universität freiburg

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[r, "\mathrm{Hom}(A,f)"] \arrow[d, "\Phi_A"'] & \mathrm{Hom}(A,X) \arrow[d, "\Phi_X"] \\
F(A) \arrow[r, "Ff"'] & F(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\
u \arrow[r, mapsto] & (Ff)u = \Phi_X(f)
\end{tikzcd}$$

What you needa know about Yoneda

Emma Bach (she/her)

Seminar on Functional Programming and Logic, Summer Semester 2025

The Yoneda Embedding

- Remember that the goal is finding out everything about an object A through its relations to other objects.

$$
\begin{tikzcd}
A \arrow[r, "f \in \mathbb{C}(A,B)"] \arrow[d, dotted, leftrightarrow, "\mathcal{Y}"'] & B \arrow[d, dotted, leftrightarrow, "\mathcal{Y}"] \\
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- Formally, we want a bijective functor

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\mathcal{Y} : \mathbb{C} &\to \mathbb{S}et^{\mathbb{C}} \\
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- Therefore, $\mathcal{Y}(f)$ is a natural transformation between $\mathbb{C}(A, -)$ and $\mathbb{C}(B, -)$

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  - Vice versa, if we know all natural transformations $\mathrm{Nat}(\mathbb{C}(A, -), F)$, we can construct the set $F(A)$.