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$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[rr, "\mathrm{Hom}(A{,}f)"] \arrow[dd, "\Phi_A"'] & & \mathrm{Hom}(A,X) \arrow[dd, "\Phi_X"] \\
& \begin{tikzcd} \mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\ u \arrow[r, mapsto] & (Ff)u = \Phi_X(f) \end{tikzcd} & \\
F(A) \arrow[rr, "Ff"'] & & F(X)
\end{tikzcd}$$

What you needa know about Yoneda

Emma Bach (she/her)

Seminar on Functional Programming and Logic, Summer Semester 2025

Motivation

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- "*Tell me your company, and I will tell you what you are.*"[1]
- The Yoneda Lemma is the result of applying this way of thinking to mathematical objects within the extremely general setting of *category theory*.
- As a result, a category $\mathbb{C}$ is often best understood by instead studying functors from that category into $\mathbb{S}et$.

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- If $\mathbb{C}(A, B)$ is a set, we call it the *homset* from A to B.
- For every category $\mathbb{C}$, there exists an *opposite category* $\mathbb{C}^{op}$, in which all morphisms are reversed.

Functors

$$
\begin{tikzcd}
A \arrow[r, "f \in \mathbb{C}(A{,}B)"] \arrow[d, dotted, "F"'] & B \arrow[d, dotted, "F"] \\
F(A) \arrow[r, "F(f)"'] & F(A)
\end{tikzcd}
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- $F(g \circ f) = F(g) \circ F(f)$

Functors from a category into itself are known as *endofunctors*.

Natural Transformations

$$A \xrightarrow{f} B$$

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- Given two functors F and G, we write the collection of all natural transformation between them as $\mathrm{Nat}(F, G)$.

Exercise 1 : Order Categories

a) Let $\leq$ be a reflexive, transitive order (a *preorder*) on a set M. Show that if we define objects by $|\mathbb{P}re(M, \leq)| = M$ and morphisms by $f_{x \leq y} \in \mathbb{P}re(x, y) \Leftrightarrow x \leq y$, then $\mathbb{P}re(M, \leq)$ forms a category.

b) Let $F : \mathbb{M} \to \mathbb{M}$ be an endofunctor on $\mathbb{M}$. Show that F defines a monotonic function $M \to M$, i.e. $\forall x, y : x \leq y \implies F(x) \leq F(y)$.

c) Let $F, G : M \to M$ be monotonic functions. Let ϕ be a natural transformation $F \to G$. Show that $\forall x \in M : F(x) \leq G(x)$.

Exercise 1 : Order Categories, Solution a)

$$\begin{tikzcd}
x \arrow[loop above, "x\leq x"] \arrow[r, "x\leq y"'] \arrow[dr, "x\leq z"'] & y \arrow[loop above, "y\leq y"] \arrow[d, "y\leq z"'] \\
& z \arrow[loop below, "z\leq z"]
\end{tikzcd}$$

- $\leq$ is reflexive, so we have $\forall x : x \leq x \implies \exists id_{x \leq x} \in \mathbb{P}re(x, x)$.
- Because of transitivity, for every pair of morphisms $f_{x \leq y}$ and $g_{y \leq z}$, we have a composed morphism $(g \circ f)_{x \leq z}$.
- Since our morphisms are just witnesses of an ordering, they dont care about the order of function application, so composition is associative.

Exercise 1 : Order Categories, Solution b)

$$\begin{tikzcd} x \arrow[r, "x\leq y"] \arrow[d, "F"] & y \arrow[d, "F"] \\ F(x) \arrow[r, "F(x)\leq F(y)"] & F(y) \end{tikzcd}$$

- By the definition of functors, F must take each morphism $f : x \leq y$ to a morphism $F(f)_{F(x)\leq F(y)}$.

Exercise 1 : Order Categories, Solution c)

$$\begin{tikzcd} F(x) \arrow[d, "\phi_x"] \\ G(x) \end{tikzcd}$$

- By the definition of natural transformations, for every object x, ϕ_x is a morphism $F(x) \to G(x)$. If such a morphism exists, we have $F(x) \leq G(x)$.

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- The naturality condition resembles an equality we saw a few weeks ago:

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- This free theorem is equivalent to the statement that r is a natural transformation.

