we can apply FTC1. Note that

$$\int_{\cos x}^{\sin x} \ln(1+2t)\,dt = \int_0^{\sin x} \ln(1+2t)\,dt + \int_{\cos x}^0 \ln(1+2t)\,dt = \int_0^{\sin x} \ln(1+2t)\,dt - \int_0^{\cos x} \ln(1+2t)\,dt.$$

Therefore,

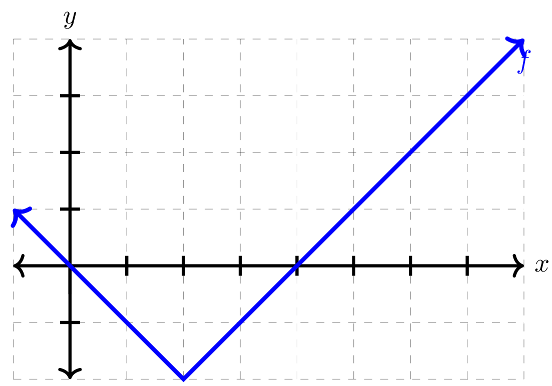
$$\begin{aligned}
\frac{d}{dx}\int_{\cos x}^{\sin x} \ln(1+2t)\,dt &= \frac{d}{dx}\left(\int_0^{\sin x} \ln(1+2t)\,dt - \int_0^{\cos x} \ln(1+2t)\,dt\right) \\
&= \frac{d}{dx}\int_0^{\sin x} \ln(1+2t)\,dt - \frac{d}{dx}\int_0^{\cos x} \ln(1+2t)\,dt \\
&= \ln(1+2\sin x)(\sin x)' - \ln(1+2\cos x)(\cos x)' \\
&= \boxed{\cos(x)\ln(1+2\sin x) + \sin(x)\ln(1+2\cos x)}
\end{aligned}$$

[i]See §5.3 The Fundamental Theorem of Calculus in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–10, let $g(x) = \int_0^x f(t)\,dt$, where the graph of f is illustrated to the right. Evaluate $g(x)$ for the given x value.



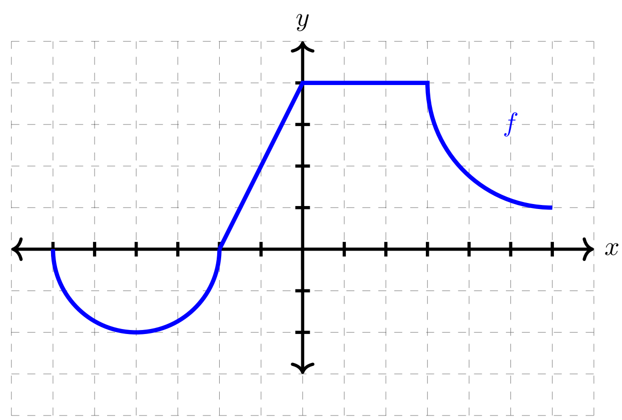
1. $g(0)$ ♠ 2. $g(1)$ 3. $g(2)$ 4. $g(3)$

♠ 5. $g(4)$ 6. $g(5)$ ♠ 7. $g(6)$ 8. $g(7)$

♠ 9. $g(8)$ 10. $g(-1)$

♠ 11. Considering still the function g from Exercises 1–10, where are the local extrema of g?

For Exercises 12–18, let $g(x) = \int_0^x f(t)\,dt$, where the graph of f is illustrated to the right. Evaluate $g(x)$ for the given x value.



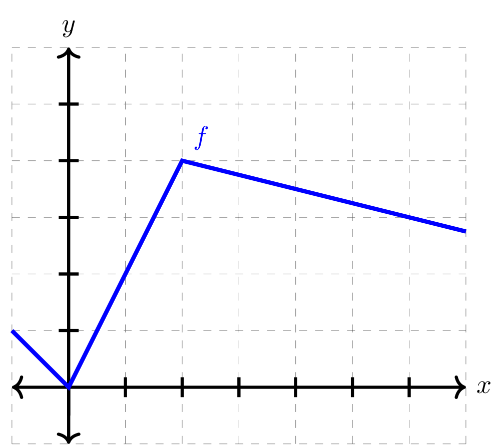
12. $g(2)$ ♠ 13. $g(3)$ ♠ 14. $g(6)$

♠ 15. $g(0)$ ♠ 16. $g(-2)$ ♠ 17. $g(-4)$

18. $g(-6)$

♠ 19. Considering still the function g from Exercises 12–18, where are the local extrema of g?

For Exercises 20–24, let $g(x) = \int_1^x f(t)\,dt$, where the graph of f is illustrated to the right. Evaluate $g(x)$ for the given x value.



20. $g(1)$ ♠ 21. $g(2)$

♠ 22. $g(6)$ ♠ 23. $g(0)$

24. $g(-1)$

♠ 25. Considering still the function g from Exercises 20–24, where are the local extrema of g?

For Exercises 26–41, find the derivative of f.

26. $y = \int_0^x \sqrt{t^2+4}\,dt$

27. $f(x) = \int_3^x e^{t^2-t}\,dt$

♠ 28. $g(x) = \int_1^x \frac{7}{t^3+1}\,dt$

♠ 29. $g(x) = \int_1^x e^{4t^2-3t}\,dt$

♠ 30. $f(x) = \int_x^\pi \sqrt{8+\sec(5t)}\,dt$

31. $y = \int_x^1 \cos\sqrt{t}\,dt$

Note $\int_x^a f(t)\,dt = -\int_a^x f(t)\,dt.$

♠ 32. $f(x) = \int_x^4 \cos(\sqrt{9t})\,dt$

33. $g(x) = \int_1^{\sqrt{x}} \frac{t^2}{t^4+1}\,dt$

34. $h(x) = \int_0^{x^4} \cos^2\theta\,d\theta$

♠ 35. $y = \int_1^{\tan x} \sqrt{2t+\sqrt{t}}\,dt$

♠ 36. $y = \int_{4-3x}^1 \frac{t^3}{1+t^2}\,dt$

♠ 37. $f(x) = \int_{2x}^{5x} \frac{t^2-1}{t^2+1}\,dt$

$$\int_{u(x)}^{v(x)} f(t)\,dt = \int_{u(x)}^{a} f(t)\,dt + \int_a^{v(x)} f(t)\,dt \text{ for any } a \in \operatorname{dom} f$$

♠ 38. $y = \int_x^{x^2} e^{t^4}\,dt$

♠ 39. $y = \int_{\cos x}^{\sin x} \ln(4+3t)\,dt$

40. $f(x) = \int_{\sin x}^1 \sqrt{1+t^2}\,dt$

41. $g(x) = \int_1^{e^x} \ln t\,dt$

"Our progress as a nation can be no swifter than our progress in education. The human mind is our fundamental resource." – John F. Kennedy

Lecture Videos

The Fundamental Theorem of Calculus - Part 2

Computing Integrals using the Fundamental Theorem of Calculus

Finding Areas using the Fundamental Theorem of Calculus

The Limitations of the Fundamental Theorem of Calculus

5.6 The Fundamental Theorem of Calculus II

The first part FTC determines that derivatives and integrals are actually inverse operations. The second part of FTC shows us how to use antiderivatives to compute definite integrals.

Theorem 5.6.1 (The Fundamental Theorem of Calculus, Part 2)**.** *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where F is any *antiderivative of f, that is, $F' = f$.*

Proof. Let $g(x) = \displaystyle\int_a^x f(t)\,dt$. We know that $g'(x) = f(x)$ by FTC1, that is, g is an antiderivative of f. We know from a previous result that the antiderivatives g and F can only differ by a constant, that is,

$$F(x) = g(x) + C$$

on (a, b). Using continuity of F and g, we also get that $F(a) = g(a) + C$ and $F(b) = g(b) + C$.

Therefore,

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] = g(b) - g(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

□

There are two important remarks to be made here. First, we should mention that the Fundamental Theorem does not give a definition of the definite integral. As discussed before, the definite integral is the area under the curve and is the limit of the areas of the rectangles as the number of rectangles tends toward infinity. The Fundamental Theorem gives us a powerful tool to compute this area without relying on the limit calculation.

Second, the Fundamental Theorem says that we may use any antiderivative of f, that is, we may choose the constant value C to be anything, like $C = 0$. Suppose that $F'(x) = f(x)$ and let C be any real number. Then

$$\int_a^b f(x)\,dx = (F(x) + C)\Big|_a^b$$

$$\begin{aligned} &= (F(b)+C)-(F(a)+C) \\ &= F(b)-F(a) \end{aligned}$$

Therefore, the constant value is irrelevant using computing definite integrals.

Example 5.6.2. Evaluate the following integrals:

(a) $\displaystyle\int_1^3 e^x\,dx = e^x\Big|_1^3 = \boxed{e^3-e}.$

(b) $\displaystyle\int_3^6 \frac{dx}{x} = \int_3^6 \frac{1}{x}\,dx = \ln|x|\Big|_3^6 = \ln 6 - \ln 3 = \ln\frac{6}{3} = \boxed{\ln 2}.$

(c) $\displaystyle\int_1^2 4t^3\,dt = t^4\Big|_1^2 = (2)^4-(1)^4 = 16-1 = \boxed{15}.$

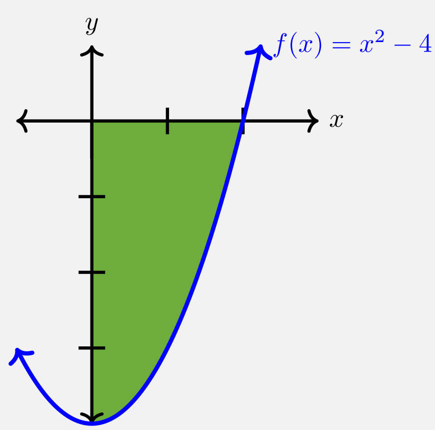
(d)

$$\begin{aligned} \int_2^5 (6x^2-3x+5)\,dx &= 6\int_2^5 x^2\,dx - 3\int_2^5 x\,dx + 5\int_2^5 dx = 2x^3\Big|_2^5 - \frac{3}{2}x^2\Big|_2^5 + 5x\Big|_2^5 \\ &= 2(5^3-2^3) - \frac{3}{2}(5^2-2^2) + 5(5-2) = 2(128-8) - \frac{3}{2}(25-4) + 5(3) \\ &= 234 - \frac{63}{2} + 15 = \boxed{\frac{435}{2}}. \end{aligned}$$

Example 5.6.2 illustrates the difference between the definite integral and the indefinite integral. A definite integral is a real number; an indefinite integral is a family of functions in which all the functions are antiderivatives of a function f.

Example 5.6.3. Find the area of the region under the curve $f(x) = x^2-4$ from 0 to 2.

Remember that by definition $\displaystyle\int_0^2 f(x)\,dx$ is the area between the curve $f(x)$ and the x-axis. So, we compute



$$\begin{aligned} \int_0^2 (x^2-4)\,dx &= \left(\frac{x^3}{3}-4x\right)\Big|_0^2 = \left(\frac{2^3}{3}-4(2)\right)-0 \\ &= \frac{8}{3}-8 = \boxed{-\frac{16}{3}} \end{aligned}$$

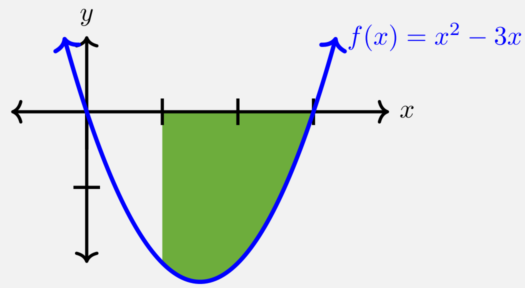
The result is a negative number because $f(x)$ is negative for values of x in the interval $[0,2]$. Since Δx is always positive, $\displaystyle\int_a^b f(x)\,dx$ is negative if $f(x)$ is negative on the interval $[a,b]$. In many practical application (such as the next section), we would want the rectangle below the x-axis to be negative. Since area should be positive, we take the absolute value. That is, the area of the region is $\boxed{16/3}$.

Example 5.6.4. Find the area of the region between the x-axis and the graph $f(x) = x^2-3x$ from $x=1$ to $x=3$.

Notice, that $f(x) = x(x-3) = 0$ when $x = 0$ and $x = 3$. Additionally, $f(1) = -2$. Therefore, the entire region is contained underneath the x-axis and $\int_1^3 f(x)\,dx < 0$. So, the area of the region is equal to $\left|\int_1^3 (x^2 - 3x)\,dx\right|$.

Now,

$$\begin{aligned}\int_1^3 (x^2-3x)\,dx &= \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_1^3 \\ &= \left(\frac{27}{3} - \frac{27}{2}\right) - \left(\frac{1}{3} - \frac{3}{2}\right) \\ &= -\frac{10}{3}.\end{aligned}$$



Therefore, the area is $\boxed{|-10/3| = 10/3}$.

Example 5.6.5. Find the area under the cosine curve from 0 to b, where $0 \le b \le \pi/2$.

Since $\int \cos x\,dx = \sin x + C$, we have

$$\int_0^{\pi/2} \cos x\,dx = \sin x\Big|_0^{\pi/2} = \sin(\pi/2) - \sin 0 = \boxed{1}.$$

Example 5.6.6. What is wrong with the following calculation?

$$\int_{-1}^3 \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Big|_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}$$

The Fundamental Theorem of Calculus applies to continuous functions. It can't be applied here because $f(x) = 1/x^2$ is not continuous on $[-1, 3]$. In fact, f has a vertical asymptote at $x = 0$ and

$$\int_{-1}^3 \frac{1}{x^2}\,dx = \boxed{\infty}.$$

[i] Additionally, a former student created the following video that explains the basics of integrals with examples: https://youtu.be/6HlsJ2IudVA. As some of the topics go beyond what we have learned so far, do not feel like you should watch the whole video. Instead, revisit it when we have learned the next topic.

[ii] See §5.3 The Fundamental Theorem of Calculus in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

For Exercises 1–19, evaluate the integral.

1. $\displaystyle\int_1^8 x^{-2/3}\,dx$

♠ 2. $\displaystyle\int_{-2}^3 (x^3-4)\,dx$

♠ 3. $\displaystyle\int_2^4 (6-2y+3y^2)\,dy$

♠ 4. $\displaystyle\int_4^9 \sqrt{t}\,dt$

5. $\displaystyle\int_{-2}^2 (3x+1)^2\,dx$

♠ 6. $\displaystyle\int_0^1 (u+3)(u-4)\,du$

♠ 7. $\displaystyle\int_1^4 \frac{u-7}{\sqrt{u}}\,du$

8. $\displaystyle\int_0^4 (4-x)\sqrt{x}\,dx$

9. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

♠ 10. $\displaystyle\int_0^\pi 7\sin\theta\,d\theta$

11. $\displaystyle\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta$

♠ 12. $\displaystyle\int_{\pi/4}^\pi \sin\theta\,d\theta$

13. $\displaystyle\int_{-\pi/3}^{\pi/3} x^4\sin x\,dx$

♠ 14. $\displaystyle\int_0^{\pi/4} 7\sec^2\theta\,d\theta$

♠ 15. $\displaystyle\int_0^7 \cosh t\,dt$

16. $\displaystyle\int_0^{\ln 2} \cosh x\,dx$

♠ 17. $\displaystyle\int_{-1}^1 e^{u+1}\,du$

♠ 18. $\displaystyle\int_{-2}^3 f(x)\,dx$ if $f(x)=\begin{cases}3, & -2\le x\le 0\\ 4-x^2, & 0<x\le 3\end{cases}$

19. $\displaystyle\int_{-1}^1 e^{x+1}\,dx$

For Exercises 20–20, find the area bounded by the given curves.

20. $y=x^2+x-2$ and the x-axis

"Progress is impossible without change, and those who cannot change their minds cannot change anything."
– George Bernard Shaw

Lecture Videos

The Net Change Theorem | Integrals and Displacement | The Net Change Theorem and Cost | The Net Change Theorem and Growth

5.7 Rates of Changes within the Sciences II

Suppose that some function f measures some scientific quantity. Then $f(b)-f(a)$ measures the *net change* of f on the interval $[a,b]$. By the the Fundamental Theorem of Calculus, we know that $f(b)-f(a)=\int_a^b f'(x)\,dx$. Hence, the net change of f can be computed by integrating its rate of change, aka derivative, f'. By this observation, we can see that the Fundamental Theorem of Calculus can be applied to all the rates of change in the sciences that we discussed in Section 3.9. For this reason, the Fundamental Theorem is sometimes called the *Net Change Theorem*. This observation about net change gives an intuitive justification of the Fundamental Theorem.

Example 5.7.1.

(a) If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into the reservoir at time t. So,

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and t_2.

(b) If $[C](t)$ is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[C]/dt$. So,

$$\int_{t_1}^{t_2} \frac{d[C]}{dt}\,dt = [C](t_2) - [C](t_1)$$

is the change in the concentration of C from time t_1 to t_2.

(c) If the mass of a rod measured from left end to a point x is $m(x)$, then the linear density (see Example 3.9.1) is $\rho(x) = m'(x)$. So,

$$\int_a^b \rho(x)\,dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

(d) If the rate of growth of a population is dn/dt (see Example 3.9.3), then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period t_1 to t_2. The population increases when births happen and decreases when deaths occur. The net population takes into account both births and deaths.

(e) If $C(x)$ is the cost of producing x units of commodity (see Example 3.9.4), then the marginal cost is the derivative $C'(x)$. So,

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from x_1 units to x_2 units.

(f) If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$. So,

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2.

(g) If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

$$\int_{t_1}^{t_2} |v(t)|\,dt = \text{total distance traveled.}$$

Example 5.7.2. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

We have

$$\begin{aligned} s(4) - s(1) &= \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt \\ &= \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Bigg|_1^4 \\ &= \left(\frac{64}{3} - \frac{16}{2} - 24\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right) \\ &= \left(\frac{64}{3} - 32\right) - \left(\frac{2}{6} - \frac{3}{6} - \frac{36}{6}\right) \\ &= -\frac{32}{3} + \frac{37}{6} = \frac{27}{6} = -\frac{9}{2}. \end{aligned}$$

This means that the particle moved $\boxed{4.5 \text{ m}}$ toward the left.

(b) Find the distance traveled during the time period.

Note that $v(t) = t^2 - t - 6 = (t-3)(t+2)$ and so $v(t) \leq 0$ on the interval $[1, 3]$ and $v(t) \geq 0$

on $[3,4]$. Thus, to compute distance we have

$$\begin{aligned}
\int_1^4 |v(t)|\,dt &= \int_1^3 |v(t)|\,dt + \int_3^4 |v(t)|\,dt = \left|\int_1^3 v(t)\,dt\right| + \left|\int_3^4 v(t)\,dt\right| \\
&= \left|\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^3\right| + \left|\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4\right| \\
&= \left|\left(\frac{27}{3} - \frac{9}{2} - 18\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right)\right| + \left|\left(\frac{64}{3} - \frac{16}{2} - 24\right) - \left(\frac{27}{3} - \frac{9}{2} - 18\right)\right| \\
&= \left|\left(\frac{54}{6} - \frac{27}{6} - \frac{108}{6}\right) + \frac{37}{6}\right| + \left|-\frac{32}{3} - \left(\frac{54}{6} - \frac{27}{6} - \frac{108}{6}\right)\right| \\
&= \left|-\frac{81}{6} + \frac{37}{6}\right| + \left|-\frac{32}{3} + \frac{81}{6}\right| \\
&= \left|-\frac{44}{6}\right| + \left|\frac{17}{6}\right| = \frac{44}{6} + \frac{17}{6} = \frac{61}{6} \approx \boxed{10.17 \text{ m}}.
\end{aligned}$$

Example 5.7.3. The marginal cost of manufacturing x yards of a certain fabric is

$$C'(x) = 3 - 0.01x + 0.000006x^2 \quad \text{(in dollars per yard).}$$

Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.

The increase of cost is equal to $\int_{2000}^{4000} C'(x)\,dx$. Thus,

$$\int_{2000}^{4000} (3 - 0.01x + 0.000006x^2)\,dx = (3x - 0.005x^2 + 0.000002x^3)\Big|_{2000}^{4000}$$

$$= (3(4000) - 0.005(4000)^2 + 0.000002(4000)^3) - (3(2000) - 0.005(2000)^2 + 0.000002(2000)^3)$$

$$= 3(4000 - 2000) - 0.005(4000^2 - 2000^2) + 0.000002(4000^3 - 2000^3)$$

$$= 3(2,000) - 0.005(12,000,000) + 0.000002(56,000,000,000) = 6,000 - 60,000 + 112,000 = 58,000$$

In other words, the cost will increase by $\boxed{\$58,000 \text{ dollars}}$.

Example 5.7.4. A bacteria population is 4000 at time $t = 0$ and its rate of growth is $1000 \cdot 2^t$ bacteria per hour after t hours. What is the population after one hour.

If $N(t)$ is the population after t hours, then $N'(t) = 1000 \cdot 2^t$. Note that

$$\int_0^1 N'(t)\,dt = N(1) - N(0) = N(1) - 4000.$$

Thus, $N(1) = 4000 + \int_0^1 N'(t)\,dt$. Thus,

$$\begin{aligned}
N(1) &= 4000 + \int_0^1 1000 \cdot 2^t\,dt = 4000 + 1000\int_0^1 2^t\,dt \\
&= 4000 + \frac{1000}{\ln 2} \cdot 2^t\Big|_0^1 = 4000 + \frac{1000}{\ln 2}(2^1 - 1)
\end{aligned}$$

$$= \quad 4000 + \frac{1000}{\ln 2} \approx 5442.700$$

In other words, there will be about $\boxed{5443}$ bacteria after one hour.

[i] See §5.4 Integration Formulas and the Net Change Theorem in OpenStax to find the corresponding section.

Exercises

(Go to Solutions)

1. Consider a particle moving along a straight line with a velocity given by $v(t) = 3t^2 - 12t + 9$, in meters per second, where t is measured in seconds. What does $\int_0^5 (3t^2 - 12t + 9)\,dt$ measure relative to position?

♠ 2. If I is the current of an electric change Q moving through a copper wire (see Exercise 3.9.15), then $I(t) = \frac{dQ}{dt}$. What does $\int_a^b I(t)\,dt$ measure relative to $Q(t)$?

♠ 3. Consider a hiker walking along a trail whose grade (the rate at which elevation climbs) at a distance x miles from the trailhead is given by $y = f(x)$. What does $\int_7^9 f(x)\,dx$ measure relative to the hiker?

4. Consider a certain bacteria culture which grows at a rate modeled by $n(t) = 50e^{0.1t}$, in bacteria per hour, where t is the time in hours. What does $\int_0^{10} 50e^{0.1t}\,dt$ measure relative to the population of the culture?

♠ 5. Consider a rancher who measures the mass of a calf, $m(t)$, in kg at month t since birth. What does $\int_8^{11} m'(t)\,dt$ measure relative to the calf?

♠ 6. Consider a non-homogeneous rod of length 4 m whose linear density (see Example 3.9.1) is given as $\rho(x) = 6 + 4\sqrt{x}$ in kilograms per meter. Find the total mass of the rod.

♠ 7. Consider a water tank which drains at a rate of $\frac{dV}{dt} = 200 - 4t$ liters per minute, for $0 \le t \le 50$. Find the amount of water which drained from the tank during the first 45 minutes, assuming the tank never emptied in that time.

8. Consider water flowing into a retention pond at a rate of $I(t) = 100\sin\left(\frac{\pi t}{12}\right) + 200$, and water flowing out of the pond at a rate of $O(t) = 150 + 5t$, both measured in gallons per hour, where t is the time in hours since midnight. Let $R(t) = I(t) - O(t)$ be the net rate of change of water in the pond. Find the net change of water in the pond from 6 am to 6 pm.

For Exercises 9–14, a particle moves along a line according to the position function $s(t)$, in meters. Find the displacement and the total distance of the particle, given its velocity or acceleration, on the time interval $[a, b]$.

9. $v(t) = 3t^2 - 12t + 9$, $[0, 5]$

♠ 10. $v(t) = 3t - 7$, $[0, 4]$

♠ 11. $v(t) = t^2 - 2t - 24$, $[1, 7]$

♠ 12. $a(t) = t + 8$, $v(0) = 6$, $[0, 10]$

♠ 13. $a(t) = 2t + 2$, $v(0) = -15$, $[0, 5]$

14. $a(t) = -\frac{4\pi^2}{3}\cos\left(\frac{2\pi}{3}t\right)$, $v(0) = 0$, $[0, 6]$

"Be brave. Take risks. Nothing can substitute experience." – Paulo Coelho

Lecture Videos

What is u-Substitution?

u-Substitution and Indefinite Integrals

Examples of Finding Antiderivatives Using u-Substitution

5.8 u-Substitution

In previous sections, we discussed ways of anti-differentiating the power rule, the sum/difference rule, and constant multiple rule of differentiation. We also discussed the antiderivatives of exponential, logarithmic functions, and a few trigonometric and inverse trigonometric functions. How might we go about reversing the product or quotient rule of derivatives? How about the chain rule? Since these derivative rules were a little more complicated, you can imagine that their antiderivatives will also be a little more complicated than what we have already seen, but nothing out of our reach. This section discusses the method of u-substitution and can be thought of as the antidifferentiation version of the chain rule.

If $u = f(x)$, then $\dfrac{du}{dx} = f'(x)$. As the notation suggests, we may multiply both sides of the equation by the differential dx, which gives

$$du = f'(x)\, dx.$$

Therefore, if $g(u)$ is a function of u, then

$$\int g(u)du = \int g(f(x)) \cdot f'(x)\, dx \tag{5.8.1}$$

Remember that the chain rule tells us that

$$\frac{d}{dx}[g(f(x))] = g'(f(x)) \cdot f'(x).$$

Multiplying both sides by the differential gives

$$d[g(f(x))] = g'(f(x)) \cdot f'(x)\, dx,$$

which justifies us thinking of u-substitution as the opposite of the chain rule. We illustrate the substitution method via example.

Example 5.8.1. Find $\displaystyle\int (2x^3+1)^4 6x^2\, dx$.

One could expand out the polynomial and use the power rule, but like differentiation and the chain rule, u-substitution simplifies the work. Let $u = 2x^3+1$. Then $du = 6x^2dx$. Using these substitutions, observe

$$\int (2x^3+1)^4(6x^2\, dx) = \int u^4 du = \left(\frac{u^5}{5}\right) + C = \boxed{\frac{1}{5}(2x^3+1)^5 + C}.$$

We can verify our result by taking the derivative:

$$\left(\frac{1}{5}(2x^3+1)^5 + C\right)' = \frac{1}{5}\left[5(2x^3+1)^4\right](6x^2) = (2x^3+1)^4 6x^2.$$

Example 5.8.2. Find the indefinite integral using u-substitution.

(a) $\displaystyle\int 6x(3x^2+4)^7\,dx$.

Let us use the substitution $u = 3x^2+4$ and $du = 6x\,dx$. Then

$$\begin{aligned}\int 6x(3x^2+4)^7\,dx &= \int (3x^2+4)^7(6x\,dx) = \int u^7\,du \\ &= \frac{u^8}{8}+C = \boxed{\frac{1}{8}(3x^2+4)^8+C}\end{aligned}$$

(b) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$.

This time use the substitution $u = x^3+1$ and $du = 3x^2\,dx$. Unfortunately, there is no $3x^2\,dx$ in the integrand. But we do have $x^2\,dx$, which is only off by a constant multiple. We can repair this problem by multiplying the integrand by a strategically chosen number 1.

$$\begin{aligned}\int x^2\sqrt{x^3+1}\,dx &= \frac{3}{3}\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int\sqrt{x^3+1}(3x^2\,dx) \\ &= \frac{1}{3}\int\sqrt{u}\,du = \frac{1}{3}\left(\frac{2}{3}u^{3/2}\right)+C \\ &= \frac{2}{9}u^{3/2}+C = \boxed{\frac{2}{9}(x^3+1)^{3/2}+C}\end{aligned}$$

The substitution method given in the examples above *will not always work*. For example, you might try to find

$$\int x^3\sqrt{x^3+1}\,dx$$

by substituting $u = x^3+1$ and $du = 3x^2\,dx$. However, this substitution cannot remove all of the x's, that is,

$$\int x^3\sqrt{x^3+1}\,dx = \frac{1}{3}\int x\sqrt{u}\,du.$$

For u-substitution to be able to work, all the variables x in the integrand must be replaced with the new variable u in order to integrate. For this reason, $\displaystyle\int x^3\sqrt{x^3+1}\,dx$ cannot be computed using u-substitution.

With practice, choosing u will become easier if you keep two principles in mind.

(i) u should equal some expression in the integrand that, when replaced with u, tends to make the integrand simpler.

(ii) u must be an expression whose derivative — disregarding any constant multiplier — is also present in the integrand.

