Approximating Irrational Numbers and Points

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Abstract

This research focuses on examining how closely we can approximate any quadratic irrational number, α, whose characteristic equation is $A\alpha^2 + B\alpha + C = 0$ where $A, B, C \in \mathbb{Z}$. α can be approximated by the convergents $\frac{p_n}{q_n}$ from the continued fraction of α. If α is purely periodic we show $A\frac{p_n^2}{q_n^2} + B\frac{p_n}{q_n} + C = \frac{c}{q_n^2}$. We also examine how closely we can approximate any quadratic irrational point, (α, β), on the unit circle. We define the characteristic equation of the point by $p\alpha + q\beta - r = 0, \exists p, q, r \in \mathbb{Z}$. (α, β) can be approximated by$(\frac{a_n}{c_n}, \frac{b_n}{c_n})$ where (a_n, b_n, c_n) are obtained from the Berggren tree. When (α, β) is purely periodic, we show $|pa_n + qb_n - rc_n| = q - 1$.

Keywords: Convergents, Characteristic Equation, Purely Periodic.

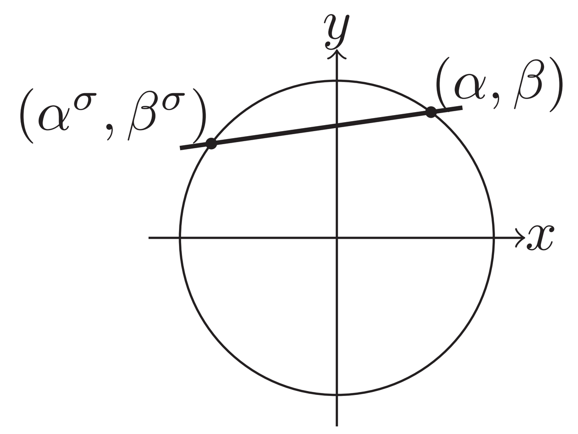
Background

For an irrational α, we can use the continued fraction $[a_0, a_1, a_2, ...]$ to approximate α by rationals. α lies between every two consecutive convergents $r_n = \frac{p_n}{q_n}, r_{n+1} = \frac{p_{n+1}}{q_{n+1}}$. When α is a quadratic irrational, which can be written as $A\alpha^2 + B\alpha + C = 0$ for $A, B, C \in \mathbb{Z}$, then $[a_0, a_1, a_2, ...]$ is eventually periodic.

Furthermore, for a quadratic irrational point, (α, β), where $p\alpha + q\beta = 1$, we investigated how $(\frac{a_n}{c_n}, \frac{b_n}{c_n})$ approximates (α, β). Every irrational point, $P = (\alpha, \beta)$, has a digit expansion with only 1,2,3. E.g. $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = [2, 2, ...] = [\overline{2}]$. To give a digit expansion analogous to a continued fraction we referred to Romik's research and examined the approximation of quadratic-irrational right triangles from the Berggren tree which are represented by the triple (a, b, c) with $a^2 + b^2 = c^2$. Given such a triangle, we can give a formula for the integer triple (a_n, b_n, c_n) which yields a more and more accurate approximation as $n \to \infty$. With this, we prove that there exists a characteristic equation $p\alpha + q\beta - r = 0, \exists p, q, r \in \mathbb{Z}$ when (α, β) is a quadratic irrational, which encompasses the ratios of sides in the triangle.

