ON THE EXPECTED VALUE OF RECURRENT GAMES OF CHANCE

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ABSTRACT. We define a recurrent game of chance, then provide formulas for calculating it's expected value.

1. RECURRENT GAME OF CHANCE DEFINITION.

A betting game that starting with an original amount being bet, a set of probabilities associated with different payoffs, then recursively applies this game conditions to the result of each successive step of the game.

What differentiates this type of game from the ones used in the regular formula of expected value is that in this type of game no new amount is added in the game, the bets are always recursively applied to the result of the each step of the game.

We could then consider two different types of this game, either the betting party can decide whether to continue the game after each successive step or the game is applied with a fixed number of steps.

2. SIMPLE EXAMPLE

Let's present a simple example:

We start with an initial amount being bet of \$100 dollars.

And we will have the following game conditions.

We will be flipping a coin and it is assumed that the process of flipping the coin is fair and we have equal probability of getting either heads or tails.

Now, if the coin lands on heads we increase the amount by 50% and if it lands on tails we decrease the amount by 40%.

We repeat this process n times applying this rules on the results of each previous step.

For clarity we define:

x_0 as the initial amount

x_n as the amount after n steps of the game.

b_h as the coefficient we multiply the amount by if it lands on heads.

P(h) as the probability of the coin landing on Heads.

In this example $b_h = 1.5$, $P(h) = 0.5$

b_t as the coefficient we multiply the amount by if it lands on tails.

P(t) as the probability of the coin landing on Heads.

In this example $b_t = 0.6$, $P(t) = 0.5$

Based on this conditions, we can conclude that this is a losing game for the betting party, and the rational decision when presented with this game is not to play the game. We will provide a proof and a general formula for this type of games in the following sections.

3. General result for two different payoffs with same probability

Let's generalize the game presented before which we will call G_*, we define:

x_0 is the original amount.

x_n as the value of the game in step n.

We define e_w and e_l as two posible events that can happen in any step of the recurrence with equal probability.

If event e_w happens at step n we calculate the value of recurrence at step n as:

$x_n = b_w * x_{n-1}$

If event e_l as:

$x_n = b_l * x_{n-1}$

Both events have equal probabilities of ocurring in any step, so $P(e_w) = P(e_l) = 0.5$.

We define e_w to be the winning outcome of the step (meaning it increases the amount at play) and e_l to be the losing outcome (meaning it decreases the amount).

We also define that the game is played over the original amount and no debt can be incurred in the game.

So another condition is that $b_w > 1 > b_l > 0$.

Sidenote (this should be a commentary at the bottom): If $b_w > b_l > 1$ or $1 > b_w > b_l$ there is nothing to prove or even reason about. If $b_w > b_l > 1$ any outcome is positive for the betting party and if $1 > b_w > b_l > 0$ any outcome is negative.

It is clear based on the definitions that the recurrence after n steps will be of the form.

$x_n = x_0 * b_w^p * b_l^q$

Where $p,q \in \mathbb{N}$, $p \geq 0$, $q \geq 0$ and $p+q = n$.

Based on this conditions the expected value of the recurrence x_n will be:

a. If $b_l > 1/b_w$ it will tend to infinity.

b. If $b_l < 1/b_w$ it will tend to 0.

c. If $b_l = 1/b_w$ it will follow a binomial distribution around x_0, the characteristics of which we will discuss next.

To start thinking about this, we'll do the following exercise.

Let's imagine a game on the number line, we are starting on 0, and then we flip a fair coin, if it comes heads (an event we will call a_w) we move one full positive integer to the right and if it comes tails (an event we will call a_l) we move a full positive integer to the left. We repeat this process n times.

Arithmetically, we start in 0, if a_w we sum 1, if a_l we sum -1.

We will represent any sequence of events by strings (sequence of characters) of the alphabet $\{a_w, a_l\}$.

We will call this game G_+.

Given a string x of n events, we will represent by $G_+(x)$ the value of the string x in the game G_+.

We will also define the value of the empty string (which we will notates as "") to be 0·

So based on our definition of the game:

$G_+("") = 0$

$G_+(a_w) = +1$

$G_+(a_l) = -1$

And given a character k and a substring l:

$G_+(kl) = G_+(k) + G_+(l)$

Now to calculate the value of $G_+(x)$ for any string x based on our definitions, we start at 0 and then add + 1 for every character a_w and -1 for every character a_l.

