Ocean Halos Model Proof

Bioeconomic model

We use a two-patch discrete-time meta population bioeconomic model to illustrate the interactions between enforcement, illegal fishing, and revenue generated from MPA spillover in a reserve system. Our system is divided into two patches, where patch F is the general fishing area and patch R is the reserve. Vessels do not need to pay in order to fish in patch F, and the fishing effort in this patch is assumed to be in open-access. Some fraction, L, of patch R can be designated as a lease area, where vessels must pay a per-unit-effort access fee in order to fish. The remaining fraction, $1 - L$, is designated as a no-take area. Fishing effort in the lease area is driven by biomass available for harvest and the per-unit-effort access fee for fishing. We also assume that illegal fishing occurs in the reserve, which is driven by biomass in the reserve and the per-unit-effort expected fine associated with being caught fishing illegally. The stock of fish in each area is stationary within a single year, but dispersal between the patches occurs at the beginning of each time step.

Fishery dynamics

Revenue for vessels in patch F is given by pqE_FX_F where p the price of fish, q is catchability, E_F is fishing effort (days) in patch F, and X_F is stock size. The cost of fishing for vessels in patch F is given by cE_F^β, where c is the variable cost of fishing, and $\beta = 1.3$ matches commonly used cost-coefficients (values of $\beta > 1$ imply that additional units of effort are increasingly costly to apply). Profits from fishing in patch F are given by:

$$\Pi_F(E_F, X_F) = pqE_FX_F - cE_F^\beta$$

In patch R, in the absence of a lease area, the only revenue for vessels comes from illegal fishing. This revenue is given by pqE_IX_R, where E_I is illegal fishing effort (days) and X_R is stock size. The cost of illegally fishing in patch R is given by $cE_I^\beta + \theta E_I\psi$, where θ is the probability of detecting each unit of illegal fishing effort and ψ is the per-unit-effort fine. Together $\theta\psi$ is the per-unit-effort expected fine for fishing illegally.

If some fraction of the reserve, L, is designated as a lease area, revenue for vessels paying to fish in the lease area is given by pqE_FX_RL. The cost of fishing in the lease area is given by $cE_L^\beta + \chi E_L$, where χ is the per-unit-effort access fee. The parameterization of lease area size implies that profit from fishing in the lease area is given by:

$$\Pi_L(E_L, X_R, L) = pqE_LX_RL - cE_L^\beta - \chi E_L$$

If a lease area exists (*i.e.* $L > 0$), we make the distinction between illegal fishing in the no-take area of the reserve, and illegal fishing in the lease area. Profits from illegal fishing in the no-take area of the reserve are given by:

$$\Pi_{I,N}(E_{I,N}, X_R, L) = pqE_{I_N}X_R(1 - L) - cE_{I_N}^\beta - \theta E_{I_N}\psi$$

We assume that some level of illegal fishing persists in the lease area, driven by the size of the per-unit-effort expected fine, and the amount of biomass left in the lease area after legal fishing has occured. Therefore profits from illegal fishing in the lease area are given by:

$$\Pi_{I,L}(E_{I,L}, X_R, L) = pqE_{I,L}X_RL(1 - qE_L) - cE_{I_L}^\beta - \theta E_{I_L}\psi$$

We can derive the marginal profits for fishing in each area by taking the derivatives of the equations above:

$$\pi_F(E_F) = \frac{\partial \Pi_F}{\partial E_F} = pqX_F - \beta c E_F^{\beta-1}$$

$$\pi_L(E_L) = \frac{\partial \Pi_L}{\partial E_L} = pqX_R L - \beta c E_L^{\beta-1} - \chi$$

$$\pi_{I_N}(E_{I,N}) = \frac{\partial \Pi_{I,N}}{\partial E_{I_N}} = pqX_R(1-L) - \beta c E_{I_N}^{\beta-1} - \theta\psi$$

$$\pi_{I_L}(E_{I,L}) = \frac{\partial \Pi_{I,L}}{\partial E_{I_L}} = pqX_R L(1-qE_L) - \beta c E_{I_L}^{\beta-1} - \theta\psi$$

Assuming that effort will continue to enter each area until the profit from the last unit of effort is exactly zero, we can then solve for equilibrium effort in each area:

$$E_F^* = \left(\frac{pqX_F}{\beta c}\right)^{\frac{1}{\beta-1}}$$

$$E_L^* = \left(\frac{pqX_R L - \chi}{\beta c}\right)^{\frac{1}{\beta-1}}$$

$$E_{I,N}^* = \left(\frac{pqX_R(1-L) - \theta\psi}{\beta c}\right)^{\frac{1}{\beta-1}}$$

$$E_{I,L}^* = \left(\frac{pqX_R L(1-qE_L^*) - \theta\psi}{\beta c}\right)^{\frac{1}{\beta-1}}$$

Stock dynamics

Harvest in each area is determined by fishing effort and stock size. Harvest from fishing in patch F is given by:

$$H_F = qE_F X_F$$

Harvest from fishing in patch R is the sum of harvest in the lease area (H_L), harvest from illegal fishing in the no-take area, and harvest from illegal fishing in the lease area:

$$H_R = H_L + H_{I,N} + H_{I,L} = qE_L X_R L + qE_I X_R(1-L) + qE_{I,L} X_R L(1-qE_L)$$

Total harvest is therefore given by:

$$\overline{H} = H_F + H_R$$

Within each patch, i, the stock grows logistically according to the Gordon-Schaefer model such that growth depends on the total initial biomass in that patch in time t ($\overline{X}_{i,t}$), the intrinsic growth rate of the stock (r), and the environmental saturation level or carrying capacity (K_i), expressed as maximum equilibrium biomass that a patch can hold. In principle, this can be patch-specific but we assume that it is constant through time and equal across space. Growth in patch i in time t is therefore given by:

$$g(X_{i,t}) = X_{i,t} + rX_{i,t}\left(1 - \frac{X_{i,t}}{K_i}\right) - H_{i,t}$$

