*Getting Proper Standard Errors From R Stan:*

The Hessian Stan's optimization program returns, results in a covariance matrix in terms of:

$$[\log(t_{p_1}), \log(t_{p_2}), \log(\sigma_1), \log(\sigma_2), p]$$

Let's use the $\Delta$ method to get a covariance matrix for: $[\mu_1, \mu_2, \sigma_1, \sigma_2, p]$. Not the following three relationships between $\mu$ and the quantiles.

$$\begin{aligned}
\Phi^{-1}_{SEV}(p) &= \log(-\log(1-p)) \\
\log(t_{pi}) &= \mu_i + \log(\sigma_i)\Phi^{-1}_{SEV}(p_i) \\
F(t;\mu;\sigma) &= \Phi_{SEV}\left[\frac{\log(t)-\mu}{\sigma}\right]
\end{aligned}$$

Solving for $\mu$ we get:

$$\mu_i = \log(t_{pi}) - \exp(\log(\sigma_i))\Phi^{-1}_{SEV}(p_i)$$

Taking partials with respect to $\frac{d\mu_i}{d\log(t_{pi})}$ and $\frac{d\mu_i}{d\log(\sigma_i)}$ we get:

$$\begin{aligned}
\frac{d\mu_i}{d\log(t_{pi})} &= 1 \\
\frac{d\mu_i}{d\log(\sigma_i)} &= -\exp(\log(\sigma_i))\Phi^{-1}_{SEV}(p_i)
\end{aligned}$$

It is easy to transform $\log(\sigma_i)$. Stan estimates the SE's for p on the unconstrained space, so we use the inverse logit transformation ($\text{logit}^{-1}(p) = \frac{1}{1+e^{-p}}$) to get the SE's on the constrained space. Putting these pieces together we get a $\Delta$ matrix:

$$\Delta = \begin{bmatrix}
1 & 0 & -\exp(\log(\sigma_1)\Phi^{-1}_{SEV}(p_1) & 0 & 0 \\
0 & 1 & 0 & -\exp(\log(\sigma_2)\Phi^{-1}_{SEV}(p_2) & 0 \\
0 & 0 & \exp(\log(\sigma_1)) & 0 & 0 \\
0 & 0 & 0 & \exp(\log(\sigma_2)) & 0 \\
0 & 0 & 0 & 0 & (\frac{1}{1+e^{-p}})(\frac{e^{-p}}{1+e^{-p}})
\end{bmatrix}$$

Now, following Meeker and Escobar (Section 8.4.3), we need to get the standard errors for the GFLP CDF to make Wald bands. Recall, the CDF: $g(\mu_1, \mu_2, \sigma_1, \sigma_2, p) = 1 - (1 - pF(\mu_1; \sigma_1)(1 - F(\mu_2; \sigma_2))$, which we can re-write as:

$$\begin{aligned}
\Pr(t < T) &= 1 - (1 - p\Phi_{SEV}(\mu_1, \sigma_1)(z_1))(1 - \Phi_{SEV}(\mu_2, \sigma_2)(z_2)) \\
&= \Phi_{SEV}(\mu_2, \sigma_2)(z_2) + p\Phi_{SEV}(\mu_1, \sigma_1)(z_1) - p\Phi_{SEV}(\mu_1, \sigma_1)(z_1)\Phi_{SEV}(\mu_2, \sigma_2)(z_2)
\end{aligned}$$

Note that the SEV distribution is standardized in terms of $z = (\log(t) - \mu)/\sigma$. Let's again take partials of the GFLP CDF with respect to the 5 parameters. Note: $z_1$ or $z_2$ should be plugged into the proper SEV function for each term.

