Homework 10

1) $f(x) = \sin(x)$. Determine Padé approx of degree 6

a. Cubic numerator & denominator: $P_3^3(x)$

$$P_3^3(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3}$$

$$T_6(x) = \sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!} x^n$$

$$= 0 + x + 0 - \frac{1}{6}x^3 + 0 + \frac{1}{120}x^5 + 0$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(1 + b_1 x + b_2 x^2 + b_3 x^3\right)$$

x_j	Eq for coeffs	
constant	$a_0 = 0$	$\longrightarrow a_0 = 0$
x	$a_1 = 1$	$\longrightarrow a_1 = 1$
x^2	$a_2 = b_1$	$\longrightarrow a_2 = 0$
x^3	$a_3 = b_2 - \frac{1}{6}$	$\longrightarrow a_3 \ \frac{1}{20} - \frac{1}{6} = -\frac{7}{60}$
x^4	$0 = b_3 - \frac{b_1}{6}$	
x^5	$0 = -\frac{b_2}{6} + \frac{1}{120}$	
x^6	$0 = -\frac{b_3}{6} + \frac{b_1}{120}$	

The last 3 equations are independent of $\{a_j\}_{j=0}^{3}$, so we can use them to find $\{b\}$:

$$\begin{bmatrix} -\frac{1}{6} & 0 & 1 \\ 0 & -\frac{1}{6} & 0 \\ \frac{1}{120} & 0 & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/120 \\ 0 \end{bmatrix}$$

$$\searrow \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/20 \\ 0 \end{bmatrix}$$

The Padé Approximation is: $P_3^3(x) = \dfrac{0 + x + 0 - \frac{7}{60}x^3}{1 + 0 + \frac{1}{20}x^2 + 0} = \dfrac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2} = \boxed{\dfrac{60x - 7x^3}{60 + 3x^2}}$

1) b. Numerator is quadratic & denominator is 4^{th} degree: $p_2^4(x)$

$$p_2^4(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$\rightarrow a_0 + a_1 x + a_2 x^2 = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4\right)$$

x_j	coeffs	
constant	$a_0 = 0$	$\rightarrow a_0 = 0$
x	$a_1 = 1$	$\rightarrow a_1 = 1$
x^2	$a_2 = b_1$	$\rightarrow a_2 = 0$
x^3	$0 = -\frac{1}{6} + b_2$	$\rightarrow b_2 = \frac{1}{6}$
x^4	$0 = b_3 - \frac{b_1}{6}$	
x^5	$0 = \frac{1}{120} + b_4 - \frac{b_2}{6}$	
x^6	$0 = -\frac{b_3}{6} + \frac{b_1}{120}$	

The last 3 equations are independent of $\{a\}$, can find $\{b\}$:

$$\begin{bmatrix} -\frac{1}{6} & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{120} & -\frac{1}{6} & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 7/360 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/6 \\ 0 \\ 7/360 \end{bmatrix}$$

The Padé Approximation is:

$$p_2^4(x) = \frac{0 + x + 0x^2}{1 + 0x + \frac{1}{6}x^2 + 0x^3 + \frac{7}{360}x^4} = \frac{x}{1 + \frac{1}{6}x^2 + \frac{7}{360}x^4} = \boxed{\frac{360x}{360 + 60x^2 + 7x^4}}$$

1) c. $P_4^2(x) = \dfrac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{1 + b_1x + b_2x^2}$

compare to $T_6(\sin(x))$:

$$\frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{1 + b_1x + b_2x^2} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$$\rightarrow a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)(1 + b_1x + b_2x^2)$$

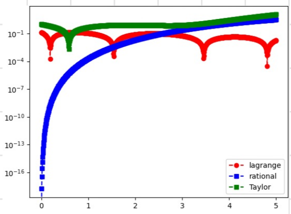
x_j	coeffs	
constant	$a_0 = 0$	
x	$a_1 = 1$	
x^2	$a_2 = b_1$	$\rightarrow a_2 = 0$
x^3	$a_3 = -\frac{1}{6} + b_2$	$\rightarrow a_3 = -\frac{7}{60}$
x^4	$a_4 = -\frac{b_1}{6}$	$\rightarrow a_4 = 0$
x^5	$0 = \frac{1}{120} - \frac{b_2}{6}$	$\rightarrow b_2 = \frac{1}{20}$
x^6	$0 = \frac{b_1}{120}$	$\rightarrow b_1 = 0$

The Padé approximation is:

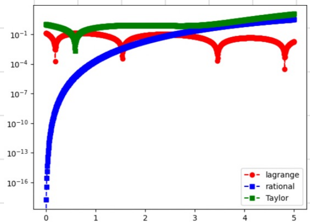
$$P_4^2(x) = \frac{0 + x + 0x^2 - \frac{7}{60}x^3 + 0x^4}{1 + 0x + \frac{1}{20}x^2} = \frac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2} = \boxed{\frac{60x - 7x^3}{60 + 3x^2}}$$

The plots below show the error of Maclaurin, Padé, & Lagrange:

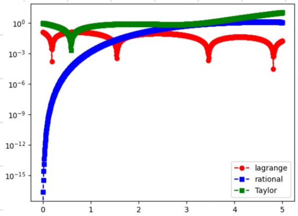
a.



c.



b.



The Padé approximation created in part (b) performs the best compared to the MacLaurin series for sin(x).

