

# Analysis for Algebraists

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# **Chapter 1**

## **Sets and Orders**

## Chapter 2

# Ordered Fields

**Definition 2.0.1.** An **ordered commutative Ring**  $(R, \leq)$  is a commutative Ring  $R$  equipped with an **ordering relation**  $\leq$ , such that for all  $a, b, c \in R$ , we have:

1.  $\leq$  defines a *total order* on  $F$ . i.e:
  - (a)  $a \leq a$  (the order is *reflexive*),
  - (b)  $a \leq b \wedge b \leq c \implies a \leq c$  (the order is *transitive*),
  - (c)  $a \leq b \wedge b \leq a \implies a = b$  (the order is *antisymmetric*),
  - (d)  $a \leq b \vee b \leq a$  (the order is *strongly connected*)
2.  $a \leq b \implies a + c \leq b + c$
3.  $0 \leq a \wedge 0 \leq b \implies 0 \leq ab$

**Lemma 2.0.2.** For every ordered commutative Ring  $(R, \leq)$  and  $a \in R$ , we have  $-a \leq 0 \leq a$  or  $a \leq 0 \leq -a$ .

*Proof.* Since the order  $\leq$  is strongly connected, we have  $a \leq 0$  or  $0 \leq a$ .

1. If  $a \leq 0$ , then we have  $-a + a \leq 0 + -a$ , i.e.  $a \leq 0 \leq -a$ ,
2. if  $0 \leq a$ , then we have  $-a + 0 \leq -a + a$ , i.e.  $-a \leq 0 \leq a$

□

**Lemma 2.0.3.** Let  $a \in (R, \leq)$ . Then  $0 \leq a^2$ .

*Proof.* Since the order  $\leq$  is strongly connected, we have  $a \leq 0$  or  $0 \leq a$ .

1. If  $0 \leq a$ , then we have  $0 \leq a \cdot a = a^2$ ,
2. if  $a \leq 0$ , then  $0 \leq -a$  and we have  $0 \leq -a \cdot (-a) = a^2$ .

□

**Lemma 2.0.4.** *Every ordered commutative Ring has characteristic 0.*

*Proof.* Assume that  $F$  is a field of characteristic  $p$ . Then an ordering relation would need to fulfill:

$$1 \leq 1 + 1 \leq \sum_{i=1}^p 1 = 0 \leq 1$$

□

## 2.1 The Archimedean Property

## 2.2 On the Importance of the Real Numbers

**Theorem 2.2.1.** *Let  $F$  be an arbitrary archimedean ordered field. Then  $F$  is isomorphic to a subfield of the real number  $\mathbb{R}$ .*

**Theorem 2.2.2.** *Let  $F$  be an arbitrary ordered field. Then  $F$  has the least-upper-bound property if and only if it is archimedean and cauchy complete.*

# Chapter 3

## Topology

**3.1 Topological Spaces**

**3.2 Metric Spaces**

## Chapter 4

# Topological Vector Spaces

4.1 Topological Vector Spaces

4.2 Normed Vector Spaces

4.3 Banach Spaces

## **Chapter 5**

# **Differentiation**

**5.1 Frechét Spaces**

**5.2 The Gateaux Derivative**

**5.3 The Frechét Derivative**

## Chapter 6

# Measure Theory

6.1 Set Algebras

6.2 Measure Spaces

6.3 The Lebesgue Measure

# Chapter 7

# Integration

## 7.1 The Bochner Integral