ON THE EXPECTED VALUE OF RECURRENT GAMES OF CHANCE

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ABSTRACT. We define a recurrent game of chance, then provide formulas for calculating it's expected value.

1. RECURRENT GAME OF CHANCE DEFINITION.

A betting game that starting with an original amount being bet, a set of probabilities associated with different payoffs, then recursively applies this game conditions to the result of each successive step of the game.

What differentiates this type of game from the ones used in the regular formula of expected value is that in this type of game no new amount is added in the game, the bets are always recursively applied to the result of the each step of the game.

We could then consider two different types of this game, either the betting party can decide whether to continue the game after each successive step or the game is applied with a fixed number of steps.

2. SIMPLE EXAMPLE

Let's present a simple example:

We start with an initial amount being bet of \$100 dollars.

And we will have the following game conditions.

We will be flipping a coin and it is assumed that the process of flipping the coin is fair and we have equal probability of getting either heads or tails.

Now, if the coin lands on heads we increase the amount by 50% and if it lands on tails we decrease the amount by 40%.

For clarity we define:

x_0 as the initial amount

x_n as the amount after n steps of the game.

b_h as the coefficient we multiply the amount by if it lands on heads.

P(h) as the probability of the coin landing on Heads.

In this example $b_h = 1.5$, $P(h) = 0.5$

b_t as the coefficient we multiply the amount by if it lands on tails.

P(t) as the probability of the coin landing on Heads.

In this example $b_t = 0.6$, $P(t) = 0.5$

Based on this conditions, we can conclude that this is a losing game for the betting party, and the rational decision when presented with this game is not to play the game. We will provide a proof and a general formula for this type of games in the following sections.

3. General result for two different payoffs with same probability

Let's generalize the game presented before, we define:

x_0 is the original amount.

And positive b, b' as the two possible outcomes with equal probabilities P(b) = P(b') = 0.5%.

Also we define b to be the winning outcome (meaning it increases the amount at play) and b' the losing outcome (meaning it decreases the amount). So another condition is that $b > 1 > b' > 0$.

Sidenote (this should be a commentary at the bottom): If b $>$b' $>$1 or 1 $>$b $>$b' there is nothing to prove or even reason about. If b $>$b' $>$1 any outcome is positive for the betting party and if 1 $>$b $>$b' $>$0 any outcome is negative.

The expected value of the recurrence x_n will be:

a. If b' $>$1/b it will tend to infinity.

b. If b' $<$1/b it will tend to 0.

c. If b'= 1/b it will follow a binomial distribution around x_0.

To start thinking about this, we'll think about case c through the following exercise.

Let's imagine a game on the number line, we are starting on 0, and then when we flip a fair coin, if it comes heads (an event we will call a) we move one full positive integer to the right and if it comes tails (an event we will call a') we move a full positive integer to the left.

Arithmetically, we start in 0, if we drew a we sum 1, if we drew a' we sum -1.

We will represent any sequence of events by strings of the alphabet {a, a'}. Strings will be enclosed by ".

We will call this game G_+.

Based on our definition, to calculate the value of the string in G_+ we start at 0 and then add +1 for every character a and -1 for every character a'.

For example:

"aaa" = 0 + 1 + 1 + 1 = 3

"aa'a" = 0 + 1 -1 + 1 = 1

"a'a'aa" = 0 -1 -1 + 1 + 1 = 0

And interesting thing is that in any sequence of events any pair of the characters a and a' will annulate themselves. So in any sequence of events the possible outcome in terms of the result will amount to just calculating the number of ”a”s or ”a’”s left after taking out the pairs of ”a” and ”a’”s in the string.

Examples, to calculate the value of the game we do:

”aa’aa’” = ”” = 0

”aaaa’” = ”aa” = 2

”aa’aa” = ”aa” = 2

”a’a’a’a” = ”a’a’” = -2

An interesting thing to notice first is that for any string of size n, if n is even then the results will be even and if n is odd then the result will be odd too.

EXPLAINS HOW BEFORE RELATES TO PASCAL TRIANGLE

THEN DO NEXT

How is this related to case c in our original game (which we will call G_*?

Let’s reformulate our original game G_*, in terms of this new one G_+.

First if b’ = 1/b which means that b’ = b^{-1}.

Let’s say that we start with an amount x_0 and when we drew a, we multiply our current x by b and if we drew a’ we multiply by b^{-1}.

So any strings of a, a’) becomes an equivalent sequence of b, b^{-1} but in G_* we are multiplying instead of summing. For clarity we’ll represent multiplication with *.

Examples:

”aa’aa” in G_+ becomes

”bb’bb’” = b * 1/b * b * 1/b = 1 = b^0

”aaaa’” in G_+ becomes

”bbbb’” = b * b * b * 1/b = b^2

”aa’aa” in G_+ becomes

”bb’bb” = b*b’*b*b = b * 1/b * b * b = b^2

”a’a’a’a” in G_+

becomes b*b*b*b’ = 1/b * 1/b * 1/b * b = b^{-2}

It is evident that the results of G_+ are equivalent to the exponentials of the results of G_* by simple arithmetics.

—– This new game might not be a clear explanation...

– Rant about b’ = 1/b is the exact definition of a fair game.. go on about pascal triangle and binomial distribution

OTHER CASES

HERE: This is the trick, and this is what I should be arriving at.

if b’ $\neq$ 1/b $\implies$ b * b’ = 1 + ϵ .

explain what that means in terms of the expected value of the recurrence

4. MULTIPLE OUTCAMES WITH EQUAL PROBABILITY

m number of possible outcames, each with same probability and associated with a coefficient b_1 to b_m

Result of the recurrence at infinity:

Let’s define s = $\prod_{i=1}^{m} b_i$

if s $>$1 tends to infinity

if s <1 tends to 0

if s = 1 tends to x_0

Prove algebraically.

5. MULTIPLE OUTCAMES EACH WITH IT'S OWN PROBABILITY

m number of possible outcames, each with it's own coefficient b_i and it's own probability p_i

Result of the recurrence at infinity:

Let's define s = $\prod_{i=1}^{m} mp_i b_i$

if s >1 tends to infinity

if s <1 tends to 0

if s = 1 tends to x_0

Examine distributions of probabilities.. this one is harsher.. to prove

Define expected value of recurrence r_n

$r_0 = x_0$

$r_n = sr_{n-1}$

Explain based on the definition of the common expected value formula

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