Example 5.8.3. Find the indefinite integrals.

(a) $\displaystyle\int \frac{x+3}{(x^2+6x)^2}\,dx$.

Let $u = x^2+6x$ and $du = (2x+6)\,dx = 2(x+3)\,dx$. We have $x+3$ in the numerator but are missing the multiplier 2. If we rescale like we did previously, we have

$$\int \frac{x+3}{(x^2+6x)^2}\,dx = \frac{1}{2}\int\frac{2(x+3)}{(x^2+6x)^2}\,dx = \frac{1}{2}\int\frac{du}{u^2}$$

$$= \frac{1}{2}\left(\frac{-1}{u}\right) + C = \boxed{-\frac{1}{2(x^2+6x)} + C}.$$

(b) $\displaystyle\int \frac{2x-3}{x^2-3x}\,dx$.

Use the substitution $u = x^2 - 3x$ and $du = (2x-3)\,dx$. Then the integral can be rewritten as

$$\int \frac{2x-3}{x^2-3x}\,dx = \int \frac{du}{u}.$$

Remember that $\dfrac{d}{dx}[\ln|f(x)|] = \dfrac{f'(x)}{f(x)}$. Hence,

$$\int \frac{du}{u} = \ln|u| + C = \boxed{\ln|x^2-3x| + C}.$$

(c) $\displaystyle\int e^{5x}\,dx$.

Use the substitution $u = 5x$, then $du = 5\,dx$. Then the integral can be rewritten as

$$\int e^{5x}\,dx = \frac{1}{5}\int e^{5x}(5\,dx) = \frac{1}{5}\int e^u\,du.$$

Therefore,

$$\frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C = \boxed{\frac{1}{5}e^{5x} + C}.$$

(d) $\displaystyle\int x^2 e^{x^3}\,dx$.

Use the substitution $u = x^3$ and $du = 3x^2\,dx$. Thus,

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du.$$

Since $\dfrac{d}{dx}[e^{f(x)}] = f'(x)e^{f(x)}$, we have that

$$\frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \boxed{\frac{1}{3}e^{x^3} + C}.$$

(e) $\displaystyle\int x^3\cos(x^4+2)\,dx$.

Use the substitution $u = x^4 + 2$, then $du = 4x^3\,dx$. Then the integral can be rewritten as

$$\int x^3\cos(x^4+2)\,dx = \frac{1}{4}\int \cos(x^4+2)(4x^3\,dx) = \frac{1}{4}\int \cos u\,du.$$

Therefore,

$$\frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \boxed{\frac{1}{4}\sin(x^4+2) + C}.$$

[i]See §5.5 Substitution and §5.6 Integrals Involving Exponential and Logarithmic Functions in OpenStax to find the corresponding sections.

Exercises

(Go to Solutions)

For Exercises 1–19, evaluate the integral.

1. $\displaystyle\int 2e^{2x}\,dx$
   $u = 2x$

♠ 2. $\displaystyle\int x^3(7+x^4)^6\,dx$
   $u = 7 + x^4$

♠ 3. $\displaystyle\int e^{-2x}\,dx$
   $u = -2x$

♠ 4. $\displaystyle\int x^2\sqrt{x^3+29}\,dx$
   $u = x^3 + 29$

♠ 5. $\displaystyle\int \frac{dx}{(1-5x)^9}$
   $u = 1 - 5x$

♠ 6. $\displaystyle\int \cos^9\theta\sin\theta\,d\theta$
   $u = \cos\theta$

♠ 7. $\displaystyle\int \frac{\sec^2(1/x^5)}{x^6}\,dx$
   $u = \dfrac{1}{x^5}$

8. $\displaystyle\int \frac{3-2x}{(x^2-3x)^{1/3}}\,dx$

9. $\displaystyle\int (\sec\theta\tan\theta + 6e^{2\theta})\,d\theta$

10. $\displaystyle\int x\sqrt{4-x^2}\,dx$

♠ 11. $\displaystyle\int x^{11}\sin(x^{12})\,dx$

♠ 12. $\displaystyle\int \frac{\ln^{10}x}{x}\,dx$

♠ 13. $\displaystyle\int \sec^2\theta\tan^4\theta\,d\theta$

14. $\displaystyle\int xe^{-x^2}\,dx$

♠ 15. $\displaystyle\int e^x\sqrt{7+e^x}\,dx$

16. $\displaystyle\int e^{\sin x}\cos(x)\,dx$

17. $\displaystyle\int e^x\cos(e^x)\,dx$

♠ 18. $\displaystyle\int \frac{\cos x}{\sin^{20}x}\,dx$

19. $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

"There is no substitute for hard work." – Thomas A. Edison

Lecture Videos

u-Substitution When the Inner Derivative Isn't Quite Right.

The Antiderivative of Tangent

u-Substitution and Definite Integrals

Definite Integrals and Symmetry

5.9 u-Substitution II

Example 5.9.1. Find the indefinite integrals.

(a) $\displaystyle\int x\sqrt{1-x}\,dx.$

Use the substitution $u = 1-x$ and $du = -dx$. Then $\int x\sqrt{1-x}\,dx = -\int x\sqrt{u}\,du$. But wait a minute; there is still an x in the integrand. Can substitution still work here?! Notice that the equation $u = 1-x$ can be explicitly solved for x, giving $x = 1-u$. If we use also this substitution, then our problem is avoided. So,

$$\begin{aligned}\int x\sqrt{1-x}\,dx &= -\int (1-u)\sqrt{u}\,du = -\int [\sqrt{u} - u\sqrt{u}]\,du \\ &= -\int \left[u^{1/2} - u^{3/2}\right]\,du = -\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C \\ &= \boxed{\frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C}\end{aligned}$$

(b) $\displaystyle\int x^5\sqrt{1+x^2}\,dx.$

Like in the previous example, let $u = 1+x^2$ and $du = 2x\,dx$. Now, we can obtain an x by factoring $x^5 = x^4 \cdot x$. But what do we do with the extra x^4? Similar to above, we note that $x^2 = u-1 \quad\Rightarrow\quad x^4 = (x^2)^2 = (u-1)^2$. Therefore, we can rewrite the integral as

$$\int x^5\sqrt{1+x^2}\,dx = \frac{1}{2}\int (x^2)^2\sqrt{1+x^2}(2x\,dx) = \frac{1}{2}\int \sqrt{u}(u-1)^2\,du.$$

Although this new integrand is much simpler than before, it still is somewhat complicated. We might be tempted to try another substitution, which will not be beneficial. Instead, we must expand the integrand in order to integrate.

$$\begin{aligned}\frac{1}{2}\int \sqrt{u}(u-1)^2\,du &= \frac{1}{2}\int \sqrt{u}(u^2-2u+1)\,du = \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)\,du \\ &= \frac{1}{2}\left[\frac{2u^{7/2}}{7} - \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3}\right] + C \\ &= \boxed{\frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C}.\end{aligned}$$

Example 5.9.2. Show that $\int \tan x\, dx = \ln|\sec x| + C$.

First note that $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$. Let $u = \cos x$ and $du = -\sin x\, dx$. Thus,

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx = -\int \frac{du}{u} = -\ln|u| + C$$
$$= -\ln|\cos x| + C = \ln(1/|\cos x|) + C = \ln|\sec x| + C.$$

Example 5.9.3. Evaluate $\int_0^5 x\sqrt{25 - x^2}\, dx$.

The first method will be to calculate the antiderivative like before and evaluate the antiderivative on the boundary of the interval. So, if we use the substitution $u = 25 - x^2$ and $du = -2x\, dx$, we have

$$\begin{aligned} \int x\sqrt{25-x^2}\, dx &= -\frac{1}{2}\int \sqrt{25-x^2}(-2x\, dx) = -\frac{1}{2}\int \sqrt{u}\, du \\ &= -\frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C = -\frac{u^{3/2}}{3} + C \\ &= -\frac{(25-x^2)^{3/2}}{3} + C \end{aligned}$$

Then

$$\int_0^5 x\sqrt{25-x^2}\, dx = -\frac{(25-x^2)^{3/2}}{3}\Bigg|_0^5 = 0 - \left(-\frac{25^{3/2}}{3}\right) = \boxed{\frac{125}{3}}.$$

Alternately, instead of substituting back in x for the indefinite integral and evaluating on the limits, we could substitute new limits into the definite integral with respect to our u-substitution. For example, again let $u = 25 - x^2$ and $du = -2x\, dx$. If $x = 0$, then $u = 25 - 0^2 = 25$. Likewise, if $x = 5$, then $u = 25 - 5^2 = 0$. Therefore,

$$\begin{aligned} \int_{x=0}^{x=5} x\sqrt{25-x^2}\, dx &= -\frac{1}{2}\int_{u=25}^{u=0} \sqrt{u}\, du = \frac{1}{2}\int_0^{25} \sqrt{u}\, du \\ &= \frac{1}{2}\frac{u^{3/2}}{3/2}\Bigg|_0^{25} = \frac{u^{3/2}}{3}\Bigg|_0^{25} \\ &= \frac{25^{3/2}}{3} - 0 = \boxed{\frac{125}{3}} \end{aligned}$$

The advantage of this second method is that we do not need to reintroduce x into the integrand after we substitute u into it and is often more efficient.

Example 5.9.4. Evaluate $\int_1^e \frac{\ln x}{x}\, dx$.

Like before, let us calculate the definite integral by changing the limits. Let $u = \ln x$ and $du = \frac{dx}{x}$. Changing the limits gives $x = 1 \Rightarrow \ln 1 = 0$ and $x = e \Rightarrow \ln e = 1$. Therefore,

$$\int_1^e \frac{\ln x}{x}\, dx = \int_0^1 u\, du = \frac{u^2}{2}\Bigg|_0^1 = \boxed{\frac{1}{2}}.$$

Theorem 5.9.5 (Integrals of Symmetric Functions)**.** *Suppose that f is continuous on $[-a,a]$.*

(i) If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

(ii) If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

Proof. First we note that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.$$

We will use a u-substitution on the first integral on the far right. Specifically, let $u = -x$ and $du = -dx$. Changing the bounds gives $x = -a \quad\Rightarrow\quad u = a$ and $x = 0 \quad\Rightarrow\quad u = 0$. Thus,

$$-\int_0^{-a} f(x)\,dx = \int_0^a f(-u)\,du.$$

Suppose now that f is even. Then $f(-u) = f(u)$. Thus,

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Suppose now that f is odd. Then $f(-u) = -f(u)$. Thus,

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx = -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0.$$

□

Example 5.9.6. Calculate the following integrals.

(a) $\displaystyle\int_{-2}^{2} (x^6+1)\,dx.$

First, we observe that $f(x) = x^6 + 1$ is even, since

$$f(-x) = (-x)^6 + 1 = x^6 + 1 = f(x).$$

Thus,

$$\begin{aligned}\int_{-2}^{2} (x^6+1)\,dx &= 2\int_0^2 (x^6+1)\,dx \\ &= 2\left[\frac{x^7}{7} + x\right]_0^2 = 2\left(\frac{2^7}{7} + 2\right) \\ &= \frac{256}{7} + 4 = \boxed{\frac{284}{7}}.\end{aligned}$$

(b) $\displaystyle\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx.$

First, we observe that $f(x) = \dfrac{\tan x}{1+x^2+x^4}$ is odd, since

$$f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4} = \frac{-\tan x}{1+x^2+x^4} = -\frac{\tan x}{1+x^2+x^4} = -f(x).$$

Therefore,

$$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx = \boxed{0}.$$

[i]See §5.5 Substitution and §5.6 Integrals Involving Exponential and Logarithmic Functions in OpenStax to find the corresponding sections.

Exercises

(Go to Solutions)

For Exercises 1–20, evaluate the integral.

1. $\displaystyle\int 2x(8x^2+4)^{100}\,dx$

2. $\displaystyle\int \frac{x^2}{(x^3-8)^{100}}\,dx$

3. $\displaystyle\int (x+1)\sqrt{2x+x^2}\,dx$

♠ 4. $\displaystyle\int (x+1)\sqrt{2x+x^2}\,dx$

♠ 5. $\displaystyle\int \frac{e^y}{(7-e^y)^2}\,dy$

♠ 6. $\displaystyle\int x^{5/2}\sin(5+x^{7/2})\,dx$

7. $\displaystyle\int \cos x\sin(\sin x)\,dx$

♠ 8. $\displaystyle\int e^{\tan(3x)}\sec^2(3x)\,dx$

♠ 9. $\displaystyle\int \frac{\sin(\ln(x))}{x}\,dx$

10. $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

♠ 11. $\displaystyle\int \frac{x^5}{1+x^{12}}\,dx$

♠ 12. $\displaystyle\int \frac{4+5x}{1+x^2}\,dx$

♠ 13. $\displaystyle\int x(8x+7)^8\,dx$

14. $\displaystyle\int \frac{x}{\sqrt{1+2x}}\,dx$

♠ 15. $\displaystyle\int x^3\sqrt{x^2+11}\,dx$

16. $\displaystyle\int \frac{x}{1+x^4}\,dx$

♠ 17. $\displaystyle\int_0^1 \sqrt[3]{1+7x}\,dx$

♠ 18. $\displaystyle\int_0^{\pi} \sec^2\left(\frac{\theta}{4}\right)\,d\theta$

♠ 19. $\displaystyle\int_1^2 \frac{e^{1/t^3}}{t^4}\,dt$

20. $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx$

Appendix A

Solutions to Select Exercises

Chapter 1 : Precalculus

1.1 Functions

1. $[-5,4]$, $[1,5]$
2. $[-4,5]$, $[-4,5]$
3. $g(0)=4$
4. $x=-4,2$
5. $x=-1,2$
6. $[-5,-3)\cup(0,1)$
7. ♠ $(-\infty,0)\cup(0,2)$
8. odd
9. even
10. ♠ odd
11. ♠ even
12. ♠ neither
13. odd
14. neither
15. odd
16. even
17. odd
18. odd
19. ♠ neither
20. ♠ odd
21. odd
22. $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$
23. $(-\infty,-2)\cup(-2,-\frac{4}{5})\cup(-\frac{4}{5},\infty)$
24. $(-\infty,-\frac{7}{2})\cup(-\frac{7}{2},\frac{1}{3})\cup(\frac{1}{3},\infty)$
25. $(-\infty,-2)\cup(-2,\infty)$
26. $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$
27. $\left[-\frac{1}{2},\infty\right)$
28. $[-2,3]$
29. $[0,4]$
30. $(-\infty,0)\cup(6,\infty)$
31. ♠ $(-\infty,0)\cup(5,\infty)$
32. $\left[-\frac{1}{2},1\right)\cup(1,\infty)$
33. $2x+h$
34. $4x+2h$
35. ♠ -9
36. ♠ $2-2a-h$
37. ♠ $10a+5h-1$
38. $4x+1+2h$
39. ♠ $-\dfrac{7}{(a+4)(a+h+4)}$
40. $3x^2+3hx+h^2$

1.2 The Library of Algebraic Functions

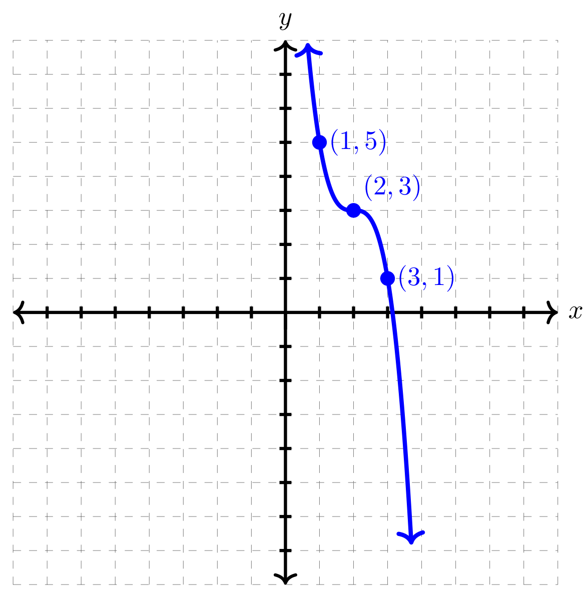
1. $y=|x|$
2. $y=\sqrt{x}$
3. $y=x^3$
4. ♠ $y=x^2$
5. ♠ $y=\dfrac{1}{x^2}$
6. ♠ $y=x^4$
7. $y=x$
8. $y=\sqrt[3]{x}$
9. $y=1$
10. $y=\dfrac{1}{x}$
11. $y=x^5$
12. $f(x)=\begin{cases}\frac{3}{2}x+\frac{1}{2}, & -3\le x<-1\\ x & -1\le x,\le 1\\ 2x-1, & 1<x\le 2\end{cases}$
13. ♠ $f(x)=\begin{cases}-x, & -1\le x<2\\ x-4, & 2\le x\le 8\end{cases}$
14. ♠ $f(x)=\begin{cases}2x+7, & x<-3\\ -x, & -3\le x\le 3\\ 2x-7, & x>3\end{cases}$
15. ♠ $f(x)=\begin{cases}-\frac{4}{3}x+1, & 0\le x<3\\ 4x-9, & x\ge 3\end{cases}$

16. $f(x) = \begin{cases} -1, & -3 \le x \le 3 \\ 2, & \text{otherwise} \end{cases}$

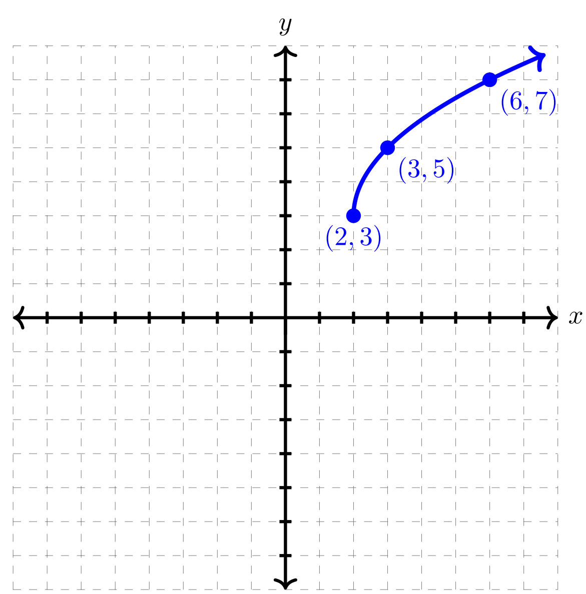
17. $f(x) = \begin{cases} -\frac{4}{3}, & 0 \le x < 3 \\ 4, & x \ge 3 \end{cases}$

1.3 Function Composition

1. 3
♠ 2. -1.5
♠ 3. 3
♠ 4. DNE
5. 4
6. $(f+g)(x) = x^3 + 8x^2 - 3, (-\infty, \infty)$; $(f-g)(x) = x^3 - 2x^2 + 3, (-\infty, \infty)$;
$(fg)(x) = 5x^5 + 15x^4 - 3x^3 - 9x^2, (-\infty, \infty)$; $(f/g)(x) = \dfrac{x^3 + 3x^2}{5x^2 - 3}, \left(-\infty, -\sqrt{\frac{3}{5}}\right) \cup \left(-\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right) \cup \left(\sqrt{\frac{3}{5}}, \infty\right)$
7. Stretch the graph vertically by a factor of 4.
8. Shrink the graph horizontally by a factor of 4.
♠ 9. Shift the graph 4 units upward.
♠ 10. Shift the graph 4 units to the left
♠ 11. First reflect the graph about the x-axis, and then shift it 1 unit downward
12. Stretch the graph horizontally and vertically by a factor of 4.
13. The graph of $y = x^3$ has been:
    - reflected across the x-axis;
    - vertically stretched by a factor of 2;
    - shifted up by 3 units;
    - shifted right by 2 units.

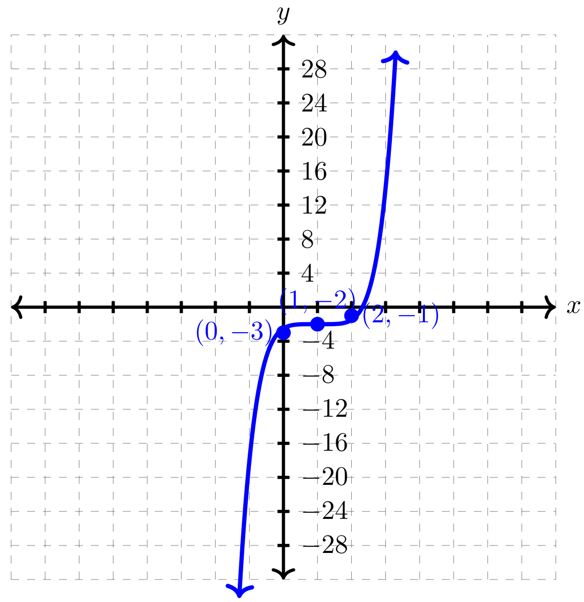


♠ 14. The graph of $y = \sqrt{x}$ has been:
    - vertically stretched by a factor of 2;
    - shifted up by 3 units;
    - shifted right by 2 units.



15. The graph of $y = x^5$ has been:

- vertically compressed by 2;
- shifted down by 3 units;
- shifted right by 1 units.



16. $(f \circ g)(x) = 14x + 8, (-\infty, \infty)$; $(g \circ f)(x) = 14x + 30, (-\infty, \infty)$;
$(f \circ f)(x) = 4x + 12, (-\infty, \infty)$; $(g \circ g)(x) = 49x + 16, (-\infty, \infty)$

♠ 17. $(f \circ g)(x) = (2x+4)^2 - 1, (-\infty, \infty)$; $(g \circ f)(x) = 2(x^2 - 1) + 4, (-\infty, \infty)$
$(f \circ f)(x) = (x^2 - 1)^2 - 1, (-\infty, \infty)$; $(g \circ g)(x) = 2(2x+4) + 4, (-\infty, \infty)$

♠ 18. $(f \circ g)(x) = \sqrt[6]{9-x}, (-\infty, 9]$; $(g \circ f)(x) = \sqrt[3]{9 - \sqrt{x}}, [0, \infty)$
$(f \circ f)(x) = \sqrt[4]{x}, [0, \infty)$; $(g \circ g)(x) = \sqrt[3]{9 - \sqrt[3]{9-x}}, (-\infty, \infty)$

♠ 19. $(f \circ g)(x) = \dfrac{x+2}{x+11} + \dfrac{x+11}{x+2}, (-\infty, -11) \cup (-11, -2) \cup (-2, \infty)$;
$(g \circ f)(x) = \dfrac{x^2 + 11x + 1}{x^2 + 2x + 1}, (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$;
$(f \circ f)(x) = x + \dfrac{1}{x} + \dfrac{x}{x^2+1}, (-\infty, 0) \cup (0, \infty)$;
$(g \circ g)(x) = \dfrac{4x+11}{x+5}, (-\infty, -5) \cup (-5, -2) \cup (-2\infty)$

20. $(f \circ g)(x) = x + 2, [2, \infty)$; $(g \circ f)(x) = \sqrt{x^2+2}, (-\infty, \infty)$;
$(f \circ f)(x) = x^4 + 8x^2 + 20, (-\infty, \infty)$; $(g \circ g)(x)\sqrt{\sqrt{x-2} - 2}, [6, \infty)$

21. $f(x) = \sqrt[4]{x}, \; g(x) = x^2 + 1$ ♠ 22. $f(x) = x^4, \; g(x) = 2x + x^2$ ♠ 23. $f(x) = \sqrt[3]{x}, \; g(x) = \dfrac{x}{6+x}$

♠ 24. $f(x) = \sqrt{x}, \; g(x) = (x^3 + 6)^2 - 1$

25. $f(x) = 1 - \sqrt{x}, \; g(x) = x^2 + 1$

26. $f(x) = \dfrac{1}{x}, \; g(x) = x^2 + 1$

27. $f(x) = \dfrac{x}{x+1}, \; g(x) = x^2$

1.4 Trigonometric Functions

1. $\dfrac{3\pi}{5}$ 2. $-\dfrac{7\pi}{6}$ ♠ 3. 22π ♠ 4. $\dfrac{3\pi}{8}$ ♠ 5. $30°$ ♠ 6. $420°$ 7. $183.3°$

8. $\dfrac{127\pi}{45}$ ♠ 9. 2 ♠ 10. $\dfrac{32\pi}{3}$ 11. 20 12. $4\sqrt{\dfrac{5}{3}}$ 13. $\sqrt{\dfrac{78}{\pi}}$

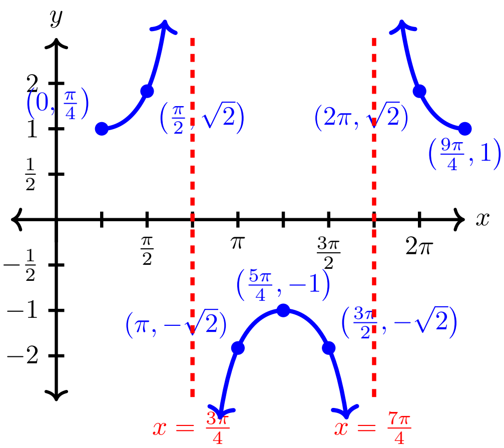
14. $\dfrac{\sqrt{3}}{2}$ ♠ 15. $-\sqrt{3}$ ♠ 16. $-\dfrac{2}{3}\sqrt{3}$ ♠ 17. 1 18. $\sqrt{3}$ 19. $-\dfrac{1}{2}$

20. $\sin\theta = \dfrac{7}{25}$, $\cos\theta = \dfrac{24}{25}$, $\csc\theta = \dfrac{25}{7}$, $\sec\theta = \dfrac{25}{24}$, $\cot\theta = \dfrac{24}{7}$, $\theta \approx 0.28379$

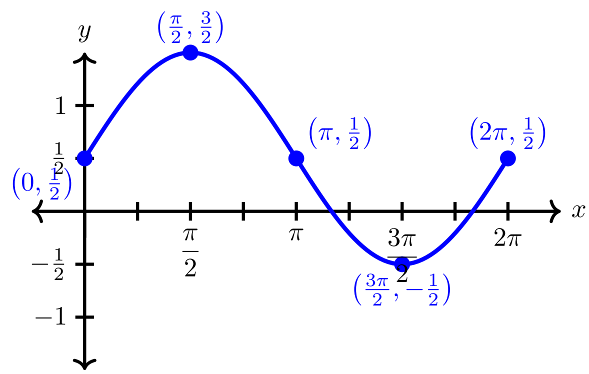
♠ 21. $\cos\theta = \dfrac{9}{41}$, $\tan\theta = \dfrac{40}{9}$, $\csc\theta = \dfrac{41}{40}$, $\sec\theta = \dfrac{41}{9}$, $\cot\theta = \dfrac{9}{40}$, $\theta \approx 1.34948$

22. $\sin\theta = \dfrac{15}{17}$, $\cos\theta = \dfrac{8}{17}$, $\csc\theta = \dfrac{17}{15}$, $\sec\theta = \dfrac{17}{8}$, $\cot\theta = \dfrac{8}{15}$, $\theta \approx 1.08084$

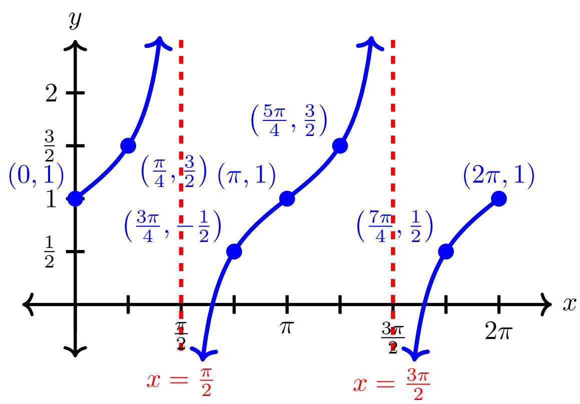
23.



♠ 24.



25.



1.5 Exponential Functions

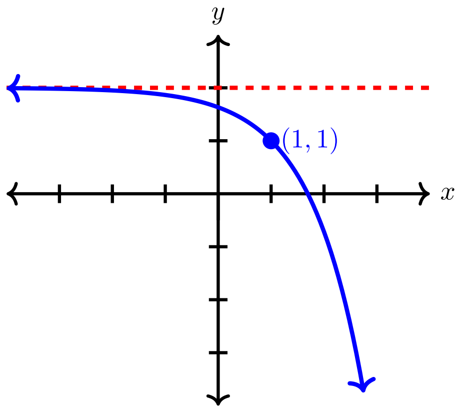
1. $128x^{15}$ 2. $648x^3$ 3. $f(x) = e^x -$ ♠ 4. $y = e^{x-4}$ ♠ 5. $y = -e^x$ 6. $f(x) = e^{-x}$

♠ 7. $f(x) = 12 - e^x$ ♠ 8. $f(x) = e^{-(x-6)}$ 9. $f(x) = -e^{-x}$ 10. $(-\infty, \infty)$

♠ 11. $(-\infty, 0]$ ♠ 12. $(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$ 13. $(-\infty, \infty)$

14. The graph of $y = e^x$ has been transformed by:

- reflection across x-axis;
- shifted up by 2 units;
- shifted right by 1 unit.



