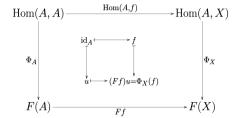
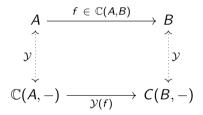
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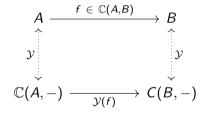
What you need aknow about Yoneda

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Seminar on Functional Programming and Logic, Summer Semester 2025

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 $\begin{array}{ccc}
A & \xrightarrow{f \in \mathbb{C}(A,B)} & B \\
\downarrow & & \downarrow \\
\mathbb{C}(A,-) & \xrightarrow{\mathcal{Y}(f)} & C(B,-)
\end{array}$

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 - ▶ Therefore, $\mathcal{Y}(f)$ is a natural transformation between $\mathbb{C}(A,-)$ and $\mathbb{C}(B,-)$

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- \triangleright So we can construct a unique \mathcal{Y} with our desired properties from any element F(A).
- Vice versa, if we know all natural transformations $Nat(\mathbb{C}(A, -), F)$, we can construct the set F(A).