What you needa know about Yoneda - Exercises

Emma Bach, Seminar on Functional Programming and Logic

Definition 1. A *category* $\mathbb{C}$ consists of:

- a collection $|\mathbb{C}|$ of *objects*;
- for all $A, B \in |\mathbb{C}|$, a collection $\mathbb{C}(A, B)$ of *morphisms* from A to B;
- for all $A \in |\mathbb{C}|$, an *identity morphism* $id_A \in \mathbb{C}(A, A)$;
- for each pair of morphisms $g \in \mathbb{C}(B, C)$, $f \in \mathbb{C}(A, B)$, a morphism $g \circ f \in \mathbb{C}(A, C)$, such that composition is associative.

Definition 2. A *functor* $F : \mathbb{C} \to \mathbb{D}$ is a structure-preserving map between two categories:

- F maps an object $A \in |\mathbb{C}|$ to an object $F(A) \in |\mathbb{D}|$
- F maps a morphism $f \in \mathbb{C}(A, B)$ to a morphism $F(f) \in \mathbb{D}(F(A), F(B))$
- $F(id_A) = id_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$

Definition 3. A *natural transformation* is a structure-preserving map between functors:

- Let $F, G : \mathbb{C} \to \mathbb{D}$ be functors.
- A natural transformation ϕ is an *indexed family of morphisms* - for every object $A \in |\mathbb{C}|$, ϕ_A is a morphism from $F(A)$ to $G(A)$.
- These morphisms satisfy the *naturality condition*:

$$\forall f \in \mathbb{C}(A, B) : \phi_B \circ F(f) = G(f) \circ \phi_A$$

Exercise 1

a) Let $\leq$ be a reflexive, transitive order (a *preorder*) on a set M. Show that if we define objects by $|\mathbb{P}re(M, \leq)| = M$ and morphisms by $\exists! f_{x \leq y} \in \mathbb{P}re(x, y) \Leftrightarrow x \leq y$, then $\mathbb{P}re(M, \leq)$ forms a category.

b) Let $F : \mathbb{M} \to \mathbb{M}$ be an endofunctor on $\mathbb{M}$. Show that F defines a monotonic function $M \to M$, i.e. $\forall x, y : x \leq y \implies F(x) \leq F(y)$.

c) Let $F, G : M \to M$ be monotonic functions. Let ϕ be a natural transformation $F \to G$. Show that $\forall x \in M : F(x) \leq G(x)$.

Definition 4. The *homfunctor* $\mathbb{C}(A, -)$ is defined as follows:

- An object B is mapped to the homset $\mathbb{C}(A, B)$.
- A morphism $f : \mathbb{C}(B, C)$ is mapped to the morphism $f\circ : \mathbb{C}(A, B) \to \mathbb{C}(A, C)$.

Definition 5. Yoneda Lemma: There exists a natural isomorphism

$$\mathrm{Nat}(\mathbb{C}(A, -), F) \simeq F(A)$$

Definition 6. Yoneda Embedding: For every object $A \in |\mathbb{C}|$, there exists a bijective functor

$$\mathcal{Y} : \mathbb{C} \to \mathbb{C}(A, -)$$

Proposition 7. *Any monoid $M = (M, *, e)$ can be represented as a category $\mathbb{M}$ with a single object $*$, where monoid elements are morphisms, such that composing two morphisms m and n gives the morphism corresponding to the element $m * n$.*

Exercise 2

Use the Yoneda embedding to show that every monoid M is isomorphic to a monoid of functions $M \to M$.