In each time step, the stock in each patch redistributes according to the following dispersal matrix:

$$D = \begin{bmatrix} d_{F,F} & d_{F,R} \\ d_{R,F} & d_{R,R} \end{bmatrix}$$

where $d_{F,F}$ is the fraction of the stock from patch F that remains in patch F (*i.e.* self-recruitment) and $d_{F,R}$ is the fraction of the stock from patch F that redistributes to patch R (*i.e.* spillover). Similarly, $d_{R,F}$ is the fraction of the stock in patch R that redistributes to patch F and $d_{R,R}$ is the fraction of the stock in patch R that remains in patch R. $d_{F,F} + d_{F,R} = 1$ and $d_{R,R} + d_{R,F} = 1$.

Biomass in patch F in time $t+1$ is therefore given by:

$$X_{F,t+1} = d_{F,F} * g(X_{F,t}) + d_{R,F} * g(X_{R,t})$$

Similarly, biomass in patch R in time $t+1$ is given by:

$$X_{R,t+1} = d_{R,R} * g(X_{R,t}) + d_{F,R} * g(X_{F,t})$$

Alternative option for dispersal

The entire stock grows together:

$$\overline{X}_{t+1} = \overline{X}_t + r\overline{X}_t\left(1 - \frac{\overline{X}_t}{K}\right) - \overline{H}_t$$

After the stock grows, a constant fraction, f, of the total stock redistributes to the reserve, and the remainder $(1-f)$ redistributes to the general fishing area. Thus biomass in time period $t+1$ in the reserve and in the fishing area respectively are given by:

$$X_{R,t+1} = f\overline{X}_{t+1}$$

$$X_{F,t+1} = (1-f)\overline{X}_{t+1}$$

Enforcement dynamics

Enforcement in patch R reduces illegal fishing. The cost of enforcement (C_E) is given by αE_E, where α is a cost coefficient and E_E is enforcement effort (days). In other models, the cost of enforcement is generally covered by an exogenous budget (*i.e.* C_E is given), where available funding dictates the level of enforcement effort. Here we consider an alternative funding-model, where enforcement effort depends on access revenue generated from the lease area and fines from vessels caught fishing illegally in the reserve:

$$E_E = \frac{\chi E_L + \theta\psi(E_{I_N} + E_{I_L})}{\alpha}$$

The relationship between enforcement effort and the probability of detecting illegal fishing activity is also dynamic. With each additional unit of enforcement effort, the probability of any given unit of fishing effort getting caught changes according to:

$$\theta(E_E) = 1 - e^{-\mu E_E}$$

where μ is the enforcement coefficient which dictates how fast the probability of getting caught changes with respect to enforcement effort. μ can also be thought of as a scaling of reserve size; in smaller reserves perfect enforcement can rapidly be achieved with the first few units of enforcement effort, while in larger reserves it takes many more units of enforcement effort to achieve perfect enforcement.

Analytical Solution? Equilibrium biomass as a function of lease-area size

Matt - this is where we would love your help. It seems like it could be possible to analytically solve for equilibrium biomass as a function of L with some simplifications to the above model. However, we get slightly stuck. Here's what we tried so far... We assume here that a constant fraction of total biomass redistributes each time step between the two patches and that the probability of detection is constant

In equilibrium, we assume that total biomass must be unchanging. Thus, $\overline{X}_{t+1} = \overline{X}_t = \overline{X}^*$. Solving for the steady state harvest then yields:

$$\overline{H^*} = r\overline{X^*}\left(1 - \frac{\overline{X^*}}{K}\right)$$

Since total harvest is equal to the sum of all harvests, we can rewrite this as

$$qE_F^* X_F^* + qE_L^* X_R^* L + qE_{I_N}^* X_R^*(1-L) + qE_{I_L}^* X_R^* L(1-qE_L^*) = r\overline{X^*}\left(1 - \frac{\overline{X^*}}{K}\right)$$

Substituting for biomass in each area in terms of total biomass, we then get

$$qE_F^*\overline{X^*}(f-1) + qE_L^*\overline{X^*}fL + qE_{I_N}^*\overline{X^*}f(1-L) + qE_{I_L}^*\overline{X^*}fL(1-qE_L^*) = r\overline{X^*}\left(1 - \frac{\overline{X^*}}{K}\right)$$

Simplifying and substituting in each of our equilibrium efforts written in terms of total biomass then gives

$$(1-f)\left(\frac{pq\overline{X^*}(1-f)}{\beta c}\right)^{\frac{1}{\beta-1}} + fL\left(\frac{pq\overline{X^*}fL-\chi}{\beta c}\right)^{\frac{1}{\beta-1}} + f(1-L)\left(\frac{pq\overline{X^*}f(1-L)-\theta\psi}{\beta c}\right)^{\frac{1}{\beta-1}} + fL\left(\frac{pq\overline{X^*}fL-\theta\psi}{\beta c}\right)^{\frac{1}{\beta-1}} + \left(1-\left(pq\right.\right.$$

Matt: We then went on to try to solve this using the implicit function theorum ($\frac{dX}{dL} = \frac{\partial/\partial X}{\partial/\partial L}$), and it got very messy very quickly. Is there a better way?

======= ## Simulations

Model parameterization