Discrete Fourier Transforms of Fractional Processes with Econometric Applications *

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Abstract

The discrete Fourier transform (dft) of a fractional process is studied. An exact representation of the dft is given in terms of the component data, leading to the frequency domain form of the model for a fractional process. This representation is particularly useful in analyzing the asymptotic behavior of the dft and periodogram in the nonstationary case when the memory parameter $d \geq \frac{1}{2}$. Various asymptotic approximations are established including some new hypergeometric function representations that are of independent interest. It is shown that smoothed periodogram spectral estimates remain consistent for frequencies away from the origin in the nonstationary case provided the memory parameter $d < 1$. When $d = 1$, the spectral estimates are inconsistent and converge weakly to random variates. Applications of the theory to log periodogram regression and local Whittle estimation of the memory parameter are discussed and some modified versions of these procedures are suggested for nonstationary cases.

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*This paper has a long history. It was presented at the Cowles Foundation Conference "New Developments in Time Series Econometrics", 23-24 October, 1999, and to the New Zealand Econometric Study Group, July 1999, Auckland, New Zealand. The paper originated in some notes on fractional processes in the nonstationary case that were written in May, 1998 and circulated at Yale. It was completed in 1999 while the author was living on Waiheke Island and visiting the University of Auckland. That version (Phillips, 1999a) was conditionally accepted subject to revision by the *Annals of Statistics* but the revise-by-date expired and the revision was never submitted. The paper was revised and updated in 2021, leaving its main results essentially unchanged. Thanks go to the Editor and referees of the *Annals of Statistics*, Chang Sik Kim, Katsumi Shimotsu, Yixiao Sun, and Karim Abadir for comments on the original paper. Katsumi and Yixiao made further helpful comments on the revision, which are greatly appreciated. Thanks are due to the NSF for research support under Grant Nos. SBR 94-22922, 97-30295, and SES 18-50860.

1 Introduction

Studies of nonstationary time series over the last four decades have produced a vast body of knowledge that has transformed the conduct of empirical research in economics. The impact of this research is now manifest in empirical work throughout the social and business sciences. A catalyst supporting these developments was the widespread recognition that real world processes in society, economics, and politics are influenced in fundamental ways by advances in technology, firm investments, and individual human decision making. These processes are rarely, if ever, stationary. Inevitably they evolve in uncertain ways over time, reflecting the arrival of new shocks to the system, some of which have persistent effects. Recognizing this reality led to an understanding that methods of data analysis need to account for the fact that the way in which memory is carried in the data differs in a fundamental manner among stationary, near-stationary and nonstationary processes.

Acknowledgement of the importance of this distinction is evident in early researches of statisticians and economists at the turn of the twentieth century (Hooker, 1901; Yule, 1926; Pearson and Elderton, 1923) on nonsense correlations[1] and the work of the mathematician Bachelier (1900) on speculative prices, which introduced the notion of a stochastic process. Methods began to emerge later that provided probabilistic underpinnings and foundations for statistical inference with data that demonstrated long range memory or dependence (Hurst, 1951, 1956; Mandelbrot and Van Ness, 1968; Granger and Joyeux, 1980; Hosking, 1981) and various types of random wandering behavior over time. In economics in the 1980s, advances in the use of function space limit theory were made that enabled the full trajectory features of nonstationary data to be reflected in regression asymptotics, leading to new understanding of such regressions, including both cointegrating and spurious regressions, and new methods of testing and inference for analyzing nonstationary data (Phillips, 1986b, 1987, 1988; Phillips and Durlauf, 1986; Durlauf and Phillips, 1988).

Joon Park played a big part in these developments, starting with his doctoral dissertation research and early research at Yale (Park and Phillips, 1988, 1989) and a sustained series of subsequent works that have helped to push out the envelope of econometric methodology for linear, nonlinear, and continuous time methods of analysis with nonstationary data. Many of these works have been jointly conducted with the present author in a longstanding collaboration that has been as pleasurable and special an academic fellowship as much as it has enriched this field of research.

My contribution to this symposium of works honoring Joon Park relates to his research on nonstationary processes and focuses on some of the defining properties of long range dependent time series. The present work has a history reaching back more than two decades and it is hoped that a good part of its value is retained amidst the considerable body of work that has emerged since the original version of the paper (Phillips, 1999a) was written. The first

[1] See Aldrich (1995) for an overview of early research on correlation, including nonsense correlations, where as Aldrich aptly puts it ‘there are more ways of going wrong than going right’.

contribution of the paper is to provide an exact representation of the discrete Fourier transform (dft) of a fractional process, which enables asymptotic analysis of its behavior and various functionals such as the periodogram in the nonstationary case when the memory parameter $d \geq \frac{1}{2}$. The methods reveal that smoothed periodogram spectral estimates remain consistent for frequencies away from the origin in the nonstationary case provided the memory parameter $d < 1$. When $d = 1$, the spectral estimates are inconsistent and converge weakly to random variates. Some useful applications of this theory are given for log periodogram regression and local Whittle estimation of the memory parameter in nonstationary cases. For an advanced textbook treatment of long memory processes, readers are referred to Surgailis et al. (2012).

The plan of the paper is as follows. Various preliminaries are given in the following Section 2. Some useful new decompositions and representations in the frequency domain are developed in Section 3 that extend related decompositions in the time domain. Section 4 develops asymptotic approximations for dfts involving special functions that help to simplify representations and enable development of limit theory for dfts of fractional processes in nonstationary cases. These results extend earlier work on the limit theory of dfts of stationary processes to the fractional case. For higher levels of dependence, when $d = 1$, the leakage from the zero frequency becomes dominant and affects the limit theory at all frequencies, so that dfts are spatially correlated across frequency asymptotically, quite unlike the stationary case. Section 5 provides some applications of the results to spectral estimation and to semiparametric estimation of the memory parameter. Particular attention in the latter case is given to log periodogram regression and local Whittle estimation. Some modified versions of these procedures are suggested which conveniently extend their range of applicability to the nonstationary case. Final remarks on long memory and autoregressive approaches to nonstationarity close out Section 5. Proofs and technical results are in the Appendix in Section 6. A notational summary is given at the end of the paper in Section 7.

A final word of introduction. While our focus is on the case where $d \in (\frac{1}{2}, 1)$, the methods introduced here are applicable when $d > 1$, and in modified form when $|d| < \frac{1}{2}$. A particularly useful approach is to combine the exact representation (3.7) that applies when $d = 1$ with results for fractional d to produce valid representations for the $d > 1$ case. The remarks and results in paragraphs 3.6 - 3.8 indicate some of these possibilities.

2 Preliminaries

We consider the fractional process X_t generated by the model

$$(1 - L)^d X_t = u_t, \quad t = 0, 1, ... \tag{2.1}$$

Our interest is primarily in the case where X_t is nonstationary and $d \geq \frac{1}{2}$, so in (2.1) we work from a given initial date $t = 0$, set $u_j = 0$ for all $j \leq 0$, and assume that u_t $(t > 0)$ is stationary with zero mean and continuous spectrum $f_u(\lambda) > 0$. This formulation corresponds to a Type II fractional process (Marinucci and Robinson, 1999; Davidson and Hashimzade, 2009). Expanding the binomial in (2.1) gives the form

$$\sum_{k=0}^{t} \frac{(-d)_k}{k!} X_{t-k} = u_t, \tag{2.2}$$

where

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = (d)(d+1)...(d+k-1)$$

is Pochhammer's symbol for the forward factorial function and $\Gamma(\cdot)$ is the gamma function. When d is a positive integer, the series in (2.2) terminates, giving the usual formulae for the model (2.1) in terms of differences and higher order differences of X_t. An alternate form for X_t is obtained by inversion of (2.1), giving

$$X_t = (1-L)^{-d} u_t = \sum_{k=0}^{t} \frac{(d)_k}{k!} u_{t-k}. \tag{2.3}$$

Throughout this paper it will be convenient to assume that the stationary component u_t in (2.1) is a linear process of the form

$$u_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j\,|c_j| < \infty, \quad C(1) \neq 0, \tag{2.4}$$

for all t and with $\varepsilon_t = iid\left(0, \sigma^2\right)$ with finite fourth moments. The summability condition in (2.4) is satisfied by a wide class of parametric and nonparametric models for u_t, enables the use of the techniques in Phillips and Solo (1992), and ensures that partial sums of u_t satisfy a functional central limit theorem, which will be needed later.

Under (2.4), the spectrum is $f_u(\lambda) = \frac{\sigma^2}{2\pi}\left|\sum_{j=0}^{\infty} c_j e^{ij\lambda}\right|^2$ and $f_u(0) = \frac{\sigma^2}{2\pi}C(1)^2 > 0$[2]. In view of (2.1), it is natural to define

$$f_x(\lambda) = |1-e^{i\lambda}|^{-2d} f_u(\lambda). \tag{2.5}$$

The function $f_x(\lambda)$ gives the spectrum of X_t when it exists and X_t is stationary (i.e. for $|d| < \frac{1}{2}$ and under infinite past initialization of X_t in (2.3)) and is the analogue of the spectrum in the nonstationary case when $d \geq \frac{1}{2}$ even though it is not integrable. In that case, Solo (1992) gave a formal justification of $f_x(\lambda)$ as a spectrum in terms of the limit of the expectation of the periodogram. Taking logarithms of (2.5) produces the equation

$$\ln(f_x(\lambda)) = -2d\ln(|1-e^{i\lambda}|) + \ln(f_u(\lambda)), \tag{2.6}$$

[2] Zeros everywhere in $f_u(\lambda)$ are ruled out if the last condition of (2.4) is strengthened to $C(e^{i\lambda}) \neq 0$ for all $\lambda \in [0, \pi]$.

which motivates a linear log periodogram regression for the estimation of d, in which $f_x(\lambda)$ is replaced by periodogram ordinates $I_x(\lambda)$ evaluated at the fundamental frequencies $\lambda_s = \frac{2\pi s}{n}$, $s = 0, 1, ..., n-1$. Here, $I_a(\lambda_s) = w_a(\lambda_s) w_a(\lambda_s)^*$, $w_a(\lambda_s)$ is the discrete Fourier transform (dft) $w_a(\lambda_s) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} a_t e^{it\lambda_s}$ of a time series a_t, and w^* is the complex conjugate of w. With this substitution (2.6) becomes

$$\ln\left(I_x\left(\lambda_s\right)\right) = -2d \ln\left|1 - e^{i\lambda_s}\right| + \ln\left(f_u\left(\lambda_s\right)\right) + U\left(\lambda_s\right), \tag{2.7}$$

where $U\left(\lambda_s\right) = \ln\left[I_x\left(\lambda_s\right)/f_x\left(\lambda_s\right)\right]$. By virtue of the continuity of f_u, $f_u\left(\lambda_s\right)$ is effectively constant for frequencies in a shrinking band around the origin, suggesting a linear least squares regression of $\ln\left(I_x\left(\lambda_s\right)\right)$ on $\ln\left|1 - e^{i\lambda_s}\right|$ over frequencies $s = 1, ..., m$ (with m a truncation number). The method has undoubted appeal, is easy to perform in practice and has been commonly employed in applications. However, (2.6) is a moment condition, not a data generating mechanism, and the analysis of this regression estimator is complicated by the difficulty of characterising the asymptotic behavior of the dft $w_x(\lambda_s)$, which is the central element in determining the properties of the regression residual $U\left(\lambda_s\right)$ in (2.7).

An important contribution by Künsch (1986) showed that, for fractional processes like (2.1), $w_x(\lambda_s)$ has quite different statistical properties from the corresponding dft, $w_u(\lambda_s)$, of the stationary process u_t for frequencies in the immediate neighbourhood of the origin. In particular, for $\lambda_s = \frac{2\pi s}{n} \to 0$, with s fixed as $n \to \infty$, the dft ordinates are asymptotically correlated, not uncorrelated. Analyses by Robinson (1995b) and Hurvich et al. (1998) for Gaussian u_t have provided an asymptotic theory in the stationary case, thereby placing log periodogram regression on a rigorous footing. More recent work has dealt with nonstationary cases where $d \geq \frac{1}{2}$ (Velasco, 1999; Kim and Phillips, 2006; Phillips, 2007). Another semiparametric estimation procedure, suggested by Künsch (1987), is the Gaussian estimator which maximises a local version of the Whittle likelihood, which is known to have a smaller variance than log periodogram regression in the stationary case (Robinson, 1995a). This estimator also relies on the behavior of $w_x(\lambda_s)$ for frequencies in the vicinity of the origin. More recent work on Whittle estimation has focused on nonstationary cases where $d \geq \frac{1}{2}$ (Velasco and Robinson, 2000; Phillips and Shimotsu, 2004; Shimotsu and Phillips, 2005, 2006; Abadir et al., 2007; Shao, 2010; Phillips, 2014) and cases of noise contaminated data (Sun and Phillips, 2003) such as in the estimation of the Fisher equation (Sun and Phillips, 2004).

The present paper provides new methods for studying the asymptotic behavior of $w_x(\lambda_s)$ for nonstationary values of d. The approach relies on an exact representation of $w_x(\lambda_s)$ in terms of the dft $w_u(\lambda_s)$ and certain residual components. This representation aids in the analysis of the properties of $w_x(\lambda_s)$ and, thereby, in the study of log periodogram regression and local Whittle estimation. The representation also provides a frequency domain version of the data generating mechanism (2.1) above. As such, it is useful in motivating some alternative approaches to inference about d that are proposed here and which have been explored in subsequent work that has appeared since the first version of this paper circulated in 1999.

3 Frequency Domain Decompositions

It is convenient to manipulate the operator $(1-L)^d$ in (2.1), with its polynomial expansion (2.2), in a form that more readily accommodates dfts. This can be done algebraically, as in Phillips and Solo (1992), by expanding the polynomial operator about its value at the complex exponential $e^{i\lambda}$, leading to the following decomposition.

3.1 Lemma *Define the fractional operator expansion* $D_n(L;d) = \sum_{k=0}^{n} \frac{(-d)_k}{k!} L^k$. *Then*

$$D_n(L;d) = D_n\left(e^{i\lambda};d\right) + \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right)\left(e^{-i\lambda}L - 1\right), \tag{3.1}$$

where $\widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right) = \sum_{p=0}^{n-1} \widetilde{d}_{\lambda p} e^{-ip\lambda} L^p$ *and* $\widetilde{d}_{\lambda p} = \sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}$.

The representation (3.1) is an immediate consequence of formula (32) in Phillips and Solo (1992) and can be obtained by straightforward algebraic manipulation. No summability conditions are required here for its validity since it is a finite sum. However, the value of d does affect the order of the terms in this expansion and, consequently, the order of magnitude of these terms when $n \to \infty$, a fact that does affect subsequent theory. Additionally, when λ depends on n, the order of these terms is affected and this too needs to be accounted for in the asymptotic theory. Much of the present paper is devoted to this accounting to assist in characterizing the limit behavior of the dft $w_x(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t e^{it\lambda}$.

Using the operator (3.1), we may write the model (2.1) in the following form for all $t \le n$

$$\begin{aligned} u_t &= D_n(L;d) X_t \\ &= D_n\left(e^{i\lambda};d\right) X_t + \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right)\left(e^{-i\lambda}L - 1\right) X_t. \end{aligned} \tag{3.2}$$

Taking dfts of the left and right sides of (3.2) now yields an exact expression for $w_x(\lambda)$ in terms of $w_u(\lambda)$. The result is stated as follows.

3.2 Theorem

$$w_u(\lambda) = w_x(\lambda) D_n\left(e^{i\lambda};d\right) + \frac{1}{\sqrt{2\pi n}}\left(\widetilde{X}_{\lambda 0}(d) - e^{in\lambda}\widetilde{X}_{\lambda n}(d)\right) \tag{3.3}$$

where $D_n\left(e^{i\lambda};d\right) = \sum_{k=0}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}$,

$$\widetilde{X}_{\lambda t}(d) = \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right) X_t = \sum_{p=0}^{n-1} \widetilde{d}_{\lambda p} e^{-ip\lambda} X_{t-p},$$

and

$$\widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right) = \sum_{p=0}^{n-1} \widetilde{d}_{\lambda p} e^{-ip\lambda} L^p, \quad \textit{with } \widetilde{d}_{\lambda p} = \sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}. \tag{3.4}$$

3.3 Remark Equation (3.3) provides an exact representation of $w_x(\lambda)$ in terms of $w_u(\lambda)$ and a residual component involving $n^{-\frac{1}{2}}\widetilde{X}_{\lambda n}(d)$. Explicitly,

$$w_x(\lambda) = D_n\left(e^{i\lambda};d\right)^{-1} w_u(\lambda) - \frac{1}{\sqrt{2\pi n}} D_n\left(e^{i\lambda};d\right)^{-1}\left(\widetilde{X}_{\lambda 0}(d) - e^{in\lambda}\widetilde{X}_{\lambda n}(d)\right). \tag{3.5}$$

In fact, (3.3) or (3.5) may be interpreted as a frequency domain version of the original model (2.1). In terms of periodogram ordinates, we have the corresponding equation

$$\begin{aligned} I_x(\lambda_s) &= |w_x(\lambda_s)|^2 = \left|D_n\left(e^{i\lambda_s};d\right)^{-1}\left[w_u(\lambda_s) - \frac{1}{\sqrt{2\pi n}}\left(\widetilde{X}_{\lambda 0}(d) - e^{in\lambda_s}\widetilde{X}_{\lambda_s n}(d)\right)\right]\right|^2 \\ &= \left|D_n\left(e^{i\lambda_s};d\right)\right|^{-2}\left[I_u(\lambda_s) - 2\,\mathrm{Re}\left\{\frac{1}{\sqrt{2\pi n}}\left(\widetilde{X}_{\lambda_s 0}(d) - \widetilde{X}_{\lambda_s n}(d)\right) w_u(\lambda_s)^*\right\}\right. \\ &\quad \left. + \frac{1}{2\pi n}\left|\left(\widetilde{X}_{\lambda_s 0}(d) - \widetilde{X}_{\lambda_s n}(d)\right)\right|^2\right], \end{aligned} \tag{3.6}$$

which may be interpreted as the data generating mechanism for the ordinates $I_x(\lambda_s)$ that are used in a log periodogram regression. Equation (3.6) reveals the model that is implicit in (2.7). To the extent that $\left|D_n\left(e^{i\lambda_s};d\right)\right|^{-2}$ can be replaced by $\left|1-e^{i\lambda_s}\right|^{-2d}$ and the component $n^{-\frac{1}{2}}\widetilde{X}_{\lambda_s n}(d)$ is small enough to be neglected, (3.6) and (2.5) might seem to suggest that $U(\lambda_s) = \ln\left[I_x(\lambda_s)/f_x(\lambda_s)\right]$ will behave like the corresponding functional, $\log\left[I_u(\lambda_s)/f_u(\lambda_s)\right]$, of the errors in (2.1). However, as will become apparent in our analysis, the residual component $n^{-\frac{1}{2}}\widetilde{X}_{\lambda_s n}(d)$ in (3.5) and (3.6) cannot be neglected, in general, and its importance grows as d increases.

3.4 Remark When $d=1$, the forward factorial $(-d)_k = 0$ for all $k>1$, so that series involving these coefficients terminate at $k=1$. In this case $D_n\left(e^{i\lambda};1\right) = \left(1-e^{i\lambda}\right)$, $\widetilde{d}_{\lambda 0} = -e^{i\lambda}$, $\widetilde{X}_{\lambda 0}(1) = -e^{i\lambda}X_0$, and $\widetilde{X}_{\lambda n}(1) = -e^{i\lambda}X_n$. Equation (3.3) then reduces to the simple form

$$w_u(\lambda) = \left(1-e^{i\lambda}\right) w_x(\lambda) + \frac{e^{i\lambda}}{\sqrt{2\pi n}}\left(e^{in\lambda}X_n - X_0\right), \tag{3.7}$$

an expression obtained by the author in earlier work and used in Corbae et al. (2002, Lemma B). In this case, it is apparent that $n^{-\frac{1}{2}}\widetilde{X}_{\lambda_s n}(d) = e^{i\lambda_s}n^{-\frac{1}{2}}X_n = O_p(1)$ for all λ_s. Thus, in the unit root case, the residual correction term $n^{-\frac{1}{2}}\widetilde{X}_{\lambda_s n}(d)$ definitely matters, plays a role in the asymptotic behavior of $w_x(\lambda_s)$ at all frequencies and thereby affects the asymptotic theory of estimators of d like those arising from log periodogram regression and local Whittle estimation. Indeed, in those cases the author has shown in other work (Phillips and Shimotsu, 2004; Phillips, 2007) that these estimators have limiting mixed normal distributions rather than normal distributions when $d=1$.

3.5 Remark When $u_t = 0$ for $t \leq 0$, in (2.1), it follows that $X_t = 0$ for $t \leq 0$ and hence $\widetilde{X}_{\lambda 0}(d) = 0$. In this event, expression (3.3) becomes

$$\begin{aligned} w_u(\lambda) &= w_x(\lambda) D_n\left(e^{i\lambda}; d\right) - \frac{e^{in\lambda}}{\sqrt{2\pi n}} \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L; d\right) X_n \\ &= w_x(\lambda) D_n\left(e^{i\lambda}; d\right) - \frac{e^{in\lambda}}{\sqrt{2\pi n}} \widetilde{X}_{\lambda n}(d), \end{aligned} \tag{3.8}$$

or, in the unit root case,

$$w_u(\lambda) = \left(1 - e^{i\lambda}\right) w_x(\lambda) + \frac{e^{i\lambda}}{\sqrt{2\pi n}} e^{in\lambda} X_n, \tag{3.9}$$

in place of (3.7). Since these initial conditions are assumed in (2.1), and since the effect of relaxing them will usually be apparent, we will henceforth use (3.8) in place of (3.3).

3.6 Remark Another useful representation for the dft of X_t can be obtained by combining the representation (3.8) with the unit root decomposition (3.9). It is especially useful when $d > 1$. Write (2.1) as

$$(1 - L) X_t = (1 - L)^{1-d} u_t := z_t \tag{3.10}$$

so that $X_t = \sum_{j=1}^t z_j + X_0$. Then, taking dfts in (3.10), we first apply (3.9) to write $w_x(\lambda_s)$ in terms of $w_z(\lambda_s)$ and then use (3.8) to reduce $w_z(\lambda_s)$ in terms of $w_u(\lambda_s)$ and a correction term. The outcome is formalized in the following theorem.

