# **Absorption Cheat Sheet**

#### MATRIX PARTITION

Suppose that the matrix of interest is partitioned as

$$egin{bmatrix} oldsymbol{U} & oldsymbol{T} \ oldsymbol{T}^ op & oldsymbol{W} \end{bmatrix}$$

where U is a square non-singular matrix.

#### **ABSORPTION**

The absorption by U is given by

$$\boldsymbol{W}^* = \boldsymbol{W} - \boldsymbol{T}^{\mathsf{T}} \boldsymbol{U}^{-1} \boldsymbol{T}.$$

Note this is a special case of **Schur complement**.

Row by row absorption is equivalent to repeated application of the formula for  $W^*$ , where U is a scalar (the pivot) and **T** is a row vector.

#### GAUSSIAN ELIMINATION

Note

$$oldsymbol{U}oldsymbol{x}+oldsymbol{T}oldsymbol{y}$$

$$oldsymbol{T}^{\!\scriptscriptstyle op}oldsymbol{x} + oldsymbol{W}oldsymbol{y}$$

Pre-multiplying (1) by  $T^{\mathsf{T}}U^{-1}$  results in  $T^{\mathsf{T}}x + T^{\mathsf{T}}U^{-1}Ty$ and Gaussian elimination in (2) results in  $(W - T^TU^TY)y$ . That is the absorption is the "coefficient" of y after elimination of x.

### MATRIX INVERSION

For inversion of symmetric positive definite matrix  $E^{n \times n}$ ,

$$egin{bmatrix} m{E} & m{I_n} \ m{I_n} & \mathbf{0}^{n imes n} \end{bmatrix}$$

then  $W^* = -E^{-1}$ .

$$oldsymbol{V}^{-1}$$
 in ML

$$\begin{bmatrix} \boldsymbol{Z}^{\!\top} \boldsymbol{R}^{\!-1} \boldsymbol{Z} + \boldsymbol{G}^{\!-1} & \boldsymbol{Z}^{\!\top} \boldsymbol{R}^{\!-1} \\ \boldsymbol{R}^{\!-1} \boldsymbol{Z} & \boldsymbol{R}^{\!-1} \end{bmatrix}$$

then  $W^* = V^{-1}$ .

#### $oldsymbol{P}$ in REML

$$\begin{bmatrix} \boldsymbol{X}^{\!\top} \boldsymbol{R}^{-1} \boldsymbol{X} & \boldsymbol{X}^{\!\top} \boldsymbol{R}^{-1} \boldsymbol{Z} & \boldsymbol{X}^{\!\top} \boldsymbol{R}^{-1} \\ \boldsymbol{Z}^{\!\top} \boldsymbol{R}^{-1} \boldsymbol{X} & \boldsymbol{Z}^{\!\top} \boldsymbol{R}^{-1} \boldsymbol{Z} + \boldsymbol{G}^{-1} & \boldsymbol{Z}^{\!\top} \boldsymbol{R}^{-1} \\ \boldsymbol{R}^{-1} \boldsymbol{X} & \boldsymbol{R}^{-1} \boldsymbol{Z} & \boldsymbol{R}^{-1} \end{bmatrix}$$

then  $W^* = P$ .

#### **ABSORPTION IN MME**

$$\begin{bmatrix} \boldsymbol{X}^{\top}\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{X}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} & \boldsymbol{X}^{\top}\boldsymbol{R}^{-1}\boldsymbol{y} \\ \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} + \boldsymbol{G}^{-1} & \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{y} \\ \boldsymbol{y}^{\top}\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{y}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} & \boldsymbol{y}^{\top}\boldsymbol{R}^{-1}\boldsymbol{y} \end{bmatrix}$$

Absorption of subset of fixed effects

Say we partition the fixed effect  $\tau = (\tau_1^T, \tau_2^T)^T$  and so the design matrix is given conformably as  $[X_1 \ X_2]$ . Then if we absorb  $\tau_2$  in the MME then the MME results in

$$\left[egin{array}{ccc} m{X_1}^ op S m{X_1} & m{X_1}^ op S m{Z} \ m{Z}^ op S m{X_1} & m{Z}^ op S m{Z} + m{G}^{-1} \end{array}
ight] \left[egin{array}{c} \hat{m{ au}} \ \hat{m{u}} \end{array}
ight] = \left[egin{array}{c} m{X_1}^ op S m{y} \ m{Z}^ op S m{y} \end{array}
ight]$$

where  $S = R^{-1} - R^{-1}X_2 (X_2^{\mathsf{T}}R^{-1}X_2)^{-1}X_2^{\mathsf{T}}R^{-1}$ .

