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What you needa know about Yoneda

Emma Bach (she/her)

Seminar on Functional Programming and Logic, Summer Semester 2025

Motivation

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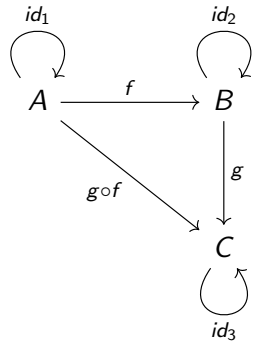
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- ▶ “*Tell me your company, and I will tell you what you are.*”¹
- ▶ The Yoneda Lemma is the result of applying this way of thinking to mathematical objects within the extremely general setting of *category theory*.
- ▶ As a result, a category \mathbb{C} is often best understood by instead studying functors from that category into \mathbf{Set} .

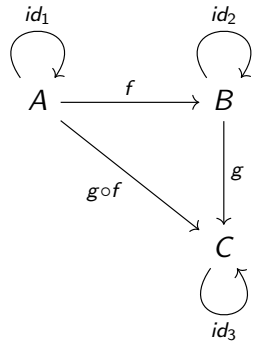
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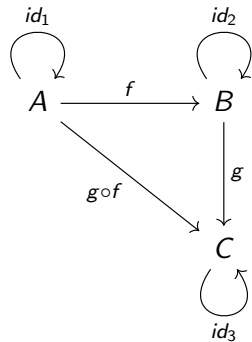
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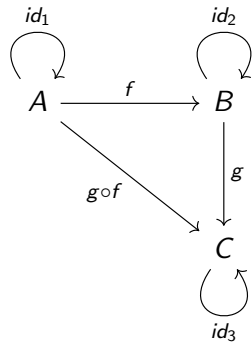
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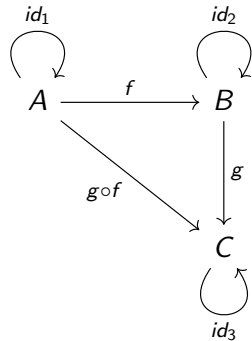
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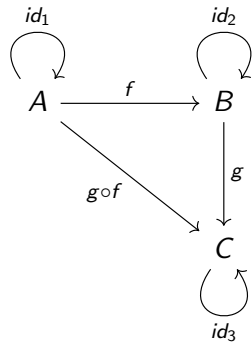
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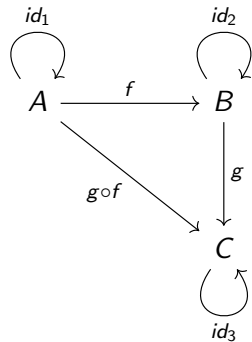
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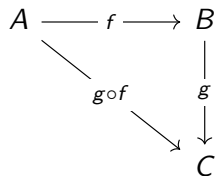
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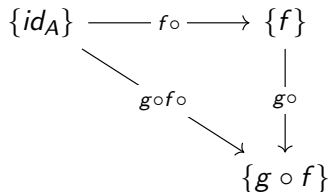


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- ▶ For every category \mathbb{C} , there exists an *opposite category* \mathbb{C}^{op} , in which all morphisms are reversed.

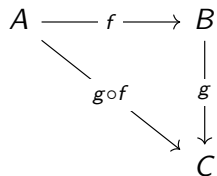
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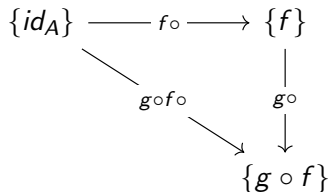
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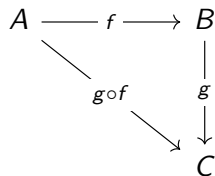
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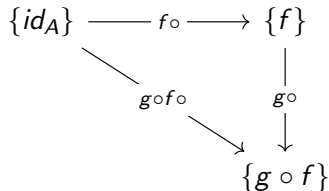
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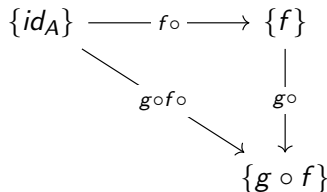
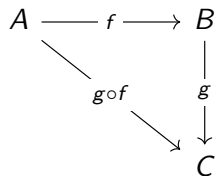
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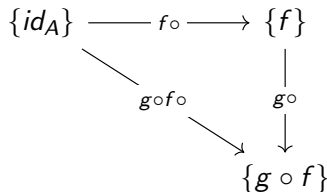
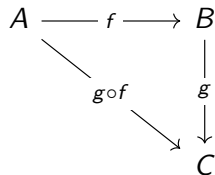