15. $f(x) = 1 - e^x$, $g(x) = x^2 + 1$

♠ 16. $f(x) = 4(2^x)$

♠ 17. 280 bacteria, $P(t) = 70\left(2^{t/4}\right)$, 1,332 bacteria, 40.63 hr

♠ 18. 14,400 bacteria, $P(t) = 900\left(4^t\right)$, 2,268 bacteria, 1.7 hr

♠ 19. $\dfrac{f(x+h) - f(x)}{h} = \dfrac{e^{x+h} - e^x}{h} = \dfrac{e^x e^h - e^x}{h} = e^x\left(\dfrac{e^h - 1}{h}\right)$

1.6 Inverse Functions and Logarithms

1. one-to-one
2. not
3. one-to-one
4. one-to-one
5. not
6. one-to-one

♠ 7. not

♠ 8. not

♠ 9. one-to-one

10. one-to-one

♠ 11. not

12. one-to-one

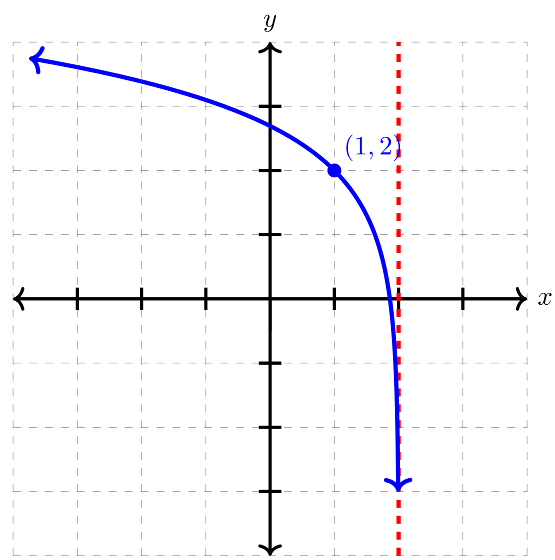
13. $(-\infty, -5) \cup (-5, -2) \cup (2, \infty)$

♠ 14. $(-2, -1) \cup (1, \infty)$

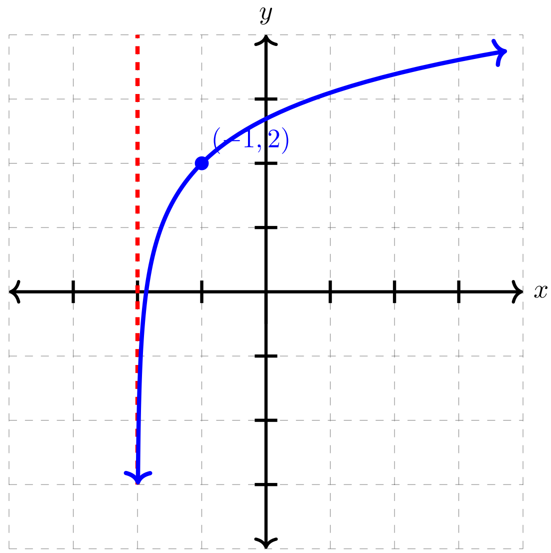
15. $(1, \infty)$

16. $(0, e) \cup (e, \infty)$

17. The graph of $y = \ln(x)$ has been transformed by:
    - reflection across y-axis;
    - shifted up by 2 units;
    - shifted right by 2 units.



18. The graph of $y = \ln(x)$ has been transformed by:
    - shifted up by 2 units;
    - shifted left by 2 units.



19. $f^{-1}(x) = \sqrt[3]{\dfrac{x+1}{3}}$ ♠ 20. $f^{-1}(x) = \dfrac{(x+1)^3}{27}$ 21. $f^{-1}(x) = \dfrac{(x-3)^3}{8}$ ♠ 22. $f^{-1}(x) = \dfrac{1}{6}(x-3)^2 - \dfrac{5}{6}$

♠ 23. $f^{-1}(x) = \dfrac{5x+1}{4-2x}$ 24. $f^{-1}(x) = \dfrac{2x-4}{x+3}$ 25. $f^{-1}(x) = \dfrac{2x}{x-3}$ ♠ 26. $f^{-1}(x) = \dfrac{1}{5}(7 + \ln x)$

♠ 27. $f^{-1}(x) = e^x - 4$ ♠ 28. $f^{-1}(x) = \sqrt[3]{\dfrac{x}{x+2}}$ 29. $f^{-1}(x) = \sqrt[3]{\dfrac{2x-4}{x+3}}$ 30. $f^{-1}(x) = \left(\dfrac{x}{x+2}\right)^3$

♠ 31. $f^{-1}(x) = -\left(\dfrac{x+2}{3x-4}\right)^3$ 32. $f^{-1}(x) = \sqrt[4]{x-5}$

1.7 Logarithms and Inverse Trigonometry

1. 3 2. -3 ♠ 3. -1 ♠ 4. $\dfrac{1}{100}$ ♠ 5. 5 ♠ 6. 3

7. -2 8. $\dfrac{1}{2}$ 9. $\frac{1}{3}\log_5(x^2+1) - \log_5(x-1) - \log_5(x+1)$

♠ 10. $3\log x + \frac{1}{2}\log(x+1) - 2\log(x-2)$ ♠ 11. $2\ln x + \frac{1}{3}\ln(x+1) - 2\ln(x+1)$

12. $\ln x + \frac{1}{2}\ln(1+x^2)$ ♠ 13. $\log_2\left(\dfrac{1}{x^3}\right)$

♠ 14. $\log\left(\dfrac{x^2-3x+2}{(x+1)^2}\right)$ 15. $\log_3\left(\dfrac{x^2}{y}\right)$ 16. $\log\left(\dfrac{\sqrt[3]{x^3+1}}{\sqrt{x^2+4}}\right)$

17. $\ln\left(\dfrac{\sqrt{x}}{x+1}\right)$ 18. $x = \dfrac{1}{4}(7 - \ln(9))$ ♠ 19. $x = 3$ ♠ 20. $x = \dfrac{1}{e^2-1}$

21. $x = 67$ 22. $x = 5$ 23. $x = 4$ 24. $x = \dfrac{13}{12}$ 25. $x = 3$ 26. $x = 6$

27. $x = 67$ 28. $x = 3$ 29. $x = \frac{1}{3}(12 + e^8)$ 30. $\dfrac{1}{2}$ 31. $2\sqrt{6}$ 32. $\dfrac{120}{169}$

33. $\dfrac{x}{\sqrt{1-x^2}}$ ♠ 34. $\dfrac{\sqrt{1-x^2}}{x}$ ♠ 35. $\sqrt{1-x^2}$ 36. $\dfrac{1}{\sqrt{1-x^2}}$

37. $\dfrac{x}{\sqrt{1-x^2}}$

♠ 38. $\dfrac{x}{\sqrt{1+x^2}}$

39. $\dfrac{1}{\sqrt{1+x^2}}$

40. $\sqrt{1+x^2}$

41. $\dfrac{\sqrt{1+x^2}}{x}$

42. $\sqrt{1+x^2}$

♠ 43. $\dfrac{1}{\sqrt{1-x^2}}$

44. $\dfrac{\sqrt{1-x^2}}{x}$

45. $\dfrac{\sqrt{x^2-1}}{x}$

46. $\sqrt{x^2-1}$

47. $\dfrac{x}{\sqrt{x^2-1}}$

48. $\dfrac{1}{\sqrt{x^2-1}}$

Chapter 2 : Limits

2.1 Error and Tolerance

1. 0.79
2. 1.44
♠ 3. 0.225
♠ 4. 0.4
5. 0.1
6. 0.049
7. 0.095
8. 1.75
9. 0.39
♠ 10. $\dfrac{1}{20}$
♠ 11. $\dfrac{1}{20}$
12. $\dfrac{1}{16}$

♠ 13. $w = 32.88$ watts, from 32.77 watts to 33.00 watts, $\delta = 0.11$ watts

♠ 14. $r = 23.26213$ cm, from 23.19361 cm to 23.33045 cm, $\delta = 0.06832$ cm

2.2 Limits of a Function

1. 0
♠ 2. 0
♠ 3. 4
♠ 4. 2
5. 4
6. 1
7. 0
♠ 8. 2
♠ 9. 2
♠ 10. 2
♠ 11. 4
♠ 12. 2
♠ 13. DNE
14. 1
15. -2
♠ 16. 3
♠ 17. 0
♠ 18. DNE
♠ 19. 5
♠ 20. 5
♠ 21. 5
♠ 22. 1
♠ 23. ∞
24. DNE

♠ 25.

x	$f(x)$
5.5	0.578947
5.1	0.560440
5.05	0.558011
5.01	0.556049
5.005	0.555802
5.001	0.555605
4.9	0.550562
4.95	0.553073
4.99	0.555061
4.995	0.555309
4.999	0.555506
$\lim\limits_{x\to 5} f(x)$	$\dfrac{5}{9}$

♠ 26.

x	$g(x)$
0	0
-0.5	-1
-0.9	-9
-0.95	-19
-0.99	-99
-0.999	-999
-2	2
-1.5	3
-1.1	11
-1.01	101
-1.001	1001
$\lim\limits_{x\to -1} g(x)$	DNE

2.3 Properties of Limits

1. -19
2. -125
3. 1
4. -1
5. DNE
6. 0
7. 8
♠ 8. DNE
♠ 9. 0
♠ 10. DNE
♠ 11. 16
12. 2
13. $-\frac{1}{3}$
♠ 14. $\dfrac{4}{11}$
♠ 15. 9
♠ 16. 528
♠ 17. $\dfrac{4}{3}$
18. 105
19. $\frac{1}{6}$
♠ 20. -4
♠ 21. $\dfrac{7}{8}$
♠ 22. DNE
♠ 23. $\dfrac{6}{5}$
24. $\dfrac{3}{2}$
25. 1
26. 1
27. 3
♠ 28. 5
♠ 29. 3
30. -1
31. 2

2.4 Properties of Limits II

1. -1
2. 1
3. DNE
4. $-\frac{1}{6}$
5. $\frac{1}{6}$
6. DNE
♠ 7. 1
8. $-\dfrac{1}{9}$
9. DNE
♠ 10. $-\dfrac{1}{81}$
11. ∞
♠ 12. $-\dfrac{1}{64}$
13. $\dfrac{1}{2}$
14. $-\dfrac{4}{5}$
15. $\dfrac{1}{2}$
♠ 16. $-\dfrac{4}{5}$
17. $\dfrac{3}{2}$
♠ 18. -4
♠ 19. 243
♠ 20. $\dfrac{1}{4}$
21. $-\dfrac{1}{25}$
22. $2x$
♠ 23. $3x^2$
24. $-\dfrac{1}{x^2}$
♠ 25. 5
♠ 26. 6
♠ 27. 2
♠ 28. 0
29. 0
30. 0
31. 0

2.5 Continuity

1. f is always continuous
♠ 2. $x = 0,\ 5$
♠ 3. f is always continuous
♠ 4. $x = 0$
♠ 5. $x = 0,\ 4$
6. $x = -2$
7. $(-\infty, \infty)$
8. $x \neq (2k+1)\pi$
♠ 9. $(-\infty, 7) \cup (7, \infty)$
♠ 10. $(-\infty, \infty)$
11. $(-\infty, 2) \cup (2, \infty)$
12. $(-\infty, 1) \cup (1, \infty)$
13. $(-\infty, 1) \cup (1, \infty)$
♠ 14. $(-\infty, 1) \cup (1, \infty)$
♠ 15. $(-\infty, 0) \cup (0, \infty)$
♠ 16. $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$
♠ 17. $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$
18. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$
♠ 19. 3

2.6 Continuity II

1. $1/3$
2. 4
♠ 3. 256
♠ 4. 3
♠ 5. 0
♠ 6. 1
♠ 7. 0.278
8. 390
9. 7
10. 24
11. 2
♠ 12. $-\dfrac{5}{4}$
13. -6
14. $\dfrac{1}{3}$
15. 2
16. 2
17. 3
♠ 18. $a = \dfrac{7}{2}$, $b = \dfrac{13}{2}$
19. *Proof.* Let $f(x) = \cos x - x$, which is a continuous function. Note that $f(0) = \cos(0) - (0) = 1 - 0 = 1 > 0$ and $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) = 0 - \left(\frac{\pi}{2}\right) = -\frac{\pi}{2} < 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 0 and $\frac{\pi}{2}$ such that $f(c) = 0$. Therefore, the above equation has a solution. □

♠ 20. *Proof.* Let $f(x) = x^4 + x - 9$, which is a continuous function. Note that $f(1) = (1)^4 + (1) - 9 = -7 < 0$ and $f(2) = (2)^4 + (2) - 9 = 9 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 1 and 2 such that $f(c) = 0$. Therefore, the above equation has a solution. □

21. *Proof.* Let $f(x) = x^4 + x - 19$, which is a continuous function. Note that $f(0) = (0)^4 + (0) - 17 = -17 < 0$ and $f(2) = (2)^4 + (2) - 17 = 18 - 17 = 1 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 0 and 2 such that $f(c) = 0$. Therefore, the above equation has a solution. □

♠ 22. *Proof.* Let $f(x) = 2x^3 - 4x^2 + 3x - 2$, which is a continuous function. Note that $f(1) = 2(1)^3 - 4(1)^2 + 3(1) - 2 = -1 < 0$ and $f(2) = 2(2)^3 - 4(2)^2 + 3(2) - 2 = 4 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 1 and 2 such that $f(c) = 0$. Therefore, the above equation has a solution. □

23. *Proof.* Let $f(x) = x^5 - 3x^3 - x^2 + 1$, which is a continuous function. Also, note that $f(1) = (1)^5 - 3(1)^3 - (1)^2 + 1 = 1 - 3 - 1 + 1 = -2 < 0$ and $f(2) = (2)^5 - 3(2)^3 - (2)^2 + 1 = 32 - 24 - 4 + 1 = 5 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 1 and 2 such that $f(c) = 0$. Therefore, f have a real root between 1 and 2. □

♠ 24. *Proof.* Let $f(x) = \sqrt[3]{x} + x - 1$, which is a continuous function. Note that $f(0) = \sqrt[3]{0} + (0) - 1 = -1 < 0$ and $f(1) = \sqrt[3]{1} + (1) - 1 = 1 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 0 and 1 such that $f(c) = 0$. Therefore, the above equation has a solution. □

25. *Proof.* Let $f(x) = 3^x + 2x - 3$, which is a continuous function. Note that $f(0) = e^0 + 2(0) - 3 = 1 - 3 = -2 < 0$ and $f(1) = e^1 + 2(1) - 3 = e - 1 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 0 and 1 such that $f(c) = 0$. Therefore, the above equation has a solution. □

26. *Proof.* Let $f(x) = \cos x - x^2 + x$, which is a continuous function. Note that $f(0) = \cos(0) - 0^2 + (0) = 1 > 0$ and $f(\pi) = \cos(\pi) - (\pi)^2 + (\pi) = -1 - \pi^2 + \pi < 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 0 and π such that $f(c) = 0$. Therefore, the above equation has a solution. □

♠ 27. *Proof.* Let $f(x) = \cos x - x^3$, which is a continuous function. Note that $f(0) = \cos(0) - (0)^3 = 1 - 0^3 = 1 > 0$ and $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^3 = 0 - \left(\frac{\pi^3}{8}\right) = -\frac{\pi^3}{8} < 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 0 and $\frac{\pi}{2}$ such that $f(c) = 0$. Therefore, the above equation has a solution. □

28. *Proof.* Let $f(x) = \ln x + 2x - 3$, which is a continuous function. Note that $f(1) = \ln 1 + 2(1) - 3 = -1 < 0$ and $f(2) = \ln(2) + 2(2) - 4 = \ln(2) + 1 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 1 and 2 such that $f(c) = 0$. Therefore, the above equation has a solution. □

2.7 Limits at Infinity

♠ 1. $-\infty$	♠ 2. $-\infty$	♠ 3. 1	♠ 4. ∞	♠ 5. 1	♠ 6. DNE
♠ 7. 1	♠ 8. ∞	♠ 9. $-\infty$	♠ 10. 0	♠ 11. 0	♠ 12. $-\infty$
♠ 13. 5	♠ 14. DNE	♠ 15. 6	♠ 16. 6	♠ 17. 6	♠ 18. ∞
19. $-\infty$	20. ∞	21. DNE	22. $-\infty$	23. ∞	24. DNE
25. DNE	♠ 26. $-\infty$	♠ 27. ∞	♠ 28. $-\infty$	♠ 29. $-\frac{7}{2}$	30. $-\infty$
♠ 31. ∞	32. $-\infty$	♠ 33. $-\infty$	♠ 34. ∞	♠ 35. 0	♠ 36. ∞

37. -1

♠ 38. $-\frac{2}{3}$

39. $-\frac{2}{3}$

40. $-\frac{1}{4}$

41. $-\frac{1}{4}$

42. $-\frac{2}{3}$

43. $-\frac{2}{3}$

44. $-\infty$

2.8 Limits at Infinity II

1. $\frac{1}{2}$

2. $\frac{1}{2}$

3. $\frac{1}{6}$

♠ 4. -1

♠ 5. 3

♠ 6. -3

♠ 7. $\frac{1}{6}$

♠ 8. -2

♠ 9. ∞

♠ 10. $\frac{1}{8}$

♠ 11. $-\frac{1}{8}$

12. 3

13. -4

14. $-\frac{3}{4}$

♠ 15. $-\frac{1}{2}$

♠ 16. 0

♠ 17. $-\frac{\pi}{2}$

♠ 18. $\frac{\pi}{2}$

19. $\frac{\pi}{2}$

20. 0

21. 1

22. 0

23. -1

♠ 24. 0

25. $-\frac{\pi}{2}$

26. $\frac{\sqrt{2}}{2}$

27. -1

♠ 28. 3

29. $x = -3,\ -\frac{1}{2}$,
$y = \frac{3}{2}$

♠ 30. $x = 5$,
$y = 1$

31. $x = 0, \sqrt[5]{10}$,
$y = -1$

♠ 32. $x = \ln(6)$,
$y = 0, 3$

33. $x = \pm\frac{2}{\sqrt{3}}$,
$y = -\frac{2}{3}$

2.9 Tangent Lines

1. -222

♠ 2. -197.0

3. -144

♠ 4. -109

5. -81.7

6. 68.5

♠ 7. 68.8

8. 69.5

9. 66

10. 4

11. 5

12. 5

13. 4

14. 6

15. 5.5

16. 4

17. 5

18. 3

19. 1

♠ 20. $y = 4x - 40$

a	$\frac{\Delta y}{\Delta x}$
8.9	4.444444
8.99	4.040404
8.999	4.004004
8.9999	4.000400
9.1	3.636364
9.01	3.960396
9.001	3.996004
9.0001	3.999600
9	4.0

♠ 21. $y = \frac{1}{2}x - 2$

a	$\frac{\Delta y}{\Delta x}$
5.5	0.585786
5.9	0.513167
5.99	0.501256
5.999	0.500125
6.5	0.449490
6.1	0.488088
6.01	0.498756
6.001	0.499875
6	0.5

♠ 22. -26 ♠ 23. -19.6 ♠ 24. -18.8 ♠ 25. -18.16 ♠ 26. -18

♠ 27. 62.42 ♠ 28. 63.35 ♠ 29. 64.09 ♠ 30. 64.26 ♠ 31. 64.28 ♠ 32. 64.28

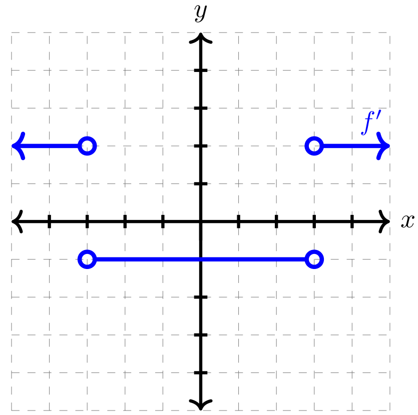
♠ 33. 6 ♠ 34. -10.89 ♠ 35. -12.42 ♠ 36. -12.55 ♠ 37. -12.57

2.10 Derivatives and Rates of Change

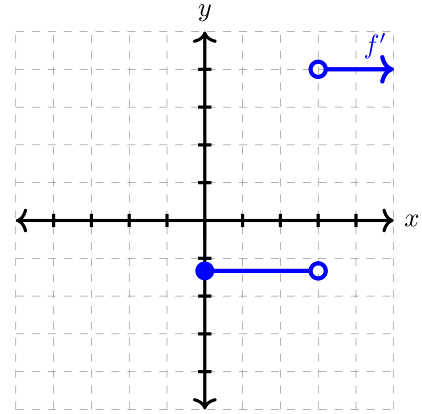
1. $y = x$
2. $y = x + \pi$
3. $y = 1$
4. $y = \frac{1}{2}x + \frac{-\pi + 3\sqrt{3}}{6}$
5. $y = ex$
6. $y = \frac{4}{e}x$
7. $y = -x + 1$
8. $y = 2x$
9. $y = 3x + 5$
♠ 10. $y = 3x + 1$
♠ 11. $y = -2x + 2$
12. $y = 2$
♠ 13. $y = 25x - 52$
14. $y = -x + 4$
♠ 15. $y = 4x + 8$
♠ 16. $y = -4x + 21$
♠ 17. $y = \frac{1}{2}x + \frac{1}{2}$
18. $y = \frac{1}{4}x + 1$
19. $y = \frac{1}{3}x + 3$
20. $y = 3x - 4$
21. $y = -4x - 4$
♠ 22. $y = 2x - 11$
23. $y = \frac{1}{4}x + 2$
24. 4
♠ 25. $\frac{1}{2}$
♠ 26. -18
27. 64.28
♠ 28. $f(6) = 2$, $f'(6) = \frac{1}{6}$
29. $f(4) = 3$, $f'(4) = \frac{1}{4}$
30. $f(x) = x^{10}$, $a = 1$
♠ 31. $f(x) = x^3$, $a = 1$
32. $f(x) = \sqrt[4]{x}$, $a = 16$
♠ 33. $f(x) = \sqrt[4]{x}$, $a = 81$
34. $f(x) = x^3 - x$, $a = 1$
35. $f(x) = x^4 + x - 2$, $a = 1$
36. $f(x) = x^{356}$, $a = 1$
37. $f(x) = 2^x$, $a = 5$
♠ 38. $f(x) = 4^x$, $a = 2$
39. $f(x) = x^x$, $a = 2$
♠ 40. $f(x) = \tan(x)$, $a = \frac{\pi}{4}$
41. $f(x) = \tan x$, $a = \frac{\pi}{4}$
42. $f(x) = \sin x$, $a = \frac{\pi}{6}$
43. $f(x) = \sin x$, $a = \frac{\pi}{3}$
44. $f(x) = \cos x$, $a = \pi$

2.11 Derivatives and Rates of Change II

1. $x = -2, 0, 2$
♠ 2. $x = 0, 4, 5$
♠ 3. $x = -6, -2, 2, 3$
♠ 4. $x = 0, 2$
♠ 5. $x = 0, 2, 4$
6. $x = -2$
7.



♠ 8.



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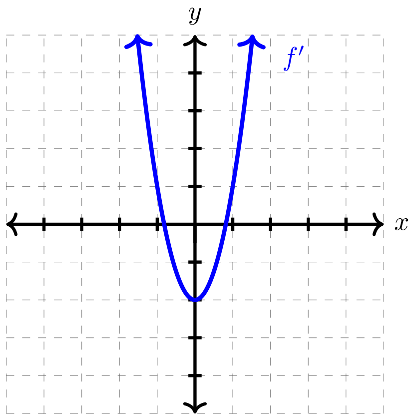

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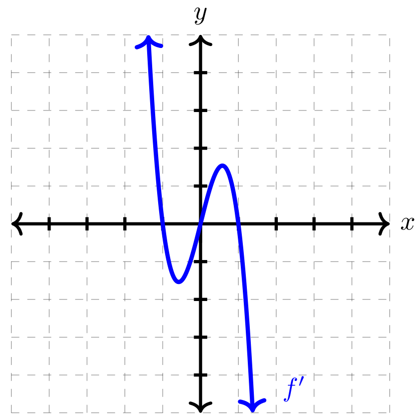

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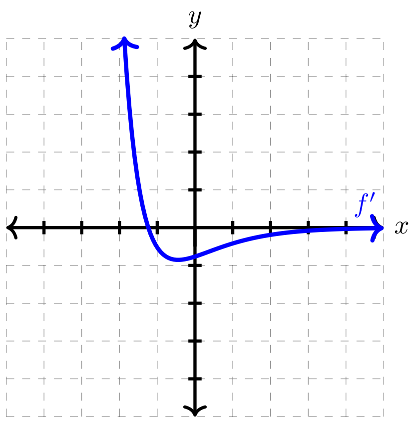
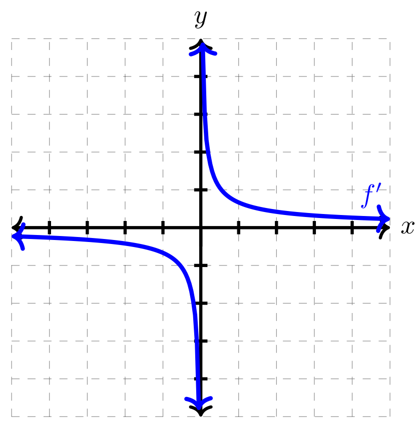
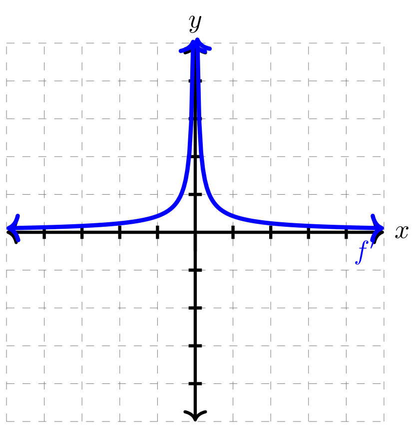


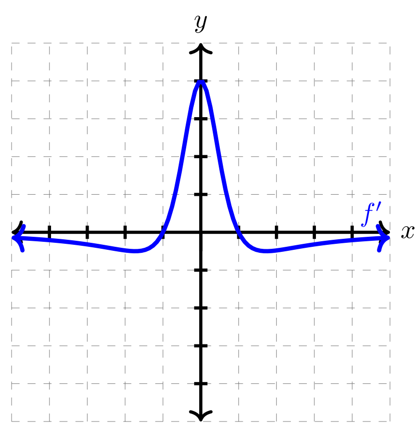
♠ 12.



♠ 13.



14.



