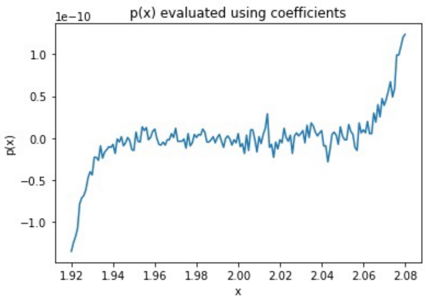
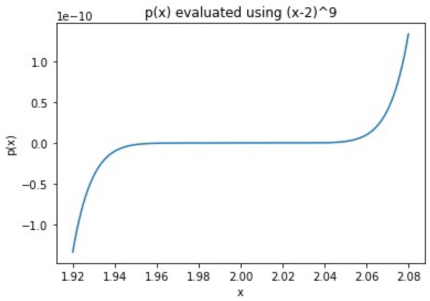
Homework 1

1) $p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$

i)



ii)



iii) The first plot, produced be evaluating p(x) by its coefficients is less "smooth" than the plot created by evaluating $p(x) = (x-2)^9$. This discrepancy is caused by the cumulative effect of rounding errors that occured during each operation when making the 1st plot. Because there are many more operations where errors could occur while making the 1st plot, the 2nd plot is the correct plot.

a)

i. $\sqrt{x+1} - 1$ for $x \approx 0$.

When x is close to 0, this expression will essentially equate to $1 \cdot 1 = 0$. Subtracting to values near 0 causes problems on the machine, so we want to calculate this in a way that will eliminate the subtraction to 0.

$$\sqrt{x+1} - 1 \times \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} = \boxed{\frac{x}{\sqrt{x+1} + 1}}$$

Multiplying by the conjugate gets rid of the cancellation

ii. $\sin(x) - \sin(y)$ for $x \approx y$.

Like above, when $x \approx y$, this expression subtracts to a value near 0. Multiplying the expression by the conjugate will eliminate this.

$$\sin(x) - \sin(y) \times \frac{\sin(x) + \sin(y)}{\sin(x) + \sin(y)} = \frac{\sin^2(x) - \sin^2(y)}{\sin(x) + \sin(y)}$$

now, use trig identity: numerator $= \sin(x+y)\sin(x-y)$:

$$\frac{\sin^2(x) - \sin^2(y)}{\sin(x) + \sin(y)} = \boxed{\frac{\sin(x+y)\sin(x-y)}{\sin(x) + \sin(y)}}$$

iii) $\frac{1-\cos(x)}{\sin(x)}$ for $x \approx 0$.

When $x \approx 0$, $\sin(x)$ nears 0. Dividing by 0, or values near 0, will cause problems on the machine. To eliminate this, we multiply the expression by the conjugate of the numerator.

$$\frac{1-\cos(x)}{\sin(x)} \times \frac{1+\cos(x)}{1+\cos(x)} = \frac{1-\cos^2(x)}{\sin(x) + \cos(x)\sin(x)} = \frac{\sin^2(x)}{\sin(x)[1+\cos(x)]}$$

$$= \boxed{\frac{\sin x}{1+\cos x}}$$

3) $f(x) = (1+x+x^3)\cos(x)$, $x_0 = 0$

- $f(0) = 1$
- $f'(x) = (1+3x^2)\cos x - (1+x+x^3)\sin x$, $f'(0) = 1$, $f'(0)\cdot x = x$
- $f''(x) = (6x)\cos x - (1+3x^2)\sin x - (1+3x^2)\sin x - (1+x+x^3)\cos x$

$= 6x\cos x - (2+6x^2)\sin x - (1+x+x^3)\cos(x)$

- $f''(0) = -1$, $\frac{f''(0)\cdot x^2}{2!} = \frac{-x^2}{2}$

$$\boxed{P_2(x) = 1 + x - \frac{x^2}{2}}$$

a. $P_2(0.5) = 1 + \frac{1}{2} - \frac{1}{8} = \boxed{\frac{11}{8} \approx 1.375}$ is the approximation for $f(0.5)$.

$$|f(0.5) - P_2(0.5)| < \frac{f^3(z)}{3!}\cdot x^3$$

- $f^{(3)}(z) = 6\cos z - 6z\sin z - 12z\sin z - 2(1+3z^2)\cos z - (1+3z^2)\cos z + (1+z-z^3)\sin z$
- $f^{(3)}(0) = 6 - 2 - 1 = 3$

$$|f(0.5) - P_2(0.5)| < \frac{3}{6}\cdot\left(\frac{1}{2}\right)^3 = \boxed{\frac{1}{16}}$$