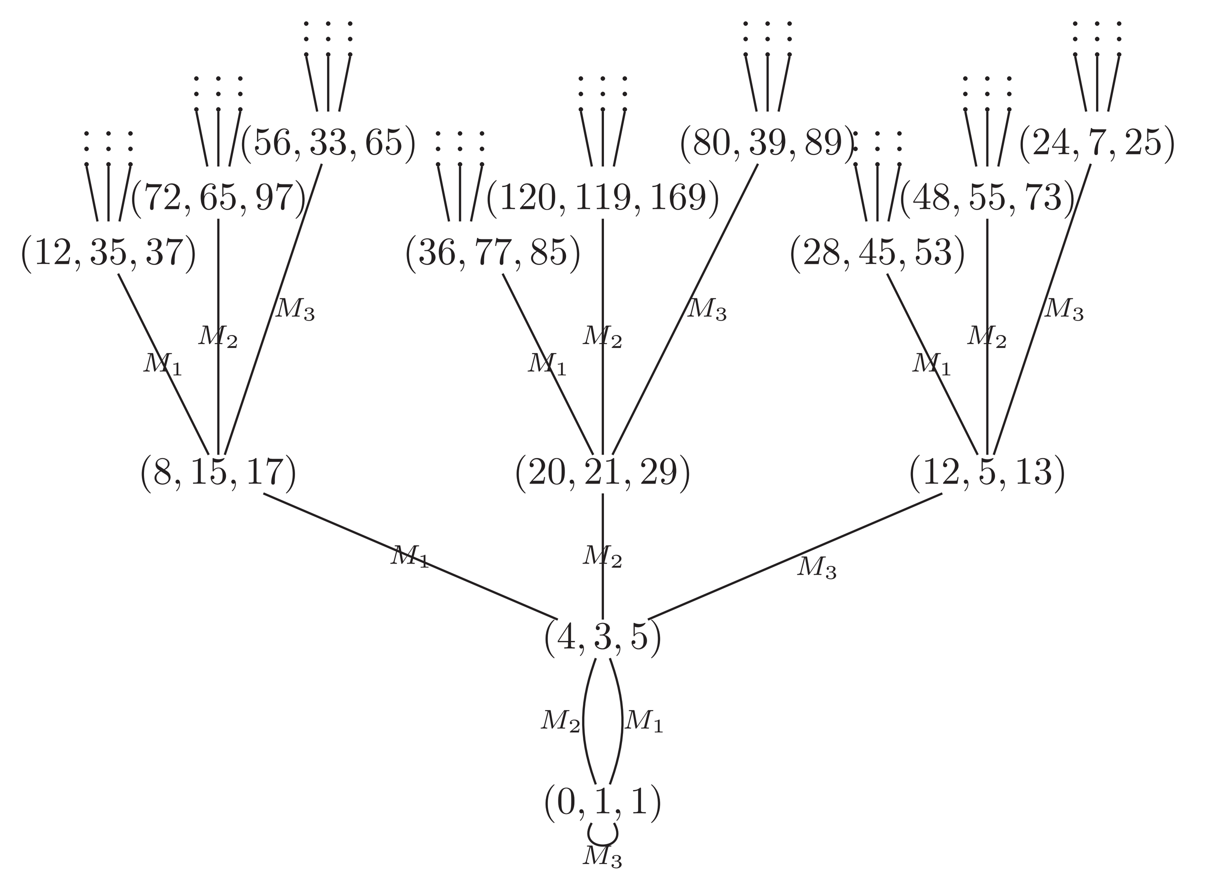
Background (Cont.)

Because the value $|pa_n + qb_n - rc_n|$ remains the same for all positive n, we know the point $pa_n + qb_n - rc_n$ is fixed so we found $p\frac{a_n}{c_n} + q\frac{b_n}{c_n} - r = \frac{q-1}{c_n}$, where $\frac{q-1}{c_n}$ is small.



The line connecting the points (α, β) and its conjugate, $(\alpha^\sigma, \beta^\sigma)$, can be found knowing the matrix, $P = \begin{bmatrix} \alpha & \alpha^\sigma & p \\ \beta & \beta^\sigma & q \\ 1 & 1 & 1 \end{bmatrix}$. Knowing that $p\alpha + q\beta = 1$, and taking the determinant, we can then find $p = -\frac{\beta - \beta^\sigma}{\alpha\beta^\sigma - \alpha^\sigma\beta}$ and $q = \frac{\alpha - \alpha^\sigma}{\alpha\beta^\sigma - \alpha^\sigma\beta}$. This helps us find the line that contains (α, β) and $(\alpha^\sigma, \beta^\sigma)$.

