Contents

0.1 Cantor Spaces and Baire Spaces

Cantor Space: 2^ω (Infinite sequences of binary Numbers)

Baire Space: ω^ω (Infinite Sequences of arbitrary Natural Numbers?)

We consider spaces X^ω, where X is an arbitrary countable ordinal.

Let $s \in X^{<\omega}$ then $N_s = \{x \in X^\omega : x \supseteq s\}$. We call **Baire Topology** the topology generated by these sets N_s.

We set a **Lebesque Measure** μ on 2^ω (and on ω^ω). In the case of a cantor space this is $\mu(N_s) = 2^{-|s|}$. In the case of ω^ω finding a measure is more complicated.

We have two notions of **small subsets** of a baire space:

- $\mathcal{N} = \{X \subset 2^\omega : \mu(X) = 0\}$
- $\mathcal{M} = \{X \subset 2^\omega : X \text{is meager}\}$

Definition 0.1. We say $A \subseteq 2^\omega$ is **nowhere dense** iff.

$$\forall s \in 2^{<\omega} : \exists t \supseteq s : N_t \cap A = \emptyset$$

Definition 0.2. A set is called **meager** if it is a countable union of closed nowhere dense sets.

0.2 Infinite Games

An **Infinite Game** is an infinite Sequence $x = (x(0), x(1), x(2), x(3), \ldots) \in X^\omega$, where player I picks an element $x(0) \in X$, then player II replies by picking an element $x(1) \in X$, etc.

Consider a subspace $A \subseteq X^\omega$, which we call the set of outputs of the game.

Given a sequence of moves $x \in X^\omega$, we say that I **wins the game** $G_X(A)$ iff. $x \in A$ and that II wins the game iff. $x \notin A$.

For player I, a **strategy** is a function

$$\sigma : \bigcup_{n \leq \omega} X^{2n} \to X$$

We analogously define a trategy for player II as a function:

$$\tau : \bigcup_{n \leq \omega} X^{2n+1} \to X$$

Intuitively, these functions assign to each point of the game (i.e. to each sequence X^k of odd / even length) a move that the given player takes at that point.

Let $y \in X^\omega$ enumerate the moves of player II. We denote $\sigma * y = (\sigma(\emptyset), \sigma(\sigma(\emptyset), y(0)), \ldots)$. Let $x \in X^\omega$ analogously denote the moves of player I.

We call σ a **winning strategy** for the game $G_X(A)$ iff. $\{\sigma * y\} \subseteq A$. We analogously call τ a winning strategy if $\{x * \tau\} \cap A = \emptyset$. If one of the players has a winning strategy, we say that the game is **determined**.

Given a line of play $x \in X^\omega$, we write $x_I(n) = \{x(2n)\}$ and $x_{II}(n) = \{x(2n+1)\}$

Is there always a winning strategy for one of the players? The answer is independent of ZF! However, it turns out that it is not independent of ZFC - The statement "every game is determined" is known as the Axiom of Determinancy, and it contradicts the full Axiom of Choice.)

Satz 0.3. *There exists A such that $G_X(A)$ (where X is countable) is not determined.*

Proof. Since X is countable, the set $X^{<\omega}$ of finite sequences of X is countable. Therefore the sets of strategies σ and τ have size $2^{\aleph_0}$. Let $\{\sigma_\alpha : \alpha < 2^{\aleph_0}\}$ enumerate all strategies for I and let τ_α enumerate all strategies for II.

Now we can define recursively two sets $A, B \subseteq X^\omega$ with $A = \{a_\alpha : \alpha < 2^{\aleph_0}\}$ and $B = \{b_\alpha : \alpha < 2^{\aleph_0}\}$ as follows:

- Assume a_i and b_i with $i < \alpha$ are already defined.
- Then define $\sigma_\alpha * y = b_\alpha$ for some $y \in X^\omega$ and set $b_\alpha \notin \{a_i : i < \alpha\}$. This is possible since $|\{\alpha_i : i < \alpha\}| < 2^{\aleph_0}$, but $|\sigma_\alpha * y| = 2^{\aleph_0}$
- Now let $x * \tau_\alpha = a_\alpha$ for some $x \in X^\omega$ s.t. $a_\alpha \notin \{b_i : i < \alpha\}$.
- Now player I cannot have a winning strategy for $G_X(A)$ and neither can player II.

(This proof implicitly uses the well-ordering theorem, so it isn't valid in ZF.) ☐

Satz 0.4. Gale-Stewart: *If X is open or closed, then X is determined.*

Proof. Let $B \subseteq X^\omega, s \in X^{<\omega}$. Define

$$B/S := \{x \in X^\omega : s \circ x \in B\}$$

Note that if I has no winning strategy in $G_X(B/S)$, then $\forall i \in X : \exists j \in X$ s.t. I has no winning strategy in $G_X(B/S \circ (i, j))$ either. We prove this by contradiction: If $\exists i \in X : \forall j \in X$ I has a winning strategy σ in $G_X(B/S \circ (i, j)$, then we would get a winning strategy for I in $G_X(B/S)$ (by playing i and the continuing according to σ).

Now, suppose that $A \subseteq X^\omega$ is open. Assume I has no winning strategy in $G_X(A)$. We can describe a strategy τ for II thanks to our lemma such that for any partial play according to τ, I has no winning strategy in $G_X(A/S)$.

By contradiction: Assume that x is a line of play according to τ such that $x \in A$. Since A is open, we have $N_{x \upharpoonright 2n} \subseteq A$. But then I should have a winning strategy in $G_X(A/x \upharpoonright 2n)$.

Therefore, we have $x \notin A$, which means τ is a winning strategy. ☐