15. $f'(x) = \dfrac{1}{4}$

16. $f'(x) = 2x$

♠ 17. $f'(x) = 5 - 16x$

18. $f'(x) = 4x$

19. $f'(x) = 6x$

20. $f'(x) = 4x + 1$

21. $f'(x) = 6x - 1$

22. $f'(x) = 3x^2$

23. $f'(x) = \dfrac{1}{2\sqrt{x}}$

♠ 24. $f'(x) = -\dfrac{1}{2\sqrt{8 - x}}$

25. $f'(x) = -\dfrac{1}{x^2}$

26. $f'(x) = 1 - \dfrac{1}{x^2}$

♠ 27. $f'(x) = -\dfrac{13}{(3 + x)^2}$

♠ 28. $f'(x) = -\dfrac{1}{2x\sqrt{x}}$

29. $f'(x) = -\dfrac{3}{2x^2\sqrt{x}}$

Chapter 3 : Derivatives

3.1 The Power Rule

1. $f'(x) = \frac{1}{2}x^{-1/2}$
2. $f'(x) = -x^{-2}$
3. $f'(x) = \frac{1}{3x^{2/3}}$
4. $f'(x) = -\frac{1}{2x^{3/2}}$
5. ♠ $f'(x) = 9x^{11}$
6. ♠ $g'(t) = -\frac{3}{2}t^{-7/4}$
7. ♠ $f'(x) = 0$
8. ♠ $y' = \frac{70}{x^6}$
9. ♠ $y' = \frac{1}{6\sqrt[6]{x^5}}$
10. ♠ $f'(x) = -\frac{9\sqrt{2}}{x^{10}}$
11. ♠ $f'(t) = -\frac{1}{3}$
12. $f'(x) = 6x - 5$
13. ♠ $f'(x) = 5x^4 - 9$
14. $y' = 21x^2 + 2$
15. $y' = -12x^2 + 3$
16. $y' = 9x^2 - 14x$
17. $y' = 15x^2 - 2x + 7$
18. $y' = -9x^{-4} - 4x^{-5}$
19. ♠ $g'(x) = 2x - 9x^2$
20. ♠ $f'(t) = \frac{3t - 10}{2\sqrt{t}}$
21. $y' = \frac{9}{2}x^{7/2} + 5x^{3/2}$
22. $f'(x) = \frac{3\sqrt{x}}{2} + \frac{\sqrt{3}}{2\sqrt{x}}$
23. ♠ $y' = 8e^x - \frac{8}{3\sqrt[3]{x^4}}$
24. $f'(x) = e^x - 3x^2$
25. ♠ $f(x) = 9\sqrt{x} + \frac{3}{\sqrt{x}} - \frac{4}{\sqrt{x^3}}$
26. ♠ $u' = \frac{1}{7\sqrt[7]{x^6}} + 14\sqrt{x^5}$
27. $f'(x) = e^x + 5x^4$
28. $f'(x) = 19x^{18} - \frac{1}{6\sqrt{x}} + e^x$
29. ♠ $y = 3x - 2$
30. ♠ $y = 19$
31. $f''(x) = \frac{-4}{25x^{6/5}}$
32. $\frac{d^2y}{dx^2} = \frac{-1}{4x^{3/2}}$
33. $f''(x) = e^x$
34. $f''(x) = e^x - 6x$
35. $\frac{d^2y}{dx^2} = \frac{36}{x^5} + 20x^{-6}$
36. $\frac{d^2y}{dx^2} = \frac{63}{4}x^{5/2} + \frac{15}{2}x^{1/2}$

3.2 The Product Rule

1. $f'(x) = x^3e^x(4 + x)$
2. ♠ $f'(x) = 1 - 2x + 12x^2 - 16x^3$
3. $f'(x) = x^{-1/2}e^x\left(\frac{1}{2} + x\right)$
4. $f'(x) = x^{3/2}e^x\left(\frac{5}{2} + x\right)$
5. ♠ $g'(x) = \frac{7e^x}{2\sqrt{x}}(2x + 1)$
6. $f'(x) = 1 - xe^x - 2e^{2x}$
7. ♠ $\frac{dy}{dx} = 2x - 1$
8. ♠ $f'(x) = 7 + \frac{62}{x^2} + \frac{27}{x^4}$
9. ♠ $y' = \frac{x^4(15 - x^4)}{(3 - x^4)^2}$
10. $f'(x) = -\frac{3(x^2 - 1)}{(x^2 + 1)^2}$
11. $f'(x) = \frac{2x^2 + 2x + 4}{(1 + 2x)^2}$
12. $f'(x) = \frac{x^2 + 4x + 3}{(x + 2)^2}$
13. ♠ $\frac{dy}{dx} = -\frac{x + 6}{(x - 6)^3}$
14. $f'(x) = \frac{-20}{(x - 10)^2}$
15. ♠ $f'(x) = -\frac{2x^3 + 12x^2 + 11}{(x^3 + x - 7)^2}$
16. ♠ $f'(t) = \frac{2t(-t^4 - 10t^2 + 22)}{(t^4 - 4t^2 + 2)^2}$
17. ♠ $f'(t) = \frac{36 + 3\sqrt{t}}{(6 + \sqrt{t})^2}$
18. ♠ $h'(x) = \frac{6}{7}x^{-1/7} - \frac{5}{14}x^{-9/14}$
19. $f''(x) = \frac{e^x(x - 1)}{x^2}$
20. ♠ $f'(x) = \frac{-x^2e^x - e^{2x} - 6e^x - 6}{(x + e^x)^2}$
21. $f''(x) = \frac{e^x}{(1 - e^x)^2}$
22. $f'(x) = -\frac{1}{e^x}$
23. $\frac{dy}{dx} = \frac{e^x(x^2 - 2x + 3)}{(3 + x^2)^2}$
24. $f'(x) = \frac{-x^2 + 2x - 1}{e^x}$

25. $f'(x) = \dfrac{(x^2+1)(xe^x+e^x)-2x(xe^x)}{(x^2+1)^2}$

26. $h'(x) = \dfrac{[1+f(x)]g'(x)-g(x)f'(x)}{[1+f(x)]^2}$

27. $y = -\dfrac{9}{25}x + \dfrac{48}{25}$

28. $y = -x+1$

29. $f'(x) = e^x x^6(x^2+16x+56)$

30. $f''(x) = e^{-x}(x^2-4x+2)$

3.3 Trigonometric Derivatives

1. 2
♠ 2. 3
3. 1
4. 1
♠ 5. $\dfrac{2}{3}$
6. 0
♠ 7. $\dfrac{5}{9}$
8. $\dfrac{2}{3}$
9. $\dfrac{5}{7}$
10. 0
11. 0
12. 0
♠ 13. 0
14. $\dfrac{9}{4}$
15. 3
16. $-\dfrac{1}{4}$
17. $\dfrac{1}{3}$
18. $f'(x) = \sec^3(x) + \tan^2(x)\sec(x)$
♠ 19. $f'(x) = \cos x - \dfrac{7}{8}\csc^2 x$
♠ 20. $y' = 30x^4 + 3\sin(x)$
♠ 21. $f'(\theta) = 8\sqrt{\theta}\cos\theta + \dfrac{4\sin\theta}{\sqrt{\theta}}$
22. $\dfrac{d^2y}{dx^2} = e^x(\cos(x)-\sin(x))$
♠ 23. $f'(x) = 9e^x\csc x(1+x-x\cot x)$
♠ 24. $y' = 7x\cos x(2\cot x - x - x\csc^2 x)$
♠ 25. $\dfrac{dy}{dx} = \dfrac{\sec x - 6\sec^2 x}{\tan^2 x}$
♠ 26. $\dfrac{dy}{dx} = \dfrac{28-4\cot x - 4x\csc^2 x}{(7-\cot x)^2}$
27. $f(\theta) = \dfrac{2\cos\theta}{(\sin\theta+1)^2}$
28. $y = -x$
♠ 29. $y = 2\sqrt{3}x + \dfrac{6-2\sqrt{3}\pi}{3}$
30. $y = \sqrt{2}x + \dfrac{\sqrt{2}(4-\pi)}{4}$
31. $y'' = \sec(x)(\tan^2(x)+\sec^2(x))$
32. $\dfrac{d^2y}{dx^2} = -2e^x\sin(x)$

33. *Proof.* Let $f(x) = \cos x$. Then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(h)\sin(x)-\cos(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{\cos(x)\cos(h)-\cos(x)}{h} - \frac{\sin(h)\sin(x)}{h}\right) \\
&= \lim_{h\to 0}\left(\cos(x)\left(\frac{\cos(h)-1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right) \\
&= \cos(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h} - \sin(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \\
&= \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x).
\end{aligned}$$

□

34. *Proof.*

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x(1') - 1(\cos x)'}{\cos^2 x} = \frac{0+\sin x}{\cos^2 x} = \frac{1}{\cos x}\left(\frac{\sin x}{\cos x}\right) = \sec x\tan x \quad \square$$

♠ 35. *Proof.*

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\frac{1}{\sin x} = \frac{\sin x(1') - 1(\sin x)'}{\sin^2 x} = \frac{0-\cos x}{\sin^2 x} = -\frac{1}{\sin x}\left(\frac{\cos x}{\sin x}\right) = -\csc x\cot x \quad \square$$

36. *Proof.*

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{\sin x(\cos x)' - \cos x(\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x \quad \square$$

3.4 The Chain Rule

1. $y' = 2x + e^{-x}$
2. $f'(x) = 4(x^2 + e^x)^3(2x + e^x)$
♠ 3. $y' = 3(x^4 + 5x^2 - 3)^2(4x^3 + 10x)$
4. $f'(x) = 231(x + 5)^{230}$
5. $g'(x) = 600x(3x^2 - 5)^{99}$
6. $h'(x) = 125(x^3+x^2)^{124}(3x^2+2x)$
7. $f'(x) = \dfrac{x}{\sqrt{x^2+1}}$
8. $\dfrac{d}{dx}f(x) = \dfrac{2x+1}{2\sqrt{x^2+x}}$
9. $y' = \dfrac{2x^3+4x}{\sqrt{x^4+4x^2+25}}$
♠ 10. $\dfrac{dy}{dx} = \dfrac{8}{3\sqrt[3]{(1+8x)^2}}$
11. $\dfrac{dy}{dx} = \dfrac{8x^2}{(8x^3+27)^{2/3}}$
12. $f'(x) = \dfrac{x^3}{(x^4+16)^{3/4}}$
♠ 13. $y' = \dfrac{16e^{4x}}{\sqrt{5+8e^{4x}}}$
14. $y' = 2\sec^2(2x)$
15. $y' = 2\sec(2x)\tan(2x)$
16. $f'(x) = -2\csc^2(2x)$
17. $g'(x) = \cos(x^3 + 5x)(3x^2 + 5)$
♠ 18. $f'(x) = -6\sin x \cos^5 x$
♠ 19. $\dfrac{d}{dx}f(x) = -8x^7\sin(a^8 + x^8)$
20. $h'(x) = 3\cos(3x + \cos x) - \sin(x)\cos(3x + \cos x)$
21. $f'(x) = 2xe^{x^2}$
♠ 22. $\dfrac{dy}{dx} = \dfrac{3e^{3\sqrt{x}}}{2\sqrt{x}}$
23. $f'(x) = -\sin(x)e^{\cos(x)} - e^x\sin(e^x)$
24. $\dfrac{dy}{dt} = -4e^{-t} - 12\sin^3 t\cos t$
♠ 25. $f'(x) = 9(1 - kx)e^{-kx}$
26. $g'(x) = e^{e^x}e^x$
♠ 27. $f'(x) = (2x - 5)^3(x^2 + x + 1)^4(28x^2 - 32x - 17)$
28. $f'(x) = 10(xe^x)^9(xe^x + e^x)$
29. $y' = e^x(\sin(x^3 - 2x) + (3x^2 - 2)\cos(x^3 - 2x))$
♠ 30. $y' = -\dfrac{48x(x^2+4)^2}{(x^2-4)^4}$
31. $f'(x) = 9\left(\dfrac{x}{x+1}\right)^8\left(\dfrac{1}{(x+1)^2}\right)$

3.5 The Chain Rule II

1. $y' = \dfrac{1}{2\sqrt{x}}(\sec(\sqrt{x}))^2e^{\tan(\sqrt{x})}$
♠ 2. $y' = 7\cos(\tan(7x))\sec^2(7x)$
♠ 3. $y' = \sin\left(\dfrac{1-e^{2x}}{1+e^{2x}}\right)\left(\dfrac{4e^{2x}}{(1+e^{2x})^2}\right)$
♠ 4. $\dfrac{dy}{d\theta} = -2\cos\theta\cot(\sin\theta)\csc^2(\sin\theta)$
♠ 5. $\dfrac{dy}{dx} = -\sin(\cos(\cos x))\sin(\cos x)\sin x$
6. $f'(x) = 3x^2 - 3^x\ln 3$
7. $f'(x) = \pi x^{\pi-1} - \pi^x\ln\pi$
8. $\dfrac{dy}{dx} = (2x\ln 3)3^{x^2}$
9. $\dfrac{d}{dx}f(x) = (\ln 2)(2x - 1)2^{x^2-x+1}$
♠ 10. $y' = \dfrac{3^{\sqrt{x}}(\ln 3)}{2\sqrt{x}}$
♠ 11. $f'(t) = 3t^2(\ln 6)6^{t^3}$
♠ 12. $f'(x) = -\dfrac{(C\ln 9)9^{C/x}}{x^2}$
13. $f'(x) = (\ln 2)\sec(x)\tan(x)2^{\sec(x)}$
14. $y = -x + \pi$
♠ 15. $y = 16x + 1$
♠ 16. $y = 3x$
17. $y = -\dfrac{3e^{\pi/4}+4}{e^{\pi/4}}x + \dfrac{3e^{\pi/4}(\pi+1)+4\pi-16}{4e^{\pi/4}}$
♠ 18. $x = 2n\pi, (2n+1)\pi$
19. 0
♠ 20. 6
♠ 21. $\dfrac{3}{\sqrt{5}}$
22. 3
23. $f''(x) = 25e^{5x}$
♠ 24. $y'' = 6e^{x+6e^x}(6e^x + 1)$
♠ 25. $y'' = -\dfrac{2\sin^2 x + \cos^2 x}{4\sin x\sqrt{\sin x}}$

26. $f''(x) = 6x - (\ln 3)^2 3^x$

3.6 Implicit Differentiation

1. $y' = -\dfrac{2x}{3y^2 - 3}$

♠ 2. $\dfrac{dy}{dx} = \dfrac{4x}{y}$

♠ 3. $\dfrac{dy}{dx} = -\dfrac{y^2}{x^2}$

♠ 4. $\dfrac{dy}{dx} = -\dfrac{8x^7}{3y^2}$

♠ 5. $\dfrac{dy}{dx} = -\dfrac{4\sqrt{y}}{\sqrt{x}}$

♠ 6. $\dfrac{dy}{dx} = \dfrac{18x + 7y}{2y - 7x}$

♠ 7. $\dfrac{dy}{dx} = \dfrac{y^3 - 2xy - 6x^2}{x^2 - 3xy^2}$

♠ 8. $\dfrac{dy}{dx} = \dfrac{4y^2 - 3x^2y - 4x^3}{x^3 - 8xy + 3y^2}$

♠ 9. $\dfrac{dy}{dx} = \dfrac{y \sin x + 8x}{\cos x - 4y}$

♠ 10. $\dfrac{dy}{dx} = \tan x \tan y$

♠ 11. $\dfrac{dy}{dx} = \dfrac{y(5y - e^{x/y})}{y^2 - xe^{x/y}}$

♠ 12. $\dfrac{dy}{dx} = \dfrac{e^y \sin x + y\cos(xy)}{e^y \cos x - x\cos(xy)}$

13. $\dfrac{dy}{dx} = \dfrac{1 - 2xy}{x^2 + \cos y - 2y}$

14. $y' = \dfrac{2y\sqrt{xy}\sin x - 2\sqrt{xy}\sin y + y}{2x\sqrt{xy}\cos y + 2\sqrt{xy}\cos x - x}$

3.7 Implicit Differentiation II

1. $f'(x) = \dfrac{2x}{\sqrt{1 - x^4}}$

♠ 2. $y' = \dfrac{2}{\sqrt{-4x^2 - 4x}}$

♠ 3. $y' = \dfrac{14\tan^{-1}(7x)}{1 + 49x^2}$

♠ 4. $f'(x) = -3 - \dfrac{9x\cos^{-1}(3x)}{\sqrt{1 - 9x^2}}$

♠ 5. $g'(x) = -\dfrac{1}{2\sqrt{1 - x^2}}$

6. $f'(x) = \tan^{-1}(x) + \dfrac{x}{1 + x^2}$

♠ 7. $-\dfrac{32}{33}$

8. $y = -\dfrac{3}{4}x + \dfrac{25}{4}$

♠ 9. $y = x + \dfrac{1}{4}$

♠ 10. $y = -2x + \dfrac{5\pi}{4}$

11. $y = -\dfrac{3}{\sqrt{5}}x - \dfrac{4}{\sqrt{5}}$

12. $\dfrac{d^2y}{dx^2} = \dfrac{2y}{x^2}$

♠ 13. $y'' = -\dfrac{16x}{25y^5}$

♠ 14. $y'' = \dfrac{2xy + 16ye^y - 8y^2e^y}{(x + 8e^y)^3}$

15. $\dfrac{d^2y}{dx^2} = -\dfrac{25}{y^3}$

♠ 16. *Proof.*

$$\begin{aligned} y &= \arccos(x) \\ \cos(y) &= x \\ \frac{d}{dx}(\cos(y)) &= \frac{d}{dx}(x) \\ -\sin(y)y' &= 1 \\ y' &= -\frac{1}{\sin(y)} \\ &= -\frac{1}{\sqrt{1 - \cos^2(y)}} \\ &= -\frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

□

17. *Proof.*

$$\begin{aligned} y &= \sec^{-1}(x) \\ \sec(y) &= x \\ \frac{d}{dx}(\sec(y)) &= \frac{d}{dx}(x) \\ \tan(y)\sec(y)y' &= 1 \\ y' &= \frac{1}{\tan(y)\sec(y)} \\ &= \frac{1}{\sec(y)\sqrt{\sec^2(y) - 1}} \\ &= \frac{1}{x\sqrt{x^2 - 1}} \end{aligned}$$

□

18. *Proof.*

$$\begin{aligned} y &= \csc^{-1}(x) \\ \csc(y) &= x \\ \frac{d}{dx}(\csc(y)) &= \frac{d}{dx}(x) \\ -\cot(y)\csc(y)y' &= 1 \\ y' &= -\frac{1}{\csc(y)\cot(y)} \\ &= -\frac{1}{\csc(y)\sqrt{\csc^2(y) - 1}} \end{aligned}$$

$$= -\frac{1}{x\sqrt{x^2-1}}$$

□

19. *Proof.*

$$\begin{aligned} y &= \cot^{-1}(x) \\ \cot(y) &= x \\ \frac{d}{dx}(\cot(y)) &= \frac{d}{dx}(x) \\ -\csc^2(y)y' &= 1 \\ y' &= -\frac{1}{\csc^2(y)} \\ &= -\frac{1}{1+\cot^2(y)} \\ &= -\frac{1}{1+x^2} \end{aligned}$$

□

3.8 Logarithmic Differentiation

1. $f'(x) = 2x + \frac{1}{x}$
2. $f'(x) = 5 + \frac{2}{x}$
3. $f'(x) = e^x + \frac{1}{x}$
4. $f'(x) = 10e^x + \frac{5}{x}$
5. $y' = \frac{1}{x} + 5e^{4-5x}$
6. $f'(x) = \frac{e^x}{5} + \frac{1}{3x}$
7. $f'(x) = \frac{5}{5x-7}$
8. $\frac{dy}{dx} = 4x^3 + 3x^2 + \frac{1}{4-x}$
9. $y' = 1 + \frac{3}{2-3x}$
10. $f'(x) = \frac{5(\ln x)^4}{x}$
11. $f'(x) = \frac{7\cos(7\ln x)}{x}$
12. $y' = 1 + \ln(x)$
13. $g'(x) = 2x\ln(x) + x$
14. $f'(t) = e^t\left(\frac{1}{t} + \ln(t)\right)$
15. $g'(t) = t^2 + 3t^2\ln(t)$

♠ 16. $f'(x) = 4\ln(3x)$

17. $f'(x) = x^2\ln(x)e^x + xe^x + 2x\ln(x)e^x$
18. $f'(x) = 2xe^{x^2}\ln(x) + \frac{e^{x^2}}{x}$
19. $g'(x) = \frac{\cos x}{x} - \sin x \ln x$
20. $f'(x) = \frac{2e^{2x-1}(1+(2x-1)\ln(2x-1))}{2x-1}$
21. $f'(x) = 7x^7e^{2-\ln(2x)}$

♠ 22. $f'(x) = 2\cot(x)$

♠ 23. $g'(x) = \frac{2x^2-1}{x(x^2-1)}$

♠ 24. $h'(x) = \frac{8x^2-x+12}{(4x+1)(x^2+1)}$

♠ 25. $f'(x) = -\frac{15}{x(\ln x)^2}$

26. $f'(x) = \frac{1-2\ln x}{x^3}$
27. $y' = \frac{e^x\ln(x^2+1) - \frac{2xe^x}{x^2+1}}{(\ln(x^2+1))^2}$
28. $f'(x) = \frac{1-3\ln(x)}{x^4}$
29. $f'(x) = \frac{2x\ln(x)-x}{(\ln(x))^2}$
30. $f'(x) = \frac{1-\ln(x)}{x^2}$
31. $f'(x) = e^{x^3\ln(x)}(x^2 + 3x^2\ln(x))$
32. $f'(x) = \frac{1-2x}{1+x-x^2}$
33. $f'(x) = \frac{1}{x\ln x}$
34. $f'(x) = \frac{1+x}{x\ln 5}$
35. $f'(x) = \frac{3x^2+1}{(\ln 2)(x^3+x+1)}$

♠ 36. $f'(x) = \frac{4x^3}{(x^4+8)\ln 10}$

♠ 37. $f'(x) = \frac{1+x}{x\ln 15}$

38. $f'(t) = \frac{1-2x}{(\ln 2)(1+x-x^2)}$
39. $g'(x) = -\frac{\tan(\sqrt{x}/2)}{(2\ln 5)\sqrt{x}}$

♠ 40. $y' = (x^3+2)^2(x^5+4)^4\left(\frac{6x^2}{x^3+2} + \frac{20x^4}{x^5+4}\right)$

41. $y' = \frac{x\sqrt{x^3-2}}{(5x+4)^6}\left(\frac{1}{x} + \frac{3x^2}{2(x^3-2)} - \frac{30}{5x+4}\right)$
42. $y' = \sqrt{\frac{e^{2x}\cos(x)}{(x+4)^3}}\left(1 - \frac{1}{2}\tan x - \frac{3}{2(x+4)}\right)$

♠ 43. $y' = \sqrt{\frac{x-2}{x^4+2}}\left(\frac{1}{2x-4} - \frac{2x^3}{x^4+2}\right)$

44. $y' = \left|\frac{(x+2)^4}{\cos(x)}\right|\left(\frac{4}{x+2} + \tan x\right)$

45. $y' = x^x(\ln x + 1)$

♠ 46. $y' = 4x^{4x}(\ln x + 1)$

47. $y' = 6x^{6x}(\ln x + 1)$

48. $y' = (1+x)^{2x}\left(\dfrac{2x}{1+x} + 2\ln(1+x)\right)$

♠ 49. $y' = 8x^{8\cos x}\left(\dfrac{\cos x}{x} - \ln x \sin x\right)$

50. $y' = (\cos x)^x\left(\ln(\cos x) - x\tan x\right)$

51. $y' = \dfrac{1}{2}(\sqrt{x+1})^x\left(\ln(x+1) + \dfrac{x}{x+1}\right)$

52. $y' = |\sin(x)|\cot(x)$

♠ 53. $y' = -2|\cos(2x)|\tan(2x)$

54. $\dfrac{dy}{dx} = \dfrac{\ln y - y/x}{\ln x - x/y}$

55. $y = x + \ln 2$

56. $y = -\dfrac{1}{2}x + (\ln 8 - 1)$

57. $y = \dfrac{1}{\ln 5}x - \dfrac{1}{\ln 5}$

58. $f''(x) = -\dfrac{1}{x^2}$

59. $f''(x) = 6x - \dfrac{1}{x^2}$

60. $f''(x) = \dfrac{1}{x}$

3.9 Rates of Changes with the Sciences

1. (a) $v(t) = 3t^2 - 18t + 24$
   (b) $a(t) = 6t - 18$
   (c) $[0,2) \cup (4,10]$

2. (a) $v(t) = 3t^2 - 3$
   (b) $9\ \dfrac{\text{m}}{\text{s}}$
   (c) $a(t) = 6t$

♠ 3. (a) $v(t) = 3t^2 - 30t + 72$
   (b) $-3\ \dfrac{\text{m}}{\text{s}}$
   (c) $t = 4, 6$
   (d) $[0,4) \cup (6,10]$
   (e) 120 m
   (f) $a(t) = 6t - 30$
   (g) $0\ \dfrac{\text{m}}{\text{s}^2}$
   (h) $(4,5) \cup (6,10]$
   (i) $[0,4) \cup (5,6)$

♠ 4. (a) $v(t) = 0.04t^3 - 0.09t^2$
   (b) $-0.04\ \dfrac{\text{m}}{\text{s}}$
   (c) $t = 0, 2.25$
   (d) $(2.25, 10]$
   (e) 43.91 m
   (f) $a(t) = 0.12t^2 - 0.18t$
   (g) $0.12\ \dfrac{\text{m}}{\text{s}^2}$
   (h) $(0, 1.5) \cup (2.25, 10]$
   (i) $(1.5, 2.25)$

♠ 5. (a) $v(t) = -\dfrac{\pi}{4}\sin\left(\dfrac{\pi t}{4}\right)$
   (b) $-0.56\ \dfrac{\text{m}}{\text{s}}$
   (c) $t = 0, 4, 8$
   (d) $(4,8)$
   (e) 4 m
   (f) $a(t) = -\dfrac{\pi^2}{16}\cos\left(\dfrac{\pi t}{4}\right)$
   (g) $-0.44\ \dfrac{\text{m}}{\text{s}^2}$
   (h) $(0,2) \cup (4,6) \cup (8,10)$
   (i) $(2,4) \cup (6,8)$

6. (a) $v(t) = \sin(\pi t) + \pi t\cos(\pi t)$
   (b) $-3\pi\ \dfrac{\text{m}}{\text{s}}$
   (c) $a(t) = 2\pi\cos(\pi t) - \pi^2 t\sin(\pi t)$

7. (a) $16\ \dfrac{\text{ft}}{\text{s}}$
   (b) $t = 6$ s
   (c) $-112\ \dfrac{\text{ft}}{\text{s}}$

♠ 8. (a) 2.9 $mpers$
   (b) $-16.7\ \dfrac{\text{m}}{\text{s}}$
   (c) 2.30 s
   (d) 27.83 m
   (e) 4.68 s

(f) $-23.35\ \frac{\text{m}}{\text{s}}$

♠ 9. $-162\ \frac{\text{gal}}{\text{min}}$
♠ 10. $-144\ \frac{\text{gal}}{\text{min}}$
♠ 11. $-108\ \frac{\text{gal}}{\text{min}}$
♠ 12. $0\ \frac{\text{gal}}{\text{min}}$

♠ 13. 0 min
♠ 14. 50 min
♠ 15. 3.63 A
♠ 16. 3 A

♠ 17. $\frac{2}{3}$ s
♠ 18. $-32\ \frac{\text{N}}{\text{km}}$
♠ 19. $\frac{dT}{dt} = 0.6090\ \frac{\text{K}}{\text{min}}$

20. (a) 124 million bacteria
(b) $\frac{39}{4}$ million bacteria per hour

♠ 21. 5.138 billion bacteria per hour

22. (a) 1000 bacteria
(b) 500 bacteria per hour

23. (a) \$550
(b) \$2.50 per widget

24. (a) $C'(x) = 8x + 100$
(b) \$14,000 per hundred barrels

♠ 25. (a) $C'(x) = 11 - 0.2x + 0.0015x^2$
(b) $C'(400) = \$171$ per yard
(c) \$171.50

♠ 26. (a) $C'(100) = \$11$ per commodity
(b) \$10.95

3.10 Related Rates

1. $60\ \frac{\text{cm}^2}{\text{s}}$
♠ 2. $20\ \frac{\text{cm}^2}{\text{s}}$
♠ 3. $88\ \frac{\text{cm}^2}{\text{s}}$
4. $-\frac{1}{4\pi}\ \frac{\text{ft}}{\text{min}}$

♠ 5. $\frac{1}{3\pi}\ \frac{\text{m}}{\text{min}}$
6. $\frac{1}{2\pi}\ \frac{\text{cm}}{\text{s}}$
♠ 7. $8000\pi\ \frac{\text{mm}^3}{\text{s}}$
8. 12 mph

♠ 9. $\frac{1710}{\sqrt{13}}$ mph
♠ 10. $\sqrt{5}$ kmph
♠ 11. 52 mph
♠ 12. $\frac{351}{\sqrt{730}}\ \frac{\text{ft}}{\text{s}}$

♠ 13. $1.091\ \frac{\text{m}^2}{\text{s}}$
14. $-\frac{3}{5}\ \frac{\text{ft}}{\text{s}}$
♠ 15. 5 m

3.11 Related Rates II

1. $\frac{4}{3}\ \frac{\text{cm}^2}{\text{min}}$
♠ 2. $\frac{14}{3}\ \frac{\text{ft}}{\text{s}}$
3. $-16\pi\ \frac{\text{in}^3}{\text{min}}$
♠ 4. $\frac{35}{3}\ \frac{\text{ft}}{\text{s}}$
♠ 5. $\frac{3}{16\pi}\ \frac{\text{cm}}{\text{min}}$

♠ 6. $1.01\ \frac{\text{m}}{\text{s}}$
♠ 7. $\frac{5}{14}\ \frac{\text{ft}}{\text{min}}$
♠ 8. $0.00115\ \frac{\text{ft}}{\text{min}}$
♠ 9. $0.11\ \frac{\text{ft}}{\text{min}}$

10. $-16\pi\ \frac{\text{in}^3}{\text{min}}$
♠ 11. $\frac{1}{50}\ \frac{\text{rad}}{\text{s}}$
♠ 12. $-\frac{13}{60}\ \frac{\text{rad}}{\text{s}}$
13. $-\frac{25}{26}\ \frac{\text{ft}}{\text{s}}$

3.12 Hyperbolic Functions

1. 0
♠ 2. 0
♠ 3. 1
♠ 4. 201.71564

5. $\frac{5}{4}$
♠ 6. $\frac{37}{12}$
7. $-\frac{4}{3}$

♠ 8. $\sinh(x) = \frac{5}{12}$, $\cosh(x) = \frac{13}{12}$, $\coth(x) = \frac{13}{5}$
$\text{sech}(x) = \frac{12}{13}$, $\text{csch}(x) = \frac{12}{5}$,

