

$$\begin{array}{ccc}
 \text{Hom}(A, A) & \xrightarrow{\text{Hom}(A, f)} & \text{Hom}(A, X) \\
 \downarrow \Phi_A & & \downarrow \Phi_X \\
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## What you needa know about Yoneda

Emma Bach (she/her)

Seminar on Functional Programming and Logic, Summer Semester 2025

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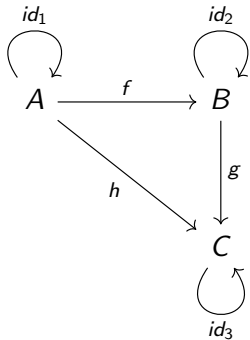
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- ▶ “*Tell me your company, and I will tell you what you are.*”<sup>1</sup>
- ▶ The Yoneda lemma is the result of applying this way of thinking to mathematical objects in category theory.
- ▶ As a result, a category  $\mathbb{C}$  is often best understood by instead studying functors from that category into  $\mathbf{Set}$ .

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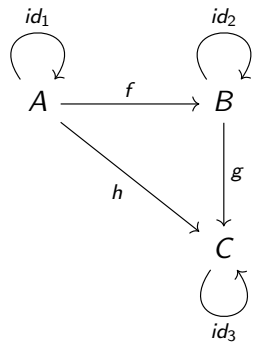
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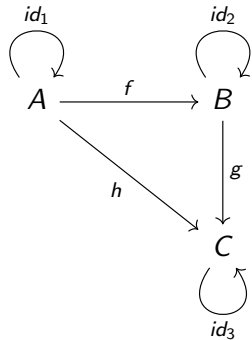


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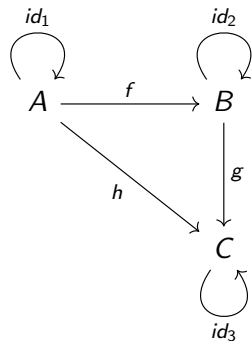
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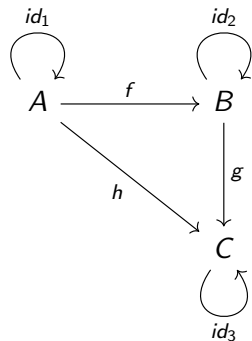


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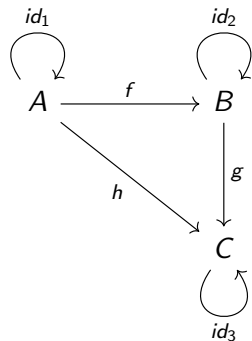
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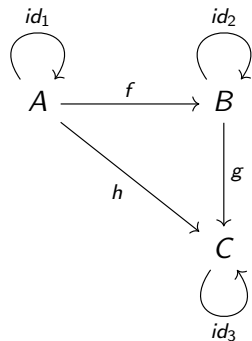
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- ▶ For every category  $\mathbb{C}$ , the *opposite category*  $\mathbb{C}^{op}$  is a category.
- ▶ For every pair of categories  $\mathbb{C}, \mathbb{D}$ , the *product category*  $\mathbb{C} \times \mathbb{D}$  is a category.

# Functors

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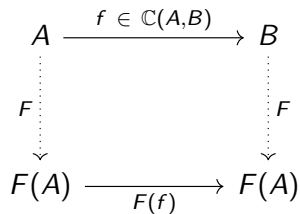
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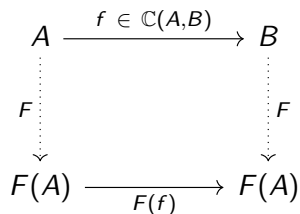
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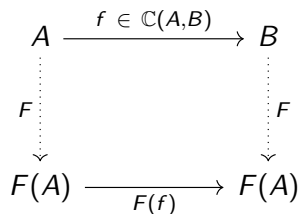


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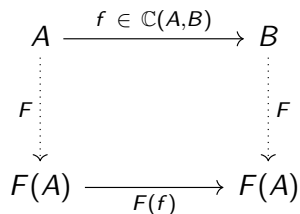
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Functors from a category into itself are known as *endofunctors*.