2) $\int_0^1 f(x)\,dx = \frac{1}{2} f(x_0) + c_1 f(x_1)$.

Unknowns are $x_0, x_1, c_1 \rightarrow$ we need 3 equations. Highest degree of precision = 2.

This means the quadrature is exact for x^k for $k = 0,1,2$. Choose to use $1, x, x^2$:

- $f(x) = 1$:

$$\int_0^1 1\,dx = \frac{1}{2}\cdot 1 + c_1$$

$$1 = \frac{1}{2} + c_1 \rightarrow \boxed{c_1 = \frac{1}{2}}$$

- $f(x) = x$:

$$\int_0^1 x\,dx = \frac{1}{2} x_0 + c_1 x_1$$

$$\frac{1}{2} = \frac{1}{2} x_0 + c_1 x_1 \rightarrow \frac{1}{2} = \frac{1}{2} x_0 + \frac{1}{2} x_1 \rightarrow \boxed{1 = x_0 + x_1}$$

$$\rightarrow x_0 = 1 - x_1$$

$$x_0 = 1 - \frac{1}{6}(3+\sqrt{3}) = \frac{1}{6}(3-\sqrt{3})$$

- $f(x) = x^2$:

$$\int_0^1 x^2\,dx = \frac{1}{2} x_0^2 + c x_1^2$$

$$\frac{1}{3} = \frac{1}{2} x_0^2 + c x_1^2 \rightarrow \frac{1}{3} = \frac{1}{2} x_0^2 + \frac{1}{2} x_1^2$$

$$\rightarrow \frac{1}{3} = \frac{1}{2}(1-x_1)^2 + \frac{1}{2} x_1^2$$

$$\rightarrow \frac{1}{3} = \frac{1}{2} x_1^2 - x_1 + \frac{1}{2} + \frac{1}{2} x_1^2$$

$$\rightarrow \frac{1}{3} = x_1^2 - x_1 + \frac{1}{2}$$

$$\rightarrow 0 = x_1^2 - x_1 + \frac{1}{6}$$

$$\rightarrow x_1 = \frac{1}{6}(3+\sqrt{3})$$

The constants are:

$c_1 = \frac{1}{2}$

$x_0 = \frac{1}{6}(3-\sqrt{3})$

$x_1 = \frac{1}{6}(3+\sqrt{3})$

3) a. $\int_{-5}^{5} \frac{1}{1+s^2}\,ds$

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b. $\left|\int_{-5}^{5} \frac{1}{1+s^2}ds - T_n\right| < 10^{-4} \rightarrow \frac{b-a}{12} \cdot h^2 f''(\mu) < 10^{-4}$

$$f''(x) = \frac{2(3x^2-1)}{(x^2+1)^3}$$

$\max_{x\in[-5,5]} |f''(x)|$ occurs at $x=0$: $|f''(0)|=2$

So, $\frac{5-(-5)}{12} \cdot \left(\frac{5-(-5)}{n}\right)^2 \cdot 2 < 10^{-4}$

$$\frac{10^3}{n^2} < 10^{-4}$$

$$\boxed{n > 10^{7/2}}$$

$\left|\int_{-5}^{5} \frac{1}{1+s^2}ds - S_n\right| < 10^{-4} \rightarrow \frac{b-a}{180} h^4 \cdot f^{(4)}(\mu) < 10^{-4}$

$$f^{(4)}\left(\frac{1}{1+x^2}\right) = \frac{24(5x^4-10x^2+1)}{(1+x^2)^5}$$

$\max_{x\in[-5,5]} f^{(4)}(x)$ occurs at $x=0 \rightarrow |f^{(4)}(0)| = 24$

So, $\frac{5-(-5)}{180}\left(\frac{5-(-5)}{\frac{n}{2}}\right)^4 \cdot 24 < 10^{-4}$

$$\frac{10^5}{\frac{180n^4}{16}} < 10^{-4}$$

$$\boxed{n > 10^2}$$

To keep the error of our trapezoidal approximation $< 10^{-4}$, we need $n \geq 10^{3.5}$ iterations. To keep the error of our Simpson's approximation $< 10^{-4}$, we need $n \geq 10^2$. So, Simpson's requires less iterations in this case.

c. For $n=100$ and $tol = 10^{-6}$: $T_n = 2.746776\ldots$ in 99 iterations

$S_n = 2.7428396\ldots$ in 49 iterations

Quad $= 2.7468015\ldots$ with an error $10^{-8} < 10^{-6}$

For $n=100$ and $tol = 10^{-4}$: $T_n = 2.746776\ldots$ in 99 iterations

$S_n = 2.74284\ldots$ in 49 iterations

Quad $= 2.746802\ldots$ with error $10^{-5} < 10^{-4}$

For $n = 10^{7/2}$ and $tol = 10^{-6}$: $T_n = 2.746802\ldots$ in 3161

$S_n = 2.7466799\ldots$ in 1580

Quad $= 2.746802\ldots$ with error $10^{-8} < 10^{-6}$

For $n = 10^{7/2}$ and $tol = 10^{-4}$: $T_n = 2.7468015\ldots$ in 3161 iterations

$S_n = 2.7466798\ldots$ in 1580 iterations

Quad $= 2.7468015\ldots$ with error $10^{-5} < 10^{-4}$