Exercise 1: Convolutions of Continuous and Discrete Variables

Random Variables: $Z = X + Y \xrightarrow{\text{PDF}} f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$

$X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$

To show: $Z = X + Y \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \quad , \; f_Y \text{ analogous}$$

$$f_Z(z) = (f_X * f_Y)(z) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(z-x-\mu_2)^2}{2\sigma_2^2}\right)$$

Consider the exponent:

$$-\frac{(x-\mu_1)^2}{2\sigma_1^2} - \frac{(z-x-\mu_2)^2}{2\sigma^2} = -\frac{x^2 - 2x\mu_1 + \mu_1^2}{2\sigma_1^2} - \frac{x^2 + 2x(\mu_2 - z) + (\mu_2 - z)^2}{2\sigma_2^2}$$

$$= -\frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) x^2 + x\left(\frac{\mu_1}{\sigma_1^2} + \frac{z-\mu_2}{\sigma_2^2}\right) - \frac{\mu_1^2}{2\sigma_1^2} - \frac{(\mu_2 - z)^2}{2\sigma_2^2}$$

$$= -A(x-B)^2 + C$$

$$A = \frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) \quad , \quad B = \frac{\frac{\mu_1}{\sigma_1^2} + \frac{z-\mu_2}{\sigma_2^2}}{2A} \quad , \quad C = -\frac{\mu_1^2}{2\sigma_1^2} - \frac{(\mu_2 - z)^2}{2\sigma_2^2} + AB^2$$

with $$AB^2 = \frac{\left(\frac{\mu_1}{\sigma_1^2} + \frac{(z-\mu_2)}{\sigma_2^2}\right)^2}{2\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)} = \frac{\sigma_1^2\sigma_2^2}{2(\sigma_1^2 + \sigma_2^2)}\left(\frac{\mu_1^2}{\sigma_1^4} + \frac{(z-\mu_2)^2}{\sigma_2^4} + \frac{2\mu_1(z-\mu_2)}{\sigma_1^2\sigma_2^2}\right)$$

$$\sigma_Z^2 = \sigma_1^2 + \sigma_2^2 \Rightarrow \frac{1}{2\sigma_Z^2}\left(\mu_1^2 \frac{\sigma_2^2}{\sigma_1^2} + (z-\mu_2)^2 \frac{\sigma_1^2}{\sigma_2^2} + 2\mu_1(z-\mu_2)\right)$$

$$\Rightarrow C = -\left(\frac{\frac{\mu_1^2}{\sigma_1^2}(\sigma_1^2 + \sigma_2^2) + \frac{(z-\mu_2)^2}{\sigma_2^2}(\sigma_1^2 + \sigma_2^2)}{2\sigma_Z^2}\right)$$

$$+ \frac{1}{2\sigma_Z^2}\left(\mu_1^2 \frac{\sigma_2^2}{\sigma_1^2} + (z-\mu_2)^2 \frac{\sigma_1^2}{\sigma_2^2} + 2\mu_1(z-\mu_2)\right)$$

$$= -\left(\frac{\mu_1^2\left(1 + \cancel{\frac{\sigma_2^2}{\sigma_1^2}}\right) + (z-\mu_2)^2\left(1 + \cancel{\frac{\sigma_1^2}{\sigma_2^2}}\right)}{2\sigma_Z^2}\right) + \frac{1}{2\sigma_Z^2}\left(\cancel{\mu_1^2 \frac{\sigma_2^2}{\sigma_1^2}} + \cancel{(z-\mu_2)^2 \frac{\sigma_1^2}{\sigma_2^2}} + 2\mu_1(z-\mu_2)\right)$$

$$= -\frac{\mu_1^2 + (z-\mu_2)^2 - 2\mu_1(z-\mu_2)}{2\sigma_Z^2}$$

$$= -\frac{\mu_1^2 + z^2 + \mu_2^2 - 2\mu_2 z - 2\mu_1 z + 2\mu_1\mu_2}{2\sigma_Z^2}$$

$$= -\frac{(z - \overbrace{(\mu_1 + \mu_2)}^{\mu_Z})^2}{2\sigma_Z^2} = -\frac{(z-\mu_Z)^2}{2\sigma_Z^2}$$

$$\Rightarrow \quad f_z(z) = \frac{e^C}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} dx \; \exp\left(-A(x-B)^2\right) = \frac{e^C}{2\pi\sigma_1\sigma_2}\sqrt{\frac{\pi}{A}}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\sqrt{\frac{2\pi}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}} \; \exp\left(-\frac{(z-\mu_z)^2}{2\sigma_z^2}\right)$$

with $\quad \mu_z = \mu_1 + \mu_2, \qquad \sigma_z^2 = \sigma_1^2 + \sigma_2^2$

2.) $\quad I \in \mathbb{R}^{H\times W}, \quad f \in \mathbb{R}^{K_H \times K_W} \quad$ with $\quad K_H < H, \; K_W < W$

and $I_{pad} \in \mathbb{R}^{(H+2p)\times(W+2p)}$, stride $s \in \mathbb{N}$

$$\mathcal{O}(i,j) = \sum_{u=0}^{K_H}\sum_{v=0}^{K_W} I_{pad}(i\cdot s+u, \, j\cdot s+v)\cdot f(u,v)$$

$$\mathcal{O} \in \mathbb{R}^{H_{out}\times W_{out}}$$

because we can't move closer to the edge than the dimension of the filter

where $\quad H_{out} = \frac{(H+2p)-K_H}{s} + 1$

$\qquad\quad W_{out} = \frac{(W+2p)-K_W}{s} + 1$

because we start counting at 0 (ie. $H_{out} = 1$ if $H+2p = K_H$)