

$$\begin{array}{ccc}
 \text{Hom}(A, A) & \xrightarrow{\text{Hom}(A, f)} & \text{Hom}(A, X) \\
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 F(A) & \xrightarrow{Ff} & F(X)
 \end{array}$$

$\begin{array}{ccc} \text{id}_A & \xrightarrow{\quad} & f \\ \downarrow & & \downarrow \\ u & \xrightarrow{\quad} & (Ff)u = \Phi_X(f) \end{array}$

What you needa know about Yoneda

Emma Bach (she/her)

Seminar on Functional Programming and Logic, Summer Semester 2025

Motivation

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- ▶ “Tell me your company, and I will tell you what you are.”¹
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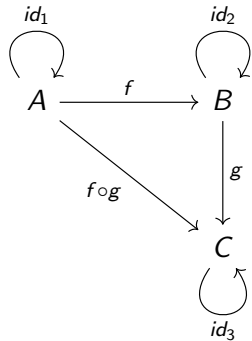
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- ▶ “Tell me your company, and I will tell you what you are.”¹
- ▶ The Yoneda Lemma is the result of applying this way of thinking to mathematical objects within the extremely general setting of category theory.
- ▶ As a result, a category \mathbb{C} is often best understood by instead studying functors from that category into \mathbf{Set} .

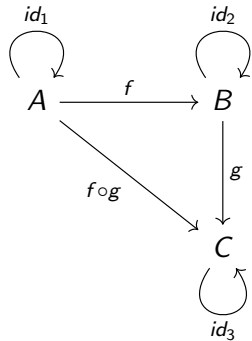
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