3.7 Theorem *If X_t follows (2.1), then*

$$w_x(\lambda)\left(1 - e^{i\lambda}\right) = w_z(\lambda) - e^{i\lambda} \frac{e^{i\lambda n} X_n}{\sqrt{2\pi n}} \tag{3.11}$$

$$= D_n\left(e^{i\lambda}; f\right) w_u(\lambda) - \frac{e^{i\lambda n}}{\sqrt{2\pi n}} \widetilde{U}_{\lambda n}(f) - e^{i\lambda} \frac{e^{i\lambda n} X_n}{\sqrt{2\pi n}}, \tag{3.12}$$

where $f = 1 - d$,

$$\widetilde{U}_{\lambda n}(f) = \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L; f\right) u_n = \sum_{p=0}^{n-1} \widetilde{f}_{\lambda p} e^{-ip\lambda} u_{n-p}, \quad \text{and } \widetilde{f}_{\lambda p} = \sum_{k=p+1}^{n} \frac{(-f)_k}{k!} e^{ik\lambda}. \tag{3.13}$$

3.8 Remark Some further decomposition beyond (3.11) and (3.12) is possible. As in Phillips and Solo (1992), we can decompose the operator $C(L)$ that appears in $u_t = C(L)\varepsilon_t$ as

$$C(L) = C\left(e^{i\lambda}\right) + \widetilde{C}_\lambda\left(e^{-i\lambda}L\right)\left(e^{-i\lambda}L - 1\right), \quad \widetilde{C}_\lambda(L) = \sum_{j=0}^{\infty} \widetilde{c}_{j\lambda} L^j, \ \widetilde{c}_{j\lambda} = e^{-i\lambda j} \sum_{k=j+1}^{\infty} c_k e^{i\lambda k},$$

where $\sum_{j=0}^{\infty}|\widetilde{c}_{j\lambda}|<\infty$ in view of the summability condition on c_j in (2.4). Then,

$$u_t = C(L)\,\varepsilon_t = C\left(e^{i\lambda}\right)\varepsilon_t + e^{-i\lambda}\varepsilon_{\lambda t-1} - \varepsilon_{\lambda t}, \tag{3.14}$$

is a valid decomposition of u_t into the *iid* component $C\left(e^{i\lambda}\right)\varepsilon_t$ and a stationary error that telescopes under the dft operation, with $\varepsilon_{\lambda t} = \widetilde{C}_\lambda\left(e^{-i\lambda}L\right)\varepsilon_t = \sum_{j=0}^{\infty}\widetilde{c}_{j\lambda}e^{-i\lambda j}\varepsilon_{t-j}$. In particular,

$$w_u(\lambda) = C\left(e^{i\lambda}\right)w_\varepsilon(\lambda) + \frac{1}{\sqrt{2\pi n}}\left(\varepsilon_{\lambda 0} - e^{in\lambda}\varepsilon_{\lambda n}\right) = C\left(e^{i\lambda}\right)w_\varepsilon(\lambda) + O_p\left(\frac{1}{\sqrt{n}}\right).$$

Using this representation in (3.12) we get

$$w_x(\lambda)\left(1-e^{i\lambda}\right) = D_n\left(e^{i\lambda};f\right)C\left(e^{i\lambda}\right)w_\varepsilon(\lambda) - \frac{e^{i\lambda n}}{\sqrt{2\pi n}}\widetilde{U}_{\lambda n}(f) - e^{i\lambda}\frac{e^{i\lambda n}X_n}{\sqrt{2\pi n}} + D_n\left(e^{i\lambda};f\right)\times O_p\left(\frac{1}{\sqrt{n}}\right). \tag{3.15}$$

Additionally, z_t in (3.10) can be written as

$$z_t = C\left(e^{i\lambda}\right)(1-L)^f\varepsilon_t + (1-L)^f\left(e^{-i\lambda}L-1\right)\varepsilon_{\lambda t}. \tag{3.16}$$

Set $\eta_t = (1-L)^f\varepsilon_t$, $\eta_{\lambda t} = (1-L)^f\varepsilon_{\lambda t}$ in (3.16) and take dfts, giving

$$\begin{aligned} w_z(\lambda) &= C\left(e^{i\lambda}\right)w_\eta(\lambda) + \frac{1}{\sqrt{2\pi n}}\left(\eta_{\lambda 0} - e^{in\lambda}\eta_{\lambda n}\right) \\ &= C\left(e^{i\lambda}\right)w_\eta(\lambda) + O_p\left(\frac{1}{\sqrt{n}}\right), \end{aligned} \tag{3.17}$$

since $\eta_{\lambda t}$ is stationary with finite variance for all $d \in \left(\frac{1}{2},\frac{3}{2}\right)$ because then $|f| < \frac{1}{2}$. (Note that $\eta_{\lambda t} = \varepsilon_{\lambda t}$ when $d = 1$). Next write

$$\eta_t = (1-L)^f\varepsilon_t = \left[D_n(L;f) + R_n(L;f)\right]\varepsilon_t \tag{3.18}$$

with

$$R_n(L;f) = \sum_{k=n+1}^{\infty}\frac{(-f)_k}{k!}L^k,$$

and note that

$$\varepsilon_{nt} := R_n(L;f)\,\varepsilon_t = O_p\left(\frac{1}{n^{\frac{1}{2}+f}}\right).$$

Applying (3.3) to the dft $w_\eta(\lambda)$ calculated from (3.18) we have

$$w_\eta(\lambda) = w_\varepsilon(\lambda)\,D_n\left(e^{i\lambda};f\right) + \frac{1}{\sqrt{2\pi n}}\left(\widetilde{\varepsilon}_{\lambda 0}(f) - e^{in\lambda}\widetilde{\varepsilon}_{\lambda n}(f)\right) + w_{n\varepsilon}(\lambda), \tag{3.19}$$

with

$$\widetilde{\varepsilon}_{\lambda n}(f) = \sum_{p=0}^{n-1} \widetilde{f}_{\lambda p} e^{-ip\lambda} \varepsilon_{n-p}, \qquad \widetilde{f}_{\lambda p} = \sum_{k=p+1}^{n} \frac{(-f)_k}{k!} e^{ik\lambda}, \tag{3.20}$$

and

$$w_{n\varepsilon}(\lambda) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \varepsilon_{nt} e^{i\lambda_t}.$$

Now $w_{n\varepsilon}(\lambda) = O_p\left(n^{-f}\right)$ because

$$E\left[w_{n\varepsilon}(\lambda) w_{n\varepsilon}(\lambda)^*\right] = \frac{1}{2\pi n} \sum_{t=1}^{n} \sum_{s=1}^{n} e^{i\frac{2\pi}{n}(t-s)} E\left(\varepsilon_{nt}\varepsilon_{ns}\right) = \frac{1}{2\pi n} \sum_{t=1}^{n} \sum_{s=1}^{n} O\left(n^{-1-2f}\right) = O\left(n^{-2f}\right). \tag{3.21}$$

Using (3.19) and (3.21) in (3.17) we get

$$w_z(\lambda) = C\left(e^{i\lambda}\right)\left[D_n\left(e^{i\lambda}; f\right) w_\varepsilon(\lambda) + \frac{1}{\sqrt{2\pi n}}\left(\widetilde{\varepsilon}_{\lambda 0}(f) - e^{in\lambda}\widetilde{\varepsilon}_{\lambda n}(f)\right)\right] + O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{n^f}\right). \tag{3.22}$$

Then, combining (3.22) with the unit root decomposition (3.11) leads to the representation

$$\begin{aligned} w_x(\lambda)\left(1 - e^{i\lambda}\right) &= C\left(e^{i\lambda}\right) D_n\left(e^{i\lambda}; f\right) w_\varepsilon(\lambda) - e^{i\lambda}\frac{X_n}{\sqrt{2\pi n}} \\ &\quad + \frac{1}{\sqrt{2\pi n}} C\left(e^{i\lambda}\right)\left(\widetilde{\varepsilon}_{\lambda 0}(f) - e^{in\lambda}\widetilde{\varepsilon}_{\lambda n}(f)\right) + O_p\left(\frac{1}{n^f}\right). \end{aligned} \tag{3.23}$$

This representation holds uniformly over λ and is likely to be most useful when $\lambda = \lambda_s = \frac{2\pi s}{n} \to 0$ and $s \to \infty$.

3.9 Remark The representations (3.8), (3.11), and (3.12) hold for all fundamental frequencies $\lambda_s = \frac{2\pi s}{n}$. They are helpful in providing asymptotic representations of $w_x(\lambda_s)$. In such expansions, it is useful to allow for situations where $s \to \infty$ as well as $n \to \infty$. In some cases, as in spectral density estimation at some frequency $\phi \neq 0$, we want the expansion rate of s to be the same as n, so that we can accommodate $\lambda_s \to \phi$ as $n \to \infty$. In other cases, as in log periodogram and Gaussian semiparametric regression, interest centers on frequencies λ_s in the vicinity of the origin, so then we consider cases where s is fixed or $s \to \infty$ and $\frac{s}{n} \to 0$ as $n \to \infty$. The following section gives results that are helpful in the determination of the asymptotic form of these representations as $n \to \infty$ under these various conditions.

4 Asymptotic Approximations

4.1 Component Approximations

We start with the sinusoidal polynomial $D_n\left(e^{i\lambda}; d\right) = \sum_{k=0}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}$ that appears in the decomposition (3.1) and theorems 3.2 and 3.7. The series can be summed in terms of hypergeometric functions and the asymptotic form taken as $n \to \infty$ depends on λ. The behavior is described in the following lemma.

4.2 Lemma[3] *Suppose $d > 0$ and is noninteger. Then*

$$D_n\left(e^{i\lambda}; d\right) = \left(1 - e^{i\lambda}\right)^d - e^{i(n+1)\lambda} \frac{(-d)_{n+1}}{(n+1)!}\, {}_2F_1\left(n+1-d, 1; n+2; e^{i\lambda}\right), \tag{4.1}$$

and, for $\cos(\lambda) < \frac{1}{2}$,

$$D_n\left(e^{i\lambda}; d\right) = \left(1 - e^{i\lambda}\right)^d + \frac{e^{i(n+1)\lambda}}{e^{i\lambda} - 1} \frac{(-d)_{n+1}}{(n+1)!}\, {}_2F_1\left(1+d, 1; n+2; \frac{e^{i\lambda}}{e^{i\lambda} - 1}\right). \tag{4.2}$$

The following asymptotic representations hold:

(a) *For fixed $\lambda \neq 0$*

$$D_n\left(e^{i\lambda}; d\right) = \left(1 - e^{i\lambda}\right)^d - \frac{1}{\Gamma(-d)\, n^{1+d}} \frac{e^{in\lambda}}{1 - e^{i\lambda}} \left[1 + O\left(\frac{1}{n}\right)\right].$$

(b) *For $\lambda = \lambda_s = \frac{2\pi i s}{n} \to 0$ and $s \to \infty$ as $n \to \infty$*

$$D_n\left(e^{i\lambda_s}; d\right) = \left(1 - e^{i\lambda_s}\right)^d + \frac{1}{2\pi i} \frac{1}{\Gamma(-d)\, n^d s} \left[1 + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{n^{1+d}}\right).$$

(c) *For $\lambda = \lambda_s = \frac{2\pi i s}{n} \to 0$ and s fixed as $n \to \infty$*

$$D_n\left(e^{i\lambda_s}; d\right) = \frac{1}{\Gamma(1-d)\, n^d}\, {}_1F_1(1, 1-d; -2\pi i s) + O\left(\frac{1}{n^{1+d}}\right).$$

(d) *For $\lambda = 0$*

$$D_n(1; d) = \frac{1}{\Gamma(1-d)} \frac{1}{n^d} \left[1 + O\left(\frac{1}{n}\right)\right].$$

In the above formulae, ${}_1F_1(a, b; z)$ and *${}_2F_1(a, b, c; z)$ denote the confluent hypergeometric function and the hypergeometric function, respectively.*

From part (d), it follows that $D_n(1; d)$ differs from zero by a term of $O\left(n^{-d}\right)$. From part (c), the same also applies to $D_n\left(e^{i\lambda_s}; d\right)$ when s is fixed and $\lambda_s = \frac{2\pi i s}{n} \to 0$. Of course, in the event that d is a positive integer, we have the following terminating polynomials

$$D_n(1; d) = \sum_{k=0}^{n} \frac{(-d)_k}{k!} = \sum_{k=0}^{d} \frac{(-d)_k}{k!} = \sum_{k=0}^{d} \binom{d}{k} (-1)^k = (1-1)^d = 0,$$

[3] Here, and elsewhere in the paper, where fractional powers of a complex variable are given they are taken to be evaluated at their principal values.

and

$$D_n\left(e^{i\lambda_s};d\right)=\sum_{k=0}^{n}\frac{(-d)_k\,e^{i\lambda_s k}}{k!}=\sum_{k=0}^{d}\binom{d}{k}\left(-e^{i\lambda_s}\right)^k=\left(1-e^{i\lambda_s}\right)^d$$

in this case.

Our next focus of interest is the correction term in (3.8) that involves $\widetilde{X}_{\lambda n}(d)$. We are especially interested in deriving an asymptotic approximation to $\widetilde{X}_{\lambda n}(d)$ at the fundamental frequencies λ_s. As in lemma 3.1, the asymptotic behavior of $\widetilde{X}_{\lambda_s,n}(d)$ is sensitive to the value of s in $\lambda_s = \frac{2\pi s}{n}$. In particular, when $d\in\left(\frac{1}{2},1\right)$, the asymptotic form of $\widetilde{X}_{\lambda_s,n}(d)$ differs, depending on whether s is fixed or whether $s\to\infty$ as $n\to\infty$. In the latter case, $n^{-\frac{1}{2}}\widetilde{X}_{\lambda_s,n}(d)=o_p(1)$, while in the former $n^{-\frac{1}{2}}\widetilde{X}_{\lambda_s,n}(d)=O_p(1)$. On the other hand, when $d=1$, $n^{-\frac{1}{2}}\widetilde{X}_{\lambda_s,n}(d)=O_p(1)$ for all $s\neq 0$. The results are given in the following theorem.

4.3 Theorem *Suppose $d\in\left(\frac{1}{2},1\right)$. Then*

(a) *For fixed $\lambda\neq 0$ as $n\to\infty$,*

$$\frac{\widetilde{X}_{\lambda,n}(d)}{\sqrt{n}}=-\frac{e^{i\lambda}}{\left(1-e^{i\lambda}\right)^{1-d}}\frac{X_n}{\sqrt{n}}+o_p\left(\frac{1}{n^{1-d}}\right)=O_p\left(\frac{1}{n^{1-d}}\right).$$

(b) *For $\lambda=\lambda_s=\frac{2\pi s}{n}\to 0$ and $\frac{s}{n^\alpha}\to\infty$, as $n\to\infty$, for some $\alpha\in\left(\frac{1}{2},1\right)$*

$$\begin{aligned}\frac{\widetilde{X}_{\lambda_s,n}(d)}{\sqrt{n}}&=-\frac{e^{i\lambda_s}}{\left(1-e^{i\lambda_s}\right)^{1-d}}\frac{X_n}{\sqrt{n}}+o_p\left(\frac{1}{s^{1-d}}\right)=-\frac{e^{i\lambda_s}}{\left(-2\pi i s\right)^{1-d}}\frac{X_n}{n^{d-\frac{1}{2}}}+o_p\left(\frac{1}{s^{1-d}}\right)\\&=O_p\left(\frac{1}{s^{1-d}}\right).\end{aligned}$$

(c) *For $\lambda=\lambda_s=\frac{2\pi s}{n}$ and s fixed, as $n\to\infty$,*

$$\begin{aligned}\frac{\widetilde{X}_{\lambda_s n}(d)}{\sqrt{n}}&=\frac{{}_1F_1\left(1,1-d;-2\pi i s\right)}{\Gamma\left(1-d\right)}\int_0^1 e^{2\pi i s r}X_{n,d}(r)dr\\&-\frac{1}{\Gamma\left(1-d\right)}\int_0^1 {}_1F_1\left(1,1-d;-2\pi i s r\right)r^{-d}X_{n,d}(1-r)dr+O_p\left(\frac{1}{n^{1-d}}\right)=O_p\left(1\right),\end{aligned}$$

where $X_{n,d}(r)=\frac{X_{\lfloor nr\rfloor}}{n^{d-\frac{1}{2}}}$.

(d) *When $d=1$, the equation*

$$\frac{\widetilde{X}_{\lambda,n}(1)}{\sqrt{n}}=-e^{i\lambda}\frac{X_n}{\sqrt{n}}=O_p\left(1\right)$$

holds for λ fixed, or $\lambda=\lambda_s=\frac{2\pi s}{n}\to 0$ with $s\to\infty$, or $\lambda_s=\frac{2\pi s}{n}\to 0$ with s fixed.

In parts (a) and (b) of theorem 4.3 the leading term in the asymptotic approximation of $n^{-\frac{1}{2}}\widetilde{X}_{\lambda,n}(d)$ is the same and so, although the error order of magnitude differs, we may write

$$\frac{\widetilde{X}_{\lambda,n}(d)}{\sqrt{n}} = -\frac{e^{i\lambda}}{\left(1-e^{i\lambda}\right)^{1-d}}\frac{X_n}{\sqrt{n}} + o_p\left(\frac{e^{i\lambda}}{\left(1-e^{i\lambda}\right)^{1-d}}\frac{X_n}{\sqrt{n}}\right),$$

for both these cases. Further, the leading term of $n^{-\frac{1}{2}}\widetilde{X}_{\lambda,n}(d)$ is $O_p(\frac{1}{n^{1-d}})$ for fixed $\lambda \neq 0$, is $O_p(\frac{1}{s^{1-d}})$ for $\lambda_s = \frac{2\pi s}{n} \to 0$ and $\frac{s}{n^{\alpha}} \to \infty$, and is $O_p(1)$ for $\lambda_s = \frac{2\pi s}{n} \to 0$ with s fixed. Thus, the correction term $n^{-\frac{1}{2}}\widetilde{X}_{\lambda,n}(d)$ is nonnegligible in a region around the origin when $d \in \left(\frac{1}{2}, 1\right)$. The asymptotic form of $n^{-\frac{1}{2}}\widetilde{X}_{\lambda,n}(d)$ in that case (i.e., case (c), with $\lambda_s = \frac{2\pi s}{n}$, and s fixed) is more complicated than the other cases and it involves hypergeometric series. The representation given in case (c) actually includes $s = 0$, for which we have the simpler form

$$\frac{\widetilde{X}_{\lambda_0 n}(d)}{\sqrt{n}} = \frac{1}{\Gamma\left(1-d\right)}\int_0^1 X_{n,d}(r)dr - \frac{1}{\Gamma\left(1-d\right)}\int_0^1 r^{-d}X_{n,d}(1-r)dr + O_p\left(\frac{1}{n^{1-d}}\right). \tag{4.3}$$

When $d = 1$, the formula given in (d) is exact, as follows directly from (3.9).

Finally, we look at the correction term $\widetilde{U}_{\lambda n}\left(f\right)$ that appears in (3.12). We concentrate on the interesting case where λ is in the vicinity of the origin and give the result corresponding to part (c) of theorem 4.3.

4.4 Theorem *Suppose $d \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $f = 1-d$. Then, for $\lambda = \lambda_s = \frac{2\pi s}{n}$ and s fixed, as $n \to \infty$*

$$\begin{aligned}\frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}} &= \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma\left(1-f\right)n^f}\left\{ {}_1F_1\left(1,1-f;-2\pi i s\right)\int_0^1 e^{-2\pi i s r}dX_n\left(1-r\right)\right. \\ &\quad \left. -\int_0^1 r^{-f}\ {}_1F_1\left(1,1-f;-2\pi i s r\right)dX_n\left(1-r\right)\right\} + O_p\left(\frac{1}{\sqrt{n}}\right),\end{aligned} \tag{4.4}$$

where $X_n(r) = n^{-\frac{1}{2}}\sum_{t=0}^{\lfloor nr \rfloor} u_t$. When $f = 0$, $\widetilde{U}_{\lambda_s n}(0) = 0$.

4.5 Approximations for $w_x\left(\lambda\right)$

Evaluating (3.8) at λ_s, we have

$$w_x\left(\lambda_s\right) = D_n\left(e^{i\lambda_s}; d\right)^{-1}\left[w_u\left(\lambda_s\right) + \frac{1}{\sqrt{2\pi n}}\widetilde{X}_{\lambda_s n}(d)\right].$$

We use lemma 4.2 and theorem 4.3 to obtain explicit expressions for $w_x\left(\lambda_s\right)$ in terms of $w_u\left(\lambda_s\right)$ and a correction term. When $d = 1$, the following exact form comes directly from (3.9)

$$w_x\left(\lambda_s\right) = \left(1-e^{i\lambda_s}\right)^{-1}w_u\left(\lambda_s\right) - \frac{e^{i\lambda_s}}{1-e^{i\lambda_s}}\frac{X_n}{\sqrt{2\pi n}}, \tag{4.5}$$

and holds for all $s = 1, 2, ...$ When $d \in (\frac{1}{2}, 1)$, it is convenient to separate the following three cases:

(a) Case $\lambda_s \to \phi \neq 0$

Here, from lemma 4.2 we have

$$\begin{aligned} D_n\left(e^{i\lambda_s}; d\right) &= \left(1 - e^{i\lambda_s}\right)^d - \frac{1}{\Gamma(-d)\, n^{1+d}} \frac{e^{in\lambda_s}}{1 - e^{i\lambda_s}} \left[1 + O\left(\frac{1}{n}\right)\right] \\ &= \left(1 - e^{i\lambda_s}\right)^d + O\left(\frac{1}{n^{1+d}}\right), \end{aligned}$$

uniformly for $\lambda_s \in \mathcal{B}_\phi = \left\{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\right\}$ where $M \to \infty$ as $n \to \infty$. Similarly, from theorem 4.3,

$$\frac{\widetilde{X}_{\lambda_s,n}(d)}{\sqrt{n}} = -\frac{e^{i\lambda_s}}{\left(1 - e^{i\lambda_s}\right)^{1-d}} \frac{X_n}{\sqrt{n}} + o_p\left(\frac{1}{n^{1-d}}\right)$$

uniformly for $\lambda_s \in \mathcal{B}_\phi$. It follows that

$$w_x(\lambda_s) = \left(1 - e^{i\lambda_s}\right)^{-d} w_u(\lambda_s) - \frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{X_n}{\sqrt{2\pi n}} + o_p\left(\frac{1}{n^{1-d}}\right), \tag{4.6}$$

uniformly for $\lambda_s \in \mathcal{B}_\phi$.

(b) Case $\lambda_s = \frac{2\pi i s}{n} \to 0$ **and** $s \to \infty$

From lemma 4.2 (b) when $s \to \infty$ as $n \to \infty$

$$D_n\left(e^{i\lambda_s}; d\right) = \left(1 - e^{i\lambda_s}\right)^d + \frac{1}{\Gamma(-d)\, n^d} \frac{1}{2\pi i s} \left[1 + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{n^{1+d}}\right).$$

And from theorem 4.3 (b) with $\frac{s}{n^\alpha} \to \infty$ for some $\alpha \in \left(\frac{1}{2}, 1\right)$, as $n \to \infty$,

$$\frac{\widetilde{X}_{\lambda_s,n}(d)}{\sqrt{n}} = -\frac{e^{i\lambda_s}}{\left(1 - e^{i\lambda_s}\right)^{1-d}} \frac{X_n}{\sqrt{n}} + o_p\left(\frac{1}{s^{1-d}}\right).$$

It follows that if $\frac{s}{n} + \frac{n^\alpha}{s} \to 0$ as $n \to \infty$, for some $\alpha \in \left(\frac{1}{2}, 1\right)$, then

$$w_x(\lambda_s) = \left(1 - e^{i\lambda_s}\right)^{-d} w_u(\lambda_s) - \frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{X_n}{\sqrt{2\pi n}} + o_p\left(\frac{\left(1 - e^{i\lambda_s}\right)^{-d}}{s^{1-d}}\right). \tag{4.7}$$

Observe that the first two terms of (4.6) and (4.7) are the same. Although the order of magnitude of the error differs in the two cases, we may write

$$w_x(\lambda_s) = \left(1 - e^{i\lambda_s}\right)^{-d} w_u(\lambda_s) - \frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{X_n}{\sqrt{2\pi n}} + o_p\left(\frac{e^{i\lambda_s}}{(1 - e^{i\lambda_s})} \frac{X_n}{\sqrt{n}}\right) \tag{4.8}$$

for both these cases, and (4.8) is valid for all $\lambda_s = \frac{2\pi s}{n}$ with $\frac{n^\alpha}{s} \to 0$.

(c) Case $\lambda_s = \frac{2\pi i s}{n} \to 0$ and s fixed

From lemma 4.2 (c) when s is fixed as $n \to \infty$, we have

$$D_n\left(e^{i\lambda_s}; d\right) = \frac{1}{\Gamma(1-d)\, n^d} \, {}_1F_1(1, 1-d; -2\pi i s) + O\left(\frac{1}{n^{1+d}}\right), \tag{4.9}$$

and it follows that

$$\begin{aligned} \frac{1}{n^d} w_x(\lambda_s) &= \frac{1}{n^d}\left[\frac{1}{\Gamma(1-d)\, n^d} \, {}_1F_1(1, 1-d; -2\pi i s) + O\left(\frac{1}{n^{1+d}}\right)\right]^{-1} \\ &\quad \times \left[w_u(\lambda_s) + \frac{1}{\sqrt{2\pi n}} \widetilde{X}_{\lambda_s n}(d)\right], \end{aligned}$$

giving

$$\frac{w_x(\lambda_s)}{n^d} = \frac{\Gamma(1-d)}{{}_1F_1(1, 1-d; -2\pi i s)} \left[w_u(\lambda_s) + \frac{1}{\sqrt{2\pi n}} \widetilde{X}_{\lambda_s n}(d)\right] + O_p\left(\frac{1}{n}\right). \tag{4.10}$$

Further, from theorem 4.3 (c),

$$\begin{aligned} \frac{\widetilde{X}_{\lambda_s n}(d)}{\sqrt{n}} &= \frac{{}_1F_1(1, 1-d; -2\pi i s)}{\Gamma(1-d)} \int_0^1 e^{2\pi i s r} X_{n,d}(r) dr \\ &\quad - \frac{1}{\Gamma(1-d)} \int_0^1 {}_1F_1(1, 1-d; -2\pi i s r)\, r^{-d} X_{n,d}(1-r) dr + O_p\left(\frac{1}{n^{1-d}}\right), \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{n^d} w_x(\lambda_s) &= \frac{\Gamma(1-d)}{{}_1F_1(1, 1-d; -2\pi i s)} w_u(\lambda_s) + \frac{1}{\sqrt{2\pi}} \int_0^1 e^{2\pi i s r} X_{n,d}(r) dr \\ &- \frac{(2\pi)^{-\frac{1}{2}}}{{}_1F_1(1, 1-d; -2\pi i s)} \int_0^1 {}_1F_1(1, 1-d; -2\pi i s r)\, r^{-d} X_{n,d}(1-r) dr \\ &+ O_p\left(\frac{1}{n^{1-d}}\right). \end{aligned} \tag{4.11}$$

Unlike (4.6) and (4.8), the term

$$\frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{X_n}{\sqrt{2\pi n}} \tag{4.12}$$

does not figure directly in (4.11). In fact, as the alternate representation in the next section shows, the term (4.12) is absorbed into the series expression in (4.11), so it is still present and figures in the leading term of the dft $w_x(\lambda_s)$ when s is fixed.

(c) Case $\lambda_s = \frac{2\pi is}{n} \to 0$ and s fixed: alternate form.

