universität freiburg

$$
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\mathrm{Hom}(A, A) \arrow[rrr, "\mathrm{Hom}(A,f)"] \arrow[ddd, "\Phi_A"'] & & & \mathrm{Hom}(A, X) \arrow[ddd, "\Phi_X"] \\
& \mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & (Ff)u = \Phi_X(f) & \\
F(A) \arrow[rrr, "Ff"'] & & & F(X)
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What you needa know about Yoneda

Emma Bach (she/her)

Seminar on Functional Programming and Logic, Summer Semester 2025

Motivation

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- A common sentiment in many cultures is the idea that things are defined by how they interact with their surroundings.
- "*Tell me your company, and I will tell you what you are.*"[1]
- The Yoneda lemma is the result of applying this way of thinking to mathematical objects in category theory.
- As a result, a category $\mathbb{C}$ is often best understood by instead studying functors from that category into $\mathbb{S}et$.

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- For every pair of categories $\mathbb{C}$, $\mathbb{D}$, the *product category* $\mathbb{C} \times \mathbb{D}$ is a category.

Functors

$$\begin{tikzcd} A \arrow[r, "f \,\in\, \mathbb{C}(A{,}B)"] \arrow[d, dotted, "F"'] & B \arrow[d, dotted, "F"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

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Functors from a category into itself are known as *endofunctors*.

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$$A \xrightarrow{f} B$$

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  - A natural transformation ϕ is an *indexed family of morphisms* - for every object $A \in |\mathbb{C}|$, ϕ_A is a morphism from $F(A)$ to $G(A)$.
  - These morphisms satisfy the *naturality condition*:

$$\forall f \in \mathbb{C}(A, B) : \phi_B \circ F(f) = G(f) \circ \phi_A$$

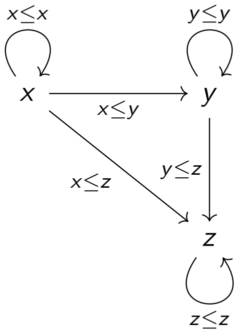
Exercise 1 : Order Categories

a) Let $\leq$ be a reflexive, transitive order (a *preorder*) on a set M. Show that if we define objects by $|\mathbb{P}re(M, \leq)| = M$ and morphisms by $\exists! f_{x \leq y} \in \mathbb{P}re(x, y) \Leftrightarrow x \leq y$, then $\mathbb{P}re(M, \leq)$ forms a category.

b) Let $F : \mathbb{M} \to \mathbb{M}$ be an endofunctor on $\mathbb{M}$. Show that F defines a monotonic function $M \to M$, i.e. $\forall x, y : x \leq y \implies F(x) \leq F(y)$.

c) Let $F, G : M \to M$ be monotonic functions. Let ϕ be a natural transformation $F \to G$. Show that $\forall x \in M : F(x) \leq G(x)$.

Exercise 1 : Order Categories, Solution a)



- $\leq$ is reflexive, so we have
  $\forall x : x \leq x \implies \exists id_{x \leq x} \in \mathbb{P}re(x, x)$.
- Because of transitivity, for every pair of morphisms $f_{x \leq y}$ and $g_{y \leq z}$, we have a composed morphism $(g \circ f)_{x \leq z}$.
- Since our morphisms are just witnesses of an ordering, they dont care about the order of function application, so composition is associative.

Exercise 1 : Order Categories, Solution b)

$$\begin{tikzcd} x \arrow[r, "x\leq y"] \arrow[d, "F"] & y \arrow[d, "F"] \\ F(x) \arrow[r, "F(x)\leq F(y)"] & F(y) \end{tikzcd}$$

- By the definition of functors, F must take each morphism $f_{x\leq y} \in \mathbb{P}re(x, y)$ to a morphism $F(f)_{F(x)\leq F(y)} \in \mathbb{P}re(F(x), F(y))$.

Exercise 1 : Order Categories, Solution c)

$$\begin{tikzcd} F(x) \arrow[d, "\phi_x \,:\, F(x) \leq G(x)"] \\ G(x) \end{tikzcd}$$

- By the definition of natural transformations, for every object x, ϕ_x is a morphism $F(x) \to G(x)$. If such a morphism exists, we have $F(x) \leq G(x)$.

