Modeling the underlying dynamics of the spread of crime

Participant: Emmy Lin

Mentor: Xuan Loc Huynh, Directed Reading Program

January 21, 2026

Solving an Ordinary Differential Equation (ODE)

In general, we want to solve:

$$\dot{x}(t) = f(x(t)), \text{ given the intital condition: } \quad x(0) = x_0$$

What's the notation?

- $x(t)$: some state of the system at time t
- $\dot{x}(t)$: rate that state changes with respect to time

Solving a simple ODE:

$$\frac{dx}{dt} = \alpha x, \quad x(0) = x_0$$

Step 1: Separate variables:

$$\frac{dx}{x} = \alpha\, dt$$

Step 2: Integrate both sides:

$$\int \frac{dx}{x} = \int \alpha\, dt \implies \ln|x| = \alpha t + C$$

Step 3: Solve for $x(t)$:

$$x(t) = e^{\alpha t + C} = C' e^{\alpha t}$$

Step 4: Apply initial condition $x(0) = x_0$:

$$x_0 = C' \implies x(t) = x_0 e^{\alpha t}$$

What if we have this System of Equations? (multiple ODEs linked together)

$$\frac{dx_1}{dt} = 2x_1 + 3x_2, \quad \frac{dx_2}{dt} = -x_1 + 4x_2$$

Since it is a linear system, we can write it in Matrix Form!

$$\dot{x}(t) = Ax(t), \quad A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Eigenvalues & Eigenvectors:

$$\det(A - \lambda I) = 0 \implies \lambda_1 = 3, \ \lambda_2 = 3$$

$$v_1 = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

General Solution:

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

Solution with Initial Condition x_0:

$$x(t) = e^{At} x_0 = c_1 e^{3t} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x_0 = c_1 v_1 + c_2 v_2$$

We can solve linear systems

We start with a general system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0.$$

If f is linear, then $f(x) = Ax$, and the system becomes

$$\dot{x}(t) = Ax(t).$$

The solution is

$$x(t) = e^{At}x_0,$$

where the matrix exponential is defined by the power series

$$e^{At} := \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

What if we have a non-linear system of differential equations such as:

$$\dot{x} = x^2 + t, \quad \dot{x}_1 = x_1 x_2$$

We can use Euler's method to give us an approximation of the solution.

What is Euler's method and what does it look like when it's specialized to the linear system (as an example, so we can understand it)?

$$\dot{x}(t) = f(x(t))$$

Using the Definition of the derivative as a limit:

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

In Euler's method, we pick a finite time step to approximate the next value of x.

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \approx \dot{x}(t) = f(x(t))$$

$$x(t + \Delta t) \approx x(t) + \Delta t \, f(x(t))$$

Substitute the linear system $\dot{x} = Ax$:

$$x_{k+1} = x_k + \Delta t \, A x_k$$

Factor out x_k

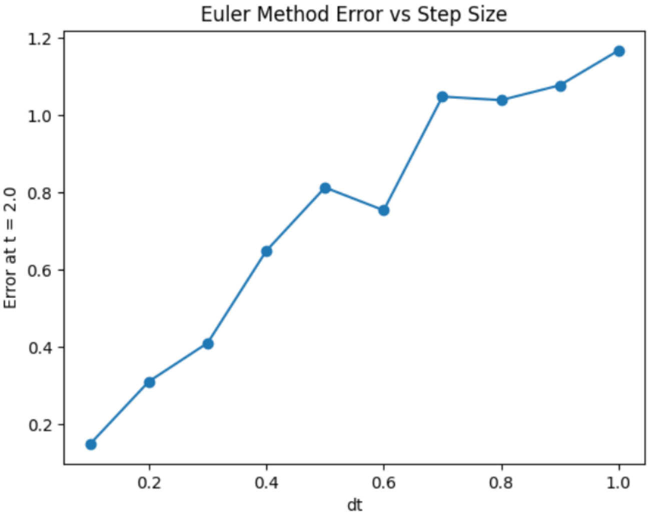
$$x_{k+1} = (I + \Delta t \, A) \, x_k$$