Theorem 3.7 gives

$$w_x(\lambda_s)\left(1-e^{i\lambda_s}\right) = D_n\left(e^{i\lambda_s};f\right) w_u(\lambda_s) - \frac{1}{\sqrt{2\pi n}}\widetilde{U}_{\lambda_s n}(f) - e^{i\lambda_s}\frac{X_n}{\sqrt{2\pi n}}, \tag{4.13}$$

with $f = 1-d$, lemma 4.2 (c) gives

$$D_n\left(e^{i\lambda_s};f\right) = \frac{1}{\Gamma(1-f)\, n^f}\ {}_1F_1(1, 1-f; -2\pi is) + O\left(\frac{1}{n^{1+f}}\right),$$

and theorem 4.4 gives

$$\begin{aligned}\frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}} &= \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(1-f)\, n^f}\left\{ {}_1F_1(1, 1-f; -2\pi is)\int_0^1 e^{-2\pi isr} dX_n(1-r)\right.\\ &\quad \left. - \int_0^1 r^{-f}\ {}_1F_1(1, 1-f; -2\pi isr)\, dX_n(1-r)\right\} + O_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Also,

$$w_u(\lambda_s) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^n e^{2\pi si\frac{t}{n}} u_t = \frac{1}{\sqrt{2\pi}}\sum_{k=1}^n e^{2\pi si\frac{n-k}{n}}\frac{u_{n-k}}{\sqrt{n}} = \frac{1}{\sqrt{2\pi}}\int_0^1 e^{-2\pi isr} dX_n(1-r) + O_p\left(\frac{1}{n}\right).$$

Combining these last three representations in (4.13), we get

$$\begin{aligned}& w_x(\lambda_s)\left(1-e^{i\lambda_s}\right)\\ =\ & \frac{1}{\Gamma(1-f)\, n^f}\ {}_1F_1(1, 1-f; -2\pi is)\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-2\pi isr} dX_n(1-r) + O_p\left(\frac{1}{n}\right)\\ & -\frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(1-f)\, n^f}\ {}_1F_1(1, 1-f; -2\pi is)\int_0^1 e^{-2\pi isr} dX_n(1-r)\\ & +\frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(1-f)\, n^f}\int_0^1 r^{-f}\ {}_1F_1(1, 1-f; -2\pi isr)\, dX_n(1-r) - e^{i\lambda_s}\frac{X_n}{\sqrt{2\pi n}}\\ & +O_p\left(\frac{1}{\sqrt{n}}\right)\\ =\ & \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(1-f)\, n^f}\int_0^1 r^{-f}\ {}_1F_1(1, 1-f; -2\pi isr)\, dX_n(1-r) - e^{i\lambda_s}\frac{X_n}{\sqrt{2\pi n}} + O_p\left(\frac{1}{\sqrt{n}}\right),\end{aligned}$$

leading to

$$\frac{1}{n^d}w_x(\lambda_s) = \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(1-f)\,n\left(1-e^{i\lambda_s}\right)}\int_0^1 r^{-f}\ {}_1F_1(1,1-f;-2\pi isr)\,dX_n(1-r)$$
$$-\frac{1}{\sqrt{2\pi}}\frac{e^{i\lambda_s}}{n\left(1-e^{i\lambda_s}\right)}\frac{X_n}{n^{d-\frac{1}{2}}}+O_p\left(\frac{1}{n^{d-\frac{1}{2}}}\right), \tag{4.14}$$

which shows how (4.12) continues to play a role in the leading term of $w_x(\lambda_s)$.

4.6 Limit Theory

Under (2.4), partial sums of u_t satisfy the functional law

$$X_n(r) = \frac{1}{\sqrt{n}}\sum_{t=0}^{\lfloor nr\rfloor} u_t \rightarrow_d B(r), \tag{4.15}$$

where B is Brownian motion with variance $\omega^2 = \sigma^2 C(1)^2$, e.g., Phillips and Solo (1992). There is a corresponding functional law for suitably standardized elements of the time series X_t. Akonom and Gouriéroux (1987) showed such a functional law for $n^{\frac{1}{2}-d}X_t$ when the components u_t follow a stationary ARMA process and the following simply extends their result to the linear process u_t.

4.7 Lemma *For u_t satisfying (2.4) and with ε_t iid $\left(0,\sigma^2\right)$ and $E\left|\varepsilon_t\right|^p < \infty$ for $p > \max\left(\frac{1}{d-\frac{1}{2}},2\right)$,*

$$X_{n,d}(r) = \frac{X_{\lfloor nr\rfloor}}{n^{d-\frac{1}{2}}} \xrightarrow{d} B_{d-1}(r) = \frac{1}{\Gamma(d)}\int_0^r (r-s)^{d-1}\,dB(s), \tag{4.16}$$

a fractional Brownian motion where $B(s)$ is Brownian motion with variance ω^2.

Like X_t, the fractional Brownian motion $B_{d-1}(r)$ is initialized at the origin, and therefore has nonstationary increments, in contrast to the other fractional process

$$W_H(r) = \frac{1}{C(H)}\int_{-\infty}^{\infty}\left[\left\{(r-s)_+\right\}^{H-\frac{1}{2}} - \left\{(-s)_+\right\}^{H-\frac{1}{2}}\right]dB(s), \quad H = d-\frac{1}{2}, \tag{4.17}$$

$$C(H) = \left\{\frac{1}{2H}+\int_0^{\infty}\left[(1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right]^2 ds\right\}^{\frac{1}{2}}, \quad 0<H<1$$

introduced by Mandelbrot and Van Ness (1968) and studied by Samorodnitsky and Taqqu (2017) in this form. Both processes reduce to Brownian motion for special cases of the parameters, viz. $d=1$ for (4.16), and $H=\frac{1}{2}$ for (4.17).

These functional laws enable us to get limit representations of the correction term $n^{-\frac{1}{2}}\widetilde{X}_{\lambda_s n}(d)$. The case where s is fixed as $n\to\infty$ is especially interesting, the other two cases following immediately from (4.16) and the respective expressions (4.6) and (4.7).

4.8 Lemma *For $\lambda_s = \frac{2\pi is}{n} \to 0$ and s fixed*

$$\frac{\widetilde{X}_{\lambda_s n}(d)}{\sqrt{n}} \xrightarrow{d} \frac{1}{\Gamma(1-d)} \int_0^1 e^{2\pi isr} B_{d-1}(r) dr \;{}_1F_1(1, 1-d; -2\pi is) - \int_0^1 e^{2\pi isr} dB(r). \tag{4.18}$$

The next result gives formulae for the stochastic Fourier integral $\int_0^r e^{2\pi siq} dB(q)$ that appears in (4.18) and (when $s = 0$) for the constituent Brownian motion B in terms of the fractional Brownian motion B_{d-1}.

4.9 Theorem *For fixed integer s*

$$\int_0^r e^{-2\pi si(r-q)} dB(q) = \frac{1}{\Gamma(1-d)} \int_0^r {}_1F_1(1, 1-d; -2\pi is(r-q))(r-q)^{-d} B_{d-1}(q) dq, \tag{4.19}$$

and, in the special case where $s = 0$,

$$B(r) = \frac{1}{\Gamma(1-d)} \int_0^r (r-q)^{-d} B_{d-1}(q) dq. \tag{4.20}$$

The equality (4.20) is the inverse (integral) transform of the fractional Brownian motion $B_{d-1}(r)$. In effect, the right side of (4.20) is the $(1-d)$'th fractional integral of the $(d-1)$'th fractional derivative of Brownian motion. Formula (4.19) extends this representation to the case $s \neq 0$. When $r = 1$, (4.19) becomes

$$\int_0^1 e^{2\pi siq} dB(q) = \frac{1}{\Gamma(1-d)} \int_0^1 {}_1F_1(1, 1-d; -2\pi is(1-q))(1-q)^{-d} B_{d-1}(q) dq.$$

4.10 Theorem *Suppose $d \in \left(\frac{1}{2}, 1\right)$. The following limit results apply.*

(a) *Let $\phi > 0$ and suppose $\lambda_{s_j} \in \mathcal{B}_\phi = \left\{\phi - \frac{\pi}{2M}, \phi + \frac{\pi}{2M}\right\}$ for a finite set of distinct integers s_j $(j = 1, ..., J)$. When $M \to \infty$ as $n \to \infty$, the family $\{w_x(\lambda_{s_j})\}_{j=1}^J$ are asymptotically independently distributed as complex normal $N_c(0, f_x(\phi))$ where $f_x(\phi) = \left|1 - e^{i\phi}\right|^{-2d} f_u(\phi)$.*

(b) *Let $\{s_j\}_{j=1}^J$ be distinct integers with $0 < l < s_j < L$ for each j and with $\frac{L}{n} + \frac{n^\alpha}{l} \to 0$ as $n \to \infty$, for some $\alpha \in \left(\frac{1}{2}, 1\right)$. The family $\{(\lambda_{s_j})^d w_x(\lambda_{s_j})\}_{j=1}^J$ are asymptotically independently distributed as $N_c(0, f_u(0))$.*

(c) *Let* $\{s_j\}_{j=1}^{J}$ *be a finite set of distinct positive integers which are fixed as* $n \to \infty$. *Then, for each* j

$$\frac{1}{n^d} w_x \left(\lambda_{s_j}\right) \xrightarrow{d} \frac{1}{\sqrt{2\pi}} \int_0^1 e^{2\pi i s_j r} B_{d-1}(r)\, dr, \tag{4.21}$$

where B_{d-1} *is the fractional Brownian motion given in* (4.16). *Joint convergence also applies.*

When $d = 1$, *the following limits apply.*

(d) *Let* $\phi > 0$ *and suppose* $\lambda_{s_j} \in \mathcal{B}_\phi = \left\{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\right\}$ *for a finite set of distinct integers* s_j $(j = 1, ..., J)$. *When* $M \to \infty$ *as* $n \to \infty$, *the family* $\{w_x(\lambda_{s_j})\}_{j=1}^{J}$ *are asymptotically distributed as*

$$\{\frac{1}{1 - e^{i\phi}} \xi_j - \frac{e^{i\phi}}{1 - e^{i\phi}} \eta\}_{j=1}^{J}, \tag{4.22}$$

where the $\{\xi_j\}_{j=1}^{J}$ *are iid* $N_c(0, f_u(\phi))$ *and are independent of*

$$\eta = \frac{B(1)}{\sqrt{2\pi}}, \tag{4.23}$$

where B *is Brownian motion with variance* ω^2.

(e) *Let* $\{s_j\}_{j=1}^{J}$ *be a finite set of distinct positive integers for which* $\frac{s_j}{n} \to 0$ *as* $n \to \infty$. *The family* $\{\lambda_{s_j} w_x(\lambda_{s_j})\}_{j=1}^{J}$ *are asymptotically distributed as*

$$i\left(\xi_j - \eta\right), \tag{4.24}$$

where ξ_j *and* η *are as in (4.22) and (4.23).*

(f) *When* s_j *is fixed as* $n \to \infty$, *the* ξ_j *in (e) have the representation*

$$\xi_j = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{2\pi i s_j r} dB(r), \tag{4.25}$$

and

$$\frac{1}{n} w_x \left(\lambda_{s_j}\right) \xrightarrow{d} \frac{1}{\sqrt{2\pi}} \int_0^1 e^{2\pi i s_j r} B(r)\, dr, \tag{4.26}$$

which also holds for $s_j = 0$.

Parts (a) and (d) show that Hannan (1973)'s result for the limit theory of dfts of stationary processes extends to fractional processes at frequencies removed from the origin when $d \in (\frac{1}{2}, 1)$ but not when $d = 1$. In the latter case, the leakage from the zero frequency is so substantial that it affects the limit theory of the dft at all frequencies, although the limit distribution is still normal. Moreover, as is apparent from the form of (4.22), the limit variates are spatially correlated across frequency by virtue of the presence of the random component η, through which the leakage is transmitted.

Part (b) shows that, when $d \in (\frac{1}{2}, 1)$, a version of Hannan's result applies to the scaled transforms $(\frac{s_j}{n})^d w_x(\lambda_{s_j})$ in a (distant) vicinity of the origin where $\lambda_{s_j} = \frac{2\pi s_j i}{n} \to 0$ but $\frac{n^\alpha}{s_j} \to 0$ as $n \to \infty$, for some $\alpha \in \left(\frac{1}{2}, 1\right)$. However, when $d = 1$, the scaled transforms $\frac{s_j}{n} w_x(\lambda_{s_j})$ are asymptotically dependent across frequency.

Part (c) shows that in the immediate vicinity of the origin (i.e., for $\lambda_{s_j} = \frac{2\pi s_j i}{n} \to 0$ with s_j fixed), the $n^{-d} w_x(\lambda_{s_j})$ are asymptotically dependent for $d \in (\frac{1}{2}, 1]$ and each converges weakly to an integral functional of fractional Brownian motion that involves the integer s_j. In earlier work, Akonom and Gouriéroux (1987) gave (4.21) in the case of ARMA u_t. An alternate expression for (4.21), which relates to (4.14) is

$$\frac{1}{n^d} w_x\left(\lambda_{s_j}\right) \xrightarrow{d} \frac{1}{\sqrt{2\pi}\Gamma\left(1+d\right)} \int_0^1 {}_1F_1\left(1, 1+d; -2\pi i s_j r\right) r^d dB\left(1-r\right)$$

and can be obtained from the formula

$$\int_0^1 e^{2\pi i s r} B_{d-1}\left(r\right) dr = \frac{1}{\Gamma\left(1+d\right)} \int_0^1 {}_1F_1\left(1, 1+d; -2\pi i s r\right) r^d dB\left(1-r\right),$$

which is proved in lemma E in the technical appendix.

The methods in the proof of theorem 4.10 are used in (Phillips, 2007, theorem 3.2) to extend existing theory showing the asymptotic independence of a finite collection of dfts of stationary time series (Hannan, 1973) to collections of a small (i.e., with less than sample size) infinity of dfts at Fourier frequencies.

5 Statistical Applications

5.1 Spectrum Estimation for Fractional Processes

The limit theory in Section 4.6 is useful in obtaining the asymptotic behavior of spectral estimates for fractional processes. We give some results for smoothed periodogram estimates for frequencies at the origin and away from the origin. The former are of interest in procedures that are used to estimate the memory parameter d. The latter reveal any leakage from low to high frequencies that occurs in spectrum estimation.

For frequencies away from the origin such as $\phi \neq 0$, the usual smoothed periodogram estimator of $f_x\left(\phi\right)$ is given by

$$\widehat{f}_{xx}\left(\phi\right) = \frac{1}{m} \sum_{\lambda_s \in \mathcal{B}(\phi)} w_x\left(\lambda_s\right) w_x\left(\lambda_s\right)^*, \tag{5.1}$$

where $\mathcal{B}_m\left(\phi\right) = (\phi - \frac{\pi}{2M}, \phi + \frac{\pi}{2M}]$ and M is the bandwidth parameter that determines the number of frequencies $m = \#\left\{\lambda_s \in \mathcal{B}_m\left(\phi\right)\right\} = [n/2M]$ used in the smoothing. At the zero frequency $\phi = 0$, we consider a one-sided average of m periodogram ordinates at the origin

$$\widehat{f}_{xx}(0) = \frac{1}{m} \sum_{s=0}^{m-1} w_x(\lambda_s) w_x(\lambda_s)^* . \tag{5.2}$$

The following theorem gives the asymptotic behavior of $\widehat{f}_{xx}(\phi)$ for these two cases and for $d \in (\frac{1}{2}, 1)$ and $d = 1$.

5.2 Theorem

(a) *For* $\phi \neq 0$ *and* $\frac{1}{2} < d < 1$

$$\widehat{f}_{xx}(\phi) \to_p \; f_x(\phi) = \frac{f_u(\phi)}{|1 - e^{i\phi}|^{2d}}.$$

(b) *For* $\phi \neq 0$ *and* $d = 1$

$$\widehat{f}_{xx}(\phi) \to_d \; f_x(\phi) + \frac{1}{2\pi} \left|1 - e^{i\phi}\right|^{-2} B(1)^2 .$$

(c) *For* $\frac{1}{2} < d < 1$ and m such that $\frac{m}{n^\alpha} \to \infty$ with $\alpha \geq \frac{1}{2d}$

$$\frac{m}{n^{2d}} \widehat{f}_{xx}(0) \to_d \; \frac{1}{2\pi} \int_0^1 B_{d-1}(r)^2 \, dr.$$

(d) *For* $d = 1$ and m such that $\frac{m}{\sqrt{n}} \to \infty$

$$\frac{m}{n^2} \widehat{f}_{xx}(0) \to_d \; \frac{1}{2\pi} \int_0^1 B(r)^2 \, dr.$$

According to part (a), spectral estimates like $\widehat{f}_{xx}(\phi)$ at frequencies removed from the origin are consistent for $f_x(\phi) = \left|1 - e^{i\phi}\right|^{-2d} f_u(\phi)$ provided $d < 1$. When $d = 1$, the estimate is inconsistent and converges weakly to a random quantity. In this case, the leakage from low frequency behavior is strong enough to persist in the limit at all frequencies $\phi > 0$. Part (d) was given in earlier work by Phillips (1991), where it was shown to be useful in analysing regression in the frequency domain with integrated time series. A new and simpler derivation is given here based on the decomposition (3.9). Part (c) can be expected to be useful in similar regression contexts with fractional processes.

5.3 Semiparametric Estimation of d

We indicate some potential applications of the above theory for the estimation of the memory parameter d in (2.1). This is a large subject which goes beyond the scope of the present paper and for which theoretical development was undertaken after the original version of this paper was completed in 1999. The main references will be reported in the following discussion. The presentation here focuses on the new ideas that led into these developments and not the technical details.

Concordant with the nonparametric approach, our concern is with the case where little is known about the short memory component u_t of (2.1) and its spectrum $f_u(\lambda)$ is treated nonparametrically. In both log periodogram estimation and local Whittle estimation, this is accomplished by working with the dft $w_x(\lambda_s)$ of the data X_t over a set of m Fourier frequencies $\{\lambda_s = \frac{2\pi s}{n} : s = 1, ..., m\}$ that shrink slowly to origin as the sample size $n \to \infty$ by virtue of a condition on m of the type $\frac{m}{n} \to 0$. It has been suggested that, in view of the asymptotic correlation of the ordinates in the vicinity of the origin (Künsch, 1986), it may be useful to trim this set of frequencies away from the origin and restrict attention to $\{\lambda_s = \frac{2\pi s}{n} : s = l, ..., m\}$ where l is a trimming number that satisfies $l \to \infty$ and $\frac{\sqrt{m}\log m}{l} \to 0$ (Robinson, 1995b), although it is now known that this trimming is not necessary (Hurvich et al., 1998).

From (4.7) we know that for $d \in (\frac{1}{2}, 1)$, the dft $w_x(\lambda_s)$

$$w_x(\lambda_s) = \left(1 - e^{i\lambda_s}\right)^{-d} w_u(\lambda_s) - \frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{X_n}{\sqrt{2\pi n}} + o_p\left(\frac{\left(1 - e^{i\lambda_s}\right)^{-d}}{s^{1-d}}\right), \tag{5.3}$$

when $\frac{s}{n} + \frac{n^\alpha}{s} \to 0$ as $n \to \infty$, for some $\alpha \in \left(\frac{1}{2}, 1\right)$. The asymptotic behavior of $w_x(\lambda_s)$ is dominated by the first two terms of (5.3), and as $d \to 1$ the importance of the second term in (5.3), which is $O_p(n^d/s)$, rivals that of the first term, which is $O_p(n^d/s^d)$. Apparently, therefore, it would seem desirable to correct the dft $w_x(\lambda_s)$ for the effects of leakage in semiparametric estimation of d simply by adding the correction term supplied by the known form of the expansion (5.3). For log periodogram regression this amounts to using the quantity

$$v_x(\lambda_s) = w_x(\lambda_s) + \frac{e^{i\lambda_s}}{1 - e^{i\lambda_s}} \frac{X_n}{\sqrt{2\pi n}} \tag{5.4}$$

in place of $w_x(\lambda_s)$ in the regression. Thus, in place of the usual least squares regression (over $s = 1, ..., m$)

$$\ln(I_x(\lambda_s)) = \widehat{c} - \widehat{d} \ln\left|1 - e^{i\lambda_s}\right|^2 + \text{error}$$

that is inspired by the form of the moment relation (2.6) in the frequency domain, the argument above suggests the linear least squares regression

$$\ln(I_v(\lambda_s)) = \widetilde{c} - \widetilde{d} \ln\left|1 - e^{i\lambda_s}\right|^2 + \text{error}, \tag{5.5}$$

in which the periodogram ordinates, $I_x(\lambda_s)$, are replaced by $I_v(\lambda_s) = v_x(\lambda_s) v_x(\lambda_s)^*$. We call this procedure *modified log periodogram regression.* This replacement is inspired by (5.3), which approximates the data generating process of the dft $w_x(\lambda_s)$ over the relevant set of frequencies as $m \to \infty$ in the regression. In place of the 'regression model'

$$\ln\left(I_x\left(\lambda_s\right)\right) = c - d\ln\left|1-e^{i\lambda_s}\right|^2 + u\left(\lambda_s\right),$$

with $c = \ln\left(f_u\left(0\right)\right)$ and

$$u\left(\lambda_s\right) = \ln\left[I_x\left(\lambda_s\right)/f_x\left(\lambda_s\right)\right] + \ln\left(f_u\left(\lambda_s\right)/f_u\left(0\right)\right),$$

as in (2.7), we now have from (5.3)

$$\begin{aligned}
I_v\left(\lambda_s\right) &= \left|\left(1-e^{i\lambda_s}\right)^{-d} w_u\left(\lambda_s\right) + o_p\left(\frac{n^d}{s}\right)\right|^2 \\
&= \left|1-e^{i\lambda_s}\right|^{-2d} I_u\left(\lambda_s\right)\left[1+\left(1-e^{i\lambda_s}\right)^{d} w_u\left(\lambda_s\right)^{-1} o_p\left(\frac{n^d}{s}\right)\right] \\
&= \left|1-e^{i\lambda_s}\right|^{-2d} I_u\left(\lambda_s\right)\left|\left[1+o_p\left(\frac{1}{s^{1-d}}\right)\right]\right|^2,
\end{aligned}$$

which leads to the new regression model

$$\ln\left(I_v\left(\lambda_s\right)\right) = c - d\ln\left|1-e^{i\lambda_s}\right|^2 + a\left(\lambda_s\right), \tag{5.6}$$

with

$$a\left(\lambda_s\right) = \ln\left[I_u\left(\lambda_s\right)/f_u\left(\lambda_s\right)\right] + \ln\left(f_u\left(\lambda_s\right)/f_u\left(0\right)\right) + O_p\left(\frac{1}{s^{1-d}}\right). \tag{5.7}$$

This relationship holds for frequencies λ_s satisfying $\frac{s}{n} + \frac{n^\alpha}{s} \to 0$ as $n \to \infty$, in view of (5.3).

The new regression (5.5) seems likely to be most useful in cases where nonstationarity is suspected. Note, however, that when $d < \frac{1}{2}$, the correction term in (5.4) is $o_p(1)$ when $\frac{\sqrt{n}}{s} \to 0$, so that use of (5.5) can also be expected to be satisfactory in the stationary case. When $d = 1$, the correction is exact for all frequencies, as is clear from (3.9). In that case, therefore, (5.6) is an exact regression relation whose error is given by

$$a\left(\lambda_s\right) = \ln\left[I_u\left(\lambda_s\right)/f_u\left(\lambda_s\right)\right] + \ln\left(f_u\left(\lambda_s\right)/f_u\left(0\right)\right). \tag{5.8}$$

It is then a relatively straightforward matter to show that the modified log periodogram estimator has the following limit theory

$$\sqrt{m}\left(\widetilde{d} - d\right) \xrightarrow{d} N\left(0, \frac{\pi^2}{24}\right), \tag{5.9}$$

i.e., the same limit distribution as the log periodogram estimator in the stationary case Robinson (1995b); Hurvich et al. (1998). By contrast, the usual log periodogram estimator $\widehat{d}$ has a mixed normal limit theory when $d = 1$, as shown in Phillips (1999b, 2007). The mixed normal limit arises in this case because of the presence of the term $(2\pi)^{-\frac{1}{2}} e^{i\lambda_s} n^{-\frac{1}{2}} X_n$ in (3.9) which is $O_p(1)$ and does not vanish as $n \to \infty$. Moreover, the usual log periodogram estimator $\widehat{d}$ is inconsistent and converges in probability to unity when $d \in (1, 2)$ as shown in Kim and Phillips (2006), which makes use of some of the present methods.

The modified regression (5.5) appears to be even more useful in the nonstationary case when $d > 1$. In that case, the usual estimator $\widehat{d}$ is inconsistent, and $\widehat{d} \to_p 1$, a fact that can be established using the expansions obtained in sections 2 and 3, whereas $\widetilde{d}$ is consistent and has the same limit distribution as that shown in (5.9). Further analysis of this modified log periodogram estimator, together with an empirical application to the Nelson-Plosser data (Nelson and Plosser, 1982), was given in Kim and Phillips (2003).

The intuition leading to the modified regression (5.5) can also be employed in the case of the local Whittle estimator Künsch (1987); Robinson (1995a). We will not go into details here. Suffice to remark that we would simply replace $I_v(\lambda_s; d)$ in the extremum estimation problem (5.16)-(5.18) given below by $I_v(\lambda_s)$, which can be computed from $v_x(\lambda_s)$ as in (5.4). The resulting estimator is a modified local Whittle estimator, and, like the modified log periodogram regression estimator in (5.5), its asymptotic properties can be expected to be the same for stationary and nonstationary values of the memory parameter, including those for which $d > 1$.

Our theory also suggests some other possibilities. In particular, we may build on the idea noted above that (5.6) gives an exact relationship when $d = 1$ with error (5.8). Indeed, the decomposition (3.8) implies the following exact relation between the transforms $w_x(\lambda_s)$ and $w_u(\lambda_s)$

$$w_x(\lambda_s) = D_n\left(e^{i\lambda_s}; d\right)^{-1}\left[w_u(\lambda_s) + \frac{1}{\sqrt{2\pi n}}\widetilde{X}_{\lambda n}(d)\right].$$

Define the new transform

$$v_x(\lambda_s; d) = w_x(\lambda_s) - D_n\left(e^{i\lambda_s}; d\right)^{-1}\frac{1}{\sqrt{2\pi n}}\widetilde{X}_{\lambda n}(d), \tag{5.10}$$

which is dependent on the memory parameter d and for which the equation

$$v_x(\lambda_s; d) = D_n\left(e^{i\lambda_s}; d\right)^{-1} w_u(\lambda_s) \tag{5.11}$$

holds exactly. Extending the ideas that led to (5.6) above, we have the exact periodogram relation

$$\ln\left(I_v(\lambda_s; d)\right) = c + \ln\left|D_n\left(e^{i\lambda_s}; d\right)\right|^{-2} + a(\lambda_s), \tag{5.12}$$

with $I_v(\lambda_s; d) = v_x(\lambda_s; d)\, v_x(\lambda_s; d)^*$, and

$$a(\lambda_s) = \ln\left[I_u(\lambda_s)/f_u(\lambda_s)\right] + \ln\left(f_u(\lambda_s)/f_u(0)\right),$$

just as in (5.8). In place of linear least squares regression, it is now possible to apply nonlinear regression directly to the regression model (5.12). Let $Y_s(d) = \ln\left(I_v(\lambda_s; d)\right)$, and $A_s =$

$\ln\left|D_n\left(e^{i\lambda_s};d\right)\right|^{-2}$. Then, nonlinear regression leads to the following extremum estimator

$$d^{\#} = \arg\min_d Q_m\left(d\right),$$

where

$$Q_m\left(d\right) = \frac{1}{m}\sum_{s=1}^{m}\left[\left\{Y_s\left(d\right) - \overline{Y_s\left(d\right)}\right\} - d\left\{A_s - \overline{A_s}\right\}\right]\left[\left\{Y_s\left(d\right) - \overline{Y_s\left(d\right)}\right\} - d\left\{A_s - \overline{A_s}\right\}\right]^{*},$$

and $\overline{A_s} = m^{-1}\sum_{s=1}^{m} A_s$, $\overline{Y_s(d)} = m^{-1}\sum_{s=1}^{m} Y_s(d)$ The advantage of $d^{\#}$ is that it is the natural estimator of d that emerges from the exact formulation of the regression model in the frequency domain, i.e., (5.12). Its disadvantage is that it is more complicated to compute than the conventional log periodogram regression estimator $\widehat{d}$ and the modified estimator $\widetilde{d}$, neither of which require numerical methods. Some simplifications in computation can be obtained by using some of the approximations developed in sections 3 and 4.