Naturality from Polymorphism

- The naturality condition resembles an equality we saw a few weeks ago:

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- This free theorem is equivalent to the statement that r is a natural transformation.

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- So our free theorem is a proof that any parametrically polymorphic function `r` is a natural transformation!
- It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors $\mathbb{H}ask \to \mathbb{H}ask$.

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- This category is known as a *functor category* and denoted $\mathbb{D}^{\mathbb{C}}$.
- A morphism $\phi \in \mathbb{D}^{\mathbb{C}}(F, G)$ is a natural transformation $F \to G$.

Functor Application as a Functor

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\begin{tikzcd}
F \arrow[r, "\eta"] \arrow[d, dotted, "-(A)"'] & G \arrow[d, dotted, "-(A)"] \\
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- Given a category $\mathbb{C}$, the operation of applying a functor $F : \mathbb{C} \to \mathbb{S}et$ to an object $A \in |\mathbb{C}|$ is itself a functor from $\mathbb{S}et^{\mathbb{C}}$ to $\mathbb{S}et$.

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- We write $-(A)$ for this functor.

Functor Application as a Functor

$$\begin{tikzcd} F \arrow[rr, "\eta"] \arrow[d, dotted, "-(A)"'] & & G \arrow[d, dotted, "-(A)"] \\ F(A) \arrow[rr, "\eta_A"'] & & G(A) \end{tikzcd}$$

- Given a category $\mathbb{C}$, the operation of applying a functor $F : \mathbb{C} \to \mathbb{S}et$ to an object $A \in |\mathbb{C}|$ is itself a functor from $\mathbb{S}et^{\mathbb{C}}$ to $\mathbb{S}et$.
- We write $-(A)$ for this functor.
- $-(A) \in \mathbb{S}et^{\mathbb{S}et^{\mathbb{C}}}$

The Yoneda Embedding

$$
\begin{tikzcd}
 & C & & D & & & \\
B & & & & E & & \\
 & & A \arrow[uul] \arrow[uur] \arrow[ull] \arrow[urr] \arrow[dll] \arrow[drr] \arrow[ddl] \arrow[ddr] & & & \Longleftrightarrow & \mathbb{C}(A,-) \\
I & & & & F & & \\
 & H & & G & & &
\end{tikzcd}
$$

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\begin{tikzcd}
A \arrow[r, "f \in \mathbb{C}(A,B)"] \arrow[d, dotted, leftrightarrow, "\mathcal{Y}"'] & B \arrow[d, dotted, leftrightarrow, "\mathcal{Y}"] \\
\mathbb{C}(A,-) \arrow[r, "\mathcal{Y}(f)"'] & \mathbb{C}(B,-)
\end{tikzcd}
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- Formally, we want a bijective functor

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\begin{aligned}
\mathcal{Y} : \mathbb{C} &\to \mathbb{S}et^{\mathbb{C}} \\
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- Therefore, $\mathcal{Y}(f)$ has to be a natural transformation between $\mathbb{C}(A, -)$ and $\mathbb{C}(B, -)$.

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- Therefore, $\mathcal{Y}(f)$ has to be a natural transformation between $\mathbb{C}(A,-)$ and $\mathbb{C}(B,-)$.
- Is it actually possible to construct all of the necessary natural transformations?

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  - Furthermore, this isomorphism is a natural transformation.
  - So we can construct the Yoneda embedding $\mathcal{Y}$ from the set $F(A)$.
  - Vice versa, if we know all natural transformations $\mathrm{Nat}(\mathbb{C}(A, -), F)$, we can construct the set $F(A)$.