The actual error is $f(0.5) - P_2(0.5) = 1.426 - 1.375 = 0.051 < 0.0625 = \frac{1}{16}$

So, the upper limit of the error holds.

b. $|f(x) - P_2(x)| < \left|\frac{p^{(3)}(x)\, x^3}{3!}\right| = \left|-\frac{x\cdot x^3}{6}\right| = \boxed{\frac{x^4}{6}}$

c. $\int_0^1 f(x)dx$

$\int_0^1 P_2(x)dx = \int_0^1 1 + x - \frac{x^2}{2}\,dx$

$= \left[x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right]_0^1 = 1 + \frac{1}{2} - \frac{1}{6} = \frac{4}{3}$

$$\boxed{\int_0^1 f(x)dx \approx \frac{4}{3}}$$

d. Using the bound for the error of the approximation found in part (b), we can estimate the error in the integral by evaluating:

$$\int_0^1 \frac{x^4}{6}\,dx = \left.\frac{x^5}{30}\right]_0^1 = \boxed{\frac{1}{30}}$$

So, the error in the integral is $\leq \frac{1}{30}$.

4) $ax^2 + bx + c = 0$, $a = 1$, $b = -56$, $c = 1$

a. Using only 3 correct decimals:

$$r_1 = \frac{56 + \sqrt{56^2 - 4}}{2} = 55.982$$

$$r_2 = \frac{56 - \sqrt{56^2 - 4}}{2} = 0.018$$

Using standard formulas:

$r_1 = 55.9821\ldots$

$r_2 = 0.01786\ldots$

$$r_1 \text{ relative error} = \frac{|55.9821\ldots - 55.982|}{55.9821\ldots} = 2.45 \times 10^{-6}$$

$$r_2 \text{ relative error} = \frac{|0.01786\ldots - 0.018|}{0.01786\ldots} = 7.68 \times 10^{-3}$$

↓

r_2 is the "bad root"

b. $(x - r_1)(x - r_2) = 0$

$\rightarrow x^2 - r_1 x - r_2 x + r_1 r_2 = 0$

$\rightarrow x^2 + (-r_1 - r_2)x + r_1 r_2 = 0$

$-r_1 - r_2 = -56$

$\boxed{r_1 + r_2 = 56}$ $\boxed{r_1 r_2 = 1}$

$1/\text{approx}(r_1)$ yields a different r_2:

$r_2 = 0.017862\ldots$

The new relative error:

$$\frac{|0.01786\ldots - 0.017862\ldots|}{0.01786\ldots} = 2.45\ldots \times 10^{-6}$$

Using the 2nd relation with our approximated r_1, we get $1/r_1 = r_2$, and this yields an r_2 with a relative error that is smaller by a factor of 10^3.

5)

a. $\tilde{x}_1 = x_1 + \Delta x_1$, $\tilde{x}_2 = x_2 + \Delta x_2$

$y = \tilde{x}_1 - \tilde{x}_2 = (x_1 - x_2) + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y}$

Absolute Error: $\boxed{|\Delta y| = |\Delta x_1 - \Delta x_2| \leq |\Delta x_1| + |\Delta x_2|}$

Relative Error: $\boxed{\frac{|\Delta y|}{y} = \frac{|\Delta x_1 - \Delta x_2|}{|x_1 - x_2|} \leq \frac{|\Delta x_1| + |\Delta x_2|}{|x_1 - x_2|}}$

- The relative error is large when x_1 and x_2 are close in value.

b. $\cos(x+\delta) - \cos(x) = \boxed{-2\sin\left(\frac{\delta}{2} + x\right)\sin\left(\frac{\delta}{2}\right)}$

- $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
- $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$
- $\cos(A+B) - \cos(A-B) = -2\sin(A)\sin(B)$

- $A+B = x+\delta \rightarrow B = \frac{\delta}{2}$
- $A-B = x \rightarrow A = \frac{\delta}{2} + x$

The new expression is more stable because it doesn't subtract close to 0. The original expression will be very close to 0 and ustable on the machine for small delta values.

c. $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(\varepsilon)$, $\varepsilon \in [x, x+\delta]$

$\cos(x+\delta) - \cos(x) = -\delta\sin(x) - \frac{\delta^2}{2!}\cos(\varepsilon)$

The algorithm I'm using to approximate $\cos(x+\delta) - \cos(x)$ is: $-\delta\sin(x)$.
I'm omitting 2nd term from the expression above because the δ^2 value in the numerator will result in the whole term being extremely small, making little difference in the approximation. The approximation in part (b) is a more accurate approximation because our algorithm is an exact relation, while the Taylor expansion algorithm has a sourse of error in the subtraction.