Absorption of subset of random effects

Say we partition the fixed effect  $u = (u_1^T, u_2^T)^T$  and so the design matrix is given conformably as  $[Z_1 \ Z_2]$ . Then if we absorb  $u_1$  and  $\tau$  in the MME then we have

$$\left(oldsymbol{Z}_2^{\! op} oldsymbol{S} oldsymbol{Z}_2 + oldsymbol{G}_2^{-1}
ight) ilde{oldsymbol{u}}_2 = oldsymbol{Z}_2^{\! op} oldsymbol{S} oldsymbol{y}$$

where  $\boldsymbol{S} = \boldsymbol{R}^{-1}$  –

$$egin{aligned} oldsymbol{R}^{-1} egin{bmatrix} oldsymbol{X}^{ op} oldsymbol{R}^{-1} oldsymbol{X} & oldsymbol{Z}_1^{ op} oldsymbol{R}^{-1} oldsymbol{X} & oldsymbol{Z}_1^{ op} oldsymbol{R}^{-1} oldsymbol{Z}_1 + oldsymbol{G}_1^{-1} \end{bmatrix}^{-1} egin{bmatrix} oldsymbol{X}^{ op} \ oldsymbol{Z}_1^{ op} \ oldsymbol{Z}_1^{ op} oldsymbol{Z}_1^{ op} oldsymbol{Z}_1^{ op} oldsymbol{R}^{-1}. \end{aligned}$$

$$m{S} = m{R}^{-1} - m{R}^{-1} egin{bmatrix} m{X} & m{Z}_1 \end{bmatrix} egin{bmatrix} m{X}^{ au} m{R}^{-1} m{X} & m{X}^{ au} m{R}^{-1} m{Z}_1 \ m{Z}_1^{ au} m{R}^{-1} m{X} & m{Z}_1^{ au} m{R}^{-1} m{Z}_1 + m{G}_1^{-1} \end{bmatrix}^{ au}$$

$$= R^{-1} - R^{-1} \begin{bmatrix} X & Z_1 \end{bmatrix} (R^{-1} - R^{-1} Z_1 (Z_1^{\mathsf{T}} R^{-1} Z_1 + G_1^{-1})^{-1} Z_1^{\mathsf{T}} R^{-1}) \begin{bmatrix} X^{\mathsf{T}} \\ Z^{\mathsf{T}} \end{bmatrix} R(Z)$$
REFERENCES

#### $oldsymbol{Z}^{\! op} oldsymbol{P} oldsymbol{Z}$ – Method 1

$$\begin{bmatrix} \boldsymbol{X}^{\top}\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{X}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} & \boldsymbol{X}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} \\ \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} + \boldsymbol{G}^{-1} & \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} \\ \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} & \boldsymbol{Z}^{\top}\boldsymbol{R}^{-1}\boldsymbol{Z} \end{bmatrix}$$

then  $\boldsymbol{W}^* = \boldsymbol{Z}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{Z}$ .

## $oldsymbol{Z}^{\! op} oldsymbol{P} oldsymbol{Z}$ – METHOD 2

$$\begin{bmatrix} \boldsymbol{X}^{\!\top} \boldsymbol{R}^{-1} \boldsymbol{X} & \boldsymbol{X}^{\!\top} \boldsymbol{R}^{-1} \boldsymbol{Z} & \boldsymbol{0} \\ \boldsymbol{Z}^{\!\top} \boldsymbol{R}^{-1} \boldsymbol{X} & \boldsymbol{Z}^{\!\top} \boldsymbol{R}^{-1} \boldsymbol{Z} + \boldsymbol{G}^{-1} & \boldsymbol{G}^{-1} \\ \boldsymbol{0} & \boldsymbol{G}^{-1} & \boldsymbol{G}^{-1} \end{bmatrix}$$

then  $\boldsymbol{W}^* = \boldsymbol{Z}^{\! op} \boldsymbol{P} \boldsymbol{Z}$ .

# $oldsymbol{y}^{\!\scriptscriptstyle op} oldsymbol{P} oldsymbol{y}$

$$\begin{bmatrix} \boldsymbol{X}^{\top} \boldsymbol{R}^{-1} \boldsymbol{X} & \boldsymbol{X}^{\top} \boldsymbol{R}^{-1} \boldsymbol{Z} & \boldsymbol{X}^{\top} \boldsymbol{R}^{-1} \boldsymbol{y} \\ \boldsymbol{Z}^{\top} \boldsymbol{R}^{-1} \boldsymbol{X} & \boldsymbol{Z}^{\top} \boldsymbol{R}^{-1} \boldsymbol{Z} + \boldsymbol{G}^{-1} & \boldsymbol{Z}^{\top} \boldsymbol{R}^{-1} \boldsymbol{y} \\ \boldsymbol{y}^{\top} \boldsymbol{R}^{-1} \boldsymbol{X} & \boldsymbol{y}^{\top} \boldsymbol{R}^{-1} \boldsymbol{Z} & \boldsymbol{y}^{\top} \boldsymbol{R}^{-1} \boldsymbol{y} \end{bmatrix}$$

then  $W^* = y^{\mathsf{T}} P y$ .

#### DETERMINANT

The determinant of U is evaluated by the product of the non-zero pivots.

# CALCULATING THE EXACT $oldsymbol{A}^{-1}$ FOR SUB-**POPULATIONS**

Partition  $A^{-1}$  into two parts: animal to be absorbed in U $S = R^{-1} - R^{-1} \begin{bmatrix} X & Z_1 \end{bmatrix} \begin{bmatrix} X^{\mathsf{T}} R^{-1} X & X^{\mathsf{T}} R^{-1} Z_1 \\ Z_1^{\mathsf{T}} R^{-1} X & Z_1^{\mathsf{T}} R^{-1} Z_1 + G_1^{-1} \end{bmatrix}$  and animals that are to remain in W. The  $W^*$  is the exact inverse relationship matrix for the remaining selected animals.