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- So our free theorem is a proof that any parametrically polymorphic function `r` is a natural transformation!
- It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors $\mathbb{S}et \to \mathbb{S}et$.

Homfunctors

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- This category is known as a *functor category* and denoted $\mathbb{D}^{\mathbb{C}}$.
- A morphism $\phi \in \mathbb{D}^{\mathbb{C}}(F, G)$ is a natural transformation $F \to G$.

The Yoneda Embedding

- Remember that the goal is finding out everything about an object A through its relations to other objects.

$$\begin{tikzcd}
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- So we want to describe an object through a collection of its homsets.
- Formally, we want a bijective functor

$$\begin{aligned} \mathcal{Y} : \mathbb{C} &\to \mathbb{S}et^{\mathbb{C}} \\ A &\mapsto \mathbb{C}(A, -) \end{aligned}$$

The Yoneda Embedding

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A \arrow[r, "f \in \mathbb{C}(A,B)"] \arrow[d, leftrightarrow, dotted, "\mathcal{Y}"'] & B \arrow[d, leftrightarrow, dotted, "\mathcal{Y}"] \\
\mathbb{C}(A,-) \arrow[r, "\mathcal{Y}(f)"'] & \mathbb{C}(B,-)
\end{tikzcd}$$

- Remember that the goal is finding out everything about an object A through its relations to other objects.
- So we want to describe an object through a collection of its homsets.
- Formally, we want a bijective functor

$$\begin{aligned} \mathcal{Y} : \mathbb{C} &\to \mathbb{S}et^{\mathbb{C}} \\ A &\mapsto \mathbb{C}(A,-) \end{aligned}$$

- We call $\mathcal{Y}$ the *Yoneda embedding*.

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- Therefore, $\mathcal{Y}(f)$ has to be a natural transformation between $\mathbb{C}(A,-)$ and $\mathbb{C}(B,-)$

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  - Vice versa, if we know all natural transformations $\mathrm{Nat}(\mathbb{C}(A, -), F)$, we can construct the set $F(A)$.
- Note that this is the *covariant* version of the Yoneda lemma. The lemma is sometimes stated equivalently in terms of the *contravariant homfunctor* $\mathbb{C}(-, A)$.

Constructing the bijection

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
\mathbb{C}(A,A) \arrow[r, "\mathbb{C}(A,f) = f\circ"] \arrow[d, "\phi_A"] & \mathbb{C}(A,B) \arrow[d, "\phi_B"] \\
F(A) \arrow[r, "F(f)"] & F(B)
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- Let $\phi \in \mathrm{Nat}(\mathbb{C}(A,-), F)$. Since ϕ is natural transformation, we have

$$F(f) \circ \phi_A = \phi_B \circ f \circ$$

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- Remember that these functors are $\mathbb{C} \to \mathbb{S}et$.
- This means our morphisms are just regular set functions.

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$$\begin{tikzcd}
id_A \arrow[r, "\mathbb{C}(A,f)=f\circ"] \arrow[d, "\phi_A"] & f \arrow[d, "\phi_B"] \\
u \in F(A) \arrow[r, "F(f)"] & \phi_B(f) = F(f)(u)
\end{tikzcd}$$

- If we apply these functions to the identity morphism id_A, we get:

$$\phi_B(f \circ id_A) = F(f)(\phi_A(id_A))$$

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Cayley's Theorem

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  - The other side sends a permutation f to the element $f(e)$
- The Yoneda lemma is often viewed as a generalization of Cayley's theorem.

Exercise 2 - Cayley's Theorem for Monoids

Use the Yoneda embedding to show that every monoid M is isomorphic to a monoid of functions $M \to M$.

Hint 1: The Yoneda embedding gives an isomorphism between objects and their homfunctors.

Hint 2: Two weeks ago we saw that every monoid M defines a category $\mathbb{M}$ with a single object $*$ and a morphism m for each element $m \in M$, where we define morphism composition to be the monoid operation.

Exercise 2 - Cayley's Theorem for Monoids

- The Yoneda embedding is an isomorphism mapping each object to its homfunctor.

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- Thus, the Yoneda embedding on M is an isomorphism between monoid objects and a set of functions $M \to M$. These functions form a monoid under composition.

$\square$