# Absorption Cheat Sheet

## Matrix Partition

Suppose that the matrix of interest is partitioned as

$$\begin{bmatrix} U & T \\ T^\top & W \end{bmatrix}$$

where $U$ is a square non-singular matrix.

## Absorption

The absorption by $U$ is given by

$$W^* = W - T^\top U^{-1} T.$$

Note this is a special case of **Schur complement**.

Row by row absorption is equivalent to repeated application of the formula for $W^*$, where $U$ is a scalar (the pivot) and $T$ is a row vector.

## Gaussian Elimination

Note

$$Ux + Ty \tag{1}$$

$$T^\top x + Wy \tag{2}$$

Pre-multiplying (1) by $T^\top U^{-1}$ results in $T^\top x + T^\top U^{-1} T y$ and Gaussian elimination in (2) results in $(W - T^\top U^{-} T)y$. That is the absorption is the "coefficient" of $y$ after elimination of $x$.

## Matrix Inversion

For inversion of symmetric positive definite matrix $E^{n\times n}$, set

$$\begin{bmatrix} E & I_n \\ I_n & 0^{n\times n} \end{bmatrix}$$

then $W^* = -E^{-1}$.

## $V^{-1}$ in ML

$$\begin{bmatrix} Z^\top R^{-1} Z + G^{-1} & Z^\top R^{-1} \\ R^{-1} Z & R^{-1} \end{bmatrix}$$

then $W^* = V^{-1}$.

## $P$ in REML

$$\begin{bmatrix} X^\top R^{-1} X & X^\top R^{-1} Z & X^\top R^{-1} \\ Z^\top R^{-1} X & Z^\top R^{-1} Z + G^{-1} & Z^\top R^{-1} \\ R^{-1} X & R^{-1} Z & R^{-1} \end{bmatrix}$$

then $W^* = P$.

## Absorption in MME

$$\begin{bmatrix} X^\top R^{-1} X & X^\top R^{-1} Z & X^\top R^{-1} y \\ Z^\top R^{-1} X & Z^\top R^{-1} Z + G^{-1} & Z^\top R^{-1} y \\ y^\top R^{-1} X & y^\top R^{-1} Z & y^\top R^{-1} y \end{bmatrix}$$

### Absorption of subset of fixed effects

Say we partition the fixed effect $\tau = (\tau_1^\top, \tau_2^\top)^\top$ and so the design matrix is given conformably as $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$. Then if we absorb $\tau_2$ in the MME then the MME results in

$$\begin{bmatrix} X_1^\top S X_1 & X_1^\top S Z \\ Z^\top S X_1 & Z^\top S Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\tau}_1 \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} X_1^\top S y \\ Z^\top S y \end{bmatrix}$$

where $S = R^{-1} - R^{-1} X_2 \left( X_2^\top R^{-1} X_2 \right)^{-1} X_2^\top R^{-1}$.

### Absorption of subset of random effects

Say we partition the fixed effect $u = (u_1^\top, u_2^\top)^\top$ and so the design matrix is given conformably as $\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}$. Then if we absorb $u_1$ and $\tau$ in the MME then we have

$$\left( Z_2^\top S Z_2 + G_2^{-1} \right) \tilde{u}_2 = Z_2^\top S y$$

where $S = R^{-1} -$
$R^{-1} \begin{bmatrix} X & Z_1 \end{bmatrix} \begin{bmatrix} X^\top R^{-1} X & X^\top R^{-1} Z_1 \\ Z_1^\top R^{-1} X & Z_1^\top R^{-1} Z_1 + G_1^{-1} \end{bmatrix}^{-1} \begin{bmatrix} X^\top \\ Z_1^\top \end{bmatrix} R^{-1}$.

$$S = R^{-1} - R^{-1} \begin{bmatrix} X & Z_1 \end{bmatrix} \begin{bmatrix} X^\top R^{-1} X & X^\top R^{-1} Z_1 \\ Z_1^\top R^{-1} X & Z_1^\top R^{-1} Z_1 + G_1^{-1} \end{bmatrix}^{-1} \begin{bmatrix} X^\top \\ Z_1^\top \end{bmatrix} R^{-1} \tag{3}$$

$$= R^{-1} - R^{-1} \begin{bmatrix} X & Z_1 \end{bmatrix} (R^{-1} - R^{-1} Z_1 (Z_1^\top R^{-1} Z_1 + G_1^{-1})^{-1} Z_1^\top R^{-1}) \begin{bmatrix} X^\top \\ Z_1^\top \end{bmatrix} R^{-1} \tag{4}$$

(5)

## $Z^\top P Z$ – Method 1

$$\begin{bmatrix} X^\top R^{-1} X & X^\top R^{-1} Z & X^\top R^{-1} Z \\ Z^\top R^{-1} X & Z^\top R^{-1} Z + G^{-1} & Z^\top R^{-1} Z \\ Z^\top R^{-1} X & Z^\top R^{-1} Z & Z^\top R^{-1} Z \end{bmatrix}$$

then $W^* = Z^\top P Z$.

## $Z^\top P Z$ – Method 2

$$\begin{bmatrix} X^\top R^{-1} X & X^\top R^{-1} Z & 0 \\ Z^\top R^{-1} X & Z^\top R^{-1} Z + G^{-1} & G^{-1} \\ 0 & G^{-1} & G^{-1} \end{bmatrix}$$

then $W^* = Z^\top P Z$.

## $y^\top P y$

$$\begin{bmatrix} X^\top R^{-1} X & X^\top R^{-1} Z & X^\top R^{-1} y \\ Z^\top R^{-1} X & Z^\top R^{-1} Z + G^{-1} & Z^\top R^{-1} y \\ y^\top R^{-1} X & y^\top R^{-1} Z & y^\top R^{-1} y \end{bmatrix}$$

then $W^* = y^\top P y$.

## Determinant

The determinant of $U$ is evaluated by the product of the non-zero pivots.

## Calculating the exact $A^{-1}$ for sub-populations

Partition $A^{-1}$ into two parts: animal to be absorbed in $U$ and animals that are to remain in $W$. The $W^*$ is the exact inverse relationship matrix for the remaining selected animals.

## References

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