Exercise 1:

1) a) $$x^T \Lambda x + w^T x + b = 0$$

Generic Formulation from the lecture for the decision boundary:

$$g_k = \underset{k}{\operatorname{argmin}} \left[\frac{1}{2} \log\left[\det(2\pi \hat{\Sigma}_k)\right] + \frac{1}{2}(x - \hat{\mu}_k)^T \hat{\Sigma}_k^{-1} (x - \hat{\mu}_k) + \log(p(y=k)) \right]$$

→ In the two class case we have then

$$g_A = \frac{1}{2} \log\left[\det(2\pi \Sigma_A)\right] + \frac{1}{2}(x - \mu_A)^T \Sigma_A^{-1} (x - \mu_A) + \log \pi_A)$$

equivalent for g_B

Then for the boundary:

$$g_A = g_B$$

$$g_A - g_B = 0$$

$$\Leftrightarrow \frac{1}{2} \log |\cancel{2\pi} \Sigma_A| + \frac{1}{2}(x-\mu_A)^T \Sigma_A^{-1} (x - \mu_A) + \log \frac{\pi_A}{\pi_B} - \left[\frac{1}{2} \log |\cancel{2\pi} \Sigma_B| \right.$$

$$\left. + \frac{1}{2} (x - \mu_B) \Sigma_B^{-1} (x - \mu_B) \right]$$

$$\Leftrightarrow \frac{1}{2} \log |\Sigma_A| - \frac{1}{2} \log |\Sigma_B| + \frac{1}{2}(x - \mu_A)^T \Sigma_A^{-1} (x - \mu_A) - \frac{1}{2}(x-\mu_B)^T \Sigma_B^{-1} (x - \mu_B)$$

$$+ \log \frac{\pi_A}{\pi_B}$$

$$\Leftrightarrow \frac{1}{2} \log \frac{|\Sigma_A|}{|\Sigma_B|} + \log \frac{\pi_A}{\pi_B}$$

1a)

given the conditions from the exercise we have

$$g_A(x) = -\frac{1}{2}(x-\mu_A)^T \Sigma_A^{-1}(x-\mu_A) + \log \pi_A$$

similar for g_B.

So then the boundary is: $g_A(x) = g_B(x)$

$$g_A(x) = g_B(x)$$

$$\Leftrightarrow \quad g_A(x) - g_B(x) = 0$$

$$\Leftrightarrow \quad -\frac{1}{2}(x-\mu_A)^T \Sigma_A^{-1}(x-\mu_A) + \log \pi_A - \left[-\frac{1}{2}(x-\mu_B)^T \Sigma^{-1}(x-\mu_B) + \log \pi_B\right]$$

$$\Leftrightarrow \quad -\frac{1}{2}\left[(x^T \Sigma_A^{-1}) - (\mu_A^T \Sigma_A^{-1})\right]\cdot(x-\mu_A) + \log\frac{\pi_A}{\pi_B} - \quad - \quad \cdot \quad -$$

$$\Leftrightarrow \quad \frac{1}{2}\left[-x^T \Sigma_A^{-1} x + x^T \Sigma_A^{-1}\mu_A + \mu_A^T \Sigma_A^{-1} x - \mu_A^T \Sigma_A^{-1}\mu_A\right]$$
$$+ \frac{1}{2}\left[x^T \Sigma_B^{-1} x - x^T \Sigma_B^{-1}\mu_B - \mu_B^T \Sigma_B^{-1} x + \mu_B^T \Sigma_B^{-1}\mu_B\right] + \log\frac{\pi_A}{\pi_B}$$

$$\Leftrightarrow \quad \frac{1}{2}\left[x^T(\Sigma_B^{-1} - \Sigma_A^{-1})x + 2x^T(\Sigma_A^{-1}\mu_A - \Sigma_B^{-1}\mu_B) + (\mu_B^T \Sigma_B^{-1}\mu_B - \mu_A^T \Sigma_A^{-1}\mu_A)\right.$$
$$+ \log\frac{\pi_A}{\pi_B}$$

$$\Rightarrow \quad \Lambda = \frac{1}{2}(\Sigma_B^{-1} - \Sigma_A^{-1})$$

$$w = \Sigma_A^{-1}\mu_A - \Sigma_B^{-1}\mu_B$$

$$b = \frac{1}{2}(\mu_B^T \Sigma_B^{-1}\mu_B - \mu_A^T \Sigma_A^{-1}\mu_A) + \log\frac{\pi_A}{\pi_B}$$

The term $x^T \Lambda x$ is a quadratic term, leading to a quadratic decision boundary.

b) If $\Sigma_A = \Sigma_B = \Sigma$:

$$\Delta = (\Sigma_A^{-1} - \Sigma_B^{-1}) = 0$$

$$\omega = \Sigma^{-1} (\mu_A - \mu_B)$$

$$b = \frac{1}{2} (\mu_B^T \Sigma^{-1} \mu_B - \mu_A^T \Sigma^{-1} \mu_A) + \log \frac{\pi_A}{\pi_B}$$

2)