Berggren Tree



$M_1 = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ $M_3 = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$

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2×2 Example

Thm : α is a quadratic irrational whose characteristic equation is $A\alpha^2 + B\alpha + C = 0$. Suppose that α is the digit expansion $\overline{[a_0, a_1, ..., a_n]}$ for a purely periodic irrational continued fraction. We can then define $\frac{p_n}{q_n}$ as the n^{th} partial convergence. Let $\frac{p_n}{q_n}$ be the approximation of α.

We can then define $A\frac{p_n^2}{q_n^2} + B\frac{p_n}{q_n} + C$ as the characteristic equation for the approximation of α. Then we have $A\frac{p_n^2}{q_n^2} + B\frac{p_n}{q_n} + C = \frac{c}{q_n^2}$, where c is defined from the matrix M, being

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$$

Finding Characteristic Equation Example:

The continued fraction $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}$, gives us the digit expansion $[\overline{1}]$. We can then say $\alpha = 1 + \frac{1}{\alpha}$. So, $\alpha - 1 = \frac{1}{\alpha}$, and $\alpha^2 - \alpha = 1$, so $\alpha^2 - \alpha - 1 = 0$. So, we can see for the characteristic equation of α that $A = 1, B = -1, C = -1$ and solving for α we see $\alpha = \frac{1 \pm \sqrt{5}}{2}$. From here, we can find $M^n = P \cdot D^n \cdot P^{-1}$, where $M^n = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^n$ $P = \begin{bmatrix} b & b \\ \lambda - a & \overline{\lambda} - a \end{bmatrix}$, $D^n = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}^n$, $P^{-1} = \frac{1}{-b(\lambda - \overline{\lambda})} \begin{bmatrix} \overline{\lambda} - a & -b \\ -\lambda + a & b \end{bmatrix}$. So, we can see that

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^n \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{-1-\sqrt{5}}{2} & -1 \\ \frac{1-\sqrt{5}}{2} & 1 \end{bmatrix}$. Thus, c is 1, so we can approximate α with the characteristic equation $\frac{p_n^2}{q_n^2} - \frac{p_n}{q_n} - 1 = \frac{1}{q_n^2}$.

3×3 Example

Thm : (α, β) is a quadratic irrational point whose characteristic equation is $p\alpha + q\beta - r = 0$. Suppose that (α, β) is the digit expansion $\overline{[d_0, d_1, ..., d_n]}$ with d_k being 1, 2 or 3 corresponding to M_1, M_2, M_3 on the Berggren Tree respectively. This is a digit expansion analogous to the one obtained from a continued fraction in our 2×2 theorem. Let $(\frac{a_n}{c_n}, \frac{b_n}{c_n})$ be the approximation of (α, β) obtained by diagonalizing $M_g = M_{d_0}M_{d_1}...M_{d_n}$.

We can then define $pa_n + qb_n - rc_n$ as the characteristic equation for the approximation of (α, β). Then we have $|pa_n + qb_n - rc_n| = q - 1$, with q being the second value of the eigenvector $v = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ of M_g associated with the eigenvalue $\lambda = 1$ or $\lambda = -1$.

Finding Characteristic Equation Example:

The quadratic irrational point $g = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, gives us the digit expansion $\overline{[3, 1]}$. We can then say $M_g = M_3M_1 = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 \\ -4 & 7 & 8 \\ -4 & 8 & 9 \end{bmatrix}$. The eigenvector of M_g associated with the eigenvalue $\lambda = 1$ is $v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. From this we obtain the characteristic equation $2\alpha - 1 = 0$ with $p = 2, q = 0$ and $r = 1$. Thus, q is 0, so we can approximate (α, β) with the characteristic equation $|2a_n - c_n| = -1$.