♠ 9. $\sinh(x) = \frac{7}{24}$, $\text{sech}(x) = \frac{24}{25}$,

$\operatorname{csch}(x) = \frac{24}{7}$, $\tanh(x) = \frac{7}{25}$, $\coth(x) = \frac{25}{7}$

♠ 10. *Proof.*

$$\sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{1}{4}((e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y})) = \frac{1}{4}(e^xe^y + e^xe^{-y} - e^{-x}e^y - e^{-x}e^{-y} + e^xe^y - e^xe^{-y} + e^{-x}e^y - e^{-x}e^{-y})$$

$$= \frac{1}{4}(2e^xe^y - 2e^{-x}e^{-y}) = \frac{2}{4}(e^{x+y} - e^{-x-y}) = \sinh(x+y) \qquad \square$$

11. ∞

12. ∞

♠ 13. ∞

♠ 14. $-\infty$

♠ 15. 1

♠ 16. -1

♠ 17. 1

♠ 18. 0

♠ 19. ∞

♠ 20. $-\infty$

21. 0

22. 0

23. $f'(x) = 3\operatorname{sech}^2(x)$

♠ 24. $y' = x\cosh x - 3\sinh x$

♠ 25. $y' = 4e^{4x}\operatorname{sech}^2(2 + e^{4x})$

♠ 26. $y' = \frac{\sinh(\ln x)}{x}$

♠ 27. $y' = 6\sinh(6x)e^{\cosh(6x)}$

♠ 28. $y' = \cosh(\cosh x)\sinh x$

29. $f'(x) = \coth(1 + x^2) - 2x^2\operatorname{csch}^2(1 + x^2)$

30. $f'(x) = \frac{\operatorname{sech}^2(x)}{1 + \tanh^2(x)}$

31. $f'(x) = \tanh(x)$

32. $f'(x) = 2e^x\operatorname{sech}^2(e^x)\tanh(e^x)$

33. $f'(x) = 2x\coth(x^2 + 1)$

34. *Proof.*

$$\begin{aligned} y &= \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \\ y' &= \left(\frac{\sinh(x)}{\cosh(x)}\right)' \\ &= \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)} \\ &= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} \\ &= \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x) \end{aligned}$$

$\square$

35. *Proof.*

$$\begin{aligned} y &= \coth(x) = \frac{\cosh(x)}{\sinh(x)} \\ y' &= \left(\frac{\cosh(x)}{\sinh(x)}\right)' \\ &= \frac{\sinh(x)\sinh(x) - \cosh(x)\cosh(x)}{\sinh^2(x)} \\ &= \frac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} \\ &= \frac{-1}{\sinh^2(x)} = -\operatorname{csch}^2(x) \end{aligned}$$

$\square$

♠ 36. *Proof.*

$$\begin{aligned} \frac{d}{dx}(\operatorname{sech} x) &= \frac{d}{dx}\left(\frac{1}{\cosh x}\right) \\ &= \frac{\cosh x(1)' - 1(\cosh x)'}{\cosh^2 x} \\ &= \frac{-\sinh x}{\cosh^2 x} \\ &= -\frac{1}{\cosh x}\left(\frac{\sinh x}{\cosh x}\right) \\ &= -\operatorname{sech} x\tanh x \end{aligned}$$

$\square$

♠ 37. *Proof.*

$$\begin{aligned} \frac{d}{dx}(\operatorname{csch} x) &= \frac{d}{dx}\left(\frac{1}{\sinh x}\right) \\ &= \frac{\sinh x(1)' - 1(\sinh x)'}{\sinh^2 x} \\ &= \frac{-\cosh x}{\sinh^2 x} \\ &= -\frac{1}{\sinh x}\left(\frac{\cosh x}{\sinh x}\right) \\ &= -\operatorname{csch} x\coth x \end{aligned}$$

$\square$

♠ 38. *Proof.*

$$\begin{aligned} y &= \cosh^{-1}(x) \\ \cosh(y) &= x \\ \frac{d}{dx}(\cosh(y)) &= \frac{d}{dx}(x) \end{aligned}$$

$$\begin{aligned} \sinh(y)y' &= 1 \\ y' &= \frac{1}{\sinh(y)} \end{aligned}$$

$$= \frac{1}{\sqrt{\cosh^2(y)-1}}$$
$$= \frac{1}{\sqrt{x^2-1}}$$

♠ 39. *Proof.*

$$\begin{aligned} y &= \tanh^{-1}(x) \\ \tanh(y) &= x \\ \frac{d}{dx}(\tanh(y)) &= \frac{d}{dx}(x) \\ \operatorname{sech}^2(y)y' &= 1 \\ y' &= \frac{1}{\operatorname{sech}^2(y)} \\ &= \frac{1}{1-\tanh^2(y)} \\ &= \frac{1}{1-x^2} \end{aligned}$$

□

40. *Proof.*

$$\begin{aligned} y &= \coth^{-1}(x) \\ \coth(y) &= x \\ \frac{d}{dx}(\coth(y)) &= \frac{d}{dx}(x) \\ -\operatorname{csch}^2(y)y' &= 1 \\ y' &= -\frac{1}{\operatorname{csch}^2(y)} \\ &= -\frac{1}{\coth^2(y)-1} \\ &= \frac{1}{1-\coth^2(y)} \\ &= \frac{1}{1-x^2} \end{aligned}$$

□

41. *Proof.*

$$\begin{aligned} y &= \operatorname{sech}^{-1}(x) \\ \operatorname{sech}(y) &= x \\ \frac{d}{dx}(\operatorname{sech}(y)) &= \frac{d}{dx}(x) \\ -\tanh(y)\operatorname{sech}(y)y' &= 1 \\ y' &= -\frac{1}{\tanh(y)\operatorname{sech}(y)} \\ &= -\frac{1}{\operatorname{sech}(y)\sqrt{1-\operatorname{sech}^2(y)}} \\ &= \frac{1}{x\sqrt{1-x^2}} \end{aligned}$$

□

Chapter 4 : Applications of Derivatives

4.1 Extrema

1. local min @ $x = 0$,
local max @ $x = \pm 1$,
global min DNE,
global max @ $y = 1$

2. local min @ $x = 6$,
local max @ $x = 2$,
global min DNE,
global max DNE

♠ 3. local min @ $x = -1$,
local max @ $x = 1$,
global min @ $y = -2$,
global max @ $y = 2$

4. local min @ $x = 3$,
local max @ $x = -3$,
global min DNE,
global max DNE

5. local min @ $x = 3$,
local max @ $x = 0$,
global min @ $y = -3$,
global max DNE

♠ 6. local min @ $x = 0$,
local max DNE,
global min DNE,
global max DNE

♠ 7. local min @ $x = -4, 2, 6$,
local max @ $x = -6, 0, 3$,
global min @ $y = -2$,
global max @ $y = 4$

♠ 8. local min @ $x = -4, 6$
or $0 < x < 3$,
local max @ $x = -6$
or $0 \le x \le 3$,
global min @ $y = -2$,
global max @ $y = 4$

9. local min @ $x = -\frac{3}{2}$,
local max @ $x = 1, 3$,
global min DNE,
global max @ $y = 4$

♠ 10. local min @ $x = 0, 7$,
local max @ $x = 2, 4$,
global min @ $y = -2$,
global max DNE

♠ 11. local min @ $x = 0$,
local max @ $x = -2, 3, 5$,
global min DNE,
global max @ $y = 4$

12. local min DNE,
local max DNE,
global min DNE,
global max DNE

13. $x = -1,\ \frac{7}{3}$

♠ 14. $x = 1, -5$

15. $x = -1,\ 2$

16. $x = -5,\ 1$

17. $x = 0,\ \frac{3}{4}$

18. $t = -2$

♠ 19. $t = 0, 4$

20. $x = -1$

21. $x = \frac{1}{e}$

22. $x = \pi k$

23. $x = -\frac{3}{2},\ \frac{3}{2}$

24. $x = 0$

25. min @ $y = -\frac{9}{4}$, max @ $y = 54$

26. min @ $y = -53$, max @ $y = 47$

27. min @ $y = -76$, max @ $y = 5$

♠ 28. min @ $y = -217$, max @ $y = 33$

29. min @ $y = -19$, max @ $y = 33$

30. min @ $y = 1$, max @ $y = 6$

31. min @ $y = 2$, max @ $y = 39$

32. min @ $y = 11$, max @ $y = 51$

33. min @ $y = -16$, max @ $y = 75$

34. min @ $y = -\frac{1}{3}$, max @ $y = \frac{1}{7}$

♠ 35. min @ $y = 0$, max @ $y = 1$

♠ 36. min @ $y = -3\sqrt{7}$, max @ $y = 32$

♠ 37. min @ $y = 0$, max @ $y = 12\sqrt{3}$

♠ 38. min @ $y = 8 + 4\pi$, max @ $y = 12\pi - 8$

♠ 39. min @ $y = \ln\left(\frac{11}{4}\right)$, max @ $y = \ln 23$

40. min @ $y = -\frac{\sqrt{3}}{4}$, max @ $y = \frac{1}{2}$

41. min @ $y \approx -0.5708$, max @ $y \approx 1.3484$

4.2 The Mean Value Theorem

1. *Proof.* Since f is differentiable on its domain, $(-\infty, \infty)$, f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. So, f satisfies the hypotheses of the Mean Value Theorem. Thus, there is a c such that

$$f'(c) = 2c = \frac{f(2) - f(0)}{2 - 0} = \frac{(2)^2 - (0)^2}{2} = 2$$
$$\Rightarrow\ c = 1$$

Since $0 < 1 < 2$, we have that $c = 1$. □

♠ 2. *Proof.* Since f is differentiable on its domain, $(-\infty, \infty)$, f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. So, f satisfies the hypotheses of the Mean Value Theorem. Thus, there is a c such that

$$f'(c) = 4c - 3 = \frac{f(2) - f(0)}{2 - 0} = \frac{(3 - 1}{2} = 1$$
$$\Rightarrow\ 4c = 4\ \Rightarrow\ c = 1$$

Since $0 < 1 < 2$, we have that $c = 1$. □

♠ 3.
Proof. Since f is differentiable on its domain, $(-\infty, \infty)$, f is continuous on $[3, 5]$ and differentiable on $(3, 5)$. So, f satisfies the hypotheses of the Mean Value Theorem. Thus, there is a c such that

$$f'(c) = -32 + 8c = \frac{f(5) - f(3)}{5 - 3}\frac{-56 - (-56)}{2} = 0$$
$$\Rightarrow\ 8c = 32\ \Rightarrow\ c = 4$$

Since $3 < 4 < 5$, we have that $c = 4$. □

4. *Proof.* Since f is differentiable on its domain, $(-\infty, \infty)$, f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. So, f satisfies the hypotheses of the Mean Value Theorem. Thus, there is a c such that

$$f'(c) = -12c^2 + 7 = \frac{f(1) - f(0)}{1 - 0} = \frac{(-4 + 7 - 3) - (-3)}{1} = 3$$
$$\Rightarrow 12c^2 = -4\ \Rightarrow\ c^2 = \frac{1}{3}\ \Rightarrow\ c = \pm\frac{1}{\sqrt{3}}$$

Since $0 < \frac{1}{\sqrt{3}} < 1$, we have that $c = \frac{1}{\sqrt{3}}$. □

♠ 5. *Proof.* Since f is differentiable on its domain, $[0, \infty)$, f is continuous on $[0, 25]$ and differentiable on $(0, 25)$. So, f satisfies the hypotheses of the Mean Value Theorem. Thus, there is a c such that

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(25) - f(0)}{25 - 0} = \frac{5}{25} = \frac{1}{5}$$

$$\Rightarrow\ 2\sqrt{c} = 5\ \Rightarrow\ \sqrt{c} = \frac{5}{2}\ \Rightarrow\ c = \frac{25}{4} = 6.25$$

Since $0 < \dfrac{25}{4} < 25$, we have that $c = \dfrac{25}{4}$. □

♠ 6. *Proof.* Since f is differentiable on its domain, $[0, \infty)$, f is continuous on $[0, 36]$ and differentiable on $(0, 36)$. So, f satisfies the hypotheses of the Mean Value Theorem. Thus, there is a c such that

$$f'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{6} = \frac{f(36) - f(0)}{36 - 0} = \frac{0 - 0}{36} = 0$$
$$\Rightarrow\ \frac{1}{2\sqrt{c}} = \frac{1}{6}\ \Rightarrow\ 2\sqrt{c} = 6\ \Rightarrow\ \sqrt{c} = 3\ \Rightarrow\ c = 9$$

Since $0 < 9 < 36$, we have that $c = 9$. □

♠ 7. *Proof.* While f is differentiable on its domain, $(-\infty, 0) \cup (0, \infty)$, but $f(0)$ DNE. Hence, f is neither differentiable on $(-1, 1)$ nor continuous on $[-1, 1]$. So, f does not satisfies the hypotheses of the Mean Value Theorem. Any number c that satisfies the conclusion of the Mean Value Theorem would satisfy

$$f'(c) = -\frac{2}{3}x^{-1/3} = \frac{f(1) - f(-1)}{1 + 1} = \frac{0 - 0}{2} = 0$$
$$\Rightarrow\ x^{-1/3} = 0\ \Rightarrow\ x^{1/3} = \frac{1}{0} \Rightarrow\Leftarrow$$

Therefore, there are no numbers c which satisfy the conclusion of the Mean Value Theorem. □

♠ 8. *Proof.* Since f is differentiable on its domain, $(-\infty, \infty)$, f is continuous on $\left[\dfrac{\pi}{12}, \dfrac{7\pi}{12}\right]$ and differentiable on $\left(\dfrac{\pi}{12}, \dfrac{7\pi}{12}\right)$. So, f satisfies the hypotheses of the Mean Value Theorem. Thus, there is a c such that

$$f'(c) = -3\sin(3c) = \frac{f(7\pi/12) - f(\pi/12)}{7\pi/12 - \pi/12} = \frac{\cos(7\pi/4) - \cos(\pi/4)}{\pi/2} = 0$$
$$\Rightarrow\ \sin(3c) = 0\ \Rightarrow\ 3c = \pi k\ \Rightarrow\ c = \frac{\pi}{3}k$$

Since $\dfrac{\pi}{12} < \dfrac{\pi}{3} < \dfrac{7\pi}{12}$, we have that $c = \dfrac{\pi}{3}$. □

♠ 9. *Proof.* While f is differentiable on its domain, $(-\infty, 3) \cup (3, \infty)$, but $f(3)$ DNE. Hence, f is neither differentiable on $(2, 5)$ nor continuous on $[2, 5]$. So, f does not satisfies the hypotheses of the Mean Value Theorem. Any number c that satisfies the conclusion of the Mean Value Theorem would satisfy

$$f'(c) = -2(c-3)^{-3} = \frac{f(5) - f(2)}{5 - 2} = \frac{1/4 - 1}{3} = -\frac{1}{4}$$
$$\Rightarrow\ (c - 3)^{-3} = \frac{1}{8}\ \Rightarrow\ (c - 3)^3 = 8$$
$$\Rightarrow\ c - 3 = 2\ \Rightarrow\ c = 5$$

But $5 \not< 5$. Therefore, there are no numbers c which satisfy the conclusion of the Mean Value Theorem. □

10. *Proof.* Since f is differentiable on its domain, $(-\infty, 0) \cup (0, \infty)$, f is continuous on $[1, 2]$ and differentiable on $(1, 2)$. So, f satisfies the hypotheses of the Mean Value Theorem. Thus, there is a c such that

$$f'(c) - \frac{1}{c^2} = \frac{f(2) - f(1)}{2 - 1} = \frac{1/2 - 1}{1} = \frac{-1/2}{1} = -\frac{1}{2}$$
$$\Rightarrow\ c^2 = 2\ \Rightarrow\ c = \pm\sqrt{2}$$

Since $1 < \sqrt{2} < 2$, we have that $c = \sqrt{2}$. □

♠ 11. 10

♠ 12. $8 \le f(5) - f(1) \le 16$

♠ 13. *Proof.* No, $\left.\dfrac{\Delta y}{\Delta x}\right|_{[2,5]} = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{17}{2} = 8.5$. The Mean Value Theorem implies that $f'(c) = 8.5$ for some $0 < c < 2$, but $f'(x) \le 6$ for all x, a contradiction. □

14. *Proof.* Let $f(x) = x^3 + 3x - 15$, which is a differentiable function. Note that $f(0) = (0)^3 + 3(0) - 15 = -15 < 0$ and $f(3) = (3)^3 + 3(3) - 15 = 21 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between 0 and 2 such that $f(c) = 0$. Therefore, the above equation has a solution.

Suppose to the contrary that f has two distinct roots a and b with $a < b$. Then the average rate of change of f on the interval $[a, b]$ is 0 since $f(a) = 0 = f(b)$. Since f is a everywhere differentiable function, the Mean Value Theorem states that there is some c between a and b such that $f'(c) = 0$.

On the other hand, $f'(x) = 3x^2 + 3 = 3(x^2 + 1)$, which is positive for all x, which is a contradiction. Therefore, f has at most one real root. □

15. *Proof.* Let $f(x) = 2x + \sin x$, which is a differentiable function. Clearly, $f(0) = 2(0) + \sin(0) = 0$. Therefore, the above equation has a solution.

    Suppose to the contrary that f has two distinct roots a and b with $a < b$. Then the average rate of change of f on the interval $[a, b]$ is 0 since $f(a) = 0 = f(b)$. Since f is a everywhere differentiable function, the Mean Value Theorem states that there is some c between a and b such that $f'(c) = 0$. On the other hand, $f'(x) = 2 + \cos x > 2 - 1 = 1$, which is positive for all x, which is a contradiction. Therefore, f has at most one real root. □

16. *Proof.* Let $f(x) = 3x + \sin(2x)$, which is a differentiable function. Clearly, $f(0) = 3(0) + \sin(0) = 0$. Therefore, the above equation has a solution.

    Suppose to the contrary that f has two distinct roots a and b with $a < b$. Then the average rate of change of f on the interval $[a, b]$ is 0 since $f(a) = 0 = f(b)$. Since f is a everywhere differentiable function, the Mean Value Theorem states that there is some c between a and b such that $f'(c) = 0$. On the other hand, $f'(x) = 3 + 2\cos(2x) \geq 3 + 2(-1) = 3 - 2 = 1$, that is, $f'(x)$ is positive for all x, which is a contradiction. Therefore, f has at most one real root. □

17. ♠ *Proof.* Let $f(x) = 7x + \cos x$, which is a differentiable function. Note that $f(-\pi) = -7\pi + \cos(-\pi) = -7\pi - 1 < 0$ and $f(0) = 7(0) + \cos(0) = 1 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between $-\pi$ and 0 such that $f(c) = 0$. Therefore, the above equation has a solution.

    Suppose to the contrary that f has two distinct roots a and b with $a < b$. Then the average rate of change of f on the interval $[a, b]$ is 0 since $f(a) = 0 = f(b)$. Since f is a everywhere differentiable function, the Mean Value Theorem states that there is some c between a and b such that $f'(c) = 0$. On the other hand, $f'(x) = 7 - \sin x \geq 7 - 1 = 6$, which is positive for all x, which is a contradiction. Therefore, f has at most one real root. □

18. *Proof.* Let $f(x) = e^x + x$, which is a differentiable function. Note that $f(-1) = e^{-1} - 1 < 0$ (note that $1/e < 1$ since $e > 1$) and $f(0) = e^0 + 0 = 1 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between -1 and 0 such that $f(c) = 0$. Therefore, the above equation has a solution.

    Suppose to the contrary that f has two distinct roots a and b with $a < b$. Then the average rate of change of f on the interval $[a, b]$ is 0 since $f(a) = 0 = f(b)$. Since f is a everywhere differentiable function, the Mean Value Theorem states that there is some c between a and b such that $f'(c) = 0$. On the other hand, $f'(x) = e^x + 1$. Since $e^x > 0$ for all x, $f'(x)$ is positive for all x, which is a contradiction. Therefore, f has at most one real root. □

19. *Proof.* Let $f(x) = \ln(x) + x$, which is a differentiable function. Note that $f(e^{-1}) = \ln(e^{-1}) + e^{-1} = -1 + \dfrac{1}{e} < 0$ (note that $1/e < 1$ since $e > 1$) and $f(1) = \ln(1) + 1 = 1 > 0$. Therefore, by the Intermediate Value Theorem, there exists some value c between -1 and 0 such that $f(c) = 0$. Therefore, the above equation has a solution.

    Suppose to the contrary that f has two distinct roots a and b with $a < b$. Then the average rate of change of f on the interval $[a, b]$ is 0 since $f(a) = 0 = f(b)$. Since f is a everywhere differentiable function, the Mean Value Theorem states that there is some c between a and b such that $f'(c) = 0$. On the other hand, $f'(x) = e^x + 1$. Since $e^x > 0$ for all x, $f'(x)$ is positive for all x, which is a contradiction. Therefore, f has at most one real root. □

4.3 The First Derivative Test

1. increase: $(-\infty, \infty)$, decrease: DNE, max DNE, min DNE, upward: DNE, downward: DNE, PoI DNE

2. increase: $\left(\dfrac{1}{2}, \infty\right)$, decrease: $\left(-\infty, \dfrac{1}{2}\right)$, max DNE, min @ $x = \dfrac{1}{2}$, upward: $(-\infty, \infty)$, downward: DNE, PoI DNE

♠ 3. increase: $\left(-\frac{9}{2},0\right)\cup\left(\frac{9}{2},\infty\right)$, decrease: $\left(-\infty,-\frac{9}{2}\right)\cup\left(0,\frac{9}{2}\right)$, max @ $x=0$, min @ $x=\pm\frac{9}{2}$, upward: $(-\infty,-3)\cup(3,\infty)$, downward: $(-3,3)$, PoI @ $x=\pm 3$

♠ 4. increase: $(0,2)$, decrease: $(-\infty,0)\cup(2,\infty)$, max @ $x=2$, min @ $x=0$, upward: $(-\infty,1)$, downward: $(1,\infty)$, PoI @ $x=1$

5. increase: $(-\infty,-1)\cup(0,1)$, decrease: $(-1,0)\cup(1,\infty)$, max @ $x=\pm1$, min @ $x=0$, upward: $\left(-\frac{3}{5},\frac{3}{5}\right)$, downward: $\left(-\infty,-\frac{3}{5}\right)\cup\left(\frac{3}{5},\infty\right)$, PoI @ $x=\pm\frac{3}{5}$

6. increase: $\left(-\frac{7}{5},0\right)\cup\left(0,\frac{7}{5}\right)$, decrease: $\left(-\infty,-\frac{7}{5}\right)\cup\left(\frac{7}{5},\infty\right)$, max @ $x=\frac{7}{5}$, min @ $x=-\frac{7}{5}$, upward: $(-\infty,-1)\cup(0,1)$, downward: $(-1,0)\cup(1,\infty)$, PoI @ $x=0,\pm1$

♠ 7. increase: $(0,7)$, decrease: DNE, max @ $x=7$, min @ $x=0$, upward: $(0,2)\cup(6,7)$, downward: $(2,6)$, PoI @ $x=2,6$

8. increase: $(0,\infty)$, decrease: $(-\infty,0)$, max DNE, min @ $x=0$, upward: $(-1,1)$, downward: $(-\infty,-1)\cup(1,\infty)$, PoI @ $x=\pm1$

♠ 9. increase: $(-2,6)$, decrease: $(-6,-2)$, max @ $x=\pm6$, min @ $x=-2$, upward: $(-4,-2)\cup(2,0)\cup(2,3)$, downward: $(-6,-4)\cup(0,2)\cup(3,6)$, PoI @ $x=-4,0,2,3$

10. increase: $(-\infty,-3)\cup(3,\infty)$, decrease: $(-3,3)$, max @ $x=-3$, min @ $x=3$, upward: $\left(-\frac{3}{2},-\frac{1}{2}\right)\cup\left(0,\frac{1}{2}\right)\cup\left(\frac{3}{2},\infty\right)$, downward: $\left(-\infty,-\frac{3}{2}\right)\cup\left(-\frac{1}{2},0\right)\cup\left(\frac{1}{2},\frac{3}{2}\right)$, PoI @ $x=0,\pm\frac{1}{2},\pm\frac{3}{2}$

11. increase: $(-2,6)$, decrease: $(-6,-2)$, max @ $x=\pm6$, min @ $x=-2$, upward: $(-4,-2)\cup(-2,0)$, downward: $(-6,-4)\cup(3,6)$, PoI @ $x=-4,0,3$

12. increase: $(-1,3)\cup(3,5)$, decrease: $(-\infty,-1)\cup(5,\infty)$, max @ $x=5$, min @ $x=-1$, upward: $(-\infty,1)\cup\left(\frac{3}{2},3\right)$, downward: $\left(1,\frac{3}{2}\right)\cup(3,-\infty)$, PoI @ $x=1,\frac{3}{2},3$

♠ 13. increase: $(-\infty,0\cup(0,4)$, decrease: $(4,7)$, max @ $x=4$, min @ $x=7$, upward: $(-\infty,0)\cup(0,2)$, downward: $(2,4)\cup(4,7)$, PoI @ $x=2$

14. increase: $(-\infty,0)\cup(1,4)$, decrease: $(0,1)\cup(4,7]$, max @ $x=0,4$, min @ $x=1,7$, upward: $(-\infty,0)\cup\left(0,\frac{13}{5}\right)\cup\left(4,\frac{23}{5}\right)$, downward: $\left(\frac{13}{5},4\right)\cup\left(\frac{23}{5},7\right)$, PoI @ $x=\frac{13}{5},4,\frac{23}{5}$

♠ 15. increase: $\left(-\frac{13}{2},-2\right)\cup(-2,3)$, decrease: $\left(-\infty,-\frac{13}{2}\right)\cup(5,\infty)$, max @ $x=3,5$, min @ $x=-\frac{13}{2}$, upward: $(-\infty,-2)\cup(0,2)$, downward: $(-2,0)\cup(2,3)\cup(5,\infty)$, PoI @ $x=\pm2,0$

16. increase: $(-\infty,0)\cup(0,5)$, decrease: $(5,\infty)$, max @ $x=5$, min DNE, upward: $(4,5)\cup(5,\infty)$, downward: $(0,4)$, PoI @ $x=0,4$

17. increase: $(-\infty,-1)\cup(1,\infty)$, decrease: $(-1,1)$, max @ $x=-1$, min @ $x=1$, upward: $(0,\infty)$, downward: $(-\infty,0)$, PoI @ $x=0$

18. increase: $(-\infty,-1)\cup(2,\infty)$, decrease: $(-1,2)$, max @ $x=-1$, min @ $x=2$, upward: $\left(\frac{1}{2},\infty\right)$, downward: $\left(-\infty,\frac{1}{2}\right)$, PoI @ $x=\frac{1}{2}$

19. increase: $(-\infty,-3)\cup(1,\infty)$, decrease: $(-3,1)$, max @ $x=-3$, min @ $x=1$, upward: $(-1,\infty)$, downward: $(-\infty,-1)$, PoI @ $x=-1$

20. increase: $(-2,0)\cup(2,\infty)$, decrease: $(-\infty,-2)\cup(0,2)$, max @ $x=0$, min @ $x=\pm 2$, upward: $\left(-\infty,-\frac{2}{\sqrt{3}}\right)\cup\left(\frac{2}{\sqrt{3}},\infty\right)$, downward: $\left(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$, PoI @ $x=\pm\frac{2}{\sqrt{3}}$

21. increase: $(-\infty,-4)\cup\left(\frac{1}{5},\infty\right)$, decrease: $\left(-4,\frac{1}{5}\right)$, max @ $x=-4$, min @ $x=\frac{1}{5}$, upward: $\left(-\frac{19}{10},\infty\right)$, downward: $\left(-\infty,-\frac{19}{10}\right)$, PoI @ $x=-\frac{19}{10}$

22. increase: $(-\sqrt{7},0)\cup(\sqrt{7},\infty)$, decrease: $(-\infty,-\sqrt{7})\cup(0,\sqrt{7})$, max @ $x=0$, min @ $x=\pm\sqrt{7}$, upward: $\left(-\infty,-\sqrt{\frac{7}{3}}\right)\cup\left(\sqrt{\frac{7}{3}},\infty\right)$, downward: $\left(-\sqrt{\frac{7}{3}},\sqrt{\frac{7}{3}}\right)$, PoI @ $x=\pm\sqrt{\frac{7}{3}}$

♠ 23. increase: $(-\infty,-4)\cup(3,\infty)$, decrease: $(-4,3)$, max @ $x=-4$, min @ $x=3$, upward: $\left(-\frac{1}{2},\infty\right)$, downward: $\left(-\infty,-\frac{1}{2}\right)$, PoI @ $x=-\frac{1}{2}$

