Analysis for Algebraists

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Sets and Orders

Chapter 2

Ordered Fields

Definition 2.0.1. An **ordered commutative Ring** $(R, \leq)$ is a commutative Ring R equipped with an **ordering relation** $\leq$, such that for all $a, b, c \in R$, we have:

1. $\leq$ defines a *total order* on F. i.e:
   (a) $a \leq a$ (the order is *reflexive*),
   (b) $a \leq b \wedge b \leq c \implies a \leq c$ (the order is *transitive*),
   (c) $a \leq b \wedge b \leq a \implies a = b$ (the order is *antisymmetric*),
   (d) $a \leq b \vee b \leq a$ (the order is *strongly connected*)
2. $a \leq b \implies a + c \leq b + c$
3. $0 \leq a \wedge 0 \leq b \implies 0 \leq ab$

Lemma 2.0.2. *For every ordered commutative Ring $(R, \leq)$ and $a \in R$, we have $-a \leq 0 \leq a$ or $a \leq 0 \leq -a$.*

Proof. Since the order $\leq$ is strongly connected, we have $a \leq 0$ or $0 \leq a$.

1. If $a \leq 0$, then we have $-a + a \leq 0 + -a$, i.e. $a \leq 0 \leq -a$,
2. if $0 \leq a$, then we have $-a + 0 \leq -a + a$, i.e. $-a \leq 0 \leq a$

□

Lemma 2.0.3. *Let $a \in (R, \leq)$. Then $0 \leq a^2$.*

Proof. Since the order $\leq$ is strongly connected, we have $a \leq 0$ or $0 \leq a$.

1. If $0 \leq a$, then we have $0 \leq a \cdot a = a^2$,
2. if $a \leq 0$, then $0 \leq -a$ and we have $0 \leq -a \cdot (-a) = a^2$.

□

Lemma 2.0.4. *Every ordered commutative Ring has characteristic* 0.

Proof. Assume that F is a field of characteristic p. Then an ordering relation would need to fulfill:

$$1 \leq 1 + 1 \leq \sum_{i=1}^{p} 1 = 0 \leq 1$$

Which implies $0 = 1$. However, by the definition of a field, we have $0 \neq 1$. □

2.1 The Archimedean Property

2.2 On the Importance of the Real Numbers

Theorem 2.2.1. *Let* F *be an arbitrary archimedean ordered field. Then* F *is isomorphic to a subfield of the real numbers* $\mathbb{R}$.

Theorem 2.2.2. *Let* F *be an arbitrary ordered field. Then* F *has the least-upper-bound property if and only if it is archimedean and cauchy complete.*

2.2.1 Alternative completeness properties

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