Finally, we remark that the exact relationship (5.11) can be used to obtain an exact form of local Whittle estimator under Gaussian assumptions about u_t. The local Whittle likelihood suggested by Künsch (1987) and studied by Robinson (1995a) has the form

$$K_m\left(G,d\right) = \frac{1}{m}\sum_{s=1}^{m}\left[\log\left(G\lambda_s^{-2d}\right) + \frac{\lambda_s^{2d}}{G}I_x\left(\lambda_s\right)\right], \tag{5.13}$$

and is minimised jointly with respect to the parameters (G,d), where $G_0 = f_u(0)$ is the true value of G. The (negative) Whittle likelihood (e.g., (Hannan and Deistler, 2012, pp. 224-225)) based on frequencies up to λ_m and up to scale multiplication is

$$\sum_{s=1}^{m}\log f_u\left(\lambda_s\right) + \sum_{s=1}^{m}\frac{I_u\left(\lambda_s\right)}{f_u\left(\lambda_s\right)}. \tag{5.14}$$

The objective function (5.13) is derived from (5.14) by using the approximate relationship

$$w_x\left(\lambda_s\right) \sim \left(1 - e^{i\lambda_s}\right)^{-d} w_u\left(\lambda_s\right) \sim \left(-\lambda_s\right)^{-d} w_u\left(\lambda_s\right),$$

or

$$I_u\left(\lambda_s\right) \sim \lambda_s^{2d} I_x\left(\lambda_s\right),$$

to transform (5.14) to be data dependent, in conjunction with the local approximation $f_u\left(\lambda_s\right) \sim G_0$. We may now proceed to transform (5.14) using the exact relationship between $w_u\left(\lambda_s\right)$ and $w_x\left(\lambda_s\right)$ that is given by (5.11) and (5.10). We get

$$\frac{1}{2}\sum_{s=1}^{m}\log\left\{\left|D_n\left(e^{i\lambda_s};d\right)\right|^{-2} f_u\left(\lambda_s\right)\right\} + \frac{1}{2}\sum_{s=1}^{m}\frac{\left|D_n\left(e^{i\lambda_s};d\right)\right|^{2} I_v\left(\lambda_s;d\right)}{f_u\left(\lambda_s\right)},$$

and this leads directly to the following 'exact' version of the local Whittle likelihood

$$L_m(G,d) = \frac{1}{m}\sum_{s=1}^{m}\left[\log\left(\left|D_n\left(e^{i\lambda_s};d\right)\right|^{-2}G\right) + \frac{\left|D_n\left(e^{i\lambda_s};d\right)\right|^2}{G} I_v\left(\lambda_s;d\right)\right]. \tag{5.15}$$

The new estimates are obtained from the joint minimization

$$\left(G^{**},d^{**}\right) = \arg\min_{d,G} L_m(G,d).$$

Concentrating out G, we find that d^{**} satisfies

$$d^{**} = \arg\min_{d} R_m(d), \tag{5.16}$$

with

$$R_m(d) = \log G^{**}(d) - 2\frac{1}{m}\sum_{j=1}^{m}\log\left|D_n\left(e^{i\lambda_s};d\right)\right|, \tag{5.17}$$

where

$$G^{**}(d) = \frac{1}{m}\sum_{j=1}^{m}\left|D_n\left(e^{i\lambda_s};d\right)\right|^2 I_v\left(\lambda_s;d\right). \tag{5.18}$$

The estimator d^{**} would seem to offer an attractive semiparametric procedure because it is based on likelihood principles and involves the exact data generating mechanism for the discrete Fourier transforms. This procedure is more computationally intensive than the usual Whittle estimator but no impediment to practical use. A full analytic investigation of the exact local Whittle estimator was conducted and reported in Shimotsu and Phillips (2005) showing that the same asymptotic properties of the local Whittle estimator apply to the exact local Whittle estimator over a full range of stationary and nonstationary values of the memory parameter d. This approach enables consistent estimation of d and the construction of valid confidence intervals for d for both stationary and nonstationary long memory time series. The procedure has proved popular in empirical research. Further work on nonstationarity-extended Whittle estimation has been done by Abadir et al. (2007) and Shao (2010).

5.4 Final Remarks

Fractional processes conveniently embody in a single memory parameter d an index that measures the extent of long range dependence in an observed time series. When a nonparametric formulation is employed for the innovations that drive the observed process, a great deal of model generality is achieved. Integer values of d include integrated processes and the special value $d = \frac{1}{2}$ provides a simple boundary between stationary and nonstationary cases. This flexibility has enabled a fundamental extension of the simple ARIMA models popularized in the 1970s wherein variate differencing became a common method of dealing with nonstationarity. The flexibility of long memory also enriched the concept of cointegration by allowing for fractional possibilities in long run equilibrium errors, thereby narrowing the differential (between variables and errors) that distinguishes a cointegrating relationship among observable integrated time series. In view of this generality, semiparametric methods and frequency domain methods such as those used in the present work have been found to be very useful in estimation, inference, and asymptotic analysis of long memory systems.

In spite of the generality that long range dependence brings to empirical analysis, it is worth remembering that some important cases are not included in its orbit. Explosive and mildly explosive time series are prime examples that have particular relevance in economics and finance where exuberance and speculation are not uncommon in real estate and financial asset markets. A simple autoregressive time series with an explosive root is not rendered stationary by differencing or fractional differencing, just as differentiating an exponential function produces a derivative that simply reproduces the exponential. Parameterizations of nonstationarity using simple autoregressive coefficients and the tests that are so enabled by such formulations therefore offer possibilities that are not encompassed in the notion of long range dependence. While autoregressions and long memory systems provide a dual parametric source of unit root dynamics, these parameterizations deliver alternative departures from unit roots that help enrich our capacity to model different types of nonstationary time series behavior and evolution.

6 Technical Appendix and Proofs

6.1 Preliminary Results

We provide some technical lemmas that are useful throughout the paper. Lemmas A and B provide results on binomial coefficients and hypergeometric functions that are either standard (e.g., Erdélyi (1953)) or follow from standard results. We give them here to facilitate our own derivations and to make the paper more accessible. Lemmas C and D provide some more specific results on sinusoidal polynomials and hypergeometric functions of sinusoids that are immediately relevant to formulae in the paper. Lemma E gives a useful inverse transform of fractional Brownian motion, an inverse transform for a hypergeometric series of fractional Brownian motion and some useful relationships between certain integral functionals of fractional Brownian motion and Brownian motion. Lemma F provides a new asymptotic expansion for hypergeometric series that allows for increasing coefficients as well as an argument that tends to unity. The expansion should be useful in other work with hypergeometric series.

Lemma A

(a) $\binom{d}{k} = (-1)^k \frac{(-d)_k}{k!}$.

(b) $(p+a)_j = \frac{(j+a)_p (a)_j}{(a)_p}$, $(a)_{j+k} = (a)_j (a+j)_k$.

(c) $\sum_{k=0}^{n} \frac{(-d)_k}{k!} = \frac{(1-d)_n}{n!} \mathbf{1}\,(d \neq 0, 1, ..) + \sum_{k=0}^{d} \frac{(-d)_k}{k!} \mathbf{1}(d = 0, 1, ..)$.

(d) $\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} = n^{\alpha-\beta}\left[1+O\left(\frac{1}{n}\right)\right].$

Proof Part (a) is immediate from the definition

$$\binom{d}{k} = \frac{d!}{(d-k)!k!} = \frac{d\,(d-1)\ldots(d-k+1)}{k!} = (-1)^k\,\frac{(-d)\ldots(-d+k-1)}{k!} = (-1)^k\,\frac{(-d)_k}{k!}.$$

The second formula in Part (b) is immediate from the definition of the forward factorial. The first formula in Part (b) follows from

$$\begin{aligned}(p+a)_j &= \frac{\Gamma(p+a+j)}{\Gamma(p+a)} = \frac{\Gamma(p+a+j)}{\Gamma(j+a)}\frac{\Gamma(j+a)/\Gamma(a)}{\Gamma(p+a)/\Gamma(a)} \\ &= (j+a)_p\,\frac{(a)_j}{(a)_p}.\end{aligned}$$

For part (c), we write the sum as a terminating hypergeometric function, and use lemma B (a) & (c) to obtain

$$\begin{aligned}\sum_{k=0}^{n}\frac{(-d)_k}{k!} &= \frac{(-d)_n}{n!}\,{}_2F_1(-n,1;d-n+1;1) \\ &= \frac{(-d)_n}{n!}\frac{\Gamma(d)\,\Gamma(d-n+1)}{\Gamma(d+1)\,\Gamma(d-n)} = \frac{\Gamma(-d+n)}{\Gamma(-d)\,n!}\frac{d-n}{d} \\ &= \frac{\Gamma(-d+n+1)}{\Gamma(-d+1)\,n!} = \frac{(1-d)_n}{n!},\end{aligned}$$

for $d \neq 0,1,2,..$, while for $d = 0,1,..$ the sum $\sum_{k=0}^{n}\frac{(-d)_k}{k!}$ simply terminates at $k = d$.

Part (d) is a standard result that follows from the Stirling approximation, e.g., Erdélyi (1953, p. 47).

Lemma B *In the following formulae,* ${}_2F_1(a,b,c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k$ *is the hypergeometric function.*

(a) $\sum_{k=0}^{n}\frac{(-d)_k}{k!}z^k = \frac{(-d)_n}{n!}z^n\,{}_2F_1\left(-n,1;d-n+1;z^{-1}\right)\mathbf{1}\,(d\neq 0,1,..) + \sum_{k=0}^{d}\frac{(-d)_k z^k}{k!}\mathbf{1}(d = 0,1,..)$.

(b) $\sum_{t=m+1}^{\infty}\frac{(-d)_t}{t!}z^t = z^{m+1}\frac{(-d)_{m+1}}{(m+1)!}\,{}_2F_1(m+1-d,1;m+2;z)$.

(c) ${}_2F_1(a,b,c;1) = \Gamma(c)\,\Gamma(c-a-b)\,/\,[\Gamma(c-a)\,\Gamma(c-b)]$ *for* $\mathrm{Re}\,(c-a-b) > 0$ and $c \neq 0,-1,-2,....$.

(d) *If* $|z| < 1$ and $|z/(z-1)| < 1$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1(a,c-b;c;z/(z-1)), \tag{6.1}$$

the right hand side giving an analytic continuation of the hypergeometric function to the half-plane $\mathrm{Re}(z) < \frac{1}{2}$.

(e) $\sum_{k=0}^{n} \frac{(-d)_k \left(e^{-i\lambda}\right)^k}{k!} = \frac{(1-d)_n e^{-i\lambda n}}{n!}\ {}_2F_1(-n,1;1-d;1-e^{i\lambda})\mathbf{1}\,(d \neq 0,1,..)+\sum_{k=0}^{d} \frac{(-d)_k e^{-i\lambda k}}{k!}\mathbf{1}(d = 0,1,..)$.

(f) *If* $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \qquad (6.2)$$

which gives an analytic continuation of ${}_2F_1(a,b;c;z)$ *to the entire* z *plane cut along* $[1,\infty]$, *i.e. to all* z *for which* $\arg(1-z) < \pi$.

Proof Part (a) is given in Erdélyi (1953, pp. 87,101) in terms of binomial coefficients. Using the form given there and lemma A (a), we have for $d \neq 0, 1, ...$

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-d)_k}{k!} z^k &= \sum_{k=0}^{n} \binom{d}{k} (-z)^k \\ &= \binom{d}{n} (-z)^n\ {}_2F_1\left(-n,1;d-n+1;z^{-1}\right) \\ &= \frac{(-d)_n}{n!} z^n\ {}_2F_1\left(-n,1;d-n+1;z^{-1}\right). \end{aligned}$$

When $d = 0, 1, ..$ the sum simply terminates at $k = d$ and the stated result follows.

For part (b) we have

$$\begin{aligned} \sum_{k=m+1}^{\infty} \frac{(-d)_k}{k!} x^k &= x^{m+1} \sum_{k=0}^{\infty} \frac{(-d)_{m+1+k}}{(m+1+k)!} x^k \\ &= x^{m+1} \sum_{k=0}^{\infty} \frac{\Gamma(m+1+k-d)}{\Gamma(-d)\,\Gamma(m+2+k)} x^k \\ &= x^{m+1} \sum_{k=0}^{\infty} \frac{(m+1-d)_k}{(m+2)_k} \frac{\Gamma(m+1-d)}{\Gamma(-d)\,\Gamma(m+2)} x^k \\ &= x^{m+1} \frac{\Gamma(m+1-d)}{\Gamma(-d)\,\Gamma(m+2)} \sum_{k=0}^{\infty} \frac{(m+1-d)_k}{(m+2)_k\, k!} x^k \\ &= x^{m+1} \frac{\Gamma(m+1-d)}{\Gamma(-d)\,\Gamma(m+2)}\ {}_2F_1(m+1-d,1;m+2;x) \qquad (6.3) \\ &= x^{m+1} \frac{(-d)_{m+1}}{(m+1)!}\ {}_2F_1(m+1-d,1;m+2;x). \end{aligned}$$

The hypergeometric function ${}_2F_1(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k$ is absolutely convergent for all $|z| \leq 1$ when $\mathrm{Re}(a+b-c) < 0$ (Erdélyi, 1953, p. 57). Hence, the series in (6.3) converges absolutely for all $|z| \leq 1$ when $d > 0$.

Part (c) is a well known summation formula (Erdélyi, 1953, p. 61). Part (d) is Euler's formula (Erdélyi, 1953, pp. 64, 105). The series for ${}_2F_1(a, b, c; z)$ converges absolutely for all $|z| < 1$ and converges absolutely for $|z| = 1$ when $\mathrm{Re}\,(c - a - b) > 0$ (Erdélyi, 1953, p. 57). The series for ${}_2F_1(a, c - b; c; z/(z-1))$ converges for $|z/(z-1)| < 1$. Since the latter inequality holds for all z for which $\mathrm{Re}(z) < \frac{1}{2}$, it follows that the right side of (6.1) gives the analytic continuation of ${}_2F_1(a, b; c; z)$ to the half plane $\mathrm{Re}(z) < \frac{1}{2}$ (Erdélyi, 1953, p. 64).

Part (e) is obtained by direct calculation. Using (a), we proceed as follows for the case $d \neq 0, 1, \ldots$:

$$\sum_{k=0}^{n} \frac{(-d)_k \left(e^{-i\lambda}\right)^k}{k!} = \frac{(-d)_n}{n!} \left(e^{-i\lambda}\right)^n \ {}_2F_1\left(-n, 1; d - n + 1; e^{i\lambda}\right) \tag{6.4}$$

$$= \frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{j=0}^{n} \frac{(-n)_j \, (1)_j \left[1 + \left(e^{i\lambda} - 1\right)\right]^j}{j! \, (d - n + 1)_j}$$

$$= \frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{(d - n + 1)_j} \sum_{q=0}^{j} \binom{j}{q} \left(e^{i\lambda} - 1\right)^q$$

$$= \frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{(d - n + 1)_j} \sum_{q=0}^{j} \frac{j!}{(j-q)!q!} \left(e^{i\lambda} - 1\right)^q$$

$$= \frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{q=0}^{n} \frac{1}{q!} \left(e^{i\lambda} - 1\right)^q \sum_{j=q}^{n} \frac{(-n)_j \, j!}{(d - n + 1)_j \, (j - q)!}$$

$$= \frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{q=0}^{n} \frac{1}{q!} \left(e^{i\lambda} - 1\right)^q \sum_{s=0}^{n-q} \frac{(-n)_{s+q} \, (s+q)!}{(d - n + 1)_{s+q} \, s!}. \tag{6.5}$$

Since $(-n)_{q+s} = (-n)_q \, (-n+q)_s$, and $(d - n + 1)_{s+q} = (d - n + 1)_q \, (d - n + 1 + q)_s$ from lemma A (b), (6.5) becomes

$$\frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{q=0}^{n} \frac{(-n)_q}{(d - n + 1)_q} \left(e^{i\lambda} - 1\right)^q \sum_{s=0}^{n-q} \frac{(q - n)_s \, (q+1)_s}{(d - n + 1 + q)_s \, s!}$$

$$= \frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{q=0}^{n} \frac{(-n)_q}{(d - n + 1)_q} \left(e^{i\lambda} - 1\right)^q \ {}_2F_1\left(q - n, q + 1; q - n + d + 1; 1\right). \tag{6.6}$$

In this expression, the ${}_2F_1$ series terminates, so lemma B (c) holds and (6.6) sums to

$$\frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{q=0}^{n} \frac{(-n)_q}{(d - n + 1)_q} \left(e^{i\lambda} - 1\right)^q \frac{\Gamma\left(q - n + d + 1\right) \Gamma\left(d - q\right)}{\Gamma\left(d + 1\right) \Gamma\left(d - n\right)}$$

$$= \frac{(-d)_n \, e^{-i\lambda n}}{n!} \sum_{q=0}^{n} \frac{(-n)_q \, (1)_q}{q!} \left(e^{i\lambda} - 1\right)^q \frac{\Gamma\left(d - n + 1\right) \Gamma\left(d - q\right)}{\Gamma\left(d + 1\right) \Gamma\left(-n + d\right)}$$

$$
\begin{aligned}
&= \frac{d-n}{d}\frac{(-d)_n e^{-i\lambda n}}{n!}\sum_{q=0}^{n}\frac{(-n)_q(1)_q}{q!}\left(e^{i\lambda}-1\right)^q\frac{\Gamma(d-q)}{\Gamma(d)} \\
&= \frac{(1-d)_n e^{-i\lambda n}}{n!}\sum_{q=0}^{n}\frac{(-n)_q(1)_q}{q!}\left(e^{i\lambda}-1\right)^q\frac{1}{(d-1)(d-2)\ldots(d-q)} \\
&= \frac{(1-d)_n e^{-i\lambda n}}{n!}\sum_{q=0}^{n}\frac{(-n)_q(1)_q}{q!}\left(e^{i\lambda}-1\right)^q\frac{(-1)^q}{(1-d)_q} \\
&= \frac{(1-d)_n e^{-i\lambda n}}{n!}\,{}_2F_1(-n,1,1-d;1-e^{i\lambda}),
\end{aligned}
$$

giving the stated result for the case $d \neq 0, 1, \ldots$. The result for $d = 0, 1, ..$ follows immediately because the series terminates at $k = d$. An alternative and more direct proof of the result makes use in (6.4) of the fact that

$$
{}_2F_1\left(-n,1;d-n+1;e^{i\lambda}\right) = \frac{(d-n)_n}{(d-n+1)_n}\,{}_2F_1\left(-n,1;1-d;1-e^{i\lambda}\right) \tag{6.7}
$$

employing the linear transformation formula ${}_2F_1(-m,b;c;z) = \frac{(c-b)_m}{(c)_m}\,{}_2F_1(-m,b;b-c-m;1-z)$ for terminating hypergeometric series - see Olver et al. (2010, Formula 15.8.7,page 390).

Part (f) is a standard result (Erdélyi, 1953, p. 59).

Lemma C *Assume* $d \neq 0, 1, ..$. Then:

(a) *For fixed* $\lambda \neq 0$ *as* $n \to \infty$

$$
\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}e^{i\lambda k} = O\left(\frac{1}{n^{1+d}}\right).
$$

(b) *For* $\lambda_s = \frac{2\pi s}{n} \to 0$ *and* $s \to \infty$ *as* $n \to \infty$

$$
\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}e^{i\lambda_s k} = -\frac{1}{2\pi i}\frac{1}{\Gamma(-d)\,n^d s}\left[1+O\left(\frac{1}{s}\right)\right]+O\left(\frac{1}{n^{1+d}}\right).
$$

(c) *For* $\lambda_s = \frac{2\pi s}{n} \to 0$ *and* s *fixed as* $n \to \infty$

$$
\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}e^{i\lambda_s k} = O\left(\frac{1}{n^d}\right).
$$

Proof Using lemma B (b), lemma A (d) and lemma F (b), given below, we get

$$
\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}e^{i\lambda k}
$$

$$
\begin{aligned}
&= \left(e^{i\lambda}\right)^{n+1} \frac{\Gamma(n+1-d)}{\Gamma(-d)\,\Gamma(n+2)}\ {}_2F_1\left(n+1-d,1;n+2,e^{i\lambda}\right) && (6.8)\\
&= e^{i\lambda(n+1)}\frac{1}{\Gamma(-d)\,n^{1+d}}\left[1+O\left(\frac{1}{n}\right)\right]\frac{1}{1-e^{i\lambda}}\left[1+O\left(\frac{1}{n}\right)\right]=O\left(\frac{1}{n^{1+d}}\right),
\end{aligned}
$$

giving part (a). For $\lambda=\lambda_s=\frac{2\pi s}{n}\to 0$ and $s\to\infty$ as $n\to\infty$ we have, using lemma F (a),

$$
\begin{aligned}
&\left(e^{i\lambda_s}\right)^{n+1}\frac{\Gamma(n+1-d)}{\Gamma(-d)\,\Gamma(n+2)}\ {}_2F_1\left(n+1-d,1;n+2;e^{i\lambda_s}\right)\\
&= \frac{e^{i\lambda_s}}{\Gamma(-d)\,n^{1+d}}\left[1+O\left(\frac{1}{n}\right)\right]\ {}_2F_1\left(n+1-d,1;n+2;e^{i\lambda_s}\right)\\
&= \frac{e^{i\lambda_s}}{\Gamma(-d)\,n^{1+d}}\left\{\frac{1}{1-e^{i\lambda_s}}\sum_{j=0}^{k-1}\frac{(1+d)_j\,(1)_j}{j!}\left(\frac{1}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right)^j\right.\\
&\quad\left.+O\left(\left(\frac{1}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right)^k\right)\right\}\left[1+O\left(\frac{1}{n}\right)\right]\\
&= \frac{1}{\Gamma(-d)\,n^{1+d}}\frac{e^{i\lambda_s}}{1-e^{i\lambda_s}}\left[1+O\left(\frac{1}{s}\right)+O\left(\frac{1}{n}\right)\right]\\
&= \frac{1}{\Gamma(-d)\,n^{1+d}}\frac{e^{i\lambda_s}}{1-e^{i\lambda_s}}\left[1+O\left(\frac{1}{s}\right)\right]+O\left(\frac{1}{n^{1+d}}\right)\\
&= -\frac{1}{\Gamma(-d)\,n^{d}}\frac{\left[1+O\left(\frac{s}{n}\right)\right]}{2\pi si}\left[1+O\left(\frac{1}{s}\right)\right]+O\left(\frac{1}{n^{1+d}}\right)\\
&= -\frac{1}{2\pi i}\frac{1}{\Gamma(-d)\,n^{d}s}\left[1+O\left(\frac{1}{s}\right)\right]+O\left(\frac{1}{n^{1+d}}\right), && (6.9)
\end{aligned}
$$

giving part (b). Finally, for s fixed as $n\to\infty$, we have

$$
\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}\left(e^{i\lambda_s}\right)^k=O\left(\sum_{k=n+1}^{\infty}\frac{1}{k^{1+d}}\right)=O\left(\frac{1}{n^d}\right),
$$

giving part (c).