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$$A \xrightarrow{f} B$$

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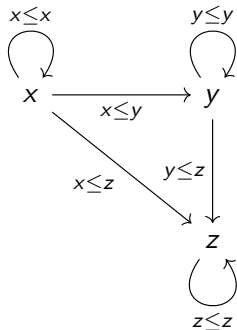
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  - ▶ These morphisms satisfy the *naturality condition*:

$$\forall f \in \mathbb{C}(A, B) : \phi_B \circ F(f) = G(f) \circ \phi_A$$

## Exercise 1 : Order Categories

- a) Let  $\leq$  be a reflexive, transitive order (a *preorder*) on a set  $M$ . Show that if we define objects by  $|\mathbb{P}re(M, \leq)| = M$  and morphisms by  $\exists! f_{x \leq y} \in \mathbb{P}re(x, y) \Leftrightarrow x \leq y$ , then  $\mathbb{P}re(M, \leq)$  forms a category.
- b) Let  $F : \mathbb{M} \rightarrow \mathbb{M}$  be an endofunctor on  $\mathbb{M}$ . Show that  $F$  defines a monotonic function  $M \rightarrow M$ , i.e.  $\forall x, y : x \leq y \implies F(x) \leq F(y)$ .
- c) Let  $F, G : M \rightarrow M$  be monotonic functions. Let  $\phi$  be a natural transformation  $F \rightarrow G$ . Show that  $\forall x \in M : F(x) \leq G(x)$ .

## Exercise 1 : Order Categories, Solution a)



- ▶  $\leq$  is reflexive, so we have  
 $\forall x : x \leq x \implies \exists id_{x \leq x} \in \mathbb{P}re(x, x).$
- ▶ Because of transitivity, for every pair of morphisms  $f_{x \leq y}$  and  $g_{y \leq z}$ , we have a composed morphism  $(g \circ f)_{x \leq z}$ .
- ▶ Since our morphisms are just witnesses of an ordering, they don't care about the order of function application, so composition is associative.



## Exercise 1 : Order Categories, Solution b)

$$\begin{array}{ccc} x & \xrightarrow{x \leq y} & y \\ \downarrow F & & \downarrow F \\ F(x) & \xrightarrow{F(x) \leq F(y)} & F(y) \end{array}$$

- By the definition of functors,  $F$  must take each morphism  $f_{x \leq y} \in \mathbb{P}re(x, y)$  to a morphism  $F(f)_{F(x) \leq F(y)} \in \mathbb{P}re(F(x), F(y))$ .

## Exercise 1 : Order Categories, Solution c)

$$\begin{array}{c} F(x) \\ \downarrow \phi_x : F(x) \leq G(x) \\ G(x) \end{array}$$

- By the definition of natural transformations, for every object  $x$ ,  $\phi_x$  is a morphism  $F(x) \rightarrow G(x)$ . If such a morphism exists, we have  $F(x) \leq G(x)$ .

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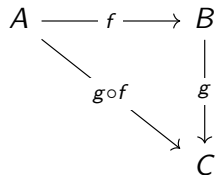
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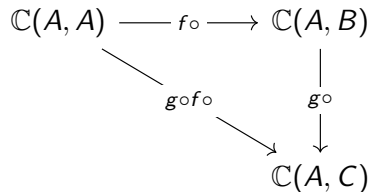
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- ▶ So our free theorem is a proof that any parametrically polymorphic function  $r$  is a natural transformation!
- ▶ It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors  $\mathbb{H}ask \rightarrow \mathbb{H}ask$ .

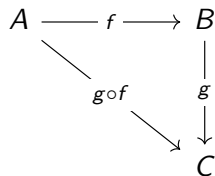
# Homfunctors



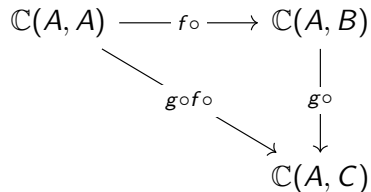
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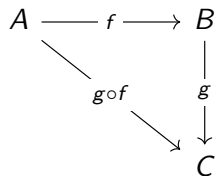
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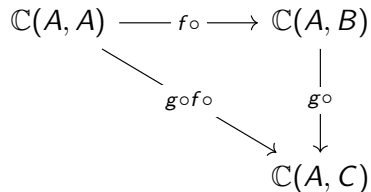
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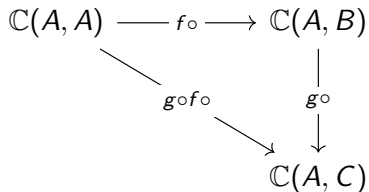
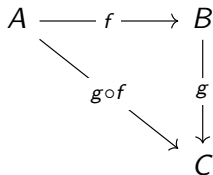


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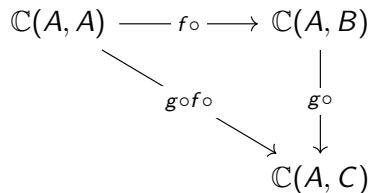
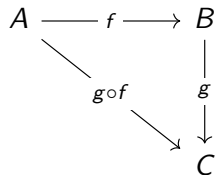