Using Python to observe the accuracy of Euler's method

```python
import numpy as np
from scipy.linalg import expm

A = np.array([[0, 1],
              [-1, 0]])

x0 = np.array([1, 0])
tfinal = 2.0

dt_list = np.arange(1, 0, -0.1)

xprecise = expm(A * tfinal) @ x0

errors = []

for dt in dt_list:
    n = int(tfinal / dt)
    x = x0.copy()
    I = np.eye(2)
    for i in range(n):
        x = (I + A * dt) @ x
    print(f"dt: {dt}")
    print(f"Analytical solution: {xprecise}")
    print(f"Numerical solution: {x}")
    print(f"Error: {np.linalg.norm(x - xprecise)}'
    print()
    errors.append(np.linalg.norm(x - xprecise))
```

```
dt: 0.6000000000000001
Analytical solution: [-0.41614684 -0.90929743]
Numerical solution: [-0.08  -1.584]
Error: 0.7538025324769726

dt: 0.5000000000000001
Analytical solution: [-0.41614684 -0.90929743]
Numerical solution: [ 0.25  -1.375]
Error: 0.8127917903761983

dt: 0.40000000000000013
Analytical solution: [-0.41614684 -0.90929743]
Numerical solution: [ 0.0656 -1.344 ]
Error: 0.6488808377873029

dt: 0.30000000000000016
Analytical solution: [-0.41614684 -0.90929743]
Numerical solution: [-0.229229 -1.27458 ]
Error: 0.4103286924945842

dt: 0.20000000000000018
Analytical solution: [-0.41614684 -0.90929743]
Numerical solution: [-0.24375296 -1.16785971]
Error: 0.31076374303563165

dt: 0.1000000000000002
Analytical solution: [-0.41614684 -0.90929743]
Numerical solution: [-0.34878537 -1.04233282]
Error: 0.1491173519290884
```

Euler method is more accurate as step size decreases

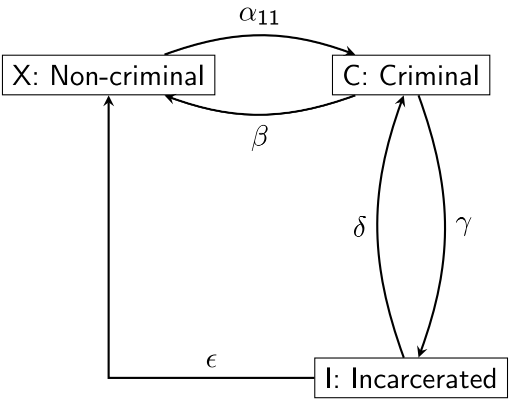


The graph appears linear. As we increase the step size, the error increases linearly, meaning the algorithm is O(n).

Modeling the underlying dynamics of the spread of crime: David McMillon1, Carl P. Simon, Jeffrey Morenoff

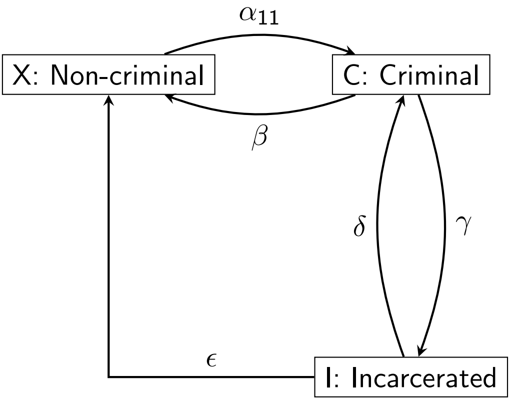
- The spread of crime is a complex, dynamic process that requires a systems-level approach.
- We build and analyze dynamical systems models for:
  - Crime, imprisonment, and recidivism
  - Using abstract transition parameters
- Analytic expressions are computed for:
  - Equilibria
  - Tipping points between high-crime and low-crime states
- These expressions are used to examine:
  - Effects of longer prison terms
  - Effects of increased incarceration rates
  - Implications of a Three-Strike Policy