Lemma D *Assume* $d\neq 1,2,\ldots$, let $r\in(0,1)$ and let $\lambda_s=\frac{2\pi s}{n}\to 0$ *with* s *fixed as* $n\to\infty$. *Then:*

$$
{}_2F_1(-\lfloor nr\rfloor,1,1-d;1-e^{i\lambda_s})= {}_1F_1\left(1,1-d;2\pi isr\right)+O\left(n^{-1}\right), \tag{6.10}
$$

$$
{}_2F_1(-\lfloor nr\rfloor,1,1-d;e^{-i\lambda_s}-1)= {}_1F_1\left(1,1-d;2\pi isr\right)+O\left(n^{-1}\right), \tag{6.11}
$$

and for nonnegative integer $p\leq n$

$$
{}_2F_1(-p,1,1-d;1-e^{-i\lambda_s})= {}_1F_1\left(1,1-d;-2\pi is\frac{p}{n}\right)+O\left(p^{-1}\right). \tag{6.12}
$$

Proof The same argument gives both results (6.10) and (6.11). We prove (6.10).

$$\begin{aligned}
& {}_2F_1(-\lfloor nr \rfloor, 1, 1-d; 1-e^{i\lambda_s}) \\
= & \sum_{j=0}^{\lfloor nr \rfloor} \frac{(-\lfloor nr \rfloor)_j}{(1-d)_j} \left(-\frac{2\pi i s}{n} + O\left(n^{-2}\right)\right)^j \\
= & \sum_{j=0}^{\lfloor nr \rfloor} \frac{(1)_j \frac{(-\lfloor nr \rfloor)_j}{(-\lfloor nr \rfloor)^j}}{(1-d)_j \, j!} \left(2\pi i s r + O\left(n^{-1}\right)\right)^j \\
= & \sum_{j=0}^{\infty} \frac{(1)_j}{(1-d)_j \, j!} (2\pi i s r)^j + O\left(n^{-1}\right) - \sum_{j=\lfloor nr \rfloor+1}^{\infty} \frac{(1)_j}{(1-d)_j \, j!} (2\pi i s r)^j \qquad (6.13) \\
= & \; {}_1F_1\left(1, 1-d; 2\pi i s r\right) + O\left(n^{-1}\right),
\end{aligned}$$

because

$$\begin{aligned}
\sum_{j=N+1}^{\infty} \frac{(1)_j \, x^j}{(1-d)_j \, j!} &= x^{N+1} \sum_{k=0}^{\infty} \frac{x^k}{(1-d)_{k+N+1}} \\
&= \frac{x^{N+1}}{\Gamma(1-d)^{-1}} \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2+N-d)} \\
&= \frac{x^{N+1}\Gamma(1-d)}{\Gamma(N+2-d)} \sum_{k=0}^{\infty} \frac{x^k (1)_k}{(2+N-d)_k \, k!} \\
&= \frac{x^{N+1}\Gamma(1-d)\, e^{N+1-d}}{\sqrt{2\pi}\,(N+2-d)^{N+1-d}} \left[\sum_{k=0}^{\infty} \frac{x^k (1)_k}{(2+N-d)_k \, k!}\right] \left[1 + O\left(\frac{1}{N}\right)\right] \\
&= O\left(\frac{1}{N^{N-\delta}}\right),
\end{aligned}$$

for all $\delta > 0$ and all finite x. Line (6.13) above follows because, for $1 \leq j \leq \lfloor nr \rfloor$,

$$\left|1 - \frac{(-\lfloor nr \rfloor)_j}{(-\lfloor nr \rfloor)^j}\right| = \left|1 - (1)\left(1 - \frac{1}{\lfloor nr \rfloor}\right) \ldots \left(1 - \frac{j-1}{\lfloor nr \rfloor}\right)\right| \leq \left|1 - \left(1 - \frac{j-1}{\lfloor nr \rfloor}\right)^j\right| = O\left(\frac{j^2}{\lfloor nr \rfloor}\right),$$

and

$$\begin{aligned}
& \frac{1}{n} \sum_{j=0}^{\lfloor nr \rfloor} \frac{(1)_j \, j^2}{(1-d)_j \, j!} (2\pi i s r)^j \qquad (6.14) \\
= & \; O\left(\frac{1}{n} \sum_{j=0}^{\lfloor nr \rfloor} \frac{(3)_j}{(1-d)_j \, j!} (2\pi i s r)^j\right) \\
= & \; O\left(\frac{1}{n} \, {}_1F_1\left(3, 1-d, 2\pi i s r\right)\right) = O\left(\frac{1}{n}\right),
\end{aligned}$$

since the ${}_1F_1$ function is everywhere convergent.

Next, for (6.12) we have

$$
\begin{aligned}
& {}_2F_1(-p,1,1-d;1-e^{-i\lambda_s}) \\
= & \sum_{j=0}^{p} \frac{(-p)_j}{(1-d)_j}\left(\frac{2\pi i s}{n}+O\left(n^{-2}\right)\right)^j \\
= & 1+\frac{(-p)}{1-d}\left(\frac{2\pi i s}{n}+O\left(n^{-2}\right)\right)+\frac{(-p)(-p+1)}{(1-d)_2}\left(\frac{2\pi i s}{n}+O\left(n^{-2}\right)\right)^2 \\
& +\ldots+\frac{(-p)_p}{(1-d)_p}\left(\frac{2\pi i s}{n}+O\left(n^{-2}\right)\right)^p \\
= & 1+\frac{(-1)}{1-d}\left(2\pi i s r+O\left(n^{-1}\right)\right)+\frac{(-1)\left(-1+O\left(p^{-1}\right)\right)}{(1-d)_2}\left(2\pi i s\frac{p}{n}+O\left(n^{-1}\right)\right)^2 \\
& +\ldots+\frac{\left(-1+O\left(p^{-1}\right)\right)^p}{(1-d)_p}\left(2\pi i s\frac{p}{n}+O\left(n^{-1}\right)\right)^p \\
= & \sum_{j=0}^{p} \frac{(1)_j\left(1+O\left(\frac{j}{p}\right)\right)^j}{(1-d)_j\, j!}\left(-2\pi i s\frac{p}{n}+O\left(n^{-1}\right)\right)^j \\
= & \sum_{j=0}^{\infty} \frac{(1)_j}{(1-d)_j\, j!}\left(-2\pi i s\frac{p}{n}\right)^j+O\left(p^{-1}\right)++O\left(n^{-1}\right) \qquad (6.15) \\
& -\sum_{j=p+1}^{\infty} \frac{(1)_j}{(1-d)_j\, j!}\left(-2\pi i s\frac{p}{n}\right)^j \\
= & {}_1F_1\left(1,1-d;-2\pi i s\frac{p}{n}\right)+O\left(n^{-1}\right)+O\left(p^{-1}\right),
\end{aligned}
$$

giving (6.12). Again, line (6.15) above follows because

$$
\frac{1}{p}\sum_{j=0}^{p}\frac{(1)_j\, j^2}{(1-d)_j\, j!}\left(-2\pi i s\frac{p}{n}\right)^j = O\left(\frac{1}{p}\sum_{j=0}^{p}\frac{(3)_j}{(1-d)_j\, j!}\left(-2\pi i s\frac{p}{n}\right)^j\right) = O\left(\frac{1}{p}\right).
$$

Lemma E

(a) *For* $j=1,2,\ldots$

$$
\Gamma\left(j+1-d\right)^{-1}\int_0^r (r-s)^{j-d} B_{d-1}(s)\, ds = \Gamma\left(j\right)^{-1}\int_{q=0}^r (r-q)^{j-1} B(q)dq
$$

and for $j=0,1,2,\ldots$

$$
\Gamma\left(j+1-d\right)^{-1}\int_0^r (r-s)^{j-d} B_{d-1}(s)\, ds = \Gamma\left(j+1\right)^{-1}\int_{q=0}^r (r-q)^{j}\, dB(q).
$$

(b)

$$\Gamma\left(1-d\right)^{-1}\int_{0}^{r}{}_{1}F_{1}\left(1,1-d;-2\pi is\left(r-q\right)\right)\left(r-q\right)^{-d}B_{d-1}\left(q\right)dq=\int_{q=0}^{r}e^{-2\pi is(r-q)}dB\left(q\right).$$

(c)

$$\begin{aligned}&\frac{1}{\Gamma\left(1+d\right)}\int_{0}^{1}{}_{1}F_{1}\left(1,1+d;-2\pi isr\right)r^{d}dB\left(1-r\right)\\=&\ \frac{1}{\Gamma\left(1-f\right)\left(-2\pi si\right)}\int_{0}^{1}r^{-f}\ {}_{1}F_{1}\left(1,1-f;-2\pi isr\right)dB\left(1-r\right)+\frac{1}{\left(-2\pi si\right)}B_{d-1}\left(1\right).\end{aligned}$$

(d)

$$\frac{1}{\Gamma\left(1+d\right)}\int_{0}^{1}{}_{1}F_{1}\left(1,1+d;-2\pi isr\right)r^{d}dB\left(1-r\right)=\int_{0}^{1}e^{2\pi isr}B_{d-1}\left(r\right)dr.$$

In the above formulae, B is Brownian motion with variance ω^2 and $B_{d-1}(r)=\frac{1}{\Gamma(d)}\int_0^r(r-s)^{d-1}dB(s)$ is a fractional Brownian motion initialized at the origin, as in lemma 3.4.

Proof To prove part (a) we use an operator approach with $D=\frac{d}{dx}$ and allow for fractional powers of D with a Weyl integral interpretation (see Lovoie et al. (1976) and Phillips (1986a) for the approach used here). The operator e^{qD} is treated at the translation operator, so that $e^{qD}f(x)=f(x+q)$. Setting $B_{d-1}\left(s\right)=0$ for all $s\leq 0$ we have

$$\begin{aligned}&\frac{1}{\Gamma\left(j+1-d\right)}\int_{0}^{r}\left(r-s\right)^{j-d}B_{d-1}\left(s\right)ds=\frac{1}{\Gamma\left(j+1-d\right)}\int_{q=0}^{\infty}q^{j-d}B_{d-1}\left(r-q\right)dq\\&=\frac{1}{\Gamma\left(j+1-d\right)}\int_{q=0}^{\infty}q^{j-d}e^{-qD}B_{d-1}\left(r\right)dq=D^{d-j-1}B_{d-1}\left(x\right)|_{x=r}\\&=D^{d-j-1}D^{1-d}B\left(x\right)|_{x=r}=D^{-j}B\left(x\right)|_{x=r}\\&=\Gamma\left(j\right)^{-1}\int_{q=0}^{r}q^{j-1}B\left(r-q\right)dq=\Gamma\left(j\right)^{-1}\int_{q=0}^{r}\left(r-q\right)^{j-1}B\left(q\right)dq,\end{aligned}\tag{6.16}$$

giving the first of the stated results and, consequently,

$$\int_{0}^{r}\left(r-s\right)^{j-d}B_{d-1}\left(s\right)ds=\frac{\Gamma\left(1-d\right)\left(1-d\right)_{j}}{\Gamma\left(j\right)}\int_{q=0}^{r}\left(r-q\right)^{j-1}B\left(q\right)dq.$$

To obtain the second form of the result we use integration by parts to give

$$\begin{aligned}\Gamma\left(j\right)^{-1}\int_{q=0}^{r}\left(r-q\right)^{j-1}B\left(q\right)dq&=j^{-1}\Gamma\left(j\right)^{-1}\int_{q=0}^{r}\left(r-q\right)^{j}dB(q)\\&=\Gamma\left(j+1\right)^{-1}\int_{q=0}^{r}\left(r-q\right)^{j}dB(q).\end{aligned}\tag{6.17}$$

Combining (6.16) and (6.17), we have

$$\frac{1}{\Gamma\left(j+1-d\right)}\int_0^r \left(r-s\right)^{j-d} B_{d-1}\left(s\right)ds = \Gamma\left(j+1\right)^{-1}\int_{q=0}^r \left(r-q\right)^j dB(q)$$

which holds also when $j=0$, giving the inverse relation

$$\frac{1}{\Gamma\left(1-d\right)}\int_0^r \left(r-s\right)^{-d} B_{d-1}\left(s\right)ds = B(r), \tag{6.18}$$

(see theorem 4.9). An alternate weak convergence proof of (6.18) is given in the proof of theorem 4.9 below and, from this result, (6.17) can alternatively be obtained by subsequent integration.

To prove part (b) we proceed as follows:

$$\begin{aligned}
&\frac{1}{\Gamma\left(1-d\right)}\int_0^r {}_1F_1\left(1,1-d;-2\pi is\left(r-q\right)\right)\left(r-q\right)^{-d} B_{d-1}\left(q\right)dq \\
=\ &\frac{1}{\Gamma\left(1-d\right)}\sum_{j=0}^{\infty}\frac{\left(1\right)_j\left(-1\right)^j}{\left(1-d\right)_j j!}\int_0^r \left(2\pi is\left(r-q\right)\right)^j\left(r-q\right)^{-d} B_{d-1}\left(q\right)dq \\
=\ &\frac{1}{\Gamma\left(1-d\right)}\sum_{j=0}^{\infty}\frac{\left(-2\pi is\right)^j}{\left(1-d\right)_j}\int_0^r \left(r-q\right)^{j-d} B_{d-1}\left(q\right)dq \\
=\ &\sum_{j=0}^{\infty}\frac{\left(-2\pi is\right)^j\left(1-d\right)_j}{\left(1-d\right)_j\Gamma\left(j\right)}\int_{q=0}^r \left(r-q\right)^{j-1} B\left(q\right)dq \\
=\ &\sum_{j=0}^{\infty}\frac{\left(-2\pi is\right)^j}{\Gamma\left(j\right)}\int_{q=0}^r \left(r-q\right)^{j-1} B\left(q\right)dq \\
=\ &\sum_{j=0}^{\infty}\frac{\left(-2\pi is\right)^j}{j!}\int_{q=0}^r \left(r-q\right)^j dB\left(q\right) = \int_{q=0}^r e^{-2\pi is(r-q)}dB\left(q\right),
\end{aligned}$$

using (6.17) in the penultimate line. This proves part (b).

To prove part (c), we expand the ${}_1F_1$ function on the right side of the formula and use

$$B_{d-1}(1) = \frac{1}{\Gamma\left(d\right)}\int_0^1 \left(1-s\right)^{d-1} dB(s) = -\frac{1}{\Gamma\left(d\right)}\int_0^1 r^{d-1}dB(1-r),$$

to get

$$\begin{aligned}
&\frac{1}{\Gamma\left(1-f\right)\left(-2\pi si\right)}\int_0^1 r^{-f}\ {}_1F_1\left(1,1-f;-2\pi isr\right)dB\left(1-r\right) + \frac{1}{\left(-2\pi si\right)}B_{d-1}\left(1\right) \\
=\ &\frac{1}{\Gamma\left(1-f\right)\left(-2\pi si\right)}\sum_{j=0}^{\infty}\frac{\left(1\right)_j\left(-2\pi si\right)^j}{j!\left(1-f\right)_j}\int_0^1 r^{j-f}dB\left(1-r\right) - \frac{1}{\left(-2\pi si\right)}\frac{1}{\Gamma\left(1-f\right)}\int_0^1 r^{-f}dB(1-r)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-f)(-2\pi si)} \sum_{j=1}^{\infty} \frac{(1)_j (-2\pi si)^j}{j!\,(1-f)_j} \int_0^1 r^{j-f} dB(1-r) \\
&= \sum_{j=1}^{\infty} \frac{(-2\pi si)^{j-1}}{\Gamma(j+1-f)} \int_0^1 r^{j-f} dB(1-r) \\
&= \sum_{k=0}^{\infty} \frac{(-2\pi si)^{k}}{\Gamma(k+1+d)} \int_0^1 r^{k+d} dB(1-r) \\
&= \frac{1}{\Gamma(1+d)} \sum_{k=0}^{\infty} \frac{(1)_k (-2\pi si)^k}{k!\,(1+d)_k} \int_0^1 r^{k+d} dB(1-r) \\
&= \frac{1}{\Gamma(1+d)} \int_0^1 {}_1F_1(1,1+d;-2\pi isr)\, r^d dB(1-r),
\end{aligned}
$$

giving the stated result.

To prove part (d) we use the exponential expansion for $e^{2\pi isr}$ in the integral on the right side, giving

$$
\begin{aligned}
\int_0^1 e^{2\pi isr} B_{d-1}(r)\,dr &= \int_0^1 e^{2\pi is(1-r)} B_{d-1}(1-r)\,dr = \int_0^1 e^{-2\pi isr} B_{d-1}(1-r)\,dr \\
&= \sum_{j=0}^{\infty} \frac{(-2\pi si)^j}{j!} \int_0^1 r^j B_{d-1}(1-r)\,dr \\
&= \sum_{j=0}^{\infty} \frac{(-2\pi si)^j}{j!} \int_0^1 (1-r)^j B_{d-1}(r)\,dr. \qquad (6.19)
\end{aligned}
$$

From part (a) we have

$$
\Gamma(j+1-d)^{-1} \int_0^r (r-s)^{j-d} B_{d-1}(s)\,ds = \Gamma(j+1)^{-1} \int_{q=0}^r (r-q)^j\, dB(q),
$$

and setting $k=j-d$ and $r=1$ gives the formula

$$
\Gamma(k+1)^{-1} \int_0^1 (1-s)^k B_{d-1}(s)\,ds = \Gamma(k+d+1)^{-1} \int_{q=0}^1 (1-q)^{k+d}\, dB(q),
$$

or

$$
\Gamma(k+1)^{-1} \int_0^1 s^k B_{d-1}(1-s)\,ds = \Gamma(k+d+1)^{-1} \int_{q=0}^1 q^{k+d} dB(1-q). \qquad (6.20)
$$

Using (6.20) in (6.19) we get

$$
\int_0^1 e^{2\pi isr} B_{d-1}(r)\,dr = \sum_{j=0}^{\infty} \frac{(-2\pi si)^j}{j!} \int_0^1 r^j B_{d-1}(1-r)\,dr
$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \frac{(-2\pi si)^j}{j!} \frac{\Gamma(j+1)}{\Gamma(j+d+1)} \int_0^1 q^{j+d} dB(1-q) \\
&= \sum_{j=0}^{\infty} \frac{(-2\pi si)^j}{j!} \frac{(1)_j}{\Gamma(j+d+1)} \int_0^1 q^{j+d} dB(1-q) \\
&= \frac{1}{\Gamma(d+1)} \int_0^1 \sum_{j=0}^{\infty} \frac{(-2\pi siq)^j}{j!} \frac{(1)_j}{(1+d)_j} q^d dB(1-q) \\
&= \frac{1}{\Gamma(d+1)} \int_0^1 {}_1F_1(1, 1+d; -2\pi isq)\, q^d dB(1-q),
\end{aligned}
$$

giving the stated result.

Lemma F *Let α and β be constants for which $\mathrm{Re}(\beta), \mathrm{Re}(\beta - \alpha) > 0$. The following asymptotic expansions to some given order k hold*

(a) *Let $\lambda_s = \frac{2\pi s}{n}$. If $\frac{s}{n} \to 0$ as $n \to \infty$ and $s \to \infty$, then*

$$
\begin{aligned}
&{}_2F_1\left(\alpha, n-\beta; n; e^{i\lambda_s}\right) \\
&= (1 - e^{i\lambda_s})^{-\alpha} \left[\sum_{j=0}^{k-1} \frac{(\alpha)_j (\beta)_j}{j!} \left(\frac{1}{2\pi is} \left[1 + O\left(\frac{s}{n}\right)\right] \right)^j + O\left(\left(\frac{1}{2\pi is} \left[1 + O\left(\frac{s}{n}\right)\right] \right)^k \right) \right] \\
&= (1 - e^{i\lambda_s})^{-\alpha} \left[\sum_{j=0}^{k-1} \frac{(\alpha)_j (\beta)_j}{j!} \left(\frac{1}{2\pi is} \left[1 + O\left(\frac{s}{n}\right)\right] \right)^j + O\left(\frac{1}{s^k} \right) \right].
\end{aligned}
$$

(b) *Let $\lambda \neq 0$ be fixed as $n \to \infty$. Then*

$$
\begin{aligned}
&{}_2F_1\left(\alpha, n-\beta; n; e^{i\lambda}\right) \\
&= (1 - e^{i\lambda})^{-\alpha} \left[\sum_{j=0}^{k-1} \frac{(\alpha)_j (\beta)_j}{j!} \left(\frac{1}{n} \frac{e^{i\lambda}}{e^{i\lambda} - 1} \left[1 + O\left(\frac{1}{n}\right)\right] \right)^j + O\left(\frac{1}{n^k} \right) \right].
\end{aligned}
$$

(c) *Let $\lambda_s = \frac{2\pi s}{n}$. If $\frac{s}{n} + \frac{n}{sp} \to 0$ as $n, s, p \to \infty$, then*

$$
\begin{aligned}
&{}_2F_1\left(\alpha, p-\beta; p; e^{i\lambda_s}\right) \\
&= (1 - e^{i\lambda_s})^{-\alpha} \left[\sum_{j=0}^{k-1} \frac{(\alpha)_j (\beta)_j}{j!} \left(\frac{n}{2\pi isp} \left[1 + O\left(\frac{s}{n}\right)\right] \right)^j + O\left(\left(\frac{n}{2\pi isp} \left[1 + O\left(\frac{s}{n}\right)\right] \right)^k \right) \right].
\end{aligned}
$$

Proof Since $\mathrm{Re}(\beta-\alpha)>0$, the series for ${}_2F_1\left(\alpha, n-\beta; n; e^{i\lambda_s}\right)$ converges absolutely for all λ_s. Using (6.1) from lemma B (d), we write

$$ {}_2F_1\left(\alpha, n-\beta; n; e^{i\lambda_s}\right) = (1-e^{i\lambda_s})^{-\alpha}\ {}_2F_1\left(\alpha, \beta; n; \frac{e^{i\lambda_s}}{e^{i\lambda_s}-1}\right), \tag{6.21} $$

where the right side has a convergent series representation for suitable λ_s, viz. when $|e^{i\lambda_s}/(e^{i\lambda_s}-1)|<1$, or $\cos(\lambda_s)<\frac{1}{2}$. Although the domain of convergence of the series on the right side series is restricted, the right hand side has a valid asymptotic expansion for large n that applies to all λ_s as we shall now show.

First observe that as $n, s\to\infty$ with $\frac{s}{n}\to 0$, the complex quantity

$$ Z_{ns} = \frac{e^{i\lambda_s}}{e^{i\lambda_s}-1} = \frac{n}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right] = \frac{n}{2\pi is}\left[1+o\left(1\right)\right] \tag{6.22} $$

lies inside the plane cut along $[1,\infty]$, i.e. $|\arg(1-Z_{ns})|<\pi$. Hence, we may use the analytic continuation of the right hand side of (6.21) based on the following integral representation (Erdélyi, 1953, p. 59; lemma B(f)):

$$ {}_2F_1\left(\beta, \alpha; n; Z_{ns}\right) = \frac{\Gamma\left(n\right)}{\Gamma\left(\alpha\right)\Gamma\left(n-\alpha\right)}\int_0^1 t^{\alpha-1}\left(1-t\right)^{n-\alpha-1}\left(1-tZ_{ns}\right)^{-\beta} dt. \tag{6.23} $$

An asymptotic series that is valid even for $|Z_{ns}|>1$ for large n may now be obtained using a method due to MacRobert (see Erdélyi (1953, p. 76)) as follows. Expand the last binomial factor in (6.23) in MacLaurin's expansion up to k terms with remainder as

$$ \left(1-tZ_{ns}\right)^{-\beta} = \sum_{j=0}^{k-1}\frac{(\beta)_j}{j!}\left(tZ_{ns}\right)^j + \frac{(\beta)_k}{k!}\left(tZ_{ns}\right)^k\int_0^1 k\left(1-q\right)^{k-1}\left(1-qtZ_{ns}\right)^{-\beta-k} dq. $$

Now scale this expansion by $\frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}t^{\alpha-1}\left(1-t\right)^{n-\alpha-1}$ and integrate term by term, using the formula

$$ \frac{\Gamma\left(n\right)}{\Gamma\left(\alpha\right)\Gamma\left(n-\alpha\right)}\int_0^1 t^{\alpha+j-1}\left(1-t\right)^{n-\alpha-1} dt = \frac{\Gamma\left(n\right)}{\Gamma\left(\alpha\right)\Gamma\left(n-\alpha\right)}\frac{\Gamma\left(\alpha+j\right)\Gamma\left(n-\alpha\right)}{\Gamma\left(n+j\right)} = \frac{(\alpha)_j}{(n)_j}. $$

This leads to

$$ \begin{aligned} {}_2F_1\left(\beta, \alpha; n; Z_{ns}\right) &= \sum_{j=0}^{k-1}\frac{(\alpha)_j(\beta)_j}{(n)_j j!}Z_{ns}^j + R_{kn} \\ &= \sum_{j=0}^{k-1}\frac{(\alpha)_j(\beta)_j}{(n)_j j!}\left(\frac{n}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right)^j + R_{kn} \end{aligned} $$

$$= \sum_{j=0}^{k-1} \frac{(\alpha)_j(\beta)_j}{j!} \left(\frac{1}{2\pi i s} \left[1 + O\left(\frac{s}{n}\right)\right] \left[1 + O\left(n^{-1}\right)\right] \right)^j + R_{kn}$$

$$= \sum_{j=0}^{k-1} \frac{(\alpha)_j(\beta)_j}{j!} \left(\frac{1}{2\pi i s} \left[1 + O\left(\frac{s}{n}\right)\right] \right)^j + R_{kn}, \tag{6.24}$$

where

$$\begin{aligned} R_{kn} &= \frac{(\beta)_k}{k! B(\alpha, n-\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{n-\alpha-1} (t Z_{ns})^k \int_0^1 k (1-q)^{k-1} (1 - qtZ_{ns})^{-\beta-k} \, dq dt \\ &= \frac{k(\alpha)_k(\beta)_k \left(\frac{n}{2\pi i s}\left[1 + O\left(\frac{s}{n}\right)\right]\right)^k}{(n)_k k! B(\alpha+k, n-\alpha)} \\ &\quad \times \int_0^1 t^{\alpha+k-1} (1-t)^{n-\alpha-1} \int_0^1 (1-q)^{k-1} \left(1 - qt\frac{n}{2\pi i s}\left[1 + O\left(\frac{s}{n}\right)\right]\right)^{-\beta-k} dq dt, \end{aligned}$$

since the beta function factors as follows

$$\begin{aligned} \frac{1}{B(\alpha, n-\alpha)} &= \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \\ &= \frac{\Gamma(\alpha+k)\Gamma(n)}{\Gamma(\alpha)\Gamma(n+k)} \frac{\Gamma(n+k)}{\Gamma(\alpha+k)\Gamma(n-\alpha)} = \frac{(\alpha)_k}{(n)_k B(\alpha+k, n-\alpha)}. \end{aligned}$$

In view of (6.22) there exists a constant $c > 0$ for which $|\mathrm{Im}(Z_{ns})| \geq c$. Then, for any given β and k, there exists an M, independent of n and s, such that

$$\sup_{t,q\in[0,1]} \left|(1 - qtZ_{ns})^{-\beta-k}\right| < M.$$

Then,

$$\begin{aligned} |R_{kn}| &\leq M \frac{k(\alpha)_k(\beta)_k \left(\frac{n}{2\pi i s}\left[1 + O\left(\frac{s}{n}\right)\right]\right)^k}{(n)_k k! B(\alpha+k, n-\alpha)} \int_0^1 t^{\alpha+k-1} (1-t)^{n-\alpha-1} \int_0^1 (1-q)^{k-1} \, dq \\ &= M \frac{k(\alpha)_k(\beta)_k \left(\frac{n}{2\pi i s}\left[1 + O\left(\frac{s}{n}\right)\right]\right)^k}{(n)_k k! B(\alpha+k, n-\alpha)} B(\alpha+k, n-\alpha) B(k, 1) \\ &= M \frac{k(\alpha)_k(\beta)_k \left(\frac{n}{2\pi i s}\left[1 + O\left(\frac{s}{n}\right)\right]\right)^k}{(n)_k k!} \frac{\Gamma(k)}{\Gamma(k+1)} \\ &= M \frac{(\alpha)_k(\beta)_k \left(\frac{1}{2\pi i s}\left[1 + O\left(\frac{s}{n}\right)\right]\left[1 + O\left(\frac{1}{n}\right)\right]\right)^k}{k!} \\ &= M \frac{(\alpha)_k(\beta)_k}{k!} \left(\frac{1}{2\pi i s}\left[1 + O\left(\frac{s}{n}\right)\right]\right)^k, \end{aligned}$$

so that R_{kn} has the same order of magnitude as the first neglected term in the expansion

(6.24). Thus, (6.24) is a valid asymptotic expansion of the form

$$\begin{aligned}
& {}_2F_1\left(\beta,\alpha;n;\frac{n}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right) \\
= & \sum_{j=0}^{k-1}\frac{(\alpha)_j(\beta)_j}{j!}\left(\frac{1}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right)^j+O\left(\left(\frac{1}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right)^k\right),
\end{aligned}$$

giving the required result for part (a). Part (b) follows in an identical manner using

$$Z=\frac{e^{i\lambda}}{e^{i\lambda}-1}$$

in place of Z_{ns}.