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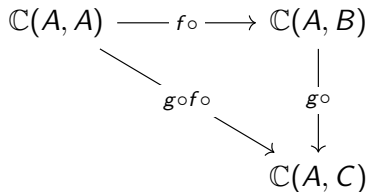
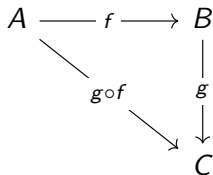
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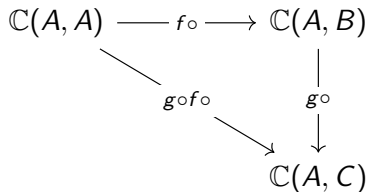
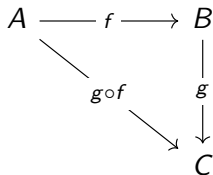
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- ▶ For any locally small category  $\mathbb{C}$ , a homset  $\mathbb{C}(A, B)$  is a set of morphisms.
- ▶ We define a functor  $\mathbb{C}(A, -) : \mathbb{C} \rightarrow \mathbf{Set}$ :
  - ▶  $\mathbb{C}(A, -)$  maps an Object  $B$  to the Homset  $\mathbb{C}(A, B)$
  - ▶ A morphism  $f : \mathbb{C}(B, C)$  is mapped to the morphism  $f \circ : \mathbb{C}(A, B) \rightarrow \mathbb{C}(A, C)$
- ▶ Similarly,  $\mathbb{C}(-, B)$  is a functor  $\mathbb{C}^{op} \rightarrow \mathbf{Set}$ :
  - ▶  $\mathbb{C}(-, B)$  maps an Object  $A$  to the Homset  $\mathbb{C}(A, B)$
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- ▶ This category is known as a *functor category* and denoted  $\mathbb{D}^{\mathbb{C}}$ .
- ▶ A morphism  $\phi \in \mathbb{D}^{\mathbb{C}}(F, G)$  is a natural transformation  $F \rightarrow G$ .

# Functor Application as a Functor

$$\begin{array}{ccc} F & \xrightarrow{\eta} & G \\ \text{\scriptsize $-(A)$} \downarrow \text{\scriptsize $\vee$} & & \downarrow \text{\scriptsize $-(A)$} \text{\scriptsize $\vee$} \\ F(A) & \xrightarrow{\eta_A} & G(A) \end{array}$$

- Given a category  $\mathbb{C}$ , the operation of applying a functor  $F : \mathbb{C} \rightarrow \mathbf{Set}$  to an object  $A \in |\mathbb{C}|$  is itself a functor from  $\mathbf{Set}^{\mathbb{C}}$  to  $\mathbf{Set}$ .



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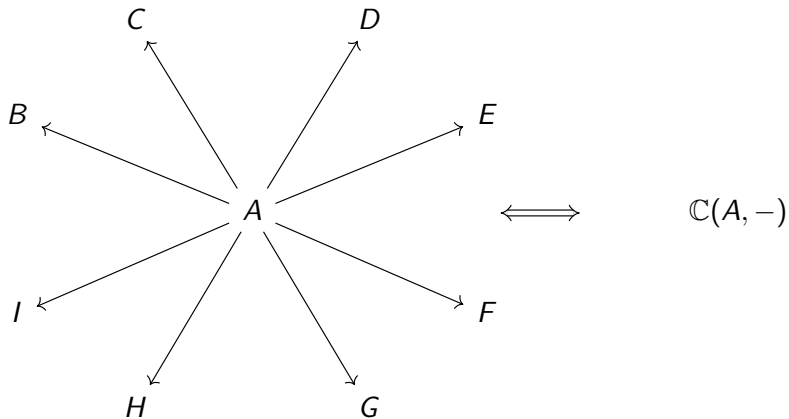
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# The Yoneda Embedding



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$$\mathcal{Y} : \mathbb{C} \rightarrow \mathbf{Set}^{\mathbb{C}}$$

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A commutative diagram illustrating the Yoneda embedding. It consists of two rows and two columns. The top row has objects  $A$  and  $B$  connected by a solid arrow labeled  $f \in \mathbb{C}(A, B)$ . The bottom row has objects  $\mathbb{C}(A, -)$  and  $\mathbb{C}(B, -)$  connected by a solid arrow labeled  $\mathcal{Y}(f)$ . On the left, a vertical dotted arrow labeled  $\mathcal{Y}$  points from  $\mathbb{C}(A, -)$  up to  $A$ . On the right, a vertical dotted arrow labeled  $\mathcal{Y}$  points from  $\mathbb{C}(B, -)$  up to  $B$ .