Constructing the bijection

$$A \xrightarrow{f} B$$

$$\begin{tikzcd}
\mathbb{C}(A, A) \arrow[r, "\mathbb{C}(A,f)=f\circ"] \arrow[d, "\phi_A"] & \mathbb{C}(A, B) \arrow[d, "\phi_B"] \\
F(A) \arrow[r, "F(f)"] & F(B)
\end{tikzcd}$$

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$$\begin{tikzcd} id_A \arrow[r, "\mathbb{C}(A{,}f) = f\circ"] \arrow[d, "\phi_A"] & f \arrow[d, "\phi_B"] \\ u \arrow[r, "F(f)"] & \phi_B(f) = F(f)(u) \end{tikzcd}$$

Application : Cayley's Theorem

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  - The other side sends a permutation f to the element $f(e)$
- The Yoneda lemma is often viewed as a generalization of Cayley's theorem.

Exercise 2 - Cayley's Theorem for Monoids

Use the Yoneda embedding to show that every monoid M is isomorphic to a monoid of functions $M \to M$.
Hint 1: The Yoneda embedding gives an isomorphism between objects and their homfunctors.
Hint 2: Two weeks ago we saw that every monoid M defines a category $\mathbb{M}$ with a single object $*$ and a morphism m for each element $m \in M$, where we define morphism composition to be the monoid operation.

Exercise 2 - Cayley's Theorem for Monoids

$$\begin{tikzcd}
* \arrow[r, "m \in \mathbb{M}(*,*)"] \arrow[d, dotted, leftrightarrow, "\mathcal{Y}"'] & * \arrow[d, dotted, leftrightarrow, "\mathcal{Y}"] \\
\mathbb{M}(*, -) \arrow[r, "\mathcal{Y}(m) = m\circ"'] & \mathbb{M}(*, -)
\end{tikzcd}$$

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\ast \arrow[r, "m \,\in\, \mathbb{M}(\ast,\ast)"] \arrow[d, dotted, mapsto, "\mathcal{Y}"'] & \ast \arrow[d, dotted, mapsto, "\mathcal{Y}"] \\
\mathbb{M}(\ast, -) \arrow[r, "\mathcal{Y}(m) = m\circ"'] & \mathbb{M}(\ast, -)
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- We only have one object $\ast$, and thus only one homfunctor $\mathbb{M}(\ast, -)$.
- Each element $m \in M$ is a morphism. By the definition of the homfunctor, this morphism is mapped to the set function

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\begin{aligned}
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- Thus, the Yoneda embedding on M is an isomorphism between monoid objects and a set of functions $M \to M$. These functions form a monoid under composition.

$\square$

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- *Profunctor optics* are neat and flexible representations of optics as individual polymorphic function.
- In particular, profunctor optics make composition of optics trivial.
- Equivalence between optics and their profunctor representations comes down to the Yoneda lemma.

Adapters

```
data Adapter a b s t = Adapter { from :: s -> a, to :: b -> t }
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- We can compose adapters with matching types and define an identity adapter `Adapter id id`.
- This lets us view the category $\mathbb{S}et^{op} \times \mathbb{S}et$ as the category $\mathbb{A}da$ where morphisms are adapters.

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- We define the category $\mathbb{P}rof$ of Profunctors to be $\mathbb{S}et^{\mathbb{S}et^{op} \times \mathbb{S}et} = \mathbb{S}et^{\mathbb{A}da}$

Profunctor Adapters

- The profunctor representation of an adapter is given by:

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type AdapterP a b s t =
    forall  p.  Profunctor  p −> p a b −> p s t
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- Specifically, the homsets in $\mathbb{A}daP$ are:

$$\begin{aligned}
\mathbb{A}daP((A, B), (S, T)) &= \mathbb{S}et^{\mathbb{S}et^{\mathbb{A}da}}(-(A, B), -(S, T)) \\
&= \mathrm{Nat}(-(A, B), -(S, T))
\end{aligned}$$

Equivalence of the representations

$$\mathbb{A}daP((A,B),(S,T)) = \mathrm{Nat}(-(A,B),-(S,T)) \overset{?}{\simeq} \mathbb{A}da((A,B),(S,T))$$

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- The Yoneda lemma enables the use of the Yoneda embedding, which is a natural isomorphism between a category $\mathbb{C}$ and the category of homfunctors on $\mathbb{C}$.
- Equivalence of adapters and profunctor adapters can be shown by applying the Yoneda embedding twice.
- Similar techniques can be used to show the equivalence of any optic and its profunctor representation.