APPROXIMATING IRRATIONAL TRIANGLES

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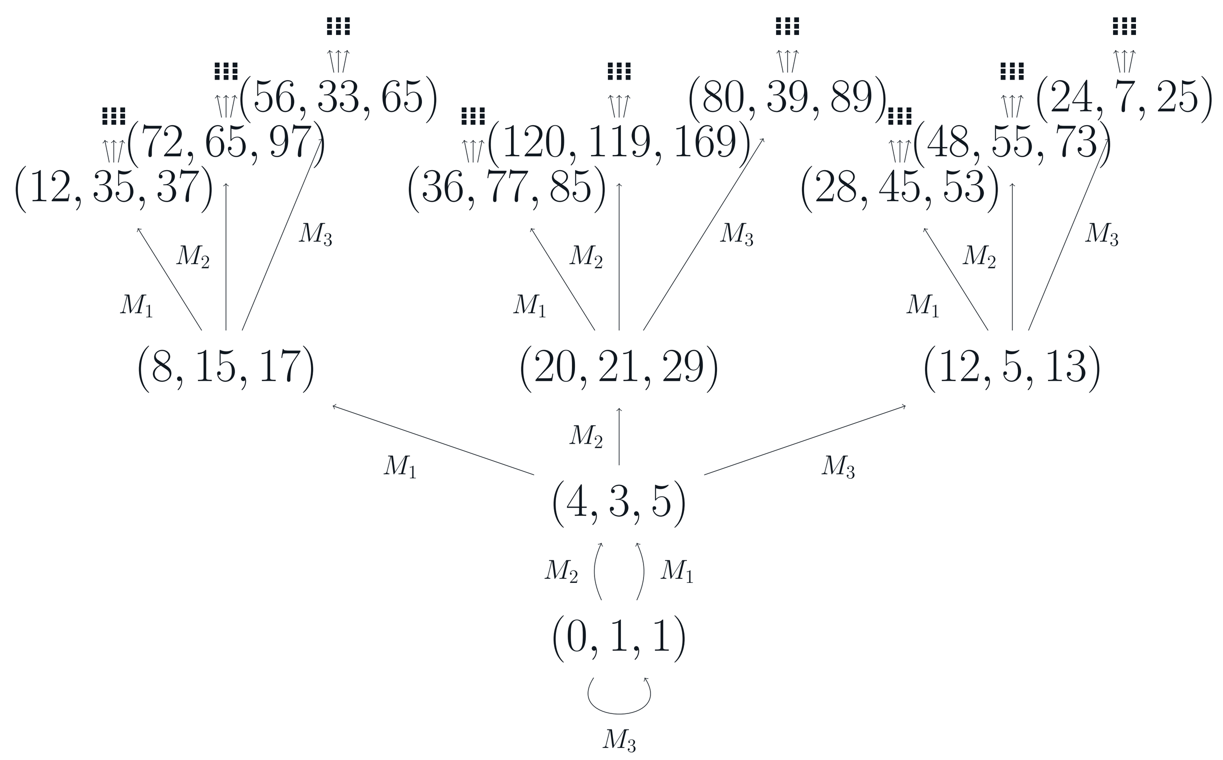
Abstract

This research examines the approximation of quadratic-irrational right triangles represented by the triple (α, β, γ) with $\alpha^2 + \beta^2 = \gamma^2$. Given such a triangle, we can give a formula for the integer triple (a_n, b_n, c_n) which yields a more and more accurate approximation as $n \to \infty$. We prove that there exists a characteristic equation $p\alpha + q\beta - r\gamma = 0$ with p, q, and r being integers which encompasses the ratios of sides in the triangle. The value $|pa_n + qb_n - rc_n|$ remains the same for all positive n.

Background

- Berggren (1934) discovered a method to generate all positive primitive integer pythagorean triples in a tree structure using (M_1, M_2, M_3) matrix multiplication.



$$M_1 = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

- Brittany Gelb ('21) showed that, for any positive n, the approximating triples (a_n, b_n, c_n) of the isosceles $(1, 1, \sqrt{2})$ can be found on the middle path, M_2, of the Berggren tree.

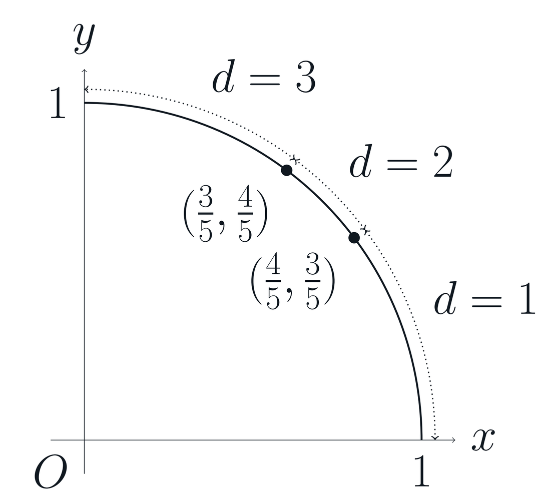
$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = {M_2}^n \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{(1+\sqrt{2})(3+2\sqrt{2})^n + (1-\sqrt{2})(3-2\sqrt{2})^n}{4} - (-1)^n \frac{1}{2} \\ \frac{(1+\sqrt{2})(3+2\sqrt{2})^n + (1-\sqrt{2})(3-2\sqrt{2})^n}{4} + (-1)^n \frac{1}{2} \\ \frac{(2+\sqrt{2})(3+2\sqrt{2})^n + (2-\sqrt{2})(3-2\sqrt{2})^n}{4} \end{pmatrix}$$

n	(a_n, b_n, c_n)
1	$(4, 3, 5)$
2	$(20, 21, 29)$
3	$(120, 119, 169)$
4	$(696, 697, 985)$
...	