The simple 3 dimensional model



- α_{11}: Rate one becomes a criminal by interacting with a criminal
- β: Rate criminals discontinue criminal habits
- γ: Rate criminals are incarcerated
- ϵ: Rate incarcerated individuals are released and reenter society
- δ: Rate incarcerated individuals are released and return to criminal life

We derive a system of ODEs



- $\dot{X} = \beta C - \alpha_{11} \frac{XC}{N-I} + \epsilon I$
- $\dot{C} = -\beta C + \alpha_{11} \frac{XC}{N-I} - \gamma C + \delta I$
- $\dot{I} = \gamma C - (\delta + \epsilon) I$
- $N = X + C + I$

Solving for high and low crime equilibrium

- Rearrange the last equation to be $X = N - C - I$ so that we can get rid of $\dot{X}$.
- Substituting X into the system, we get a 2-dimensional system:

$$\dot{C} = -\beta C + \alpha_{11} \frac{(N - C - I)C}{N - I} - \gamma C + \delta I$$
$$\dot{I} = \gamma C - (\delta + \epsilon) I$$

- To find equilibrium states, set $\dot{C} = 0$ and $\dot{I} = 0$:

$$0 = -\beta C + \alpha_{11} \frac{(N - C - I)C}{N - I} - \gamma C + \delta I$$
$$0 = \gamma C - (\delta + \epsilon) I$$

- There are two equilibriums:
  - $C = 0, I = 0$
  - $I = \frac{\gamma}{\delta + \epsilon} C$ and

$$-\beta C + \alpha_{11} \frac{(N - C - I)C}{N - I} - \gamma C + \delta \frac{\gamma}{\delta + \epsilon} C = 0$$

Solving for high and low crime equilibrium

- Solving for C in the second equilibrium:

$$\frac{C}{N-I} = 1 - \frac{1}{\alpha_{11}}\left(\beta + \gamma - \frac{\gamma\delta}{\delta + \epsilon}\right)$$

$$C = (N-I)\frac{\alpha_{11} - (\beta+\gamma) + \frac{\gamma\delta}{\delta+\epsilon}}{\alpha_{11}}$$

- Thus, the zeros are:

$$C = 0, I = 0 \quad \text{or} \quad C = \frac{N(\alpha_{11} - (\beta+\gamma))}{\alpha_{11}}, I = \frac{\gamma}{\delta+\epsilon}C$$

Use Phase Portraits to see how the system behaves for the following three cases

- **Case (a):** $\alpha_{11} - (\beta + \gamma) > 0$
  - Origin $(C, I) = (0, 0)$ is **unstable**
  - All trajectories go to the endemic (high-crime) equilibrium
- **Case (b):** $\alpha_{11} - (\beta + \gamma) < 0$ and $(\beta + \gamma) - \alpha_{11} < \frac{\gamma\delta}{\delta+\epsilon}$
  - Origin $(0, 0)$ is **unstable**
  - Endemic equilibrium is **stable**
  - Two equilibria exist, but only the endemic one attracts trajectories
- **Case (c):** $\alpha_{11} - (\beta + \gamma) < 0$ and $(\beta + \gamma) - \alpha_{11} > \frac{\gamma\delta}{\delta+\epsilon}$
  - Origin $(0, 0)$ is the **only equilibrium**
  - Globally **stable**
  - System converges to crime-free state