♠ 24. increase: $(-\infty,-2)\cup(1,\infty)$, decrease: $(-2,1)$, max @ $x=-2$, min @ $x=1$, upward: $\left(-\frac{1}{2},\infty\right)$, downward: $\left(-\infty,-\frac{1}{2}\right)$, PoI @ $x=-\frac{1}{2}$

25. increase: $\left(\frac{1-2\sqrt{7}}{9},\frac{1+2\sqrt{7}}{9}\right)$, decrease: $\left(-\infty,\frac{1-2\sqrt{7}}{9}\right)\cup\left(\frac{1+2\sqrt{7}}{9},\infty\right)$, max @ $x=\frac{1+2\sqrt{7}}{9}$, min @ $x=\frac{1-2\sqrt{7}}{9}$, upward: $\left(-\frac{1}{9},\infty\right)$, downward: $\left(-\infty,-\frac{1}{9}\right)$, PoI @ $x=-\frac{1}{9}$

♠ 26. increase: $(-4,0)\cup(4,\infty)$, decrease: $(-\infty-,4)\cup(0,4)$, max @ $x=0$, min @ $x=\pm 4$, upward: $\left(-\infty,-\frac{4}{\sqrt{3}}\right)\cup\left(\frac{4}{\sqrt{3}},\infty\right)$, downward: $\left(-\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)$, PoI @ $x=\pm\frac{4}{\sqrt{3}}$

♠ 27. increase: $\left(0,\frac{\pi}{4}\right)\cup\left(\frac{5\pi}{4},2\pi\right)$, decrease: $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$, max @ $x=\frac{\pi}{4}$, min @ $x=\frac{5\pi}{4}$, upward: $\left(\frac{3\pi}{4},\frac{7\pi}{4}\right)$, downward: $\left(0,\frac{3\pi}{4}\right)\cup\left(\frac{7\pi}{4},2\pi\right)$, PoI @ $x=\frac{3\pi}{4},\frac{7\pi}{4}$

♠ 28. increase: $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, decrease: $\left(0,\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right)$, max @ $x=\frac{3\pi}{2}$, min @ $x=\frac{\pi}{2}$, upward: $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$, downward: $\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right)$, PoI @ $x=\frac{\pi}{6},\frac{5\pi}{6}$

♠ 29. increase: $\left(-\frac{\ln 2}{3},\infty\right)$, decrease: $\left(-\infty,-\frac{\ln 2}{3}\right)$, max DNE, min @ $x=-\frac{\ln 2}{3}$, upward: $(-\infty,\infty)$, downward: DNE, PoI DNE

30. increase: $(-\infty,0)\cup\left(0,\frac{1}{4}\right)$, decrease: $\left(\frac{1}{4},\infty\right)$, max @ $x=\frac{1}{4}$, min DNE, upward: $(-\infty,0)$, downward: $(0,\infty)$, PoI @ $x=0$

4.4 The Second Derivative Test

1. max @ $x=4$, min @ $x=-\frac{4}{3}$

♠ 2. max @ $x=-1$, min @ $x=1$

♠ 3. max @ $x=0$, min @ $x=\frac{12}{11}$

4. max @ $x=2$, min @ $x=0,3$

5. -2 ♠ 6. 2 ♠ 7. 4 ♠ 8. $\frac{4}{3}$ ♠ 9. $-\infty$ 10. 0

11. ∞ 12. $\frac{1}{24}$ 13. 1 14. 2 15. 0 16. $-\frac{1}{2}$

17. 1 18. 2 19. 0 20. $\frac{9}{2}$ 21. $\frac{1}{2}$ 22. ∞

23. 1 24. 2 25. 1 26. $-\frac{2}{\pi}$

4.5 l'Hospital's Rule

1. 0 ♠ 2. $-\frac{1}{2}$ ♠ 3. 0 4. 0 ♠ 5. $\frac{19}{2}$ ♠ 6. $\frac{49}{2}$ ♠ 7. 1

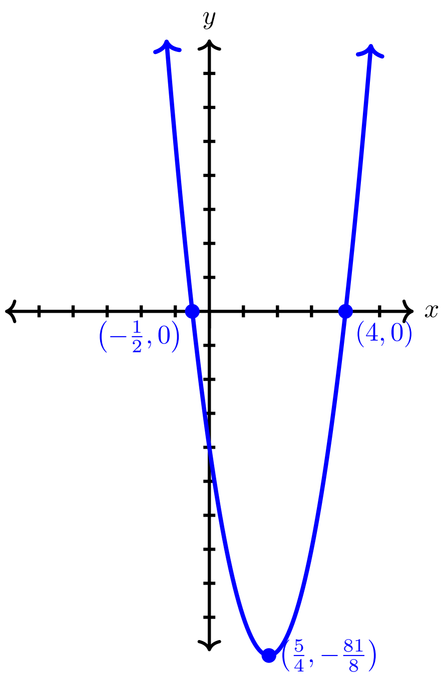
♠ 8. $\frac{1}{7}$ 9. 15 ♠ 10. 8π ♠ 11. 2 ♠ 12. 0 13. π 14. -1

15. $\frac{1}{2}$ ♠ 16. $\frac{9}{2}$ ♠ 17. 1 ♠ 18. e^{-8} 19. e^{-2} 20. 1 21. 1

22. 1 23. 1 24. ∞

4.6 Curve Sketching

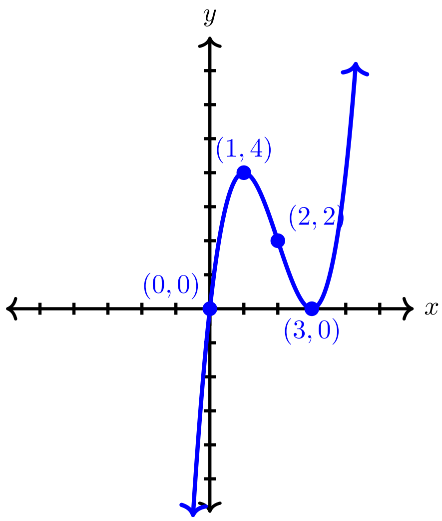
1.



domain: dom $f = (-\infty, \infty)$
intercepts: y-intercept @ -4,
x-intercepts @ $-\frac{1}{2}, 4$
symmetry: DNE
discontinuities: DNE
end behavior: $x \to \pm\infty$, $y \to \infty$
first derivative/extrema:
increase on $\left(-\frac{5}{4}, \infty\right)$, decrease on $\left(-\infty, -\frac{5}{4}\right)$
min @ $\left(\frac{5}{4}, -\frac{81}{8}\right)$
second derivative/concavity:
upward on $(-\infty, \infty)$

	$\left(-\infty, -\frac{5}{4}\right)$	$\left(-\frac{5}{4}, \infty\right)$
$f'(x)$	$-$	$+$
$f''(x)$	$+$	$+$

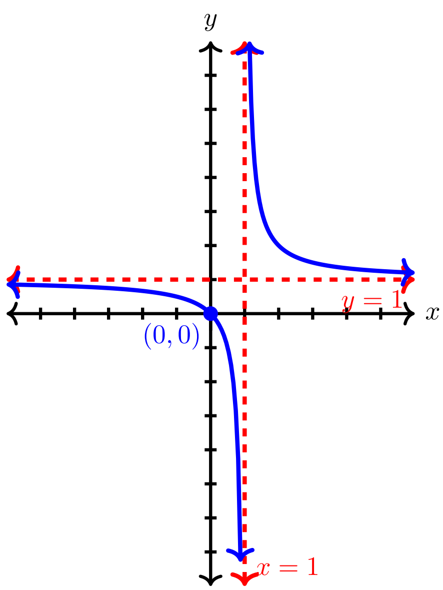
♠ 2.



domain: dom $f = (-\infty, \infty)$
intercepts: y-intercept @ 0, x-intercepts @ 0,3
symmetry: DNE
discontinuities: DNE
end behavior: $x \to \infty$, $y \to \infty$;
$x \to -\infty$, $y \to -\infty$
first derivative/extrema:
increase on $(-\infty, 0) \cup (3, \infty)$, decrease on $(1, 3)$
max @ $(1, 4)$, min @ $(3, 0)$
second derivative/concavity:
upward on$(2, \infty) \cup (-\infty, 2)$,
downward on $(1, 3)$, PoI @ $(2, 2)$

	$(-\infty, 1)$	$(1, 2)$	$(2, 3)$	$(3, \infty)$
$f'(x)$	$+$	$-$	$-$	$+$
$f''(x)$	$-$	$-$	$+$	$+$

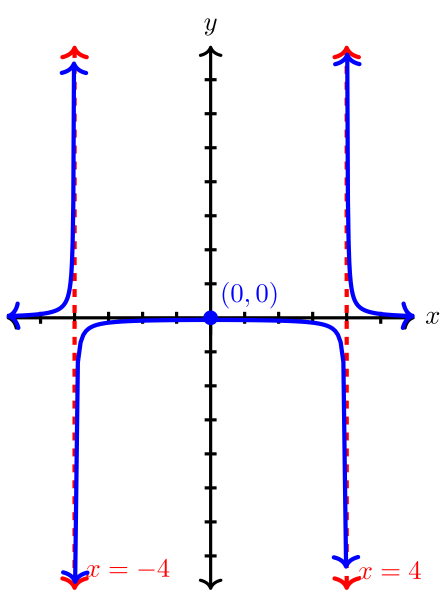
♠ 3.



domain: dom $f = (-\infty, 1) \cup (1, \infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: DNE
discontinuities: vertical asymptote @ $x = 1$
end behavior: horizontal asymptote @ $y = 1$
first derivative/extrema:
decrease on $(-\infty, 1) \cup (1, \infty)$
second derivative/concavity: upward on $(1, \infty)$,
downward on $(-\infty, 1)$

	$(-\infty, 1)$	$(1, \infty)$
$f'(x)$	$-$	$-$
$f''(x)$	$-$	$+$

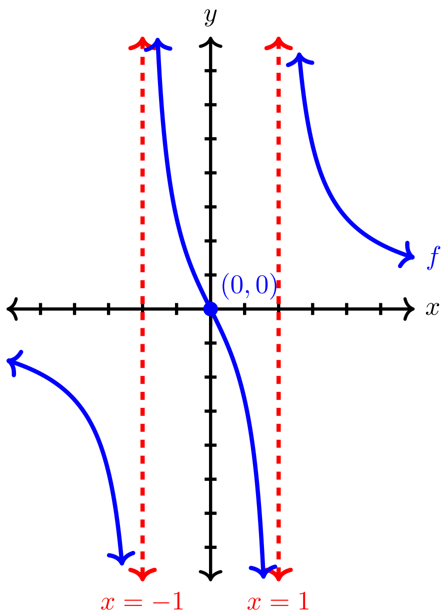
♠ 4.



domain: dom $f = (-\infty, -4) \cup (-4, 4) \cup (4, \infty)$
intercepts: y-intercept @ $-\dfrac{1}{16}$
symmetry: even
discontinuities: vertical asymptotes @ $x = \pm 4$
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema:
increase on $(-\infty, -4) \cup (-4, 0)$,
decrease on $(0, 4) \cup (4, \infty)$
second derivative/concavity:
upward on $(-\infty, -4) \cup (4, \infty)$,
downward on $(-4, 4)$

	$(-\infty, -4)$	$(-4, 0)$	$(0, 4)$	$(4, \infty)$
$f'(x)$	$+$	$+$	$-$	$-$
$f''(x)$	$+$	$-$	$-$	$+$

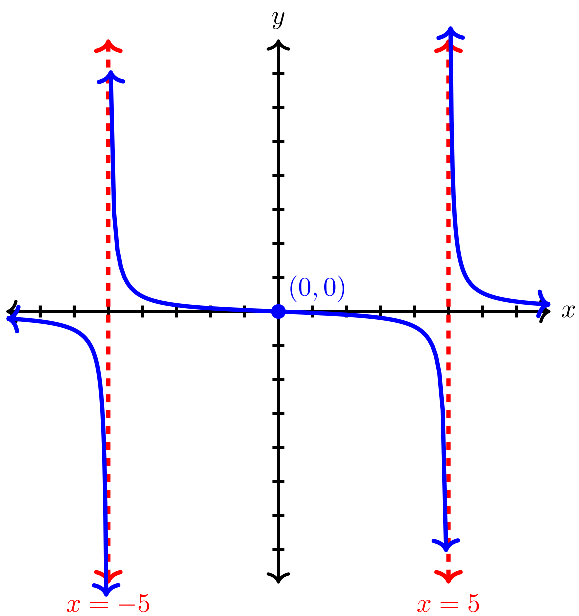
5.



domain: dom $f = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: odd
discontinuities: vertical asymptotes @ $x = \pm 1$
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema:
decrease on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
second derivative/concavity:
upward on $(-1, 0) \cup (1, \infty)$,
downward on $(-\infty, -1) \cup (0, 1)$, PoI @ $x = 0$

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$-$	$-$	$-$	$-$
$f''(x)$	$-$	$+$	$-$	$+$

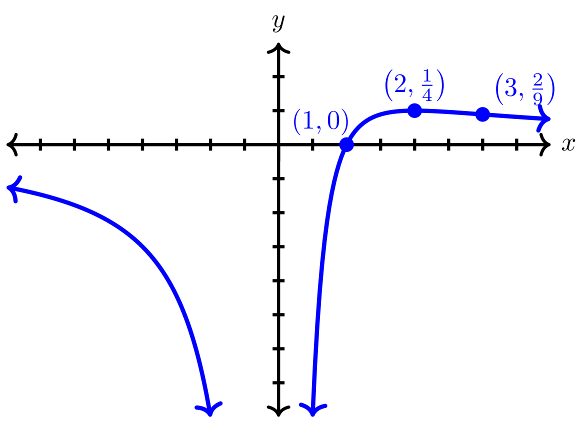
♠ 6.



domain: dom $f = (-\infty, -5) \cup (-5, 5) \cup (5, \infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: odd
discontinuities: vertical asymptotes @ $x = \pm 5$
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema:
decrease on $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$
second derivative/concavity:
upward on $(-5, 0) \cup (5, \infty)$,
downward on $(-\infty, -5) \cup (0, 5)$, PoI @ $(0, 0)$

	$(-\infty, -5)$	$(-5, 0)$	$(0, 5)$	$(5, \infty)$
$f'(x)$	$-$	$-$	$-$	$-$
$f''(x)$	$-$	$+$	$-$	$+$

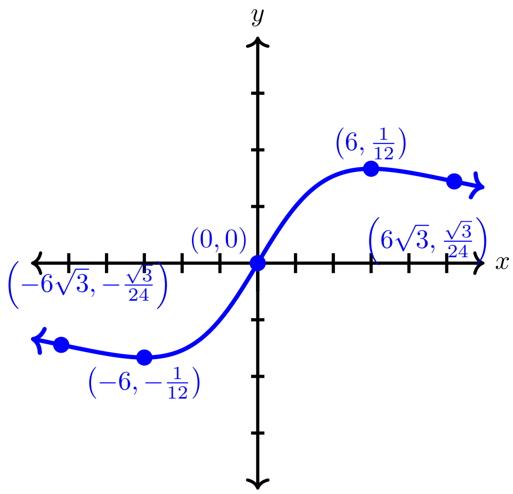
7.



domain: dom $f = (-\infty, 0) \cup (0, \infty)$
intercepts: x-intercept @ 1
symmetry: DNE
discontinuities: vertical asymptote @ $x = 0$
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema: increase on $(0, 2)$,
decrease on $(-\infty, 0) \cup (2, \infty)$, max @ $\left(2, \dfrac{1}{4}\right)$
second derivative/concavity: upward on $(3, \infty)$,
downward on $(-\infty, 0) \cup (0, 3)$, PoI @ $\left(3, \frac{2}{9}\right)$

	$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$

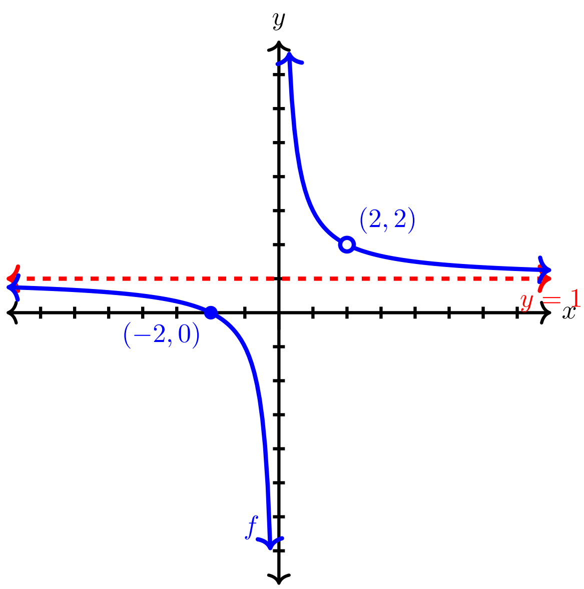
♠ 8.



domain: dom $f = (-\infty, \infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: odd
discontinuities: DNE
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema: increase on $(-6, 6)$,
decrease on $(-\infty, -6) \cup (6, \infty)$,
second derivative/concavity:
upward on $(-6\sqrt{3}, 0) \cup (6\sqrt{3}, \infty)$,
downward on $(-\infty, -6\sqrt{3}) \cup (0, 6\sqrt{3})$
PoI @ $(0,0)$, $\left(\pm 6\sqrt{3}, \pm\dfrac{\sqrt{3}}{24}\right)$

	$(-\infty, -6\sqrt{3})$	$(-6\sqrt{3}, -6)$	$(-6, 0)$	$(0, 6)$	$(6, 6\sqrt{3})$	$(6\sqrt{3}, \infty)$
$f'(x)$	$-$	$-$	$+$	$+$	$-$	$-$
$f''(x)$	$-$	$+$	$+$	$-$	$-$	$+$

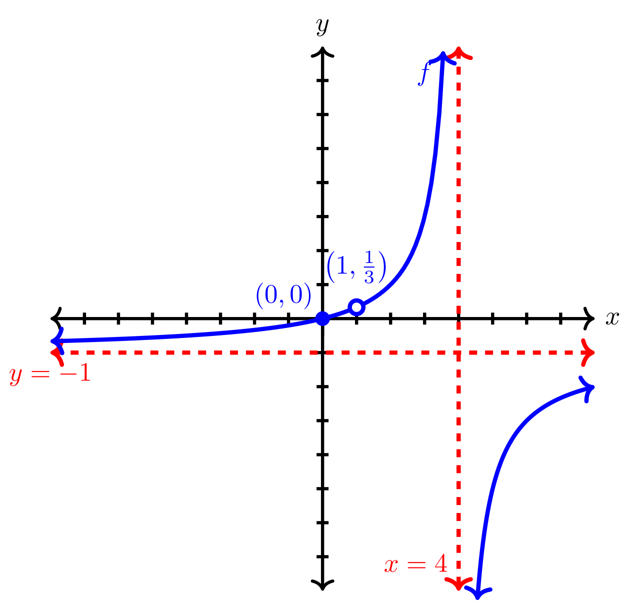
♠ 9.



domain: dom $f = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$
intercepts: x-intercepts @ $x = -2$
symmetry: DNE
discontinuities: vertical asymptote @ $x = 0$,
removed point @ $(2, 2)$
end behavior: horizontal asymptote @ $y = 1$
first derivative/extrema:
decrease on $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$
second derivative/concavity:
upward on $(0, 2) \cup (2, \infty)$,
downward on $(-\infty, 0)$

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	$-$	$-$	$-$
$f''(x)$	$-$	$+$	$+$

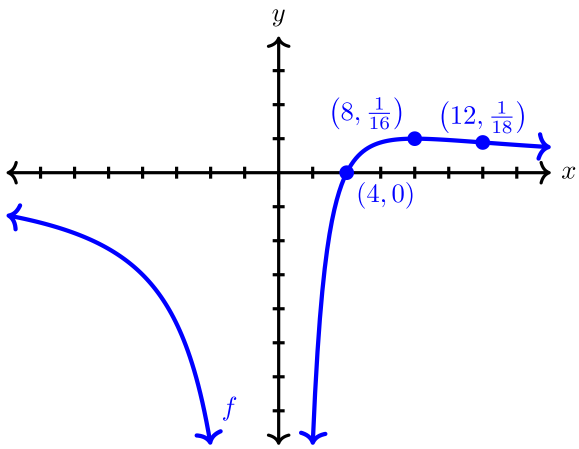
♠ 10.



domain: dom $f = (-\infty, 1) \cup (1, 4) \cup (4, \infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: DNE
discontinuities: vertical asymptote @ $x = 4$,
removed point @ $\left(1, \frac{1}{3}\right)$
end behavior: horizontal asymptote @ $y = -1$
first derivative/extrema:
increase on $(-\infty, 1) \cup (1, 4) \cup (4, \infty)$
second derivative/concavity:
upward on $(-\infty, 1) \cup (1, 4)$, downward on $(4, \infty)$

	$(-\infty, 1)$	$(1, 4)$	$(4, \infty)$
$f'(x)$	$+$	$+$	$+$
$f''(x)$	$+$	$+$	$-$

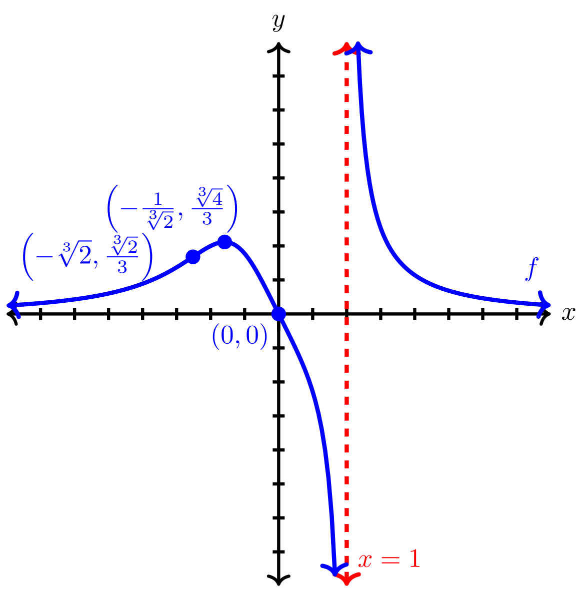
♠ 11.



domain: dom $f = (-\infty, 0) \cup (0, \infty)$
intercepts: DNE
symmetry: DNE
discontinuities: vertical asymptote @ $x = 0$
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema: increase on $(0, 8)$,
decrease on $(-\infty, 0) \cup (8, \infty)$, max @ $\left(8, \frac{1}{16}\right)$
second derivative/concavity:
upward on $(12, \infty)$,
downward on $(-\infty, 0) \cup (0, 12)$, PoI @ $\left(12, \frac{1}{18}\right)$

	$(-\infty, 0)$	$(0, 8)$	$(8, 12)$	$(12, \infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$

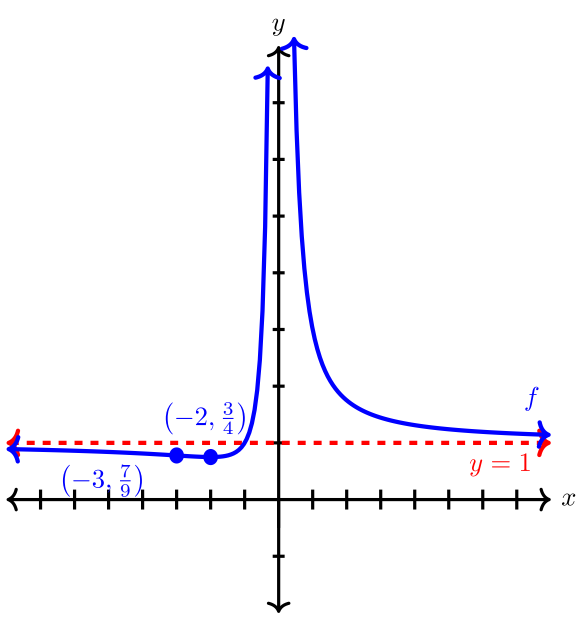
♠ 12.



domain: dom $f = (-\infty, 1) \cup (1, \infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: DNE
discontinuities: vertical asymptote @ $x = 1$
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema:
increase on $\left(-\infty, -\frac{1}{\sqrt[3]{2}}\right)$,
decrease on $\left(-\frac{1}{\sqrt[3]{2}}, 1\right) \cup (1, \infty)$, max @ $\left(8, \frac{1}{16}\right)$
second derivative/concavity:
upward on $(-\infty, -\sqrt[3]{2}) \cup (1, \infty)$,
downward on $(-\sqrt[3]{2}, 0)$, PoI @ $\left(-\sqrt[3]{2}, \frac{\sqrt[3]{2}}{3}\right)$

	$(-\infty, -\sqrt[3]{2})$	$\left(-\sqrt[3]{2}, -\frac{1}{\sqrt[3]{2}}\right)$	$\left(-\frac{1}{\sqrt[3]{2}}, 0\right)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$+$	$+$	$-$	$-$	$-$
$f''(x)$	$+$	$-$	$-$	$-$	$-$

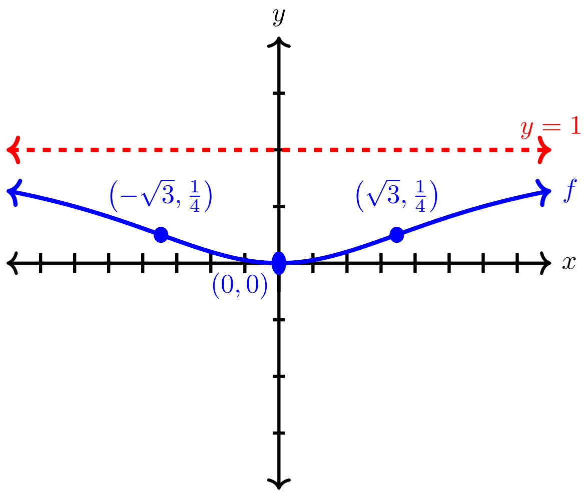
♠ 13.



domain: dom $f = (-\infty, 0) \cup (0, \infty)$
intercepts: DNE
symmetry: DNE
discontinuities: vertical asymptote @ $x = 0$
end behavior: horizontal asymptote @ $y = 1$
first derivative/extrema: increase on $(-2, 0)$,
decrease on $(-\infty, -2) \cup (0, \infty)$, min @ $\left(-2, \frac{3}{4}\right)$
second derivative/concavity:
upward on $(-\infty, -3) \cup (0, \infty)$,
downward on $(-3, 0)$, PoI @ $\left(-3, \frac{7}{9}\right)$

	$(-\infty, -3)$	$(-3, -2)$	$(-2, 0)$	$(0, \infty)$
$f'(x)$	$-$	$-$	$+$	$-$
$f''(x)$	$+$	$-$	$-$	$+$

14.