To prove part (c) we proceed as in the proof of part (a), setting $Z_{ns}=\frac{e^{i\lambda_s}}{e^{i\lambda_s}-1}$ as in (6.22). Then

$$\begin{aligned}
{}_2F_1\left(\beta,\alpha;p;Z_{ns}\right) &= \sum_{j=0}^{k-1}\frac{(\alpha)_j(\beta)_j}{(p)_j j!}Z_{ns}^j+R_{knp} \\
&= \sum_{j=0}^{k-1}\frac{(\alpha)_j(\beta)_j}{(p)_j j!}\left(\frac{n}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right)^j+R_{knp} \\
&= \sum_{j=0}^{k-1}\frac{(\alpha)_j(\beta)_j}{j!}\left(\frac{n}{2\pi isp}\left[1+O\left(\frac{s}{n}\right)\right]\left[1+O\left(p^{-1}\right)\right]\right)^j+R_{knp}, \\
&= \sum_{j=0}^{k-1}\frac{(\alpha)_j(\beta)_j}{j!}\left(\frac{n}{2\pi isp}\left[1+O\left(\frac{s}{n}\right)\right]\right)^j+R_{knp}
\end{aligned}$$

since $\frac{ps}{n}\to\infty$. The remainder is

$$\begin{aligned}
R_{knp} &= \frac{(\beta)_k}{k!B\left(\alpha,p-\alpha\right)}\int_0^1 t^{\alpha-1}\left(1-t\right)^{p-\alpha-1}\left(tZ_{ns}\right)^k\int_0^1 k\left(1-q\right)^{k-1}\left(1-qtZ_{ns}\right)^{-\beta-k}dqdt \\
&= \frac{k(\alpha)_k(\beta)_k\left(\frac{n}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right)^k}{(p)_k k!B\left(\alpha+k,p-\alpha\right)} \\
&\quad\times\int_0^1 t^{\alpha+k-1}\left(1-t\right)^{p-\alpha-1}\int_0^1\left(1-q\right)^{k-1}\left(1-qt\frac{n}{2\pi is}\left[1+O\left(\frac{s}{n}\right)\right]\right)^{-\beta-k}dqdt.
\end{aligned}$$

As in the case of R_{kn}, we have

$$\begin{aligned}
\left|R_{knp}\right| &\leq M\frac{(\alpha)_k(\beta)_k}{k!}\left(\frac{n}{2\pi isp}\left[1+O\left(\frac{s}{n}\right)\right]\left[1+O\left(p^{-1}\right)\right]\right)^k \\
&= M\frac{(\alpha)_k(\beta)_k}{k!}\left(\frac{n}{2\pi isp}\left[1+O\left(\frac{s}{n}\right)\right]\right)^k,
\end{aligned}$$

again since $\frac{ps}{n}\to\infty$. Thus, R_{knp} has the same order as the first neglected term in the series and we get the asymptotic expansion

$$
\begin{aligned}
&{}_2F_1\left(\beta,\alpha;p;\frac{e^{i\lambda_s}}{e^{i\lambda_s}-1}\right)\\
&= \sum_{j=0}^{k-1}\frac{(\alpha)_j(\beta)_j}{j!}\left(\frac{n}{2\pi isp}\left[1+O\left(\frac{s}{n}\right)\right]\right)^j+O\left(\left(\frac{n}{2\pi isp}\left[1+O\left(\frac{s}{n}\right)\right]\right)^k\right),
\end{aligned}
$$

which leads to the stated result.

6.2 Proofs of Main Lemmas and Theorems

6.1 Proof of Lemma 3.1 See (Phillips and Solo, 1992, formula (32)).

6.2 Proof of Theorem 3.2 From (3.2) we have the following alternate form for the model (2.1) for all $t \leq n$

$$
u_t=(1-L)^d X_t = D_n(L;d)X_t = D_n\left(e^{i\lambda};d\right)X_t+\widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right)\left(e^{-i\lambda}L-1\right)X_t. \quad (6.25)
$$

Observe that

$$
\widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right)\left(e^{-i\lambda}L-1\right)X_t=\left(e^{-i\lambda}L-1\right)\widetilde{X}_{\lambda t}=e^{-i\lambda}\widetilde{X}_{\lambda t-1}(d)-\widetilde{X}_{\lambda t}(d), \quad (6.26)
$$

where $\widetilde{X}_{\lambda t}(d)=\widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right)X_t=\sum_{p=0}^{n-1}\widetilde{d}_{\lambda p}e^{-ip\lambda}X_{t-p}$. Since the right side of (6.26) is a telescoping Fourier sum, taking dfts of (6.26) leaves us with $\frac{1}{\sqrt{2\pi n}}\left(\widetilde{X}_{\lambda 0}(d)-e^{in\lambda}\widetilde{X}_{\lambda n}(d)\right)$. It follows that when we take dfts of expression (6.25) we have

$$
\left[D_n\left(e^{i\lambda};d\right)\right]w_x(\lambda_s)+\frac{1}{\sqrt{2\pi n}}\left(\widetilde{X}_{\lambda 0}(d)-e^{in\lambda}\widetilde{X}_{\lambda n}(d)\right)=w_u(\lambda), \quad (6.27)
$$

giving the required formula (3.3).

6.3 Proof of Theorem 3.7 Equation (3.11) follows immediately from the definition $(1-L)X_t = z_t$ and (3.9). Equation (3.12) follows by applying (3.8) to $z_t=(1-L)^{1-d}u_t$.

6.4 Proof of Lemma 4.2 Using the hypergeometric series representation from lemma B (b), and the asymptotic expansion in lemma A (d), we have for $d > 0$

$$
\begin{aligned}
&D_n\left(e^{i\lambda};d\right)\\
&= \sum_{k=0}^{n}\frac{(-d)_k}{k!}e^{ik\lambda}=\left(\sum_{k=0}^{\infty}-\sum_{k=n+1}^{\infty}\right)\frac{(-d)_k}{k!}e^{ik\lambda}\\
&= \left(1-e^{i\lambda}\right)^d-e^{i(n+1)\lambda}\frac{\Gamma(n+1-d)}{\Gamma(-d)(n+1)!}\,{}_2F_1\left(n+1-d,1;n+2;e^{i\lambda}\right)
\end{aligned}
$$

$$= \quad \left(1-e^{i\lambda}\right)^{d}-\frac{e^{i(n+1)\lambda}}{\Gamma\left(-d\right)n^{1+d}}\left[1+O\left(\frac{1}{n}\right)\right]\ {}_{2}F_{1}\left(n+1-d,1;n+2;e^{i\lambda}\right), \qquad (6.28)$$

giving (4.1). Formula (4.2) follows immediately from lemma B (d), noting that $|e^{i\lambda}/(e^{i\lambda}-1)| < 1$ when $2\cos(\lambda) < 1$.

Next, using lemma F (b), we have for fixed $\lambda \neq 0$,

$${}_{2}F_{1}\left(n+1-d,1;n+2;e^{i\lambda}\right)=(1-e^{i\lambda})^{-1}\left[1+O\left(\frac{1}{n}\right)\right]. \qquad (6.29)$$

It follows from (6.28) and (6.29) that as $n\to\infty$ and for fixed $\lambda\neq 0$

$$D_{n}\left(e^{i\lambda};d\right)=\left(1-e^{i\lambda}\right)^{d}-\frac{1}{\Gamma\left(-d\right)n^{1+d}}\frac{e^{i(n+1)\lambda}}{1-e^{i\lambda}}\left[1+O\left(\frac{1}{n}\right)\right],$$

giving part (a).

When $\lambda_s = \frac{2\pi i s}{n} \to 0$ as $n\to\infty$ and $s\to\infty$, we proceed as follows. Using lemma F (a) in the hypergeometric factor in the second term of (6.28), we have

$$\begin{aligned}&{}_{2}F_{1}\left(n+1-d,1;n+2;e^{i\lambda_s}\right)\\ =\;&\frac{1}{1-e^{i\lambda_s}}\sum_{j=0}^{k-1}\frac{(1+d)_j\,(1)_j}{j!}\left(\frac{1}{2\pi i s}\left[1+O\left(\frac{s}{n}\right)\right]\right)^{j}+O\left(\left(\frac{1}{2\pi i s}\left[1+O\left(\frac{s}{n}\right)\right]\right)^{k}\right).\end{aligned} \qquad (6.30)$$

Then, as in the argument leading to (6.9), the second term of (6.28) admits the following valid asymptotic expansion for $\lambda=\lambda_s\to 0$ as $n\to\infty$ and $s\to\infty$:

$$\begin{aligned}&\frac{e^{i\lambda_s}}{\Gamma\left(-d\right)n^{1+d}}\left[1+O\left(\frac{1}{n}\right)\right]\ {}_{2}F_{1}\left(n+1-d,1;n+2;e^{i\lambda_s}\right)\\ =\;&-\frac{1}{2\pi i}\frac{1}{\Gamma\left(-d\right)n^{d}s}\left[1+O\left(\frac{1}{s}\right)\right]+O\left(\frac{1}{n^{1+d}}\right),\end{aligned} \qquad (6.31)$$

and so from (6.28) and (6.31) we get

$$D_{n}\left(e^{i\lambda_s};d\right)=\left(1-e^{i\lambda_s}\right)^{d}+\frac{1}{2\pi i}\frac{1}{\Gamma\left(-d\right)n^{d}s}\left[1+O\left(\frac{1}{s}\right)\right]+O\left(\frac{1}{n^{1+d}}\right),$$

giving part (b). The result can also be shown directly by noting from Lemma C(b) that

$$\begin{aligned}D_{n}\left(e^{i\lambda_s};d\right)&=\left(1-e^{i\lambda_s}\right)^{d}-\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}e^{ik\lambda_s}\\ &=\left(1-e^{i\lambda_s}\right)^{d}+\frac{1}{2\pi i}\frac{1}{\Gamma(-d)n^{d}s}\left(1+O\left(\frac{1}{s}\right)\right)+O\left(\frac{1}{n^{1+d}}\right).\end{aligned}$$

For part (c), we start by using the following summation formula from lemma B (e)

$$\sum_{k=0}^{n} \frac{(-d)_k \left(e^{i\lambda_s}\right)^k}{k!} = \frac{(1-d)_n \, e^{i\lambda_s n}}{n!} \; {}_2F_1(-n,1,1-d;1-e^{-i\lambda_s}).$$

Since s is fixed, we have from lemma D (6.12) with $p=n$

$${}_2F_1(-n,1,1-d;1-e^{-i\lambda_s}) = \; {}_1F_1\left(1,1-d;-2\pi is\right) + O\left(n^{-1}\right).$$

It follows that

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-d)_k \left(e^{i\lambda_s}\right)^k}{k!} &= \frac{(1-d)_n \, e^{i\lambda_s n}}{n!} \left[\, {}_1F_1\left(1,1-d;-2\pi is\right) + O\left(n^{-1}\right)\right] \\ &= \frac{(1-d)_n}{n!} \; {}_1F_1\left(1,1-d;-2\pi is\right) + O\left(\frac{1}{n^{1+d}}\right), \end{aligned} \tag{6.32}$$

and, then, for fixed s as $n\to\infty$, we have

$$D_n\left(e^{i\lambda_s};d\right) = \sum_{k=0}^{n} \frac{(-d)_k \, e^{i\lambda_s k}}{k!} = \frac{1}{\Gamma\left(1-d\right) n^d} \; {}_1F_1\left(1,1-d;-2\pi is\right) + O\left(\frac{1}{n^{1+d}}\right), \tag{6.33}$$

as required for part (c).

Part (d) follows as a special case of formula (6.33) with $s=0$. We also get the result directly from lemma A (c), viz.

$$D_n\left(1;d\right) = \sum_{k=0}^{n-1} \frac{(-d)_k}{k!} = \frac{(1-d)_{n-1}}{(n-1)!} = \frac{1}{\Gamma\left(1-d\right)} \frac{1}{n^d} \left[1+O\left(\frac{1}{n}\right)\right].$$

It follows that $D_n\left(1;d\right)$ differs from zero by a term of $O\left(n^{-d}\right)$.

6.5 Proof of Theorem 4.3 Parts (a) and (b). We write $\widetilde{X}_{\lambda n}(d)$ as the sum of two components, the first involving $L+1$ components, with $1<L<n$ and where the choice of L will be discussed below. We have:

$$\begin{aligned} \widetilde{X}_{\lambda n}(d) = \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right) X_n = \sum_{p=0}^{n-1} \widetilde{d}_{\lambda p} e^{-ip\lambda} X_{n-p} = \sum_{p=0}^{n-1} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}\right) e^{-ip\lambda} X_{n-p} \\ = \sum_{p=0}^{L} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}\right) e^{-ip\lambda} X_{n-p} + \sum_{p=L+1}^{n-1} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}\right) e^{-ip\lambda} X_{n-p}. \end{aligned}$$

Then

$$
\begin{aligned}
\frac{\widetilde{X}_{\lambda n}(d)}{\sqrt{n}} &= \frac{1}{n^{1-d}}\sum_{p=0}^{L}\left(\sum_{k=p+1}^{n}\frac{(-d)_k}{k!}e^{ik\lambda}\right)e^{-ip\lambda}\frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
&\quad+\frac{1}{n^{1-d}}\sum_{p=L+1}^{n-1}\left(\sum_{k=p+1}^{n}\frac{(-d)_k}{k!}e^{ik\lambda}\right)e^{-ip\lambda}\frac{X_{n-p}}{n^{d-\frac{1}{2}}}. \qquad (6.34)
\end{aligned}
$$

Next, look at the sinusoidal sum $\sum_{k=p+1}^{n}\frac{(-d)_k}{k!}e^{ik\lambda}$ that appears in (6.34). We use the truncated binomial series formula from lemma B (b) in this sum, giving

$$
\begin{aligned}
\sum_{k=p+1}^{n}\frac{(-d)_k}{k!}e^{i\lambda k} &= \sum_{k=p+1}^{\infty}\frac{(-d)_k}{k!}e^{i\lambda k}-\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}e^{i\lambda k} \\
= \quad & \left(e^{i\lambda}\right)^{p+1}\frac{(-d)_{p+1}}{(p+1)!}\,{}_2F_1\left(1+p-d,1;p+2,e^{i\lambda}\right) \\
& -\left(e^{i\lambda}\right)^{n+1}\frac{(-d)_{n+1}}{(n+1)!}\,{}_2F_1\left(n+1-d,1;n+2,e^{i\lambda}\right). \qquad (6.35)
\end{aligned}
$$

For large n and fixed $\lambda \neq 0$ we have, using lemma C (a),

$$
\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}e^{i\lambda k}=O\left(\frac{1}{n^{1+d}}\right), \qquad (6.36)
$$

while for $\lambda=\lambda_s=\frac{2\pi is}{n}\to 0$ and $s\to\infty$ as $n\to\infty$ we have from lemma C (b)

$$
\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}e^{i\lambda_s k}=-\frac{1}{\Gamma(-d)\,n^d}\frac{1}{2\pi is}\left[1+O\left(\frac{1}{s}\right)\right]+O\left(\frac{1}{n^{1+d}}\right). \qquad (6.37)
$$

So, neglecting the second term of (6.35) in view of (6.37), we get

$$
\sum_{t=p+1}^{n}\frac{(-d)_t}{t!}\left(e^{i\lambda_s}\right)^t=\left(e^{i\lambda_s}\right)^{p+1}\frac{\Gamma(p+1-d)}{\Gamma(-d)\,(p+1)!}\,{}_2F_1\left(1+p-d,1;p+2,e^{i\lambda_s}\right)+O\left(\frac{1}{n^d s}\right) \qquad (6.38)
$$

for all $s\to\infty$, as $n\to\infty$. Finally, for s fixed as $n\to\infty$, we have from lemma C (c)

$$
\sum_{k=n+1}^{\infty}\frac{(-d)_k}{k!}\left(e^{i\lambda_s}\right)^k=O\left(\frac{1}{n^d}\right),
$$

so that (6.38) also holds with s fixed.

Using (6.38), we deduce that

$$
\begin{aligned}
& \frac{1}{n^{1-d}} \sum_{p=0}^{L} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \\
= \; & e^{i\lambda_s} \frac{1}{n^{1-d}} \sum_{p=0}^{L} \frac{(-d)_{p+1}}{(p+1)!} \; {}_2F_1\left(1+p-d, 1; p+2, e^{i\lambda_s}\right) + O\left(\frac{L}{ns}\right) \\
= \; & e^{i\lambda_s} \frac{1}{n^{1-d}} \sum_{p=0}^{\infty} \frac{(-d)_{p+1}}{(p+1)!} \; {}_2F_1\left(1+p-d, 1; p+2, e^{i\lambda_s}\right) \\
& - e^{i\lambda_s} \frac{1}{n^{1-d}} \sum_{p=L+1}^{\infty} \frac{(-d)_{p+1}}{(p+1)!} \; {}_2F_1\left(1+p-d, 1; p+2, e^{i\lambda_s}\right) + O\left(\frac{L}{ns}\right).
\end{aligned} \tag{6.39}
$$

Now

$$
\begin{aligned}
& \sum_{p=0}^{\infty} \frac{(-d)_{p+1}}{(p+1)!} \; {}_2F_1\left(1+p-d, 1; p+2, e^{i\lambda}\right) \\
= \; & \sum_{p=0}^{\infty} \frac{(-d)_{p+1}}{(p+1)!} \sum_{k=0}^{\infty} \frac{(1+p-d)_k \, (1)_k}{k! \, (p+2)_k} e^{i\lambda k} \\
= \; & \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-d)_{p+1}}{(p+1)!} \frac{(1+p-d)_k}{(p+2)_k} e^{i\lambda k} \\
= \; & \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-d)_{p+1}}{(p+1)!} \frac{(1-d+k)_p \, (2)_p}{(1-d)_p \, (k+2)_p} \frac{(1-d)_k}{(2)_k} e^{i\lambda k} \\
= \; & \sum_{k=0}^{\infty} \left[\sum_{p=0}^{\infty} \frac{(-d)_{p+1}}{(p+1)!} \frac{(1-d+k)_p \, (2)_p}{(1-d)_p \, (k+2)_p} \right] \frac{(1-d)_k}{(2)_k} e^{i\lambda k}.
\end{aligned} \tag{6.40}
$$

Next, since $(2)_p = (p+1)!$ and

$$
(-d)_{p+1} = \frac{\Gamma(1-d+p)}{\Gamma(-d)} = \frac{(-d)\,\Gamma(1-d+p)}{\Gamma(1-d)} = (-d)\,(1-d)_p
$$

we have

$$
\begin{aligned}
& \sum_{p=0}^{\infty} \frac{(-d)_{p+1}}{(p+1)!} \frac{(1-d+k)_p \, (2)_p}{(1-d)_p \, (k+2)_p} = (-d) \sum_{p=0}^{\infty} \frac{(1-d+k)_p}{(k+2)_p} \\
& = (-d) \sum_{p=0}^{\infty} \frac{(1-d+k)_p \, (1)_p}{(k+2)_p \, p!} = (-d) \; {}_2F_1\left(k+1-d, 1; k+2; 1\right) \\
& = (-d) \frac{\Gamma(k+2)\,\Gamma(d)}{\Gamma(k+1)\,\Gamma(1+d)} = -\left(k+1\right),
\end{aligned} \tag{6.41}
$$

where the explicit representation in the last line follows by the summation formula of lemma B (c). Using (6.41) in (6.40) we get

$$\sum_{k=0}^{\infty}\left[\sum_{p=0}^{\infty}\frac{(-d)_{p+1}}{(p+1)!}\frac{(1-d+k)_p\,(2)_p}{(1-d)_p\,(k+2)_p}\right]\frac{(1-d)_k}{(2)_k}e^{i\lambda k}=-\sum_{k=0}^{\infty}\frac{(k+1)\,(1-d)_k}{(2)_k}e^{i\lambda k}$$

$$= -\sum_{k=0}^{\infty}\frac{(1-d)_k}{k!}e^{i\lambda k}=-\frac{1}{\left(1-e^{i\lambda}\right)^{1-d}}. \tag{6.42}$$

Thus,

$$\sum_{p=0}^{\infty}\frac{(-d)_{p+1}}{(p+1)!}\,{}_2F_1\left(1+p-d,1;p+2,e^{i\lambda_s}\right)$$

$$= \sum_{k=0}^{\infty}\left[\sum_{p=0}^{\infty}\frac{(-d)_{p+1}}{(p+1)!}\frac{(1-d+k)_p\,(2)_p}{(1-d)_p\,(k+2)_p}\right]\frac{(1-d)_k}{(2)_k}e^{i\lambda_s k}=-\frac{1}{\left(1-e^{i\lambda_s}\right)^{1-d}}. \tag{6.43}$$

Next, using lemma F (c) we find that for $\frac{s}{n}+\frac{n}{Ls}\to 0$ (which holds under the conditions on s and L that are given below),

$$\sum_{p=L+1}^{\infty}\frac{(-d)_{p+1}}{(p+1)!}\,{}_2F_1\left(1+p-d,1;p+2,e^{i\lambda_s}\right)$$

$$= O\left(\sum_{p=L+1}^{\infty}\frac{(-d)_{p+1}}{(p+1)!}\frac{1}{1-e^{i\lambda_s}}\left[1+O\left(\frac{n}{sp}\right)\right]\right)$$

$$= O\left(\frac{1}{1-e^{i\lambda_s}}\sum_{p=L+1}^{\infty}\frac{1}{p^{1+d}}\left[1+O\left(p^{-1}\right)\right]\left[1+O\left(\frac{n}{sp}\right)\right]\right)$$

$$= O\left(\frac{1}{L^d}\frac{1}{1-e^{i\lambda_s}}\right). \tag{6.44}$$

It follows from (6.39), (6.43) and (6.44) that

$$\frac{1}{n^{1-d}}\sum_{p=0}^{L}\left(\sum_{k=p+1}^{n}\frac{(-d)_k}{k!}e^{ik\lambda}\right)e^{-ip\lambda_s} = -\frac{1}{n^{1-d}}\frac{e^{i\lambda_s}}{\left(1-e^{i\lambda_s}\right)^{1-d}}+O\left(\frac{L}{ns}\right)$$

$$+\frac{1}{n^{1-d}}\sum_{p=L+1}^{\infty}\frac{(-d)_{p+1}}{(p+1)!}\,{}_2F_1\left(1+p-d,1;p+2,e^{i\lambda_s}\right)$$

$$= -\frac{1}{n^{1-d}}\frac{e^{i\lambda_s}}{\left(1-e^{i\lambda_s}\right)^{1-d}}+O\left(\frac{L}{ns}\right)+O\left(\frac{n^d}{L^d}\frac{1}{s}\right). \tag{6.45}$$

The first term in (6.45) is $O\left(\frac{1}{s^{1-d}}\right)$ and dominates the second term. The first term also dominates the third term when $\frac{n}{Ls}\to 0$, which will be the case when $\frac{s}{n^{\alpha}}\to\infty$, as $n\to\infty$, for some $\alpha \in (0,1)$ and $L = \lfloor n^{1-\alpha} \rfloor$ and when $d < 1$. (Note that for s fixed the last term of (6.45) does matter, and this distinguishes the s fixed case, which will be considered below in the proof of part (c)). Hence, when $n \to \infty$, $\lambda_s \to 0$ and $\frac{s}{n^\alpha} \to \infty$ (with L chosen as $L = \lfloor n^{1-\alpha} \rfloor$), we have

$$\begin{aligned}
& \frac{1}{n^{1-d}} \sum_{p=0}^{L} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
= & \frac{1}{n^{1-d}} \sum_{p=0}^{L} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \left[\frac{X_n}{n^{d-\frac{1}{2}}} + o_p(1) \right] \qquad (6.46) \\
= & -\frac{1}{n^{1-d}} \frac{e^{i\lambda_s}}{(1-e^{i\lambda_s})^{1-d}} \frac{X_n}{n^{d-\frac{1}{2}}} + o_p\left(\frac{1}{s^{1-d}}\right) \\
= & -\frac{e^{i\lambda_s}}{(1-e^{i\lambda_s})^{1-d}} \frac{X_n}{\sqrt{n}} + o_p\left(\frac{1}{s^{1-d}}\right). \qquad (6.47)
\end{aligned}$$