Condition for Crime-Free Equilibrium

- The system converges to a crime-free equilibrium if the parameters satisfy:

$$(\beta + \gamma) - \alpha_{11} > \frac{\gamma\delta}{\delta + \epsilon}$$

- Algebraically, this threshold can also be written as:

$$\frac{\alpha_{11}}{(\beta + \gamma)(\frac{\epsilon}{\epsilon + \delta})} < 1$$

- Interpretation:
  - If the inequality holds, the system goes to $(C, I) = (0, 0)$ (crime-free equilibrium)
  - Otherwise, it will converge to a high-crime endemic equilibrium

What if you want to find the threshold without graphing the phase portraits

Theorem (Lyapunov Stability)

Let $x = 0$ be an equilibrium point for the system

$$\dot{x} = f(x)$$

and let $D \subset \mathbb{R}^n$ be a domain containing $x = 0$.
Suppose there exists a continuously differentiable function $V : D \to \mathbb{R}$ such that:

- $V(0) = 0$ *and* $V(x) > 0$ *for all* $x \in D \setminus \{0\}$
- $\dot{V}(x) \leq 0$ *for all* $x \in D$

Then $x = 0$ is **stable**.
Moreover, if $\dot{V}(x) < 0$ for all $x \in D \setminus \{0\}$, then $x = 0$ is **asymptotically stable**.

Using the Lyapunov Function to solve for threshold

- Consider the Lyapunov function:

$$V(C, I) = C + AI, \quad A > 0, \quad (C, I) \in [0, \infty) \times [0, \infty)$$

- Its derivative along system trajectories:

$$\begin{aligned}
\dot{V}(C, I) &= \dot{C} + A\dot{I} \\
&= \alpha_{11} \frac{C(N - C - I)}{N - I} - (\beta + \gamma)C + \delta I \\
&\quad + A\big(\gamma C - (\delta + \epsilon)I\big) \\
&= \underbrace{\left(\alpha_{11} \frac{(N - C - I)}{N - I} - (\beta + \gamma) + A\gamma\right)}_{\text{terms with } C} C + \underbrace{(\delta - A\delta - A\epsilon)}_{\text{terms with } I} I
\end{aligned}$$

- Since $A > 0$, $V(C, I) > 0$ for all $(C, I) \neq (0, 0)$.

Using the Lyapunov Function to solve for threshold

- For stability, require $\dot{V} < 0$ for all $C, I \geq 0$:

$$\alpha_{11} \frac{(N - C - I)}{N - I} - (\beta + \gamma) + A\gamma < 0, \quad \delta - A\delta - A\epsilon < 0$$

- Solve for A:

$$\frac{\delta}{\delta + \epsilon} < A < \frac{\beta + \gamma - \alpha_{11}}{\gamma}$$

- This yields the threshold condition for convergence to the crime-free equilibrium:

$$\frac{\alpha_{11}}{(\beta + \gamma)(\frac{\epsilon}{\epsilon + \delta})} < 1$$

- **Conclusion:** We recover the same threshold from the phase portrait analysis.

Interpreting this threshold

- α_{11}: Contagion parameter of criminal behavior
- β: Rate at which criminals discontinue criminal habits
- γ: Rate at which criminals are incarcerated
- δ: Rate at which incarcerated individuals return to criminal life
- ϵ: Rate at which incarcerated individuals are released and re-enter society

$$\frac{\alpha_{11}}{(\beta+\gamma)(\frac{\epsilon}{\epsilon+\delta})} < 1$$

When analyzing the threshold, it makes sense for it to be less than 1. The numerator represents the rate at which crime is committed through contact with criminals, and the denominator represents the rate at which individuals discontinue crime, go to prison, or are released. As the denominator becomes larger and the numerator smaller, the system will converge to a low-crime equilibrium.

References

McMillon, D., Simon, C.P., Morenoff, J. (2014). *Modeling the Underlying Dynamics of the Spread of Crime*. PLoS ONE, 9(4), e88923. doi:10.1371/journal.pone.0088923

Hirsch, M.W., Smale, S., & Devaney, R.L. (2013). *Differential Equations, Dynamical Systems, and an Introduction to Chaos*. Academic Press.

Questions and Answers