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$$
\begin{tikzcd}
\mathrm{Hom}(A, A) \arrow[r, "\mathrm{Hom}(A,f)"] \arrow[d, "\Phi_A"'] & \mathrm{Hom}(A, X) \arrow[d, "\Phi_X"] \\
F(A) \arrow[r, "Ff"'] & F(X)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\
u \arrow[r, mapsto] & (Ff)u = \Phi_X(f)
\end{tikzcd}
$$

What you needa know about Yoneda

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Seminar on Functional Programming and Logic, Summer Semester 2025

Motivation

- A common sentiment in many cultures is the idea that people are defined by how they interact with their surroundings.

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- The Yoneda Lemma is the result of applying this way of thinking to mathematical objects within the extremely general setting of *category theory*.

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- A common sentiment in many cultures is the idea that people are defined by how they interact with their surroundings.
- "*Tell me your company, and I will tell you what you are.*"[1]
- The Yoneda Lemma is the result of applying this way of thinking to mathematical objects within the extremely general setting of *category theory*.
- As a result, a category $\mathbb{C}$ is often best understood by instead studying functors from that category into $\mathbb{S}et$.

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Categories

$$\begin{tikzcd}
A \arrow[loop above, "id_1"] \arrow[r, "f"] \arrow[dr, "f \circ g"'] & B \arrow[loop above, "id_2"] \arrow[d, "g"] \\
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A *category* $\mathbb{C}$ consists of:

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- for all $A \in |\mathbb{C}|$, an *identity morphism* $id_A \in \mathbb{C}(A, A)$;

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- for all $A, B \in |\mathbb{C}|$, a collection $\mathbb{C}(A, B)$ of *morphisms* from A to B;
- for all $A \in |\mathbb{C}|$, an *identity morphism* $id_A \in \mathbb{C}(A, A)$;
- an associative *composition morphism* $f \circ g \in \mathbb{C}(A, C)$ for each pair of morphisms $f \in \mathbb{C}(A, B)$, $g \in \mathbb{C}(A, B)$.

If $\mathbb{C}(A, B)$ is a set, we call it the *homset* from A to B.

Homfunctors

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f \circ g"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

$$\begin{tikzcd}
\{id_A\} \arrow[r, "f\circ"] \arrow[dr, "f \circ g \circ"'] & \{f\} \arrow[d, "g\circ"] \\
& \{f \circ g\}
\end{tikzcd}$$

- For any category $\mathbb{C}$, a homset $\mathbb{C}(A, B)$ is a set of morphisms.
- We define a functor $\mathbb{C}(A, -) : \mathbb{C} \to \mathbb{S}et$:
  - $\mathbb{C}(A, -)$ maps an Object B to the Homset $\mathbb{C}(A, B)$
  - A morphism $f : \mathbb{C}(B, C)$ is mapped to the morphism $f\circ : \mathbb{C}(A, B) \to \mathbb{C}(A, C)$

Natural Transformations

- A structure-preserving map between functors.
  - Let $F, G : \mathbb{C} \to \mathbb{D}$ be functors.
  - A natural transformation ϕ is an indexed family of morphisms $\phi_A \in \mathbb{D}(F(A), G(A))$ from $F(A)$ to $G(A)$
  - These morphisms satisfy the following *naturality condition*:

$$\forall f \in \mathbb{C}(A, B) : \phi_B \circ F(f) = G(f) \circ \phi_A$$

- Given two functors F and G, we write the collection of all natural transformation between them as $\mathrm{Nat}(F, G)$.

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- This is the free theorem we got for a parametrically polymorphic function `r :: [X] -> [X]` and an arbitrary function `a : A -> B`.
- This free theorem proves that r is a natural transformation from the list functor to itself.

Naturality from Polymorphism

- In general, assume we have:
  - two functors `F` and `G`,
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- So our free theorem is a proof that any parametrically polymorphic function `r` is a natural transformation!
- It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors $\mathbb{S}et \to \mathbb{S}et$.

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- Furthermore, this isomorphism is a natural transformation.

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- The Yoneda Lemma states that:

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- Furthermore, this isomorphism is a natural transformation.
- So the set of natural transformations from the homfunctor $\mathbb{C}(A, -)$ is in a one-to-one correspondence with the elements of the set $F(A)$.

The Yoneda Lemma

Proof

...

Instances of the Yoneda Lemma

- Cayley's theorem in group theory
- Countless theorems in algebra, particulary in algebraic topology
- Proofs by indirect inequality: $b \preceq a$ iff. $\forall c : (a \preceq c) \implies (b \preceq c)$
- Profunctor optics in functional programming

Profunctor Optics