$$\begin{aligned}
\frac{dg}{d\mu_1} &= -p\phi_{SEV}(\mu_1, \sigma_1)\frac{1}{\sigma_1} + p\phi_{SEV}(\mu_1, \sigma_1)\Phi_{SEV}(\mu_2, \sigma_2)\frac{1}{\sigma_1} \\
\frac{dg}{d\mu_2} &= -\phi_{SEV}(\mu_2, \sigma_2)\frac{1}{\sigma_2} + p\Phi_{SEV}(\mu_1, \sigma_1)\phi_{SEV}(\mu_2, \sigma_2)\frac{1}{\sigma_2} \\
\frac{dg}{d\sigma_1} &= p\phi_{SEV}(\mu_1, \sigma_1)\left(\frac{-log(t) + \mu_1}{\sigma_1^2}\right) - p\Phi_{SEV}(\mu_2, \sigma_2)\phi_{SEV}(\mu_1, \sigma_1)\left(\frac{-log(t) + \mu_1}{\sigma_1^2}\right) \\
\frac{dg}{d\sigma_2} &= \phi_{SEV}(\mu_2, \sigma_2)\left(\frac{-log(t) + \mu_2}{\sigma_2^2}\right) - p\Phi_{SEV}(\mu_1, \sigma_1)\phi_{SEV}(\mu_2, \sigma_2)\left(\frac{-log(t) + \mu_2}{\sigma_2^2}\right) \\
\frac{dg}{dp} &= \Phi_{SEV}(\mu_1, \sigma_1) - \Phi_{SEV}(\mu_1, \sigma_1)\Phi_{SEV}(\mu_2, \sigma_2)
\end{aligned}$$

Putting these into a 5 x 5 matrix we can once again use the $\Delta$ method for the standard error of g. Then with this standard error we can use a normal approximation and get $\hat{g} \pm z_{1-\alpha/2}\hat{g_{se}}$

Taking this a step further (Meeker and Hong, p. 169) we can use a tangent method to get a CI that respects the bounds of a CDF.

Let $y_e = log(t)$. The lower and upper bounds are calculated as:

$$y_l = y_e - \frac{z_{1-\alpha/2}\hat{g_{se}}}{\frac{dF(t)}{log(t)}}$$
$$y_u = y_e + \frac{z_{1-\alpha/2}\hat{g_{se}}}{\frac{dF(t)}{log(t)}}$$

Taking the derivative of the GFLP in terms of $log(t)$, which it is already parameterized in terms of is:

$$\frac{dg(t)}{dlog(t)} = p\phi_{SEV}(\mu_1, \sigma_1)(z_1)(1 - \Phi_{SEV}(\mu_2, \sigma_2)(z_2))(1/\sigma_1) + \phi_{SEV}(\mu_2, \sigma_2)(z_2)(1 - p\Phi_{SEV}(\mu_1, \sigma_1)(z_1))(1/\sigma_2)$$

Finally, we plug these end points back into the GFLP CDF to get our final confidence intervals.

$$[\hat{F}(\exp(y_l)), \hat{F}(\exp(y_u))]$$

Figure 1: Wald Band 1 for Drive Model 3

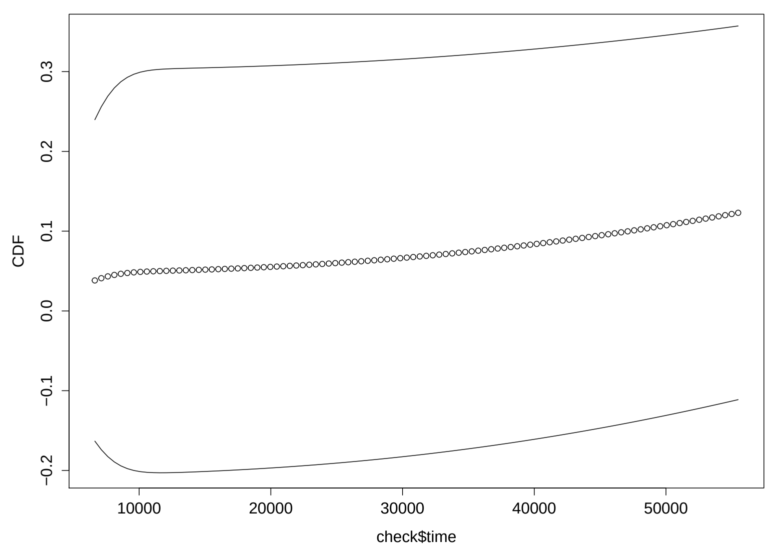

Figure 2: Wald Band for Drive Model 3 Hong and Meeker