Approximating Irrational Triangles

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Math/CS Colloquium

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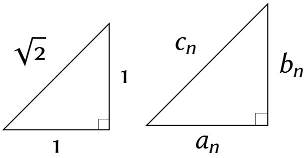
Brittany Gelb's ('21) formula

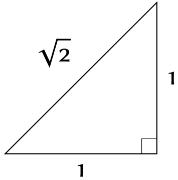
$$a_n = \frac{(1+\sqrt{2})(3+2\sqrt{2})^n + (1-\sqrt{2})(3-2\sqrt{2})^n}{4} - (-1)^n\frac{1}{2},$$
$$b_n = \frac{(1+\sqrt{2})(3+2\sqrt{2})^n + (1-\sqrt{2})(3-2\sqrt{2})^n}{4} + (-1)^n\frac{1}{2},$$
$$c_n = \frac{(2+\sqrt{2})(3+2\sqrt{2})^n + (2-\sqrt{2})(3-2\sqrt{2})^n}{4}.$$

n	(a_n, b_n, c_n)
1	(3, 4, 5)
2	(21, 20, 29)
3	(119, 120, 169)
4	(697, 696, 985)
...	



How do we characterize $1 - 1 - \sqrt{2}$ and those approximating integer triples?

$$\frac{a}{b} = \frac{1}{1}$$
$$a = b$$
$$a - b = 0$$



n	(a_n, b_n, c_n)
1	$(3, 4, 5)$
2	$(21, 20, 29)$
3	$(119, 120, 169)$
4	$(697, 696, 985)$
...	

What is the next best thing? $|a - b| = 1$.

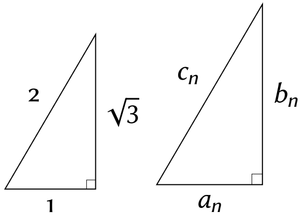
My interest in a different triangle

$$a_n = \frac{(2+\sqrt{3})(7+4\sqrt{3})^n + (2-\sqrt{3})(7-4\sqrt{3})^n}{6} - \frac{2}{3},$$

$$b_n = \frac{(3+2\sqrt{3})(7+4\sqrt{3})^n + (3-2\sqrt{3})(7-4\sqrt{3})^n}{6},$$

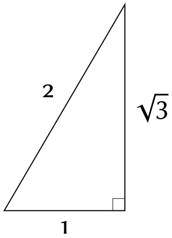
$$c_n = \frac{(2+\sqrt{3})(7+4\sqrt{3})^n + (2-\sqrt{3})(7-4\sqrt{3})^n}{3} - \frac{1}{3}.$$

n	(a_n, b_n, c_n)
1	(8, 15, 17)
2	(120, 209, 241)
3	(1680, 2911, 3361)
4	(23408, 40545, 46817)
...	



Characterizing $1 - \sqrt{3} - 2$

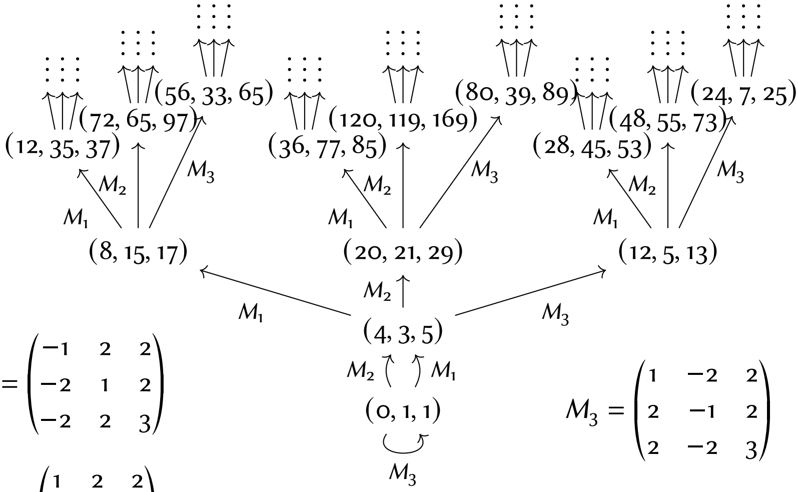
$$\frac{c}{a} = \frac{2}{1}$$
$$2a = c$$
$$c - 2a = 0$$



Next best thing: $c - 2a = 1$.

n	(a_n, b_n, c_n)
1	$(8, 15, 17)$
2	$(120, 209, 241)$
3	$(1680, 2911, 3361)$
4	$(23408, 40545, 46817)$
...	

Berggren (1934) Tree

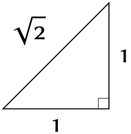


$$M_1 = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

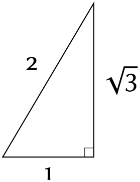
$$M_3 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

How to find approx. integer triples for a specific irrational triangle



$$\begin{array}{c}(119, 120, 169)\\ M_2 \big\uparrow \\ (21, 20, 29)\\ M_2 \big\uparrow \\ (3, 4, 5)\end{array}$$

$$M^n \begin{bmatrix}1\\0\\1\end{bmatrix} = (M_2)^n \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}a_n\\b_n\\c_n\end{bmatrix}$$



$$\begin{array}{c}(1680, 2911, 3361)\\ \cdot \xrightarrow{M_1} \\ \xleftarrow{M_3} \\ (120, 209, 241)\\ \cdot \xrightarrow{M_1} \\ \xleftarrow{M_3} \\ (8, 15, 17)\end{array}$$

$$M^n \begin{bmatrix}0\\1\\1\end{bmatrix} = (M_3 M_1)^n \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}a_n\\b_n\\c_n\end{bmatrix}$$

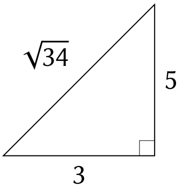
What if the k value isn't 1?

$$\frac{a}{b} = \frac{3}{5}$$

$$5a = 3b$$

$$5a - 3b = 0$$

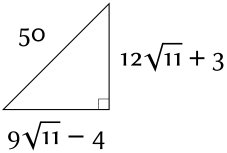
$$5a - 3b = -3$$



n	(a_n, b_n, c_n)
1	(2340, 3901, 4549)
2	(11463480, 19105801, 22281001)
3	(56148124860, 93580208101, 109132338349)
4	(275013504102960, 458355840171601, 534530170952401)
...	

My theorem:

For any right, quadratic-irrational triangle, a characteristic equation can be written in the form of $pa + qb + rc = 0$, where p, q, and r are integers.

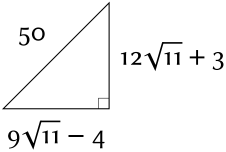


$$-8a + 6b - c = 0$$

Furthermore...

A formula for the approximating integer triangle (a_n, b_n, c_n) can be written as $|pa + qb + rc| = k$, with p, q, r, and k being integers.

AND, there exists a minumum value for k.



$$-8a + 6b - c = 5$$

Thank you!