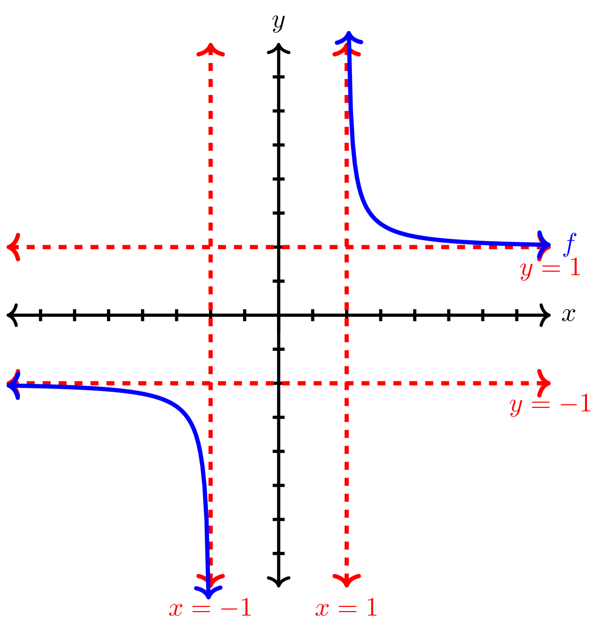


domain: dom $f = (-\infty, \infty)$
intercepts: y-intercept @ 0, x-intercepts @ 0
symmetry: even
discontinuities: DNE
end behavior: horizontal asymptote @ $y = 1$
first derivative/extrema: increase on $(0, \infty)$, decrease on $(-\infty, 0)$, min @ $(0, 0)$
second derivative/concavity:
upward on $(-\sqrt{3}, \sqrt{3})$,
downward on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$, PoI @ $(\pm\sqrt{3}, \frac{1}{4})$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	$-$	$-$	$+$	$+$
$f''(x)$	$-$	$+$	$+$	$-$

4.7 Curve Sketching II

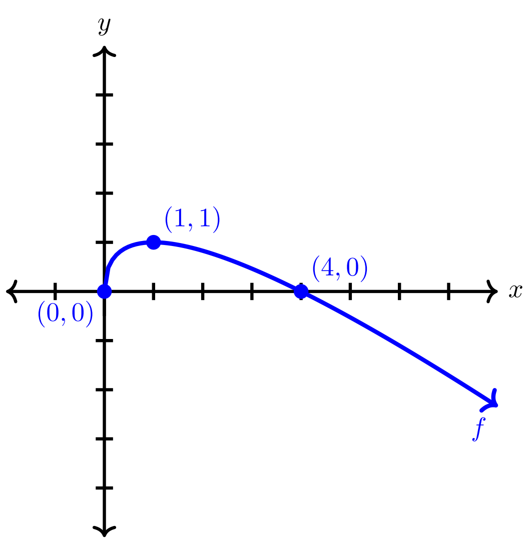
1.



domain: dom $f = (-\infty, -1) \cup (1, \infty)$
intercepts: DNE
symmetry: odd
discontinuities: vertical asymptotes @ $x = \pm 1$
end behavior: horizontal asymptotes @ $y = \pm 1$
first derivative/extrema:
decrease on $(-\infty, -1) \cup (1, \infty)$
second derivative/concavity: upward on $(1, \infty)$, downward on $(-\infty, -1)$

	$(-\infty, -1)$	$(1, \infty)$
$f'(x)$	$-$	$-$
$f''(x)$	$-$	$+$

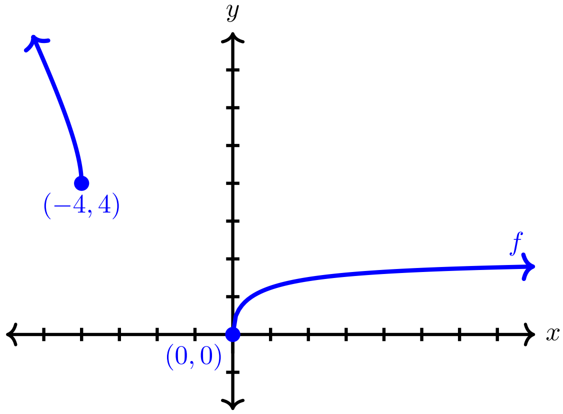
♠ 2.



domain: dom $f = [0, \infty)$
intercepts: y-intercept @ 0, x-intercepts @ 0, 4
symmetry: DNE
discontinuities: DNE
end behavior: $x \to 0^+$, $y \to 0^+$,
$x \to \infty$, $y \to -\infty$
first derivative/extrema: increase on $[0, 1)$, decrease on $(1, \infty)$, max @ $(1, 1)$, min @ $(0, 0)$
second derivative/concavity:
downward on $[0, \infty)$

	$[0, 1)$	$(1, \infty)$
$f'(x)$	$+$	$-$
$f''(x)$	$-$	$-$

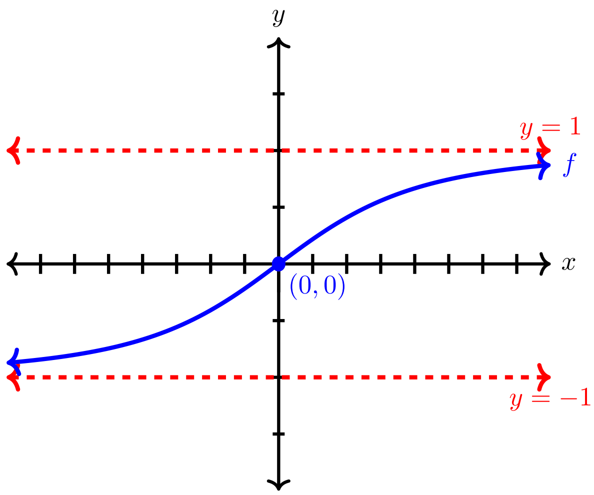
♠ 3.



domain: dom $f = (-\infty,-4] \cup [0,\infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: DNE
discontinuities: DNE
end behavior: $x \to \pm\infty$, $y \to \infty$
first derivative/extrema: increase on $[0,\infty)$, decrease on $(-\infty,-4]$, min @ $(-4,4)$, $(0,0)$
second derivative/concavity:
downward on $(-\infty,-4] \cup [0,\infty)$

	$(-\infty,-4]$		$[0,\infty)$
$f'(x)$	$-$		$+$
$f''(x)$	$-$		$-$

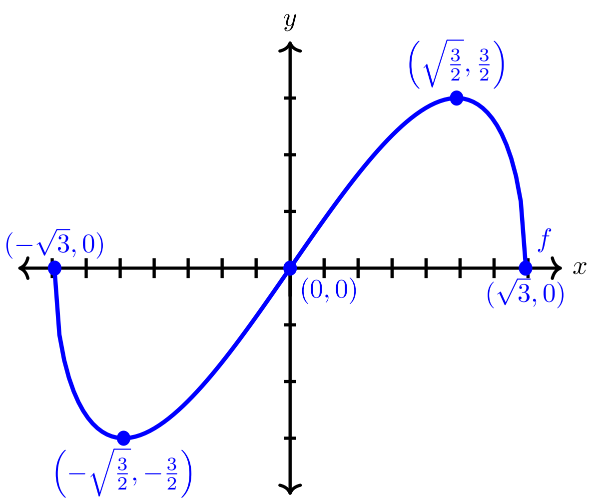
♠ 4.



domain: dom $f = (-\infty,\infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: odd
discontinuities: DNE
end behavior: horizontal asymptotes @ $y = \pm 1$
first derivative/extrema: increase on $(-\infty,\infty)$
second derivative/concavity: upward on $(-\infty,0)$,
downward on $(0,\infty)$, PoI @ $(0,0)$

	$(-\infty,0)$	$(0,\infty)$
$f'(x)$	$+$	$+$
$f''(x)$	$+$	$-$

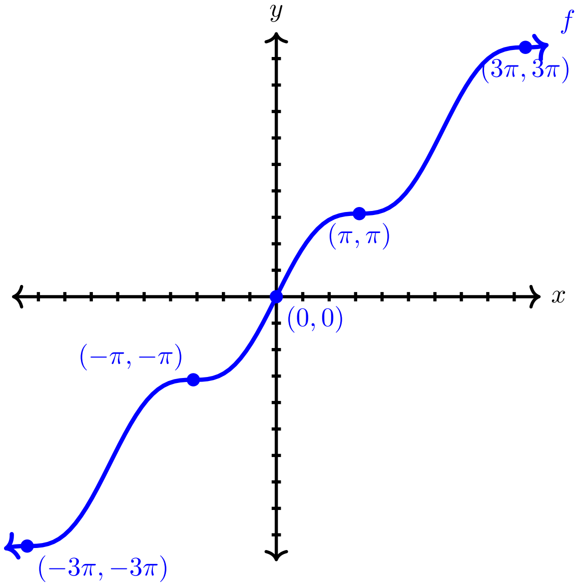
5.



domain: dom $f = [-\sqrt{3},\sqrt{3}]$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: odd
discontinuities: DNE
end behavior: $x \to \pm\sqrt{3}^{\pm}$, $y \to 0^{\pm}$
first derivative/extrema:
increase on $\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)$,
decrease on $\left[-\sqrt{3},-\sqrt{\frac{3}{2}}\right) \cup \left(\sqrt{\frac{3}{2}},\sqrt{3}\right]$,
max @ $(-\sqrt{3},0)$, min @ $(\sqrt{3},0)$
second derivative/concavity:
upward on $(-\sqrt{3},0)$, downward on $(0,\sqrt{3})$, PoI @ $(0,0)$

	$\left[-\sqrt{3},-\sqrt{\frac{3}{2}}\right)$	$\left(-\sqrt{\frac{3}{2}},0\right)$	$\left(0,\sqrt{\frac{3}{2}}\right)$	$\left(\sqrt{\frac{3}{2}},\sqrt{3}\right]$
$f'(x)$	$-$	$+$	$+$	$-$
$f''(x)$	$-$	$-$	$+$	$+$

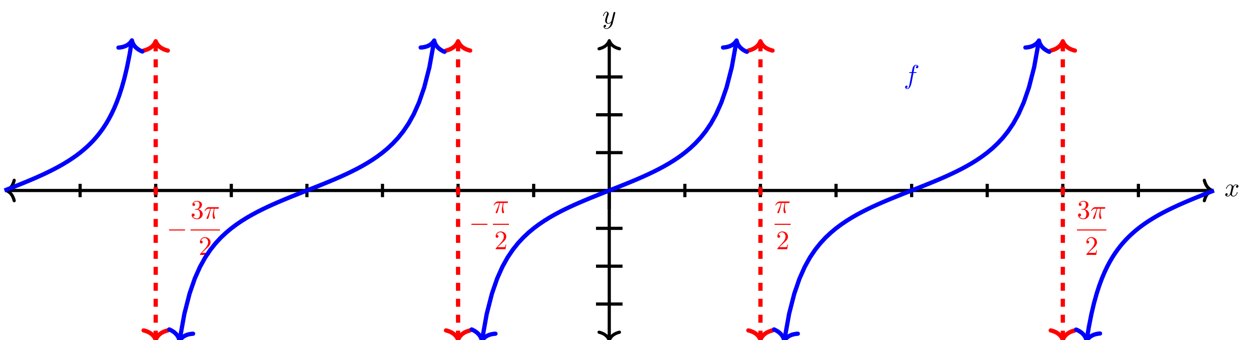
♠ 6.



domain: dom $f = (-\infty, \infty)$
intercepts: y-intercept @ 0, x-intercept @ 0
symmetry: odd
discontinuities: DNE
end behavior: $x \to \pm\infty$, $y \to \pm\infty$
first derivative/extrema: increase on $(-\infty, \infty)$
second derivative/concavity:
upward on $((2k-1)\pi, 2k\pi)$,
downward on $(2k\pi, (2k+1)\pi)$,
PoI @ $((2k+1)\pi, (2k+1)\pi)$

	$((2k-1)\pi, 2k\pi)$	$(2k\pi, (2k+1)\pi)$
$f'(x)$	$+$	$+$
$f''(x)$	$+$	$-$

7.



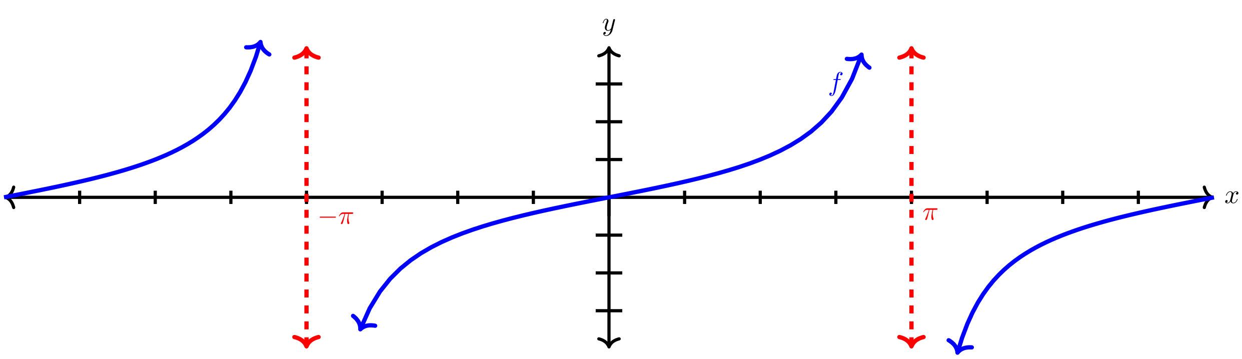
domain: $x \neq \pm\dfrac{\pi}{2} + \pi k$
intercepts: y-intercept @ 0,
x-intercepts @ πk
symmetry: odd
discontinuities: vertical asymptote
@ $x = \pm\dfrac{\pi}{2} + \pi k$

end behavior: π-periodic
first derivative/extrema:
increase on domain
second derivative/concavity:
upward on $\left(\pi k, \frac{\pi}{2} + \pi k\right)$,
downward on $\left(\frac{\pi}{2} + \pi k, \pi k\right)$,

PoI @ x-intercepts

	$\left(-\frac{\pi}{2} + \pi k, \pi k\right)$	$\left(\pi k, \frac{\pi}{2} + \pi k\right)$
$f'(x)$	$+$	$+$
$f''(x)$	$-$	$+$

♠ 8.



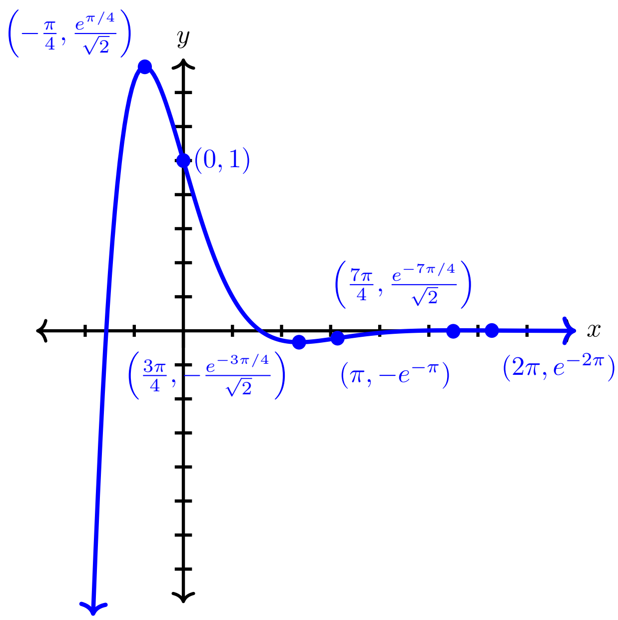
domain: $x \neq (2k+1)\pi$
intercepts: y-intercept @ 0,
x-intercepts @ $2k\pi$
symmetry: odd
discontinuities: vertical asymptote
@ $x = (2k+1)\pi$

end behavior: 2π-periodic
first derivative/extrema:
increase on domain
second derivative/concavity:
upward on $(2k\pi, (2k+1)\pi k)$,
downward on $((2k-1)\pi, 2k\pi)$,

PoI @ x-intercepts

	$((2k-1)\pi, 2k\pi)$	$(2k\pi, (2k+1)\pi k)$
$f'(x)$	$+$	$+$
$f''(x)$	$-$	$+$

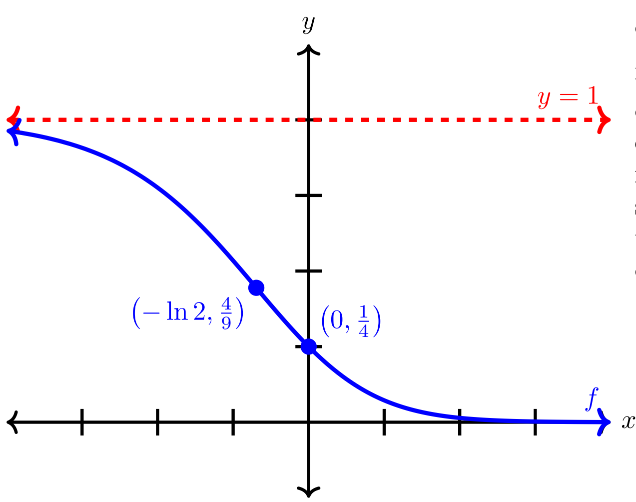
♠ 9.



domain: dom $f = (-\infty, \infty)$
intercepts: y-intercept @ 1,
x-intercepts @ $\dfrac{\pi}{2} + \pi k$
symmetry: DNE
discontinuities: DNE
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema:
increase on $\left(\frac{3\pi}{4} + \pi k, \frac{7\pi}{4} + \pi k\right)$,
decrease on $\left(-\frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k\right)$,
max @ $x = -\frac{\pi}{4} + \pi k$, min @ $x = \frac{3\pi}{4} + \pi k$
second derivative/concavity:
upward on $(\pi k, (k+1)\pi)$,
downward on $\left(-\frac{\pi}{4} + \pi k, \pi k\right) \cup \left((k+1)\pi, \frac{7\pi}{4} + \pi k\right)$,
PoI @ $x = \pi k$

	$\left(-\frac{\pi}{4}+\pi k, \pi k\right)$	$\left(\pi k, \frac{3\pi}{4}+\pi k\right)$	$\left(\frac{3\pi}{4}+\pi k, (k+1)\pi\right)$	$\left((k+1)\pi, \frac{7\pi}{4}+\pi k\right)$
$f'(x)$	$-$	$-$	$+$	$+$
$f''(x)$	$-$	$+$	$+$	$-$

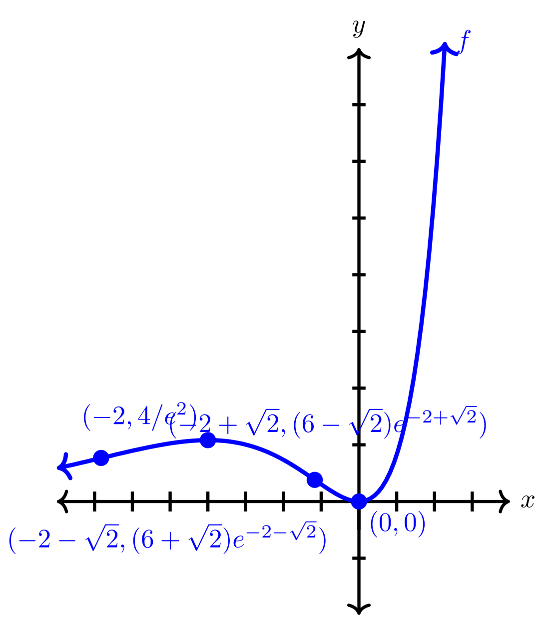
♠ 10.



domain: dom $f = (-\infty, \infty)$
intercepts: y-intercept @ $\dfrac{1}{4}$ symmetry: DNE
discontinuities: DNE
end behavior: horizontal asymptotes @ $y = 0, 1$
first derivative/extrema: decrease on $(-\infty, \infty)$
second derivative/concavity:
upward on $(-\ln 2, \infty)$,
downward on $(-\infty, -\ln 2)$, PoI @ $\left(-\ln 2, \frac{4}{9}\right)$

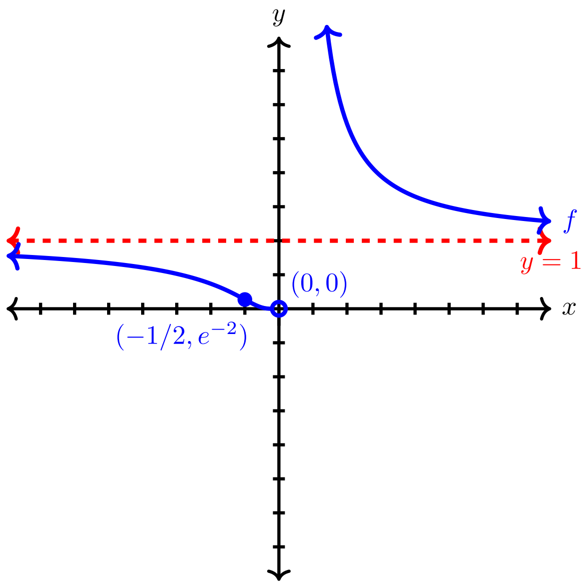
	$(-\infty, -\ln 2)$	$(-\ln 2, \infty)$
$f'(x)$	$-$	$-$
$f''(x)$	$-$	$+$

11.



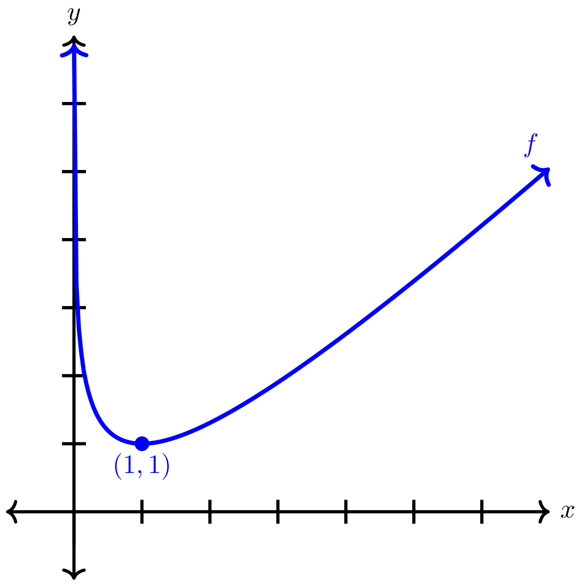
domain: dom $f = (-\infty, \infty)$
intercepts: y-intercept @ 0,
x-intercepts @ 0
symmetry: DNE
discontinuities: DNE
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema:
increase on $(-\infty, -2) \cup (0, \infty)$,
decrease on $(-2, 0)$, max @ $\left(-2, \frac{4}{e^2}\right)$, min @ $(0, 0)$
second derivative/concavity:
upward on $(-\infty, -2-\sqrt{2}) \cup (-2+\sqrt{2}, \infty)$,
downward on $(-2-\sqrt{2}, -2+\sqrt{2})$,
PoI @ $\left(-2 \pm \sqrt{2}, (6 \mp 4\sqrt{2})e^{-2\pm\sqrt{2}}\right)$

	$(-\infty, -2-\sqrt{2})$	$(-2-\sqrt{2}, -2)$	$(-2, -2+\sqrt{2})$	$(-2+\sqrt{2}, 0)$	$(0, \infty)$
$f'(x)$	$+$	$+$	$-$	$-$	$+$
$f''(x)$	$+$	$-$	$-$	$+$	$+$

12. 

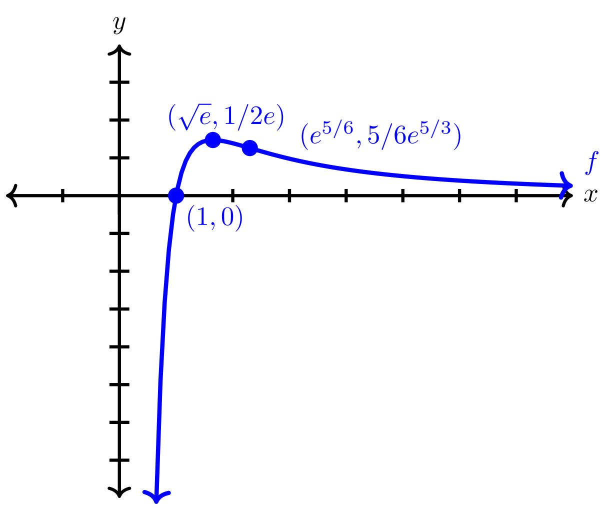
domain: dom $f = (-\infty, 0) \cup (0, \infty)$
intercepts: DNE
symmetry: DNE
discontinuities: vertical asymptote @ $x = 0$,
removed point @ $(0, 0)$
end behavior: horizontal asymptote @ $y = 1$
first derivative/extrema:
decrease on $(-\infty, 0) \cup (0, \infty)$
second derivative/concavity:
upward on $\left(-\frac{1}{2}, 0\right) \cup (0, \infty)$,
downward on $\left(-\infty, -\frac{1}{2}\right)$,
PoI @ $\left(-\frac{1}{2}, \frac{1}{e^2}\right)$

	$\left(-\infty, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, 0\right)$	$(0, \infty)$
$f'(x)$	$-$	$-$	$-$
$f''(x)$	$-$	$+$	$+$

♠ 13. 

domain: dom $f = (0, \infty)$
intercepts: DNE
symmetry: DNE
discontinuities: vertical asymptote @ $x = 0$
end behavior: $x \to 0^+, \infty$, $y \to \infty$
first derivative/extrema: increase on $(0, 1)$
decrease on $(1, \infty)$, min @ $(1, 1)$
second derivative/concavity:
upward on $(0, \infty)$

	$(0, 1)$	$(1, \infty)$
$f'(x)$	$-$	$+$
$f''(x)$	$+$	$+$

14. 

domain: dom $f = (0, \infty)$
intercepts: x-intercept @ 1
symmetry: DNE
discontinuities: vertical asymptote @ $x = 0$
end behavior: horizontal asymptote @ $y = 0$
first derivative/extrema: increase on $(0, e^{1/2})$,
decrease on $(e^{1/2}, e^{5/6}) \cup (e^{5/6}, \infty)$,
max @ $\left(e^{1/2}, \frac{1}{2e}\right)$
second derivative/concavity:
upward on $(0, e^{1/2})$,
downward on $(e^{1/2}, e^{5/6}) \cup (e^{5/6}, \infty)$,
PoI @ $\left(e^{5/6}, \frac{5e^{5/3}}{6}\right)$

	$(0, e^{1/2})$	$(e^{1/2}, e^{5/6})$	$(e^{5/6}, \infty)$
$f'(x)$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$+$

4.8 Optimization

1. -625
♠ 2. $\dfrac{529}{4}$
♠ 3. -576
♠ 4. 16
5. 9
6. 16
♠ 7. 225
♠ 8. $24\sqrt{6}$
9. 20
10. 40
♠ 11. $\dfrac{225}{4}$
♠ 12. $\left(-\dfrac{8}{5}, \dfrac{4}{5}\right)$
♠ 13. $\left(-\dfrac{1}{3}, \pm\dfrac{4\sqrt{2}}{3}\right)$
♠ 14. $2r^2$
♠ 15. $2ab$
♠ 16. 6 cm^2
17. 4000

4.9 Optimization II

1. 120,000 ft^2
♠ 2. 4000 ft $\times$ 6000 ft
♠ 3. 20 cm $\times$ 20 cm $\times$ 10 cm
♠ 4. \$81.77
5. 62.5 ft^3
6. 60 cents
7. \$120
8. \$94.25
♠ 9. $\dfrac{4}{27}\pi r^2 h$
♠ 10. $\dfrac{48}{4+\pi}$ ft
11. 10 in $\times$ 30 in
♠ 12. 24 cm $\times$ 36 cm
♠ 13. 7 m, $\dfrac{28\sqrt{3}}{9+4\sqrt{3}}$ m
♠ 14. ≈ 6.65 ft
♠ 15. $h \approx 3.96$ cm, $r \approx 2.81$ cm
♠ 16. 3 km
♠ 17. 2.26 hr
18. $(9^{2/3}+6^{2/3})^{3/2} \approx 21.1$ ft
19. 45 million
20. 1350

4.10 Newton's Method

1. $L(x) = \dfrac{x+9}{6}; \sqrt{10} \approx \dfrac{19}{6}$
2. $L(x) = \dfrac{x+64}{16}; \sqrt{65} \approx \dfrac{129}{16}$
♠ 3. $L(x) = 16x - 9; f(1.1) \approx 8.6$
4. $L(x) = \dfrac{x+16}{12}; \sqrt[3]{7} \approx \dfrac{23}{12}$
5. $L(x) = \dfrac{x+48}{32}; \sqrt[4]{17} \approx \dfrac{65}{32}$
♠ 6. $L(x) = \dfrac{1}{3}x + \dfrac{4}{3}; \sqrt[3]{65.61} \approx 4.0333$
♠ 7. $L(x) = -\dfrac{1}{4}x + 2; \sqrt{3.9} \approx 1.975, \sqrt{3.99} \approx 1.9975$
♠ 8. $L(x) = \dfrac{1}{3}x + 1; \sqrt[3]{0.95} \approx 0.983, \sqrt[3]{1.1} \approx 1.033$
9. $L(x) = 2x - 1; \sqrt{10.261} \approx \dfrac{16}{5}$
10. $L(x) = 3x - 4; (2.1)e^{0.1} \approx 2.3$
11. $L(x) = x; \sin(0.1) \approx 0.1$
12. $L(x) = -\dfrac{1}{2}x + \dfrac{6\sqrt{3}+\pi}{12}; \cos\left(\dfrac{1}{2}\right) \approx \dfrac{6\sqrt{3}+\pi-3}{12} \approx 0.8778248$
13. $L(x) = 1 + 2\left(x - \dfrac{\pi}{4}\right); \tan\left(\dfrac{\pi}{5}\right) \approx 1 - \dfrac{\pi}{10} \approx 0.6858407$
14. $x_2 = 1.41667$
15. $x_2 = 1.66667$
♠ 16. $x_2 = -1.21429$
17. $x_2 = -0.81579$
18. $x_2 = 2.93204$
♠ 19. $x_2 = 1.64516$
20. $x_2 = 3.72806$
21. $x_2 = 1.17846$
♠ 22. $x_2 = 1.17846$
♠ 23. $x_2 = -3.71030$
24. $x_2 = -1050.19260$
♠ 25. 1.87217123
♠ 26. 1.04675433
♠ 27. 1.25665999
♠ 28. -1.50311740
29. -1
30. $\dfrac{9}{2}$