Line (6.46) above is justified by a separate argument, which we now develop. We use the fact, from lemma 4.7, that $n^{\frac{1}{2}-d} X_{n-p} = O_p(1)$ and $p \le L = \lfloor n^{1-\alpha} \rfloor$. We proceed as follows. Select $K = \lfloor n^{1-\eta} \rfloor \to \infty$ with $0 < \eta < \alpha$ (we will place a further condition on η below). Then, $\frac{L}{K} + \frac{K}{n} \to 0$ and we may write (for large n)

$$\begin{aligned}
\frac{X_{n-p}}{n^{d-\frac{1}{2}}} &= \frac{1}{n^{d-\frac{1}{2}}} \sum_{j=0}^{n-p} \frac{(d)_j}{j!} u_{n-p-j} = \frac{1}{n^{d-\frac{1}{2}}} \sum_{j=K+1}^{n-p} \frac{(d)_j}{j!} u_{n-p-j} + \frac{1}{n^{d-\frac{1}{2}}} \sum_{j=0}^{K} \frac{(d)_j}{j!} u_{n-p-j} \\
&= \sum_{j=K+1}^{n-p} \frac{1}{\left(\frac{j}{n}\right)^{1-d}} \frac{u_{n-p-j}}{\sqrt{n}} \left[1 + O_p\left(\frac{1}{K}\right) \right] + \left(\frac{K}{n}\right)^{d-\frac{1}{2}} \frac{1}{K^{d-\frac{1}{2}}} \sum_{j=0}^{K} \frac{(d)_j}{j!} u_{n-p-j} \\
&= \sum_{j=K+1}^{n-p} \frac{1}{\left(\frac{j}{n}\right)^{1-d}} \frac{u_{n-p-j}}{\sqrt{n}} + O_p\left(\frac{1}{K}\right) + O_p\left(\left(\frac{K}{n}\right)^{d-\frac{1}{2}}\right) \\
&= \sum_{j=K+1}^{n-p} \frac{1}{\left(\frac{j+p}{n}\right)^{1-d}} \frac{u_{n-p-j}}{\sqrt{n}} \left(\frac{j+p}{j}\right)^{1-d} + O_p\left(\frac{1}{K}\right) + O_p\left(\left(\frac{K}{n}\right)^{d-\frac{1}{2}}\right) \\
&= \sum_{k=K+p+1}^{n} \frac{1}{\left(\frac{k}{n}\right)^{1-d}} \frac{u_{n-k}}{\sqrt{n}} \left(\frac{k}{k-p}\right)^{1-d} + O_p\left(\frac{1}{K}\right) + O_p\left(\left(\frac{K}{n}\right)^{d-\frac{1}{2}}\right) \\
&= \sum_{k=K+p+1}^{n} \frac{1}{\left(\frac{k}{n}\right)^{1-d}} \frac{u_{n-k}}{\sqrt{n}} \left(1 + O\left(\frac{p}{k}\right)\right)^{d-1} + O_p\left(\frac{1}{K}\right) + O_p\left(\left(\frac{K}{n}\right)^{d-\frac{1}{2}}\right) \\
&= \sum_{k=1}^{n} \frac{1}{\left(\frac{k}{n}\right)^{1-d}} \frac{u_{n-k}}{\sqrt{n}} - \left(\frac{K+p}{n}\right)^{d-\frac{1}{2}} \sum_{k=1}^{K+p} \frac{1}{\left(\frac{k}{K+p}\right)^{1-d}} \frac{u_{n-k}}{\sqrt{K+p}}
\end{aligned}$$

$$+O_p\left(\frac{p}{n^{d-\frac{1}{2}}}\sum_{k=K+p+1}^{n}\frac{1}{k^{2-d}}u_{n-k}\right)+O_p\left(\frac{1}{K}\right)+O_p\left(\left(\frac{K}{n}\right)^{d-\frac{1}{2}}\right). \tag{6.48}$$

Observe that for any $\delta>0$, $\sum_{k=1}^{\infty}\frac{1}{k^{1+\delta}}u_{n-k}$ converges almost surely since $\sum_{k=1}^{\infty}\frac{1}{k^{1+\delta}}E|u_{n-k}|<\infty$. Then,

$$\begin{aligned}
E\left|\sum_{k=K+p+1}^{n}\frac{1}{k^{2-d}}u_{n-k}\right| &\le \sum_{k=K+p+1}^{n}\frac{1}{k^{2-d}}E\left|u_{n-k}\right|\le\sum_{k=K+p+1}^{\infty}\frac{1}{k^{2-d}}E\left|u_{n-k}\right| \\
&\le \frac{1}{K^{1-d-\delta}}\sum_{k=K+p+1}^{\infty}\frac{1}{k^{1+\delta}}E\left|u_{n-k}\right|=o\left(\frac{1}{K^{1-d+\delta}}\right),
\end{aligned}$$

and so

$$\sum_{k=K+p+1}^{n}\frac{1}{k^{2-d}}u_{n-k}=o_p\left(\frac{1}{K^{1-d-\delta}}\right).$$

It follows that

$$\begin{aligned}
\frac{p}{n^{d-\frac{1}{2}}}\sum_{k=K+p+1}^{n}\frac{1}{k^{2-d}}u_{n-k} &= o_p\left(\frac{p}{n^{d-\frac{1}{2}}}\frac{1}{n^{(1-\eta)(1-d-\delta)}}\right)=o_p\left(\frac{L}{n^{\frac{1}{2}-\eta(1-d-\delta)-\delta}}\right) \\
&= o_p\left(\frac{\sqrt{n}}{n^{\alpha-\eta(1-d-\delta)-\delta}}\right)
\end{aligned}$$

uniformly for $p\le L$. For $K=\lfloor n^{1-\eta}\rfloor$ and with η satisfying

$$0<\eta<\min\left(\alpha,\frac{\alpha-\frac{1}{2}-\delta}{1-d-\delta}\right),$$

and choosing δ such that $0<\delta<\alpha-\frac{1}{2}$, we have

$$\frac{p}{n^{d-\frac{1}{2}}}\sum_{k=K+p+1}^{n}\frac{1}{k^{2-d}}u_{n-k}=o_p\left(1\right), \tag{6.49}$$

uniformly for $p\le L$.

Using (6.49), we find that (6.48) can be written as

$$\begin{aligned}
\frac{X_{n-p}}{n^{d-\frac{1}{2}}} &= \left[\frac{1}{n^{d-\frac{1}{2}}}\sum_{k=0}^{n}\frac{(d)_k}{k!}u_{n-k}+o_p\left(1\right)\right]+O_p\left(\frac{K+p}{n}\right)^{d-\frac{1}{2}}+o_p\left(1\right)+O_p\left(\frac{1}{K}\right)+O_p\left(\left(\frac{K}{n}\right)^{d-\frac{1}{2}}\right) \\
&= \frac{X_n}{n^{d-\frac{1}{2}}}+O_p\left(\frac{K}{n}\right)^{d-\frac{1}{2}}+O_p\left(\frac{1}{K}\right)+o_p(1)=\frac{X_n}{n^{d-\frac{1}{2}}}+o_p(1),
\end{aligned}$$

uniformly for $p\le L=n^{1-\alpha}$ with $\alpha>\frac{1}{2}$, thereby establishing (6.46).

When $n \to \infty$ with fixed $\lambda \neq 0$, we have, in view of the use of (6.36) rather than (6.37) in the above arguments, the same expression but with an $o_p\left(n^{-(1-d)}\right)$ error. Specifically,

$$
\begin{aligned}
& \frac{1}{n^{1-d}} \sum_{p=0}^{L} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
= \quad & -\frac{1}{n^{1-d}} \frac{e^{i\lambda}}{(1-e^{i\lambda})^{1-d}} \frac{X_n}{n^{d-\frac{1}{2}}} + o_p\left(\frac{1}{n^{1-d}}\right) + O\left(\frac{1}{n}\frac{n^d}{L^d}\frac{1}{1-e^{i\lambda}}\right) + O\left(\frac{1}{n^{1-d}}\frac{1}{n^d}\right) \\
= \quad & -\frac{1}{n^{1-d}} \frac{e^{i\lambda}}{(1-e^{i\lambda})^{1-d}} \frac{X_n}{n^{d-\frac{1}{2}}} + o_p\left(\frac{1}{n^{1-d}}\right). \qquad (6.50)
\end{aligned}
$$

In both cases the dominant approximation is given by the first term and we can write

$$
\frac{1}{n^{1-d}} \sum_{p=0}^{L} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda} \right) e^{-ip\lambda} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} = -\frac{e^{i\lambda}}{(1-e^{i\lambda})^{1-d}} \frac{X_n}{\sqrt{n}} + o_p\left(\frac{e^{i\lambda}}{(1-e^{i\lambda})^{1-d}} \frac{X_n}{\sqrt{n}}\right).
$$

It remains to show that we may neglect the second term of (6.34). Using lemma C(b), lemma 4.7, (6.38) and lemma F (c), we have, when $n \to \infty$, $\lambda_s \to 0$ and $\frac{s}{n^\alpha} \to \infty$ and $L = n^{1-\alpha}$

$$
\begin{aligned}
& \frac{1}{n^{1-d}} \sum_{p=L+1}^{n-1} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
= \quad & \frac{1}{n^{1-d}} \sum_{p=L+1}^{n-1} \left(\frac{e^{i\lambda_s(p+1)}\,(-d)_{p+1}}{(p+1)!}\ {}_2F_1\left(1+p-d, 1; p+2, e^{i\lambda_s}\right) + O\left(\frac{1}{n^d s}\right) \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
= \quad & \frac{e^{i\lambda_s}}{n^{1-d}} \sum_{p=L+1}^{n-1} \left(\frac{(-d)_{p+1}}{(p+1)!}\ {}_2F_1\left(1+p-d, 1; p+2, e^{i\lambda_s}\right) \right) \frac{X_{n-p}}{n^{d-\frac{1}{2}}} + O_p\left(\frac{1}{s}\right) \\
= \quad & O\left(\frac{e^{i\lambda_s}}{1-e^{i\lambda_s}} \frac{1}{n^{1-d}} \sum_{p=L+1}^{n-1} \left(\frac{(-d)_{p+1}}{(p+1)!} \left[1 + O\left(\frac{n}{sp}\right)\right] \right) \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \right) + O_p\left(\frac{1}{s}\right) \\
= \quad & O_p\left(\frac{n^d}{L^d s}\right) + O_p\left(\frac{1}{s}\right), \qquad (6.51)
\end{aligned}
$$

which is $o_p(\frac{1}{s^{1-d}})$ since $\frac{n}{Ls} \to 0$.

For the case of fixed $\lambda \neq 0$ and with $L = n^{1-\alpha}$ we get

$$
\begin{aligned}
& \frac{1}{n^{1-d}} \sum_{p=L+1}^{n-1} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} = O_p\left(\frac{1}{n^{1-d}} \sum_{p=L+1}^{n-1} \frac{1}{p^{1+d}} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \right) \\
= \quad & O_p\left(\frac{1}{n^{1-d}L^d}\right) = O_p\left(\frac{1}{n^{1-\alpha d}}\right) = o_p\left(\frac{1}{n^{1-d}}\right). \qquad (6.52)
\end{aligned}
$$

In both cases (6.51) and (6.52), the order is smaller than the leading term of (6.47) and (6.50), respectively. Hence, for both fixed $\lambda \neq 0$ and $\lambda_s \to 0$ and $\frac{s}{n^\alpha} \to \infty$ as $n \to \infty$, we have

$$\begin{aligned}
\frac{\widetilde{X}_{\lambda n}(d)}{\sqrt{n}} &= \frac{1}{n^{1-d}} \sum_{p=0}^{n} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
&= -\frac{e^{i\lambda}}{\left(1-e^{i\lambda}\right)^{1-d}} \frac{X_n}{\sqrt{n}} + o_p\left(\frac{e^{i\lambda}}{\left(1-e^{i\lambda}\right)^{1-d}} \frac{X_n}{\sqrt{n}} \right),
\end{aligned}$$

giving the required results.

Part (c). Our interest is in

$$\frac{\widetilde{X}_{\lambda_s n}(d)}{\sqrt{n}} = \frac{1}{n^{1-d}} \sum_{p=0}^{n-1} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}}.$$

From lemma B (e) we have

$$\sum_{k=0}^{m} \frac{(-d)_k \left(e^{i\lambda_s}\right)^k}{k!} = \frac{(1-d)_m e^{i\lambda_s m}}{m!} \; {}_2F_1(-m, 1, 1-d; 1-e^{-i\lambda_s}). \tag{6.53}$$

Since s is fixed, $1 - e^{-i\lambda_s} = \frac{2\pi i s}{n} + O\left(n^{-2}\right)$ and using lemma D and (6.53) we get

$$\sum_{k=0}^{n} \frac{(-d)_k \left(e^{i\lambda_s}\right)^k}{k!} = \frac{(1-d)_n}{n!} \; {}_1F_1\left(1, 1-d; -2\pi i s\right) + O\left(\frac{1}{n^{1+d}}\right). \tag{6.54}$$

Using (6.53) with $m = p$ and lemma D again we obtain

$$\begin{aligned}
\sum_{k=0}^{p} \frac{(-d)_k \left(e^{i\lambda_s}\right)^k}{k!} &= \frac{(1-d)_p e^{i\lambda_s p}}{p!} \; {}_2F_1(-p, 1, 1-d; 1-e^{-i\lambda_s}) \\
&= \frac{(1-d)_p e^{i\lambda_s p}}{p!} \; {}_1F_1\left(1, 1-d; -2\pi i s \frac{p}{n}\right) + O\left(\frac{1}{p^{1+d}}\right). \qquad (6.55)
\end{aligned}$$

Now $n^{\frac{1}{2}-d} X_{n-p} = O_p(1)$, uniformly in $p \leq n$, so that

$$\frac{1}{n^{1-d}} \sum_{p=0}^{n} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} = \left[\frac{1}{n^{1-d}} \sum_{p=0}^{n} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \right] O_p(1).$$

Using (6.54) and (6.55) and noting that $\sum_{p=0}^{n} p^{-1-d} = O(1)$, we have

$$\frac{1}{n^{1-d}} \sum_{p=0}^{n} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}}$$

$$
\begin{aligned}
= \quad & \frac{1}{n^{1-d}} \sum_{p=0}^{n} \left\{ \frac{(1-d)_n \; {}_1F_1(1,1-d;-2\pi i s)}{n!} - \frac{(1-d)_p \, e^{i\lambda_s p} \; {}_1F_1\left(1,1-d;-2\pi i s \frac{p}{n}\right)}{p!} \right\} e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
& + O_p\left(\frac{1}{n^{1-d}}\right).
\end{aligned}
$$

Next observe that, since s is fixed as $n \to \infty$,

$$
\begin{aligned}
& \frac{1}{n^{1-d}} \frac{(1-d)_n}{n!} \sum_{p=0}^{n} e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} = \frac{1}{\Gamma(1-d)} \frac{1}{n} \sum_{p=0}^{n} e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} + O_p\left(\frac{1}{n}\right) \\
= \quad & \frac{1}{\Gamma(1-d)} \frac{1}{n} \sum_{p=0}^{n} e^{-2\pi i s \frac{p}{n}} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} + O_p\left(\frac{1}{n}\right) = \frac{1}{\Gamma(1-d)} \int_0^1 e^{-2\pi i s r} X_{n,d}(1-r) dr + O_p\left(\frac{1}{n}\right) \\
= \quad & \frac{1}{\Gamma(1-d)} \int_0^1 e^{2\pi i s r} X_{n,d}(r) dr + O_p\left(\frac{1}{n}\right).
\end{aligned}
$$

Further,

$$
\begin{aligned}
& \frac{1}{n^{1-d}} \sum_{p=0}^{n} \frac{(1-d)_p \; {}_1F_1\left(1,1-d;-2\pi i s \frac{p}{n}\right)}{p!} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
= \quad & \frac{1}{\Gamma(1-d)} \frac{1}{n^{1-d}} \sum_{p=1}^{n} \frac{{}_1F_1\left(1,1-d;-2\pi i s \frac{p}{n}\right)}{p^d} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} + O_p\left(\frac{1}{n^{1-d}}\right) \\
= \quad & \frac{1}{\Gamma(1-d)} \frac{1}{n} \sum_{p=1}^{n} \frac{{}_1F_1\left(1,1-d;-2\pi i s \frac{p}{n}\right)}{\left(\frac{p}{n}\right)^d} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} + O_p\left(\frac{1}{n^{1-d}}\right) \\
= \quad & \frac{1}{\Gamma(1-d)} \int_0^1 {}_1F_1(1,1-d;-2\pi i s r)\, r^{-d} X_{n,d}(1-r)\, dr + O_p\left(\frac{1}{n^{1-d}}\right).
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
& \frac{1}{n^{1-d}} \sum_{p=0}^{n} \left(\sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda_s} \right) e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
= \quad & \frac{1}{n^{1-d}} \sum_{p=0}^{n} \left\{ \frac{(1-d)_n \; {}_1F_1(1,1-d;-2\pi i s)}{n!} - \frac{(1-d)_p \, e^{i\lambda_s p} \; {}_1F_1(1,1-d;-2\pi i s r)}{p!} \right\} e^{-ip\lambda_s} \frac{X_{n-p}}{n^{d-\frac{1}{2}}} \\
& + O_p\left(\frac{1}{n^{1-d}}\right) \\
= \quad & \frac{1}{\Gamma(1-d)} \left[\int_0^1 e^{-2\pi i s r} \; {}_1F_1(1,1-d;-2\pi i s)\, X_{n,d}(1-r) dr \right. \\
& \left. - \int_0^1 {}_1F_1(1,1-d;-2\pi i s r)\, r^{-d} X_{n,d}(1-r) dr \right] + O_p\left(\frac{1}{n^{1-d}}\right) \\
= \quad & \frac{{}_1F_1(1,1-d;-2\pi i s)}{\Gamma(1-d)} \int_0^1 e^{2\pi i s r} X_{n,d}(r) dr
\end{aligned}
$$

$$-\frac{1}{\Gamma(1-d)}\int_0^1 {}_1F_1(1,1-d;-2\pi isr)\, r^{-d}X_{n,d}(1-r)dr + O_p\left(\frac{1}{n^{1-d}}\right),$$

giving the stated result.

Part (d). When $d=1$ the series expression for $n^{-\frac{1}{2}}\widetilde{X}_{\lambda n}(d)$ terminates because $(-d)_k = 0$ for all $k>1$, so that only the term involving $p=0$ is retained. We then have

$$\frac{\widetilde{X}_{\lambda n}(1)}{\sqrt{n}} = -e^{i\lambda}\frac{X_n}{\sqrt{n}},$$

which holds for all λ.

6.6 Proof of Theorem 4.4 By definition, $z_t = (1-L)^{1-d}u_t = (1-L)^f u_t$, and from theorem 3.7 we have

$$\begin{aligned} w_x(\lambda)\left(1-e^{i\lambda}\right) &= w_z(\lambda) - e^{i\lambda}\frac{X_n}{\sqrt{2\pi n}} \\ &= D_n\left(e^{i\lambda}; f\right) w_u(\lambda) - \frac{e^{i\lambda n}}{\sqrt{2\pi n}}\widetilde{U}_{\lambda n}(f) - e^{i\lambda}\frac{X_n}{\sqrt{2\pi n}}, \end{aligned}$$

where

$$\widetilde{U}_{\lambda n}(f) = \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L; f\right) u_n = \sum_{p=0}^{n-1}\widetilde{f}_{\lambda p}e^{-ip\lambda}u_{n-p}, \quad \text{and } \widetilde{f}_{\lambda p} = \sum_{k=p+1}^{n}\frac{(-f)_k}{k!}e^{ik\lambda}.$$

Now, as in lemma B (e), we have

$$\begin{aligned} \frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}} &= \frac{1}{\sqrt{2\pi n}}\sum_{p=0}^{n-1}\widetilde{f}_{\lambda_s p}e^{-ip\lambda_s}u_{n-p} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{p=0}^{n-1}\left(\sum_{k=p+1}^{n}\frac{(-f)_k}{k!}e^{ik\lambda_s}\right)e^{-ip\lambda_s}\frac{u_{n-p}}{\sqrt{n}} \\ &= \frac{1}{\sqrt{2\pi}}\sum_{p=0}^{n}\left\{\frac{(1-f)_n e^{i\lambda_s n}}{n!}\ {}_2F_1(-n,1,1-f;1-e^{-i\lambda_s}) \right. \\ &\quad \left. -\frac{(1-f)_p e^{i\lambda_s p}}{p!}\ {}_2F_1(-p,1,1-f;1-e^{-i\lambda_s})\right\} e^{-ip\lambda_s}\frac{u_{n-p}}{\sqrt{n}}. \end{aligned}$$

As in the proof of theorem 4.3 and using the fact that $\sum_{p=1}^{n} p^{-1-f}u_{n-p} = O_p(1)$ as $n\to\infty$, we proceed as follows

$$\frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \sum_{p=0}^{n} \left\{ \frac{(1-f)_n \ {}_1F_1(1, 1-f; -2\pi i s)}{n!} \right. \\
&\quad \left. - \frac{(1-f)_p \, e^{i\lambda_s p} \ {}_1F_1\left(1, 1-f; -2\pi i s \frac{p}{n}\right)}{p!} + O\left(\frac{1}{p^{1+f}}\right) \right\} e^{-ip\lambda_s} \frac{u_{n-p}}{\sqrt{n}} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{p=0}^{n} \left\{ \frac{(1-f)_n \ {}_1F_1(1, 1-f; -2\pi i s)}{n!} \right. \\
&\quad \left. - \frac{(1-f)_p \, e^{i\lambda_s p} \ {}_1F_1\left(1, 1-f; -2\pi i s \frac{p}{n}\right)}{p!} \right\} e^{-ip\lambda_s} \frac{u_{n-p}}{\sqrt{n}} + O_p\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{(1-f)_n \ {}_1F_1(1, 1-f; -2\pi i s)}{n!} \sum_{p=0}^{n} e^{-ip\lambda_s} \frac{u_{n-p}}{\sqrt{n}} \\
&\quad - \frac{1}{\sqrt{2\pi}} \sum_{p=0}^{n} \frac{(1-f)_p \ {}_1F_1\left(1, 1-f; -2\pi i s \frac{p}{n}\right)}{p!} \frac{u_{n-p}}{\sqrt{n}} + O_p\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{(1-f)_n \ {}_1F_1(1, 1-f; -2\pi i s)}{n!} \int_0^1 e^{-2\pi i s r} dX_n(1-r) \\
&\quad - \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(1-f)} \sum_{p=0}^{n} \frac{1}{p^f} \left[1 + O\left(\frac{1}{p}\right)\right] \ {}_1F_1\left(1, 1-f; -2\pi i s \frac{p}{n}\right) \frac{u_{n-p}}{\sqrt{n}} + O_p\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{(1-f)_n \ {}_1F_1(1, 1-f; -2\pi i s)}{n!} \int_0^1 e^{-2\pi i s r} dX_n(1-r) \\
&\quad - \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(1-f)\, n^f} \sum_{p=0}^{n} \frac{1}{\left(\frac{p}{n}\right)^f} \ {}_1F_1\left(1, 1-f; -2\pi i s \frac{p}{n}\right) \frac{u_{n-p}}{\sqrt{n}} + O_p\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{(1-f)_n \ {}_1F_1(1, 1-f; -2\pi i s)}{n!} \int_0^1 e^{-2\pi i s r} dX_n(1-r) \\
&\quad - \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(1-f)\, n^f} \int_0^1 r^{-f} \ {}_1F_1(1, 1-f; -2\pi i s r)\, dX_n(1-r) + O_p\left(\frac{1}{\sqrt{n}}\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(1-f)\, n^f} \left\{ {}_1F_1(1, 1-f; -2\pi i s) \int_0^1 e^{-2\pi i s r} dX_n(1-r) \right. \\
&\quad \left. - \int_0^1 r^{-f} \ {}_1F_1(1, 1-f; -2\pi i s r)\, dX_n(1-r) \right\} + O_p\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

So we have

$$
\begin{aligned}
\frac{\widetilde{U}_{\lambda_s n}(f)}{\sqrt{2\pi n}} &= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(1-f)\, n^f} \left\{ {}_1F_1(1, 1-f; -2\pi i s) \int_0^1 e^{-2\pi i s r} dX_n(1-r) \right. \\
&\quad \left. - \int_0^1 r^{-f} \ {}_1F_1(1, 1-f; -2\pi i s r)\, dX_n(1-r) \right\} + O_p\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}
$$

as required. Note that when $f = 0$, we get

$$
{}_1F_1(1, 1; -2\pi i s) = e^{-2\pi s i} = 1, \qquad {}_1F_1(1, 1; -2\pi i s r) = e^{-2\pi i s r},
$$

and $\widetilde{U}_{\lambda_s n}(0) = 0$.