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- Therefore,  $\mathcal{Y}(f)$  has to be a natural transformation between  $\mathbb{C}(A, -)$  and  $\mathbb{C}(B, -)$ .
- Is it actually possible to construct all of the necessary natural transformations?



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- ▶ Furthermore, this isomorphism is a natural transformation.
- ▶ So we can construct the Yoneda embedding  $\mathcal{Y}$  from the set  $F(A)$ .
- ▶ Vice versa, if we know all natural transformations  $\mathbf{Nat}(\mathbb{C}(A, -), F)$ , we can construct the set  $F(A)$ .

## Constructing the bijection

$$A \xrightarrow{f} B$$

► Let  $\phi \in \text{Nat}(\mathbb{C}(A, -), F)$ . Since  $\phi$  is natural transformation, we have

$$\begin{array}{ccc} \mathbb{C}(A, A) & \xrightarrow{\mathbb{C}(A, f) = f \circ} & \mathbb{C}(A, B) \\ \downarrow \phi_A & & \downarrow \phi_B \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

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- ▶ Remember that these functors are  $\mathbb{C} \rightarrow \text{Set}$ .
- ▶ This means our morphisms are just regular set functions.

# Constructing the bijection

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ id_A & \xrightarrow{\mathbb{C}(A,f)=f \circ} & f \\ \downarrow \phi_A & & \downarrow \phi_B \\ u \in F(A) & \xrightarrow{F(f)} & \phi_B(f) = F(f)(u) \end{array}$$

- If we apply these functions to the identity morphism  $id_A$ , we get:

$$\phi_B(f \circ id_A) = F(f)(\phi_A(id_A))$$

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  - ▶ The other side sends a permutation  $f$  to the element  $f(e)$
- ▶ The Yoneda lemma is often viewed as a generalization of Cayley's theorem.

## Exercise 2 - Cayley's Theorem for Monoids

Use the Yoneda embedding to show that every monoid  $M$  is isomorphic to a monoid of functions  $M \rightarrow M$ .

**Hint 1:** The Yoneda embedding gives an isomorphism between objects and their homfunctors.

**Hint 2:** Two weeks ago we saw that every monoid  $M$  defines a category  $\mathbb{M}$  with a single object  $*$  and a morphism  $m$  for each element  $m \in M$ , where we define morphism composition to be the monoid operation.

## Exercise 2 - Cayley's Theorem for Monoids

- The Yoneda embedding is an isomorphism mapping each object to its homfunctor.

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- ▶ Thus, the Yoneda embedding on  $M$  is an isomorphism between monoid objects and a set of functions  $M \rightarrow M$ . These functions form a monoid under composition.



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- ▶ *Profunctor optics* are neat and flexible representations of optics as individual polymorphic function.
- ▶ In particular, profunctor optics make composition of optics trivial.
- ▶ Equivalence between optics and their profunctor representations comes down to the Yoneda lemma.

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- ▶ We can compose adapters with matching types and define an identity adapter `Adapter id id`.
- ▶ This lets us view the category  $\text{Set}^{op} \times \text{Set}$  as the category  $\mathbb{A}da$  where morphisms are adapters.

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- ▶ We define the category  $\mathbb{P}rof$  of Profunctors to be  $\mathbf{Set}^{\mathbf{Set}^{op} \times \mathbf{Set}} = \mathbf{Set}^{Ada}$



# Profunctor Adapters

- The profunctor representation of an adapter is given by:

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type AdapterP a b s t =  
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- ▶ Specifically, the homsets in  $\mathbb{A}daP$  are:

$$\begin{aligned}\mathbb{A}daP((A, B), (S, T)) &= \mathbf{Set}^{\mathbf{Prof}}(-(A, B), -(S, T)) \\ &= \mathbf{Set}^{\mathbf{Set}^{\mathbb{A}da}}(-(A, B), -(S, T))\end{aligned}$$

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- ▶ This involves expanding the functors on the left using the Yoneda lemma and then summarizing everything by applying the Yoneda embedding twice.
- ▶ This proof is left as an exercise to the reader :) (a short version of the proof is uploaded alongside the slides on ilias).

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- ▶ The Yoneda lemma enables the use of the Yoneda embedding, which is a natural isomorphism between a category  $\mathbb{C}$  and the category of homfunctors on  $\mathbb{C}$ .
- ▶ Equivalence of adapters and profunctor adapters can be shown by applying the Yoneda embedding twice.
- ▶ Similar techniques can be used to show the equivalence of any optic and its profunctor representation.