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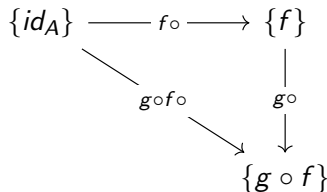
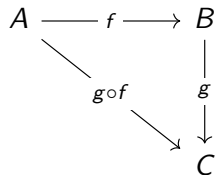
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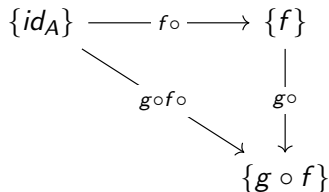
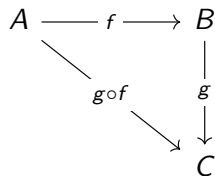
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Natural Transformations

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \\ F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow \phi_A & & \downarrow \phi_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

- ▶ A structure-preserving map between functors.
 - ▶ Let $F, G : \mathbb{C} \rightarrow \mathbb{D}$ be functors.
 - ▶ A natural transformation ϕ is an indexed family of morphisms, such that for every object $A \in |\mathbb{C}|$, ϕ_A is a morphism from $F(A)$ to $G(A)$.
 - ▶ These morphisms satisfy the following *naturality condition*:

$$\forall f \in \mathbb{C}(A, B) : \phi_B \circ F(f) = G(f) \circ \phi_A$$

- ▶ Given two functors F and G , we write the collection of all natural transformation between them as $\text{Nat}(F, G)$.

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- ▶ It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors $\mathcal{Set} \rightarrow \mathcal{Set}$.

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- ▶ Note that this is the *covariant* version of the Yoneda lemma. The lemma is sometimes stated equivalently in terms of the *contravariant homfunctor* $\mathbb{C}(-, A)$.

Constructing the bijection

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► Let $\phi \in \text{Nat}(\mathbb{C}(A, -), F)$. Since ϕ is natural transformation, we have

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$$\begin{array}{ccc} \mathbb{C}(A, A) & \xrightarrow{\mathbb{C}(A, f) = f \circ} & \mathbb{C}(A, B) \\ \downarrow \phi_A & & \downarrow \phi_B \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

- Remember that these functors are $\mathbb{C} \rightarrow \text{Set}$.

Constructing the bijection

$$A \xrightarrow{f} B$$

- ▶ Let $\phi \in \text{Nat}(\mathbb{C}(A, -), F)$. Since ϕ is natural transformation, we have

$$\mathbb{C}(A, A) \xrightarrow{\mathbb{C}(A, f) = f \circ} \mathbb{C}(A, B)$$

$$F(f) \circ \phi_A = \phi_B \circ f \circ$$

$$\begin{array}{ccc} \mathbb{C}(A, A) & \xrightarrow{\mathbb{C}(A, f) = f \circ} & \mathbb{C}(A, B) \\ \downarrow \phi_A & & \downarrow \phi_B \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

- ▶ Remember that these functors are $\mathbb{C} \rightarrow \text{Set}$.
- ▶ This means our morphisms are just regular set functions.

Constructing the bijection

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \\ id_A & \xrightarrow{\mathbb{C}(A,f)=f \circ} & f \\ \downarrow \phi_A & & \downarrow \phi_B \\ u \in F(A) & \xrightarrow{F(f)} & \phi_B(f) = F(f)(u) \end{array}$$

- If we apply these functions to the identity morphism id_A , we get:

$$\phi_B(f \circ id_A) = F(f)(\phi_A(id_A))$$

$$\phi_B(f) = F(f)(\phi_A(id_A)) := F(f)(u)$$

Cayley's Theorem

- ▶ Every group $(G, *, e)$ is isomorphic to a subgroup of the group of permutations of G .

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- ▶ Specifically:
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 - ▶ The other side sends a permutation f to the element $f(e)$
- ▶ The Yoneda lemma is often viewed as a generalization of Cayley's theorem.

Exercise 1 - Cayley's Theorem for Monoids

Use the Yoneda embedding to show that every monoid M is isomorphic to a monoid of functions $M \rightarrow M$.

Hint 1: The Yoneda embedding gives an isomorphism between objects and their homfunctors.

Hint 2: Two weeks ago we saw that every monoid M defines a category with a single object $*$ and a morphism m for each element $m \in M$.

Exercise 1 - Cayley's Theorem for Monoids

$$\begin{array}{ccc} A & \xrightarrow{f \in \mathbb{C}(A,B)} & B \\ \uparrow \scriptstyle \gamma & & \uparrow \scriptstyle \gamma \\ \mathbb{C}(A, -) & \xrightarrow{\gamma(f)} & \mathbb{C}(B, -) \end{array}$$

- The Yoneda embedding is an isomorphism mapping each object to its homfunctor.