Formula for (a_n, b_n, c_n)

- By normalizing (α, β, γ) we represent the triangle as a point $P = (\frac{\alpha}{\gamma}, \frac{\beta}{\gamma})$ on the unit quarter circle.
- Romik (2008) discovered a system to translate a point P on the unit circle into a digit expansion format $P = [d_1, d_2, ...]$ where $d_k = d(T^{k-1}(P))$.

$$T(x, y) = \left(\frac{|2 - x - 2y|}{3 - 2x - 2y}, \frac{|2 - 2x - y|}{3 - 2x - 2y} \right)$$



- Lagrange's theorem: P has repeating tails if and only if P is defined over $\mathbb{Q}(\sqrt{D})$.
- When P is defined over $\mathbb{Q}(\sqrt{D})$, $P = [\overline{d_1, \ldots, d_m}]$.
- Based on the digits of P, we can define M_P as

$$M_P = M_{d_1} M_{d_2} \cdots M_{d_m}.$$

- We define (a_n, b_n, c_n) by

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = {M_P}^n \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}.$$

- By diagonalizing M_P we obtain a formula for (a_n, b_n, c_n).

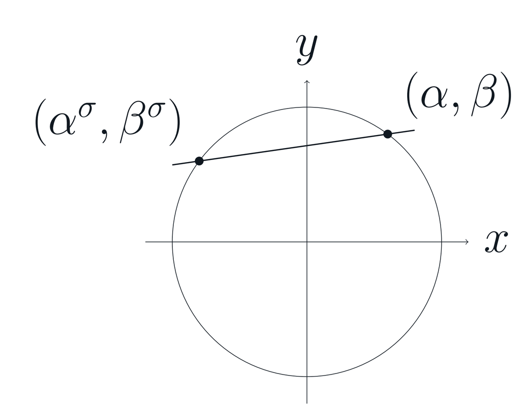
Characteristic Equation

- Theorem: Given a quadratic-irrational right triangle $(\alpha, \beta, 1)$, a characteristic equation

$$p\alpha + q\beta - r = 0$$

can be written, with p, q, and r being integers without common factor. Such integers are unique for each triangle $(\alpha, \beta, 1)$.

- Graphically, the characteristic equation can be respresented as the unique line intersecting the two points (α, β) and its conjugate $(\alpha^\sigma, \beta^\sigma)$.



Characterisitc Equation (Cont.)

- (p, q, r) is an eigenvector of M_P associated with the eigenvalue 1 or -1.
- With the pairing method and M_P being orthogonal, we can write

$$p\alpha + q\beta - r = \left\langle \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}, \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right\rangle = \left\langle M_P \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}, M_P \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right\rangle$$

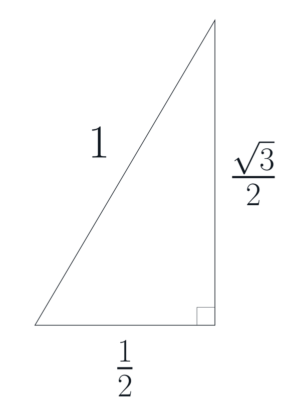
And because λ is an eigenvalue of M_P and $(\alpha, \beta, 1)$ is an eigenvector

$$= \left\langle \lambda \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}, \pm 1 \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right\rangle = \pm\lambda \left\langle \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}, \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}, \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right\rangle$$

From this, we obtain $p\alpha + q\beta - r = 0$

- Theorem: $|pa_n + qb_n - rc_n|$ is non-zero and the same for all $n \geq 1$.

Examples

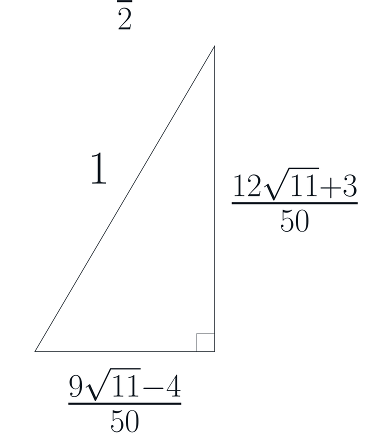


$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) = [\overline{3, 1}]$$

$$-2\alpha + 1 = 0$$

$$-2a_n + c_n = 1$$

n	(a_n, b_n, c_n)
1	$(8, 15, 17)$
2	$(120, 209, 241)$
3	$(1680, 2911, 3361)$
4	$(23408, 40545, 46817)$
...	



$$\left(\frac{9\sqrt{11} - 4}{50}, \frac{12\sqrt{11} + 3}{50} \right) = [\overline{3, 1, 2, 2}]$$

$$-8\alpha + 6\beta - 1 = 0$$

$$-8a_n + 6b_n - c_n = 5$$

n	(a_n, b_n, c_n)
1	$(180, 299, 349)$
2	$(71800, 118881, 138881)$
3	$(28576380, 47314219, 55274269)$
4	$(11373327600, 18830940161, 21999020161)$
...	

Acknowledgements

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