6.7 Proof of Lemma 4.7 Akonom and Gouriéroux (1987) prove the result when u_t follows a stationary and invertible ARMA process. Using the device in Phillips and Solo (1992), we write

$$u_t = C(L)\varepsilon_t = C(1)\varepsilon_t + \widetilde{\varepsilon}_{t-1} - \widetilde{\varepsilon}_t$$

where $\widetilde{\varepsilon}_t = \widetilde{C}(L)\varepsilon_t = \sum_{j=0}^{\infty} \widetilde{c}_j \varepsilon_{t-j}$ and $\widetilde{c}_j = \sum_{k=j+1}^{\infty} c_k$. Under (2.4), $\widetilde{\varepsilon}_t$ is stationary with mean zero and finite variance $\sigma^2 \sum_{j=0}^{\infty} \widetilde{c}_j^2$. Then

$$X_t = (1-L)^{-d} u_t = C(1)(1-L)^{-d}\varepsilon_t - (1-L)^{1-d}\widetilde{\varepsilon}_t.$$

Now for $\frac{1}{2} < d \leq 1$, $\xi_t = (1-L)^{1-d}\widetilde{\varepsilon}_t$ is stationary with mean zero and finite variance, so that $n^{\frac{1}{2}-d}\xi_{\lfloor nr \rfloor} \to_p 0$. On the other hand, $X_t^{\varepsilon} = (1-L)^{-d}\varepsilon_t$ is a fractional process constructed from $iid\left(0,\sigma^2\right)$ innovations with $E|\varepsilon_t|^p < \infty$, and so from Akonom and Gouriéroux (1987)

$$X_{n,d}^{\varepsilon}(r) = \frac{1}{n^{d-\frac{1}{2}}} X_{\lfloor nr \rfloor}^{\varepsilon} \xrightarrow{d} \frac{\sigma}{\Gamma(d)} \int_0^r (r-s)^{d-1} dW(s).$$

It follows that

$$\begin{aligned} X_{n,d}(r) &= \frac{1}{n^{d-\frac{1}{2}}} X_{\lfloor nr \rfloor} \xrightarrow{d} B_{d-1}(r) = \frac{\sigma C(1)}{\Gamma(d)} \int_0^r (r-s)^{d-1} dW(s) \\ &= \frac{1}{\Gamma(d)} \int_0^r (r-s)^{d-1} dB(s), \end{aligned}$$

as stated.

6.8 Proof of Lemma 4.8 By theorem 4.3 (c), lemma 4.7 and the continuous mapping theorem we have

$$\begin{aligned} \frac{\widetilde{X}_{\lambda n}(d)}{\sqrt{n}} &= \frac{{}_1F_1(1,1-d;-2\pi i s)}{\Gamma(1-d)} \int_0^1 e^{2\pi i s r} X_{n,d}(r) dr \\ &\quad - \frac{1}{\Gamma(1-d)} \int_0^1 {}_1F_1(1,1-d;-2\pi i s r)\, r^{-d} X_{n,d}(1-r) dr + O_p\left(\frac{1}{n^{1-d}}\right) \\ &\xrightarrow{d} \frac{{}_1F_1(1,1-d;-2\pi i s)}{\Gamma(1-d)} \int_0^1 e^{-2\pi i s r} B_{d-1}(1-r) dr \\ &\quad - \frac{1}{\Gamma(1-d)} \int_0^1 {}_1F_1(1,1-d;-2\pi i s r)\, r^{-d} B_{d-1}(1-r) dr. \end{aligned} \tag{6.56}$$

In the above, we can replace $X_{n,d}(r)$ by a continuous polygonal version up to an $o_p(1)$ error uniformly over $r \in [0,1]$. The continuous mapping theorem then applies since the mapping $f \longmapsto \int_0^1 r^{-d} f(1-r) dr$ is continuous when $d < 1$ for all continuous functions f, and since the confluent hypergeometric function ${}_1F_1(a,c;x)$ is an entire function of x.

Now observe from lemma E that

$$\Gamma\left(1-d\right)^{-1}\int_0^1 {}_1F_1\left(1,1-d;-2\pi is\left(1-q\right)\right)\left(1-q\right)^{-d}B_{d-1}\left(q\right)dq=\int_{q=0}^1 e^{-2\pi is(1-q)}dB\left(q\right).$$

It follows that (6.56) is

$$\begin{aligned}
&\frac{{}_1F_1\left(1,1-d;-2\pi is\right)}{\Gamma\left(1-d\right)}\int_0^1 e^{-2\pi isr}B_{d-1}(1-r)dr\\
&-\frac{1}{\Gamma\left(1-d\right)}\int_0^1 {}_1F_1\left(1,1-d;-2\pi isr\right)r^{-d}B_{d-1}(1-r)dr\\
=&\ \frac{{}_1F_1\left(1,1-d;-2\pi is\right)}{\Gamma\left(1-d\right)}\int_0^1 e^{2\pi is(1-r)}B_{d-1}(1-r)dr-\int_{q=0}^1 e^{-2\pi is(1-q)}dB\left(q\right)\\
=&\ \frac{{}_1F_1\left(1,1-d;-2\pi is\right)}{\Gamma\left(1-d\right)}\int_0^1 e^{2\pi isr}B_{d-1}(r)dr-\int_{q=0}^1 e^{2\pi isq}dB\left(q\right). \qquad (6.57)
\end{aligned}$$

Then,

$$\frac{\widetilde{X}_{\lambda n}(d)}{\sqrt{n}}\overset{d}{\to}\frac{{}_1F_1\left(1,1-d;-2\pi is\right)}{\Gamma\left(1-d\right)}\int_0^1 e^{2\pi isr}B_{d-1}(r)dr-\int_{q=0}^1 e^{2\pi isq}dB\left(q\right), \qquad (6.58)$$

giving the first stated result.

6.9 Proof of Theorem 4.9 We offer two proofs of (4.20). The first is by operational techniques and is given in the proof of lemma E (a) - see (6.18). The second is by way of weak convergence of the two sides of (3.8) as $n\to\infty$. At $\lambda_s=0$, (3.8) is

$$\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^n u_t=\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^n X_tD_n\left(1,d\right)-\frac{1}{\sqrt{2\pi n}}\widetilde{X}_{\lambda_0,n}(d). \qquad (6.59)$$

From lemma A (c) for $d\in(\frac{1}{2},1]$

$$\begin{aligned}
D_n\left(1,d\right)&=\sum_{k=0}^n\frac{(-d)_k}{k!}=\frac{(1-d)_n}{n!}\\
&=\frac{1}{\Gamma\left(1-d\right)n^d}\left[1+O\left(n^{-1}\right)\right],
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{t=1}^n X_tD_n\left(1,d\right)&=\frac{1}{\Gamma\left(1-d\right)}\frac{1}{n}\sum_{t=1}^n\frac{X_t}{n^{d-\frac{1}{2}}}\left[1+O\left(n^{-1}\right)\right]\\
&\overset{d}{\to}\frac{1}{\Gamma\left(1-d\right)}\int_0^1 B_{d-1}(r)dr. \qquad (6.60)
\end{aligned}$$

From theorem 4.3 (c), (4.3), lemma 4.7 and the continuous mapping theorem we have

$$\begin{aligned}\frac{\widetilde{X}_{\lambda_0 n}(d)}{\sqrt{n}} &= \frac{1}{\Gamma(1-d)}\int_0^1 X_{n,d}(r)dr - \frac{1}{\Gamma(1-d)}\int_0^1 r^{-d}X_{n,d}(1-r)dr + O_p\left(\frac{1}{n^{1-d}}\right) \\ &\xrightarrow{d} \frac{1}{\Gamma(1-d)}\left[\int_0^1 B_{d-1}(r)dr - \int_0^1 r^{-d}B_{d-1}(1-r)dr\right]. \qquad (6.61)\end{aligned}$$

It follows from (6.59), (6.60) and (6.61) that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} u_t \xrightarrow{d} B(1) = \frac{1}{\Gamma(1-d)}\int_0^1 (1-r)^{-d} B_{d-1}(r)dr, \qquad (6.62)$$

Applying the same argument to the relation

$$\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{\lfloor nr\rfloor} u_t = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{\lfloor nr\rfloor} X_t D_n(1,d) - \frac{1}{\sqrt{2\pi n}}\widetilde{X}_{\lambda_0,\lfloor nr\rfloor}(d),$$

instead of (6.59), we obtain the more general formula

$$B(r) = \frac{1}{\Gamma(1-d)}\int_0^r (r-q)^{-d} B_{d-1}(q)dq.$$

To prove (4.19), we can proceed in the same way using (3.8) and theorem 4.3 (c). Or we can employ operational techniques, as in the proof of lemma E (b), which gives the stated result directly.

6.10 Proof of Theorem 4.10 Part (a) follows from the representation (4.6) and standard results on the asymptotic behavior of the dft of a stationary process whose spectrum is continuous. Indeed, from (4.6) and using lemma 4.7 we have

$$\begin{aligned} w_x(\lambda_{s_j}) &= \left(1-e^{i\lambda_{s_j}}\right)^{-d} w_u(\lambda_{s_j}) - \frac{e^{i\lambda_{s_j}}}{1-e^{i\lambda_{s_j}}}\frac{X_n}{\sqrt{2\pi n}} + o_p\left(\frac{1}{n^{1-d}}\right) \\ &= \left(1-e^{i\phi}\right)^{-d} w_u(\lambda_{s_j})\left[1+O\left(\frac{1}{M}\right)\right] + O_p\left(\frac{1}{n^{1-d}}\right)\end{aligned}$$

where the error magnitudes hold uniformly for $\lambda_{s_j} \in \mathcal{B}_\phi = \left\{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\right\}$. Theorem 3 of Hannan (1973) implies that the quantities $\{w_u(\lambda_{s_j})\}_{j=1}^{J}$ are asymptotically independent and distributed with the same complex normal distribution $N_c(0, f_u(\phi))$ as $n \to \infty$. The stated result for the quantities $\{w_x(\lambda_{s_j})\}_{j=1}^{J}$ follows directly.

Part (b) proceeds as follows. From (4.7) we have

$$w_x(\lambda_{s_j}) = \left(1-e^{i\lambda_{s_j}}\right)^{-d} w_u(\lambda_{s_j}) - \frac{e^{i\lambda_{s_j}}}{1-e^{i\lambda_{s_j}}}\frac{X_n}{\sqrt{2\pi n}} + o_p\left(\frac{\left(1-e^{i\lambda_{s_j}}\right)^{-d}}{s_j^{1-d}}\right).$$

Then,

$$
\begin{aligned}
\left(\lambda_{s_j}\right)^d w_x\left(\lambda_{s_j}\right) &= \left(\lambda_{s_j}\right)^d\left(1-e^{i\lambda_{s_j}}\right)^{-d} w_u\left(\lambda_{s_j}\right)-\left(\lambda_{s_j}\right)^d \frac{e^{i\lambda_{s_j}}}{1-e^{i\lambda_{s_j}}}\frac{X_n}{\sqrt{2\pi n}}+o_p\left(\frac{s_j^d\left(1-e^{i\lambda_{s_j}}\right)^{-d}}{n^d s_j^{1-d}}\right) \\
&= \left(-\frac{1}{i}\right)^d w_u\left(\lambda_{s_j}\right)\left[1+O\left(\frac{L}{n}\right)\right] \\
&\quad +\left(\frac{2\pi s_j}{n}\right)^d \frac{n}{2\pi i s_j}\left[1+O\left(\frac{L}{n}\right)\right]\frac{1}{\sqrt{2\pi}n^{1-d}}\frac{X_n}{n^{d-\frac{1}{2}}}+o_p\left(\frac{1}{n^{\alpha(1-d)}}\right) \\
&= e^{\frac{\pi d i}{2}} w_u\left(\lambda_{s_j}\right)+O_p\left(\frac{L}{n}\right)+o_p\left(\frac{1}{n^{\alpha(1-d)}}\right)
\end{aligned}
$$

uniformly over s_j. It follows that the family $\left\{\left(\lambda_{s_j}\right)^d w_x\left(\lambda_{s_j}\right)\right\}_{j=1}^J$ are asymptotically distributed as

$$\left\{e^{\frac{\pi d i}{2}} w_u\left(\lambda_{s_j}\right)\right\}_{j=1}^J,$$

that is, the members of the family are asymptotically independent and have the same complex normal distribution, $e^{\frac{\pi d i}{2}}N_c(0,f_u(0))$ or simply $N_c(0,f_u(0))$, as $n\to\infty$.

For part (c) note that for each j

$$\frac{1}{n^d}w_x\left(\lambda_{s_j}\right)=\frac{1}{\sqrt{2\pi}}\frac{1}{n}\sum_{t=1}^n\frac{X_t}{n^{d-\frac{1}{2}}}e^{2\pi s_j i\frac{t}{n}}=\frac{1}{\sqrt{2\pi}}\int_0^1 e^{2\pi i s_j r}X_{n,d}(r)dr+o_p\left(1\right),$$

and so, by the continuous mapping theorem,

$$\frac{1}{n^d}w_x\left(\lambda_{s_j}\right)\xrightarrow{d}\frac{1}{\sqrt{2\pi}}\int_0^1 e^{2\pi i s_j r}B_{d-1}(r)dr,$$

giving the stated result for each s_j. It is clear from the Cramér-Wold device that joint convergence for $\{n^{-d}w_x(\lambda_{s_j}):j=1,...,J\}$ also applies. Another approach to this result is to note from (4.10) that (dropping the subscript on s_j)

$$\frac{w_x\left(\lambda_s\right)}{n^d}=\frac{\Gamma\left(1-d\right)}{{}_1F_1\left(1,1-d;-2\pi i s\right)}\left[w_u\left(\lambda_s\right)+\frac{1}{\sqrt{2\pi n}}\widetilde{X}_{\lambda_s n}(d)\right]+O_p\left(\frac{1}{n}\right). \tag{6.63}$$

Now

$$w_u\left(\lambda_s\right)=\frac{1}{\sqrt{2\pi n}}\sum_{t=1}^n u_t e^{2\pi s i\frac{t}{n}}=\frac{1}{\sqrt{2\pi}}\int_0^1 e^{2\pi s i r}dX_n\left(r\right)+o_p\left(1\right), \tag{6.64}$$

where $X_n\left(r\right)=\frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor nr\rfloor}u_t$, and from (6.58) it follows that we may write

$$\frac{\widetilde{X}_{\lambda_s n}(d)}{\sqrt{n}}=\frac{{}_1F_1\left(1,1-d;-2\pi is\right)}{\Gamma\left(1-d\right)}\int_0^1 e^{2\pi isr}X_{n,d}(r)dr-\int_{q=0}^1 e^{2\pi isq}dX_n\left(q\right)+o_p\left(1\right). \tag{6.65}$$

Combining (6.64) and (6.65) in (6.63) we get

$$\frac{w_x\left(\lambda_s\right)}{n^d}=\int_0^1 e^{2\pi isr}X_{n,d}(r)dr+o_p\left(1\right)\xrightarrow{d}\frac{1}{\sqrt{2\pi}}\int_0^1 e^{2\pi isr}B_{d-1}(r)dr,$$

as above.

Part (d) follows from (3.9) and (4.15). Explicitly,

$$\begin{aligned} w_x\left(\lambda_{s_j}\right) &= \left(1-e^{i\lambda_{s_j}}\right)^{-1}w_u\left(\lambda_{s_j}\right)-\frac{e^{i\lambda_{s_j}}}{1-e^{i\lambda_{s_j}}}\frac{X_n}{\sqrt{2\pi n}} \\ &= \left[\left(1-e^{i\phi}\right)^{-1}w_u\left(\lambda_{s_j}\right)-\frac{e^{i\phi}}{1-e^{i\phi}}\frac{X_n}{\sqrt{2\pi n}}\right]\left[1+O\left(\frac{1}{M}\right)\right] \\ &\xrightarrow{d}\left(1-e^{i\phi}\right)^{-1}\xi_j-\frac{e^{i\phi}}{1-e^{i\phi}}\eta \end{aligned} \tag{6.66}$$

where the family $\{\xi_j\}_{j=1}^J$ are *iid* $N_c\left(0,f_u\left(\phi\right)\right)$ as in part (a), and the ξ_j are independent of

$$\eta=\frac{B(1)}{\sqrt{2\pi}}, \tag{6.67}$$

where B is the Brownian motion in (4.15), since the ordinates $w_u(\lambda_{s_j})$ are asymptotically independent of $w_u(\lambda_0)$ for all $s_j\neq 0$.

For part (e), (3.9) yields

$$\begin{aligned} \left(\lambda_{s_j}\right)w_x\left(\lambda_{s_j}\right) &= \left(\lambda_{s_j}\right)\frac{1}{1-e^{i\lambda_{s_j}}}w_u\left(\lambda_{s_j}\right)-\left(\lambda_{s_j}\right)\frac{e^{i\lambda_{s_j}}}{1-e^{i\lambda_{s_j}}}\frac{X_n}{\sqrt{2\pi n}} \\ &= -\frac{1}{i}w_u\left(\lambda_{s_j}\right)\left[1+O\left(\frac{1}{n}\right)\right]+\frac{1}{i}\left[1+O\left(\frac{1}{n}\right)\right]\frac{1}{\sqrt{2\pi}}\frac{X_n}{\sqrt{n}} \\ &\xrightarrow{d} i\left(\xi_j-\eta\right), \end{aligned}$$

where the family $\{\xi_j\}_{j=1}^J$ are *iid* $N_c\left(0,f_u\left(0\right)\right)$, and the ξ_j are independent of η, which has the same form as in (6.67) above. Finally, when s_j is fixed, (4.15) and the continuous mapping theorem imply that

$$\frac{1}{n}w_x\left(\lambda_{s_j}\right)=\frac{1}{\sqrt{2\pi}}\frac{1}{n}\sum_{t=1}^n\frac{X_t}{\sqrt{n}}e^{2\pi is_j\frac{t}{n}}\xrightarrow{d}\frac{1}{\sqrt{2\pi}}\int_0^1 e^{2\pi is_jr}B\left(r\right)dr, \tag{6.68}$$

which gives (4.26). Since $e^{2\pi is_jr}$ is continuously differentiable we may apply by integration by parts to (6.68), giving

$$\frac{1}{\sqrt{2\pi}}\left[\left.\frac{e^{2\pi i s_j r}B(r)}{2\pi i s_j}\right|_0^1-\frac{1}{2\pi i s_j}\int_0^1 e^{2\pi i s_j r}dB(r)\right]=\frac{1}{\sqrt{2\pi}}\frac{1}{2\pi i s_j}\left[B(1)-\int_0^1 e^{2\pi i s_j r}dB(r)\right],$$

which leads to the representation

$$\xi_j=\frac{1}{\sqrt{2\pi}}\int_0^1 e^{2\pi i s_j r}dB(r),$$

giving (4.25). Obviously, (6.68) also holds for $s_j=0$, and part (f) is proved.

6.11 Proof of Theorem 5.2 From (4.6) and lemma 4.7 we have

$$\begin{aligned}
w_x(\lambda_s) &= \left(1-e^{i\lambda_s}\right)^{-d}w_u(\lambda_s)-\frac{e^{i\lambda_s}}{1-e^{i\lambda_s}}\frac{X_n}{\sqrt{2\pi n}}+o_p\left(\frac{1}{n^{1-d}}\right)\\
&= \left(1-e^{i\phi}\right)^{-d}w_u(\lambda_s)\left[1+O\left(\frac{1}{M}\right)\right]+O_p\left(\frac{1}{n^{1-d}}\right),
\end{aligned}$$

where the error magnitudes hold uniformly for $\lambda_s\in\mathcal{B}_\phi=\left\{\phi-\frac{\pi}{M},\phi+\frac{\pi}{M}\right\}$. Then, as $n\to\infty$ with $\frac{M}{n}\to 0$, we have

$$\begin{aligned}
\widehat{f}_{xx}(\phi) &= \frac{1}{m}\sum_{\lambda_s\in\mathcal{B}(\phi)}w_x(\lambda_s)w_x(\lambda_s)^*\\
&= \frac{1}{|1-e^{i\phi}|^{2d}}\frac{1}{m}\sum_{\lambda_s\in\mathcal{B}(\phi)}w_u(\lambda_s)w_u(\lambda_s)^*+O_p\left(\frac{1}{M}\right)+O_p\left(\frac{1}{n^{1-d}}\right)\\
&\overset{p}{\to} \frac{1}{|1-e^{i\phi}|^{2d}}f_u(\phi),
\end{aligned}\tag{6.69}$$

by virtue of the consistency of the smoothed periodogram estimate in the stationary (linear process) case (e.g., (Hannan, 1970, ch. IV)), giving part (a).

For part (b), when $d=1$ we have from (6.66)

$$w_x(\lambda_s)=\left[\left(1-e^{i\phi}\right)^{-1}w_u(\lambda_s)-\frac{e^{i\phi}}{1-e^{i\phi}}\frac{X_n}{\sqrt{2\pi n}}\right]\left[1+O\left(\frac{1}{M}\right)\right],$$

and, as $n\to\infty$ with $\frac{M}{n}\to 0$, we have

$$\begin{aligned}
\widehat{f}_{xx}(\phi) &= \frac{1}{m}\sum_{\lambda_s\in\mathcal{B}(\phi)}w_x(\lambda_s)w_x(\lambda_s)^*\\
&= \frac{1}{|1-e^{i\phi}|^{2}}\frac{1}{m}\sum_{\lambda_s\in\mathcal{B}(\phi)}w_u(\lambda_s)w_u(\lambda_s)^*-\frac{2}{|1-e^{i\phi}|^{2}}\mathrm{Re}\left(\frac{1}{m}\sum_{\lambda_s\in\mathcal{B}(\phi)}w_u(\lambda_s)\frac{e^{-i\phi}X_n}{\sqrt{2\pi n}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\left|1-e^{i\phi}\right|^{2}}\left(\frac{X_{n}}{\sqrt{2\pi n}}\right)^{2}+O_{p}\left(\frac{1}{M}\right) \\
&\xrightarrow{d} \frac{1}{\left|1-e^{i\phi}\right|^{2d}} f_{u}(\phi)+\frac{1}{\left|1-e^{i\phi}\right|^{2}}\left(\frac{B(1)}{\sqrt{2\pi}}\right)^{2},
\end{aligned}
$$

in view of (6.69) and (4.15).

To prove part (c), we write the sum (5.2) as the sum over the full set of frequencies $\{\lambda_s\}_{s=0}^{n-1}$ and a residual, i.e.,

$$
\begin{aligned}
\frac{m}{n^{2d}}\widehat{f}_{xx}(0) &= \sum_{s=0}^{m-1}\frac{w_x(\lambda_s)}{n^d}\frac{w_x(\lambda_s)^*}{n^d} \\
&= \sum_{s=0}^{n-1}\frac{w_x(\lambda_s)}{n^d}\frac{w_x(\lambda_s)^*}{n^d}-\sum_{s=m}^{n-1}\frac{w_x(\lambda_s)}{n^d}\frac{w_x(\lambda_s)^*}{n^d} \\
&= \frac{1}{2\pi}\sum_{t=1}^{n}\left(\frac{X_t}{n^d}\right)^2-\sum_{s=m}^{n-1}\frac{w_x(\lambda_s)}{n^d}\frac{w_x(\lambda_s)^*}{n^d} \\
&= \frac{1}{2\pi}\frac{1}{n}\sum_{t=1}^{n}\left(\frac{X_t}{n^{d-\frac{1}{2}}}\right)^2-\sum_{s=m}^{n-1}\frac{w_x(\lambda_s)}{n^d}\frac{w_x(\lambda_s)^*}{n^d}. \qquad (6.70)
\end{aligned}
$$

Since $\frac{m}{n^\alpha}\to\infty$ we have by (4.8)

$$
\begin{aligned}
\frac{1}{n^d}w_x(\lambda_s) &= \frac{1}{n^d}\left[\left(1-e^{i\lambda_s}\right)^{-d}w_u(\lambda_s)-\frac{e^{i\lambda_s}}{1-e^{i\lambda_s}}\frac{X_n}{\sqrt{2\pi n}}+o_p\left(\frac{e^{i\lambda_s}}{\left(1-e^{i\lambda_s}\right)^{1-d}}\frac{X_n}{\sqrt{n}}\right)\right] \\
&= O_p\left(\frac{1}{m^d}\right),
\end{aligned}
$$

uniformly for $s\geq m$. When m is such that $\frac{m}{n^\alpha}\to\infty$, it follows that

$$
\frac{1}{n^d}w_x(\lambda_s)=o_p\left(\frac{1}{n^{\alpha d}}\right),
$$

and then

$$
\sum_{s=m}^{n-1}\frac{w_x(\lambda_s)}{n^d}\frac{w_x(\lambda_s)^*}{n^d}=o_p\left(\frac{n}{n^{2\alpha d}}\right)=o_p(1) \qquad (6.71)
$$

for α chosen such $\alpha\geq\frac{1}{2d}$. We deduce from (6.70), (6.71), (4.15) and the continuous mapping theorem that

$$
\frac{m}{n^{2d}}\widehat{f}_{xx}(0)=\frac{1}{2\pi}\frac{1}{n}\sum_{t=1}^{n}\left(\frac{X_t}{n^{d-\frac{1}{2}}}\right)^2+o_p(1)\to_d \frac{1}{2\pi}\int_0^1 B_{d-\frac{1}{2}}(r)^2\,dr,
$$

giving the stated result in part (c). Part (d) follows in an analogous fashion with $d=1$ and $\alpha\geq\frac{1}{2}$.

7 Notation

$\rightarrow_{\text{a.s.}}$	almost sure convergence
$=_d$	distributional equivalence
$:=$	definitional equality
$o_{\text{a.s.}}(1)$	tends to zero almost surely
$o_p(1)$	tends to zero in probability
$\rightarrow_p$	convergence in probability
$\overset{d}{\rightarrow}, \rightarrow_d$	weak convergence
$\lfloor \cdot \rfloor$	integer part of
$(a)_k$	$(a)(a+1)\ldots(a+k-1)$ forward factorial
${}_1F_1(a, c; z)$	$\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k k!} z^k$, confluent hypergeometric function
${}_2F_1(a, b, c; z)$	$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$ hypergeometric function
$\mathbf{1}(A)$	indicator of A
$X_n(r)$	$n^{-\frac{1}{2}} \sum_{t=0}^{\lfloor nr \rfloor} u_t$
$X_{n,d}(r)$	$n^{\frac{1}{2}-d} X_{\lfloor nr \rfloor}$
$\Gamma(z)$	$\int_0^{\infty} e^{-t} t^{z-1} dt$ gamma function ($\text{Re}(z) > 0$)
$B(z, w)$	$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ beta function
$w_a(\lambda)$	$\frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} a_t e^{it\lambda}$ discrete Fourier transform

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