31. $x_{n+1} = x_n - \dfrac{x_n^2 - x_n - 10}{2x_n - 1}$

32. $x_{n+1} = x_n - \dfrac{e^{x_n} - 7x_n}{e^{x_n} - 7}$

4.11 Antiderivatives

1. $\frac{1}{2}x^2 - 7x + C$
♠ 2. $\frac{1}{6}x^3 - x^2 + 8x + C$
♠ 3. $C + \frac{1}{2}x + \frac{2}{9}x^3 - \frac{3}{14}x^4$
♠ 4. $\frac{7}{10}x^{10} - \frac{4}{7}x^7 + \frac{11}{4}x^4 + C$
♠ 5. $\frac{5}{3}x^3 - \frac{4}{x} + C$
♠ 6. $\frac{2}{3}x^3 + \frac{3}{2}x^2 - 9x + C$
♠ 7. $\frac{4}{5}x - 7\ln|x| + C$
8. $5\ln|x| - 4x^{3/2} + C$
♠ 9. $C + 4\sqrt{x} + \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2}$
10. $xe^5 + C$
11. $\tan^{-1}(x) + 7x + C$
12. $e^x + 2\sin^{-1}(x) + C$
♠ 13. $9e^t - 6\sinh(t) + C$
♠ 14. $-3\cot\theta - 4e^\theta + C$
♠ 15. $\frac{4}{3}x^3 + 6x + 5\tan^{-1}x + C$
16. $4\theta^{3/2} + 8\sin\theta + C$
17. $\frac{x^3}{3} + \cos(x) + e^x + C$
18. $\frac{4}{3}x^{3/2} + 6\sin(x) + C$
19. $-3\cos x + \frac{1}{3}x^3 - e^x + C$
20. $-\csc\theta + e^{2\theta} + C$
21. $\tan(x) + x + C$
22. $\ln|x| + \cosh(x) + C$
23. $s(t) = t^2 + t + 1$
24. $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3$
25. $s(t) = -4.9t^2 - 25t + 1000$
♠ 26. $s(t) = t^5 - \frac{1}{3}t^6 + 6$
♠ 27. $s(t) = \frac{6}{5}t^5 - \frac{5}{4}t^4 + \frac{5}{3}t^3 + Ct + D$
♠ 28. $s(t) = \frac{1}{3}t^3 + \frac{9}{2}t^2 - 4t + 8$
29. $s(t) = t^{3/2}$
30. $s(t) = 2e^{2t} + 2t^3 - 1$
31. $s(t) = t^2 + 3\cos t + 2$
32. $s(t) = -\cos(t) + t^2 + t + 5$
33. $s(t) = -\sin t - 3\cos t + 3t + 6$

Chapter 5 : Integrals

5.1 Sigma Notation

♠ 1. $\frac{7}{8} + \frac{7}{9} + \frac{7}{10} + \frac{7}{11} + \frac{7}{12} + \frac{7}{13}$
♠ 2. $-1 + \frac{5}{7} + \frac{11}{13} + \frac{17}{19} + \frac{23}{25}$
♠ 3. $\sum_{i=1}^{10} 3i$
♠ 4. $\sum_{i=1}^{6} \frac{1}{i^2}$
♠ 5. $\sum_{i=1}^{19} \frac{i}{i+9}$
♠ 6. $\sum_{k=0}^{n} (-1)^k 2x^k$
7. 400
8. -25050
♠ 9. 324
10. 959
11. 122
♠ 12. 7600
13. 3310
♠ 14. 3276
♠ 15. 0
16. 160,000
17. $\frac{99}{100}$
♠ 18. 221
19. $\frac{63}{64}$
20. $-\frac{910}{729}$
21. $\frac{127}{2}$

5.2 Area Under the Curve

1. under, 16.125
♠ 2. under, 16.125
3. under, 16.125
♠ 4. over, 21.1875
♠ 5. under, 18
6. under, 18
♠ 7. under, 18
♠ 8. over, 20.25

9. under, 19.125
♠ 10. under, 19.125
11. under, 19.125
12. over, 19.6875
13. under, 2.828427
14. over, 6.828427
15. under, 4.828427
16. over, 5.464102
♠ 17. under, 4.146264
♠ 18. over, 6.146264
♠ 19. under, 5.146264
♠ 20. over, 5.383819
21. under, 4.7650418
22. over, 5.765042
23. under, 5.265042
24. over, 5.352257
25. over, 1.340759
26. under, 0.555360
27. under, 0.948059
28. over, 1.026172
♠ 29. over, 1.238848
♠ 30. under, 0.715249
♠ 31. under, 0.977049
♠ 32. over, 1.011515
33. over, 1.183465
34. under, 0.790766
35. under, 0.987116
36. over, 1.006455
37. over, 1.570796
38. under, 0
39. over, 1.110721
40. under, 0.785398
♠ 41. over, 1.340759
♠ 42. under, 0.555360
♠ 43. over, 1.026172
♠ 44. under, 0.948059
45. over, 1.238848
46. under, 0.715249
47. over, 1.011515
48. under, 0.977049
49. under, 2
50. over, 5
51. over, 3.5
52. under, 2.75
♠ 53. under, 2.28
♠ 54. over, 4.08
♠ 55. over, 3.18
♠ 56. under, 2.91
57. under, 2.375
58. over, 3.875
59. over, 3.125
60. under, 2.9375

5.3 Riemann Sums

1. under, 8.1816
2. over, 15.4754
3. under, 11.8285
4. under, 9.8641
5. over, 14.8748
6. under, 12.36945
7. under, 11.5074
8. over, 14.0127
9. under, 12.76005
10. under, 4.712389
11. under, 4.712389
12. under, 4.712389
13. over, 5.363034
♠ 14. under, 3.141593
♠ 15. over, 6.283185
♠ 16. under, 4.712389
17. under, 5.037712
18. under, 5.037712
♠ 19. under, 5.037712
♠ 20. over, 5.193937
♠ 21. under, 4.252313
♠ 22. over, 5.823110
♠ 23. under, 5.037712
24. under, 5.095690
25. under, 5.095690
26. under, 5.095690
27. over, 5.164623
♠ 28. under, 4.572091
♠ 29. over, 5.619289
30. under, 5.095690
31. -110
♠ 32. 130
♠ 33. 10
♠ 34. 30
♠ 35. -40
36. 80
37. 20
38. $\int_0^1 x^2\,dx$
♠ 39. $\int_0^2 (7+x)^{10}\,dx$
♠ 40. $\int_0^{\pi/5} \tan x\,dx$
41. $\int_3^5 (x^2+\sqrt{1+2x})\,dx$
♠ 42. $\int_{2\pi}^{3\pi} \frac{\cos x}{x}\,dx$
43. $\int_2^7 x\ln(1+x^2)\,dx$
♠ 44. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3+\frac{3i}{n}}{2+\left(3+\frac{3i}{n}\right)^4}\left(\frac{3i}{n}\right)$
45. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{2i}{n}-5\ln\left(2+\frac{2i}{n}\right)\right)\left(\frac{2i}{n}\right)$
46. $\lim_{n\to\infty} \sum_{i=1}^{n} e^{i\ln 2/n}\left(\frac{\ln 2}{n}\right)$

47. $\frac{25}{2}$ ♠ 48. 100 ♠ 49. 26 ♠ 50. $\frac{37}{2}$ ♠ 51. $\frac{9}{2}$ 52. $28 - \frac{17}{4}\pi$

5.4 Definite Integrals

1. 21 2. 39 3. 3 4. -3 5. $-\frac{8}{3}$ ♠ 6. $-\frac{80}{3}$ ♠ 7. $\frac{464}{3}$

♠ 8. $\frac{14}{3}$ 9. $\frac{215}{3}$ 10. 20 11. 2 ♠ 12. $27 + \frac{81\pi}{4}$ ♠ 13. 0 ♠ 14. $108 - 40\sqrt{10}$

♠ 15. 6 ♠ 16. $2e^3 - 2e - 4$ ♠ 17. $5e^7 - 5e^5$ ♠ 18. 8.1 ♠ 19. 134

5.5 The Fundamental Theorem of Calculus

1. 0 ♠ 2. $-\frac{1}{2}$ 3. -2 4. $-\frac{7}{2}$ ♠ 5. -4 6. $-\frac{7}{2}$ ♠ 7. -2

8. $\frac{1}{2}$ ♠ 9. 4 10. $-\frac{1}{2}$ ♠ 11. max @ $x = 0$, min @ $x = 4$ 12. 8

♠ 13. 12 ♠ 14. $24 - \frac{9}{4}\pi$ ♠ 15. 0 ♠ 16. -4 ♠ 17. $\pi - 4$ 18. $2\pi - 4$

♠ 19. max @ $x = \pm 6$, min @ $x = -2$ 20. 0 ♠ 21. 3 ♠ 22. 17 ♠ 23. -1

24. $-\frac{3}{2}$ ♠ 25. max @ $x = 7$, min @ $x = -1$ 26. $\frac{dy}{dx} = \sqrt{x^2+4}$ 27. $f'(x) = e^{x^2-x}$ ♠ 28. $g'(x) = \frac{7}{x^3+1}$

♠ 29. $g'(x) = e^{4x^2-3x}$ ♠ 30. $f'(x) = -\sqrt{8+\sec(5x)}$ 31. $y' = -\cos(\sqrt{x})$ ♠ 32. $f'(x) = -\cos(\sqrt{9x})$

33. $g'(x) = \frac{x}{2\sqrt{x}(x^2+1)}$ 34. $h'(x) = 4x^3\cos^2(x^4)$ ♠ 35. $\frac{dy}{dx} = \sec^2 x\sqrt{2\tan x + \sqrt{\tan x}}$

♠ 36. $y' = \frac{3(4-3x)^3}{1+(4-3x)^2}$ ♠ 37. $f'(x) = \frac{5(25x^2-1)}{25x^2+1} - \frac{2(4x^2-1)}{4x^2+1}$ ♠ 38. $y' = 2xe^{x^8} - e^{x^4}$

♠ 39. $\frac{dy}{dx} = \cos x\ln(4+3\sin x) + \sin x\ln(4+3\cos x)$ 40. $f'(x) = -\cos x\sqrt{1+\sin^2 x}$ 41. $g'(x) = xe^x$

5.6 The Fundamental Theorem of Calculus II

1. 3 ♠ 2. $-\frac{15}{4}$ ♠ 3. 56 ♠ 4. $\frac{38}{3}$ 5. 52 ♠ 6. $-\frac{73}{6}$ ♠ 7. $-\frac{28}{3}$

8. $\frac{128}{15}$ 9. $\frac{\pi}{4}$ ♠ 10. 14 11. $\sqrt{2} - 1$ ♠ 12. $1 + \frac{\sqrt{2}}{2}$ 13. 0 ♠ 14. 7

♠ 15. $\sinh(7)$ 16. $\frac{3}{4}$ ♠ 17. $e^2 - 1$ ♠ 18. 9 19. $e^2 - 1$ 20. $\frac{9}{2}$

5.7 Rates of Changes within the Sciences II

1. The integral measures the displacement of the particle for the first 5 seconds.

♠ 2. The integral measures the net change of charge from time $t = a$ to time $t = b$.

♠ 3. The integral measures the change of elevation of the hiker from mile marker 7 to mile marker 9.

4. The integral measures the increase of the bacteria population during the first 10 hours of growth.

♠ 5. The integral measures the change of the calf's mass between month 8 and month 11.

♠ 6. $\frac{136}{3}$ kg

♠ 7. 4,950 L

8. -120 G

9. 20 m, 28 m

♠ 10. -4 m, $\frac{37}{3}$ m

♠ 11. -78 m, $\frac{266}{3}$ m

♠ 12. $\frac{1880}{3}$ m , $\frac{1880}{3}$ m

♠ 13. $-\frac{25}{3}$ m, $\frac{137}{3}$ m

14. 0 m, 24 m

5.8 u-Substitution

1. $e^{2x} + C$

♠ 2. $\frac{1}{28}(x^4 + 7)^7 + C$

♠ 3. $-\frac{1}{2}e^{-2x} + C$

♠ 4. $\frac{2}{9}(x^3 + 29)^{3/2} + C$

♠ 5. $\frac{1}{40(1 - 5x)^8} + C$

♠ 6. $-\frac{1}{10}\cos^{10}\theta + C$

♠ 7. $-\frac{1}{5}\tan\left(\frac{1}{x^5}\right) + C$

8. $-\frac{3}{2}(x^2 - 3x)^{2/3} + C$

9. $\sec(\theta) + 3e^{2\theta} + C$

10. $-\frac{1}{3}\sqrt{4 - x^2}^3 + C$

♠ 11. $-\frac{1}{12}\cos(x^{12}) + C$

♠ 12. $\frac{1}{11}\ln^{11} x + C$

♠ 13. $\frac{1}{5}\tan^5\theta + C$

14. $-\frac{1}{2}e^{-x^2} + C$

♠ 15. $\frac{2}{3}(e^x + 7)^{3/2} + C$

16. $e^{\sin x} + C$

17. $\sin(e^x) + C$

♠ 18. $-\frac{1}{19}\csc^{19} x + C$

19. $\frac{1}{2}(\sin^{-1} x)^2 + C$

5.9 u-Substitution II

1. $\frac{1}{808}(8x^2 + 4)^{101} + C$

2. $-\frac{1}{297(x^3 - 8)^{99}} + C$

3. $\frac{1}{3}(2x + x^2)^{3/2} + C$

♠ 4. $\frac{1}{3}(x^2 + 2x)^{3/2} + C$

♠ 5. $\frac{1}{7 - e^y} + C$

♠ 6. $-\frac{2}{7}\cos(x^{7/2} + 5) + C$

7. $-\cos(\sin x) + C$

♠ 8. $\frac{1}{3}e^{\tan(3x)} + C$

♠ 9. $-\cos(\ln(x)) + C$

10. $-2\cos(\sqrt{x}) + C$

♠ 11. $\frac{1}{6}\tan^{-1}(x^6) + C$

♠ 12. $4\tan^{-1} x + \frac{5}{2}\ln(1 + x^2) + C$

♠ 13. $\frac{1}{640}(8x + 7)^{10} - \frac{7}{576}(8x + 7)^9 + C$

14. $\frac{1}{6}(1 + 2x)^{3/2} - \frac{1}{2}(1 + 2x)^{1/2} + C$

♠ 15. $\frac{1}{5}(x^2 + 11)^{5/2} - \frac{11}{3}(x^2 + 11)^{3/2} + C$

16. $\frac{1}{2}\tan^{-1}(x^2) + C$

♠ 17. $\frac{45}{28}$

♠ 18. 4

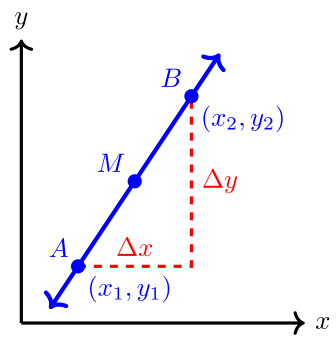
♠ 19. $\frac{1}{3}(e - \sqrt[8]{e})$

20. $\frac{\pi}{8}$

Appendix B

Useful Formulas

Trigonometry/Geometry



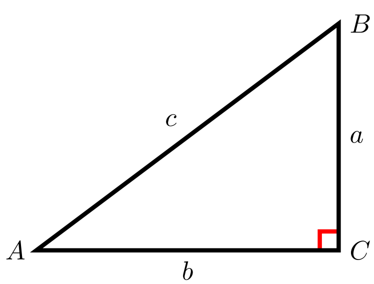
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ (Distance Formula)

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$ (Midpoint Formula)

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$ (Slope)

$$y = mx + b$$ (Slope-Intercept Form)

$$y - y_1 = m(x - x_1)$$ (Point-Slope Form)



$$a^2 + b^2 = c^2$$ (Pythagorean Equation)

(Trigonometric Ratios via Right Triangles)

$$\sin A = \frac{a}{c} \qquad \cos A = \frac{b}{c} \qquad \tan A = \frac{a}{b}$$

$$\csc A = \frac{c}{a} \qquad \sec A = \frac{c}{b} \qquad \cot A = \frac{b}{a}$$

(Complementary Identities)

$$\sin A = \cos B \qquad \cos A = \sin B \qquad \tan A = \cot B$$

$$\csc A = \csc B \qquad \sec A = \csc B \qquad \cot A = \tan B$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$ (Ratio Identities)

(Reciprocal Identities)

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

(Symmetry Identities)

$$\sin(-\theta) = -\sin\theta \quad \cos(-\theta) = \cos\theta \quad \tan(-\theta) = -\tan\theta \quad \csc(-\theta) = -\csc\theta \quad \sec(-\theta) = \sec\theta \quad \cot(-\theta) = -\cot\theta$$

$$\cos^2 + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$ (Pythagorean Identities)

(Angle Sum/Difference Identities)

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

(Double Angle Identites)

$$\sin(2A) = 2\sin(A)\cos(A) \qquad \begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A)\\ &= 2\cos^2(A) - 1\\ &= 1 - 2\sin^2(A)\end{aligned} \qquad \tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

(Half Angle Identites)

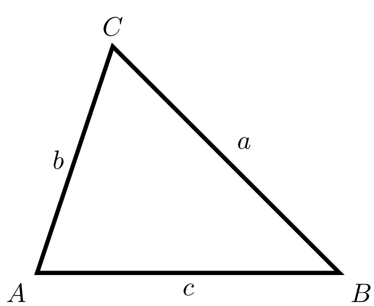
$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}} \qquad \cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}} \qquad \begin{aligned}\tan\left(\frac{A}{2}\right) &= \frac{1-\cos A}{\sin A}\\ &= \frac{\sin A}{1+\cos A}\end{aligned}$$

(Product-to-Sum Identities)

$$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B)) \qquad \cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$
$$\cos A\cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \qquad \sin A\sin B = \frac{1}{2}(-\cos(A+B) + \cos(A-B))$$

(Sum-to-Product Identities)

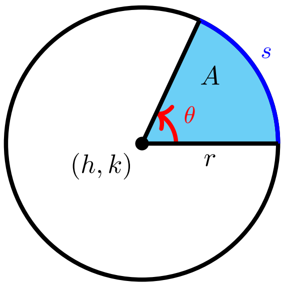
$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ (Law of Sines)

$$a^2 = b^2 + c^2 - 2bc\cos A$$ (Law of Cosines)
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\begin{aligned}\text{Area} &= \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C\\ &= \frac{a^2\sin B\sin C}{2\sin A} = \frac{b^2\sin A\sin C}{2\sin B} = \frac{c^2\sin A\sin B}{2\sin C}\\ &= \sqrt{s(s-a)(s-b)(s-c)}, \quad s = \frac{1}{2}(a+b+c)\end{aligned}$$ (Area of a Triangle)



$$(x-h)^2+(y-k)^2=r^2$$
(Equation of Circle)

$$C=2\pi r$$
(Circumference)

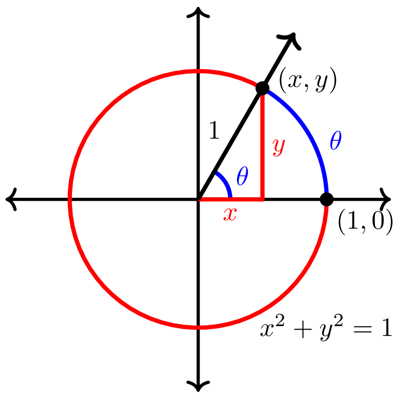
$$A=\pi r^2$$
(Area of Circle)

$$s=r\theta$$ (Arc Length)

$$A=\frac{1}{2}r^2\theta$$ (Sector Area)

(Angular Velocity)

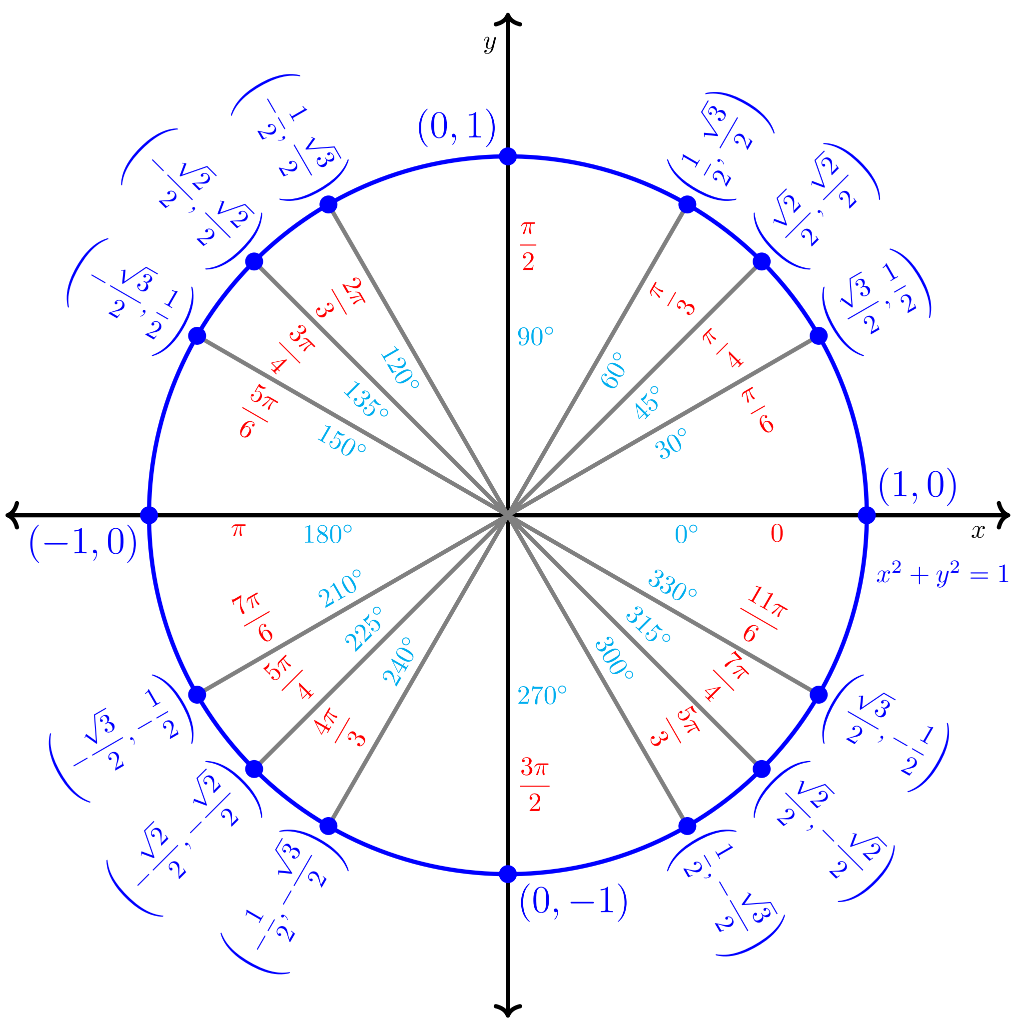
$$v=r\omega,\quad \omega=\frac{\theta}{t},\quad v=\frac{s}{t}$$



(Trigonometric Ratios via Unit Circle)

$\sin\theta=y$ $\qquad$ $\tan\theta=\dfrac{y}{x}$ $\qquad$ $\sec\theta=\dfrac{1}{x}$

$\cos\theta=x$ $\qquad$ $\cot\theta=\dfrac{x}{y}$ $\qquad$ $\csc\theta=\dfrac{1}{y}$



$$V = \frac{4}{3}\pi r^3 \qquad A = 4\pi r^2 \qquad \text{(Sphere)}$$
$$V = \pi r^2 h \qquad A = 2\pi r(r+h) \qquad \text{(Cylinder)}$$
$$V = \frac{1}{3}\pi r^2 h \qquad A = \pi r\sqrt{r^2+h^2} \qquad \text{(Cone)}$$

Algebra

$$a^2 - b^2 = (a-b)(a+b) \qquad a^2 \pm 2ab + b^2 = (a \pm b)^2 \qquad \text{(Factorization)}$$
$$a^3 - b^3 = (a-b)(a^2+ab+b^2) \qquad a^3 + b^3 = (a+b)(a^2-ab+b^2)$$

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad \text{(Quadratic Equation)}$$

If $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, then

$$(a+b)^n = a^n + na^{n-1}b + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + nab^{n-1} + b^n$$

(Binomial Theorem)

$$a^m a^n a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn} \qquad \text{(Exponent Laws)}$$
$$\log_a(rs)\log_a r + \log_a s \qquad \log_a\left(\frac{r}{s}\right) = \log_a r - \log_a s \qquad \log_a(r^n) = n\log_a r \qquad \text{(Logarithm Laws)}$$

Limits

$$\lim_{x\to\infty} x^n = \infty \qquad \lim_{x\to-\infty} x^{2n} = \infty \qquad \lim_{x\to-\infty} x^{2n+1} = -\infty$$
$$\lim_{x\to\pm\infty}(a_n x^n + \ldots + a_1 x + a_0) = \lim_{x\to\pm\infty} a_n x^n \qquad \lim_{x\to\pm\infty}\frac{1}{x^n} = 0 \quad (n>0)$$
$$\lim_{x\to\pm\infty}\frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0} = \lim_{x\to\pm\infty}\frac{a_n x^n}{b_m x^m} \qquad \lim_{x\to-\infty} a^x = 0 \quad (a>1) \qquad \lim_{x\to 0}\frac{e^x-1}{x} = 1$$
$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0 \qquad \lim_{x\to\pm\infty}\arctan(x) = \pm\frac{\pi}{2}$$

Suppose $r \in \mathbb{R}$, $c > 0$, $a > 1$ (Arithmetic at Infinity)

$r + \infty = \infty$	$r + (-\infty) = -\infty$	$\infty + \infty = \infty$	$(-\infty) + (-\infty) = -\infty$
$c\infty = \infty$	$(-c)(-\infty) = \infty$	$c(-\infty) = -\infty$	$(-c)(\infty) = -\infty$
$\infty(\infty) = \infty$	$(-\infty)(-\infty) = \infty$	$\infty(-\infty) = -\infty$	$(-\infty)\infty = -\infty$
$r/\infty = 0$	$r/(-\infty) = 0$	$a^\infty = \infty$	$a^{-\infty} = 0$

(Indeterminant Forms)

$$\infty - \infty \qquad \frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

Derivatives

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \qquad \frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2} \qquad \frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}f(g(x)) \cdot \frac{d}{dx}g(x)$$

(Power Functions)

$$\frac{d}{dx}u^n = nu^{n-1}u' \qquad \frac{d}{dx}(\sqrt{u}) = \frac{u'}{2\sqrt{u}} \qquad \frac{d}{dx}\left(\frac{1}{u}\right) = \frac{u'}{u^2}$$

(Exponentials and Logarithms)

$$\frac{d}{dx}(e^u) = u'e^u \qquad \frac{d}{dx}(a^u) = u'a^u(\ln a) \qquad \frac{d}{dx}(\ln u) = \frac{u'}{u}$$

$$\frac{d}{dx}(\ln|u|) = \frac{u'}{u} \qquad \frac{d}{dx}(\log_a u) = \frac{u'}{(\ln a)u} \qquad \frac{d}{dx}(\log_a|u|) = \frac{u'}{(\ln a)u}$$

(Trigonometric Functions)

$$\frac{d}{dx}(\sin u) = u'\cos u \qquad \frac{d}{dx}(\tan u) = u'\sec^2 u \qquad \frac{d}{dx}(\sec u) = u'\sec u \tan u$$

$$\frac{d}{dx}(\cos u) = -u'\sin u \qquad \frac{d}{dx}(\cot u) = -u'\csc^2 u \qquad \frac{d}{dx}(\csc u) = -u'\csc\frac{du}{dx}\cot u$$

(Inverse Trigonometric Functions)

$$\frac{d}{dx}(\sin^{-1} u) = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}(\tan^{-1} u) = \frac{u'}{1+u^2} \qquad \frac{d}{dx}(\sec^{-1} u) = \frac{u'}{x\sqrt{u^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} u) = -\frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}(\cot^{-1} u) = -\frac{u'}{1+u^2} \qquad \frac{d}{dx}(\csc^{-1} u) = -\frac{u'}{x\sqrt{u^2-1}}$$

(Hyperbolic Functions)

$$\frac{d}{dx}(\sinh u) = u'\cosh u \qquad \frac{d}{dx}(\tanh u) = u'\operatorname{sech}^2 u \qquad \frac{d}{dx}(\operatorname{sech} u) = -u'\operatorname{sech} u \tanh u$$

$$\frac{d}{dx}(\cosh u)u'\sinh u \qquad \frac{d}{dx}(\operatorname{csch} u) = -u'\operatorname{csch} u \coth u \qquad \frac{d}{dx}(\coth u) = -u'\operatorname{csch}^2 u$$

(Inverse Hyperbolic Functions)

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{u'}{\sqrt{1+u^2}} \qquad \frac{d}{dx}(\tanh^{-1} u) = \frac{u'}{1-u^2} \qquad \frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{u'}{u\sqrt{1-u^2}}$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{u'}{\sqrt{u^2-1}} \qquad \frac{d}{dx}(\coth^{-1} u) = \frac{u'}{1-u^2} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{u'}{|u|\sqrt{u^2+1}}$$