For example:

$G_+(a_wa_wa_w) = 0 + 1 + 1 + 1 = 3$

$G_+(a_wa_la_w) = 0 + 1 -1 + 1 = 1$

$G_+(a_la_la_wa_w) = 0 -1 -1 + 1 + 1 = 0$

It is immediately clear that in any sequence of events any pair of the characters a_w and a_l will annulate themselves.

$G_+(a_w a_l) = G_+(a_l a_w) = G_+(\text{""}) = 0$

Also due to the properties of addition, in any sequence of events the possible outcome in terms of the result will amount to just calculating the number of a_ws or a_ls left after taking out the pairs of a_w and a_l''s in the string.

Examples of calculating the value of the game:

$G_+(a_w a_l a_w a_l) = G_+(\text{""}) = 0$

$G_+(a_l a_l a_w a_w) = G_+(\text{""}) = 0$

$G_+(a_w a_w a_w a_l) = G_+(a_w a_w) = 2$

$G_+(a_w a_w a_l a_w) = G_+(a_w a_w) = 2$

$G_+(a_l a_l a_l a_w) = G_+(a_l a_l) = -2$

An interesting thing to notice first is that for any string of size n, if n is even then the results will be even and if n is odd then the result will be odd too.

EXPLAINS HOW EVERYTHING BEFORE RELATES TO PASCAL TRIANGLE

THEN DO NEXT

How is this related to case c in our original game G_*?

Let's reformulate our original game G_*.

We could also represent any sequence of events in $G_(*)$ as a string from alphabet $\{b_w, b_l\}$ for representing events e_w and e_l respectively.

So if we were to calculate a sequence of events in a game of G_* we would do the following:

$G_*(\text{""}) = x_0$

$G_x(b_w) = x_0 * b_w$

$G_x(b_l) = x_0 * b_l$

$G_x(b_w b_l) = x_0 * b_w * b_l$

... ETC.. EXPLAIN THIS BETTER..

Now if $b_l = 1/b_w \implies b_l = b_w^{-1}$.

Which means that for this case we can reformulate the recurrence formula as:

$x_n = x_0 * b_w^r$

Where $r \in \mathbb{N}$.

Also note that if $b_l = 1/b_w \implies b_w * b_l = 1$.

EXPLAIN BELOW BETTER

Let's reformulate our original game G_* in terms of this new one G_+.

It is evident that in this case the results of G_+ are equivalent to the exponentials of b_w in the results of G_* for this case.

EXPLAIN HOW DISTRIBUTION ON G_+ relate to the distribution on G_*.

– Rant about how is the exact definition of a fair game.. go on about pascal triangle and binomial distribution

Also please consider the following, we could also represent any possible result of the game as:

$x_n = x_0 * (b_w * b_l)^x * b_w^y * b_l^z$

Where $2x + z + y = n$ and either z or n equal zero.

Considering the distribution of the results the expected value would be where $2x = n$ and $z, y = 0$.

Considering that in our case $b_l = 1/b_w$ that means that $b_l * b_w = 1$.

Then:

$x_n = x_0 * (b_w * b_l)^x * b_w{}^y * b_l{}^z = x_0 * (1)^x * b_w{}^y * b_l{}^z$

Now if $b_l \neq 1/b_w \implies b_w * b_l = 1 + \epsilon$.

The distribution would be the same as the other case and our central value would be where 2x = n.

But now,

$x_n = x_0 * (b_w * b_l)^x * b_w{}^y * b_l{}^z = x_0 * (1 + \epsilon)^x * b_w{}^y * b_l{}^z$

If $\epsilon > 0$ then the expected value will tend to infinity as n increases and if $0 > \epsilon$ it will tend to zero.

4. MULTIPLE OUTCAMES WITH EQUAL PROBABILITY

m number of possible outcames, each with same probability and associated with a coefficient b_1 to b_m

Result of the recurrence at infinity:

Let's define $s = \prod_{i=1}^{m} b_i$

if $s > 1$ tends to infinity

if $s < 1$ tends to 0

if s = 1 tends to x_0

Prove algebraically.

5. MULTIPLE OUTCAMES EACH WITH IT'S OWN PROBABILITY

m number of possible outcames, each with it's own coefficient b_i and it's own probability p_i

Result of the recurrence at infinity:

Let's define s = $\prod_{i=1}^{m} mp_i b_i$

if s $>$1 tends to infinity

if s $<$1 tends to 0

if s = 1 tends to x_0

Examine distributions of probabilities.. this one is harsher.. to prove

Define expected value of recurrence r_n

$r_0 = x_0$

$r_n = sr_{n-1}$

Explain based on the definition of the common expected value formula

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