Analysis for Algebraists

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Chapter 1

Sets and Orders

Chapter 2

Ordered Fields

Definition 2.0.1. An **ordered commutative Ring** $(R, \leq)$ is a commutative Ring R equipped with an **ordering relation** $\leq$, such that for all $a, b, c \in R$, we have:

1. $\leq$ defines a *total order* on F. i.e:
   (a) $a \leq a$ (the order is *reflexive*),
   (b) $a \leq b \wedge b \leq c \implies a \leq c$ (the order is *transitive*),
   (c) $a \leq b \wedge b \leq a \implies a = b$ (the order is *antisymmetric*),
   (d) $a \leq b \vee b \leq a$ (the order is *strongly connected*)
2. $a \leq b \implies a + c \leq b + c$
3. $0 \leq a \wedge 0 \leq b \implies 0 \leq ab$

Lemma 2.0.2. *For every ordered commutative Ring* $(R, \leq)$ *and* $a \in R$*, we have* $-a \leq 0 \leq a$ *or* $a \leq 0 \leq -a$*.*

Proof. Since the order $\leq$ is strongly connected, we have $a \leq 0$ or $0 \leq a$.

1. If $a \leq 0$, then we have $-a + a \leq 0 + -a$, i.e. $a \leq 0 \leq -a$,
2. if $0 \leq a$, then we have $-a + 0 \leq -a + a$, i.e. $-a \leq 0 \leq a$

□

Lemma 2.0.3. *Let* $a \in (R, \leq)$*. Then* $0 \leq a^2$*.*

Proof. Since the order $\leq$ is strongly connected, we have $a \leq 0$ or $0 \leq a$.

1. If $0 \leq a$, then we have $0 \leq a \cdot a = a^2$,
2. if $a \leq 0$, then $0 \leq -a$ and we have $0 \leq -a \cdot (-a) = a^2$.

$\square$

Lemma 2.0.4. *Every ordered commutative Ring has characteristic* 0.

Proof. Assume that F is a field of characteristic p. Then an ordering relation would need to fulfill:

$$1 \leq 1 + 1 \leq \sum_{i=1}^{p} 1 = 0 \leq 1$$

Which implies $0 = 1$. However, by the definition of a field, we have $0 \neq 1$. $\square$

2.1 The Archimedean Property

2.2 On the Importance of the Real Numbers

Theorem 2.2.1. *Let F be an arbitrary archimedean ordered field. Then F is isomorphic to a subfield of the real numbers $\mathbb{R}$.*

Theorem 2.2.2. *Let F be an arbitrary ordered field. Then F has the least-upper-bound property if and only if it is archimedean and cauchy complete.*

2.2.1 Alternative completeness properties

Chapter 3

Topology

3.1 Metric Spaces

3.2 Topological Spaces

3.3 Uniform Spaces

Many theorems in analysis require a notion of *uniform convergence*, *uniform continuity*, and so on. These ideas can be easily expressed in a metric space - recall that, for example, a function $f : M \to N$ between metric spaces is uniformly continuous if for every $\varepsilon > 0$, a $\delta > 0$ exists such that if $d_M(x, y) < \delta$, then $d_N(f(x), f(y)) < \varepsilon$.

Meanwhile, we wouldn't be able to refine the definition of continuity like this in a topological space, since the general structure of the neighborhoods of a topological space might vary wildly at different locations in the space - the important quality of a metric space here is that the notion of distance in a metric space can be applied "uniformly" to pairs of points, no matter where they are located. In this section, we want to define a set of spaces more general than metric spaces, but less general than topological spaces, which shares this important property of "uniformity", which will allow us to generalize many useful properties of metric spaces.

3.3.1 Diagonal Uniformity

Definition 3.3.1. For any set X, we denote by $\Delta(X)$ the diagonal $\{(x, x) \mid x \in X\}$ in $X \times X$.

Our first definition of a *uniform structure* on a set X is based on the observation that in a metric space, x and y are close together if and only if (x, y) is close to $\Delta(X)$.

Definition 3.3.2. For any pair of subsets U, V of $X \times X$ (which by definition can be viewed as relations on X), we can extend the notion of function composition to these arbitrary relations by defining $U \circ V$ to be the set

$$\{(x, y) \in X \times X \mid \exists z \in X : ((x, z) \in V, (z, y) \in U\}$$

Definition 3.3.3. A **diagonal uniformity** on a set X is a collection $\mathcal{D}(X)$ of subsets of $X \times X$, called **surroundings**, such that:

1. If $D \in \mathcal{D}$, then $\Delta(X) \subset D$,
2. If $D_1, D_2 \in \mathcal{D}$, then $D_1 \cap D_2 \in \mathcal{D}$,
3. If $D \in \mathcal{D}$, then there exists an $E \in \mathcal{D}$ such that $E \circ E \subset D$,
4. If $D \in \mathcal{D}$, then there exists an $E \in \mathcal{D}$ such that $E^{-1} \subset D$
5. If $D \in \mathcal{D}$ and $D \subset E$, then $E \in \mathcal{D}$.

We call a set X equipped with such a structure a **uniform space**.

Example 3.3.4. For any metric space (M, d), the metric d generates a *metric uniformity* by having a surrounding

$$D_\varepsilon^d = \{(x, y) \in M \times M \mid d(x, y) < \varepsilon\}$$

for every $\varepsilon > 0$. Uniformities that can be generated in this way from metrics are called **metrizable**.

Comment 3.3.5. For an arbitrary metric d, the uniformity generated by d is identical to the one generated by a scaled version λd (with $\lambda \in \mathbb{R}^\times$). Therefore different metrics may generate the same uniformity.

Chapter 4

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4.1 Normed Vector Spaces

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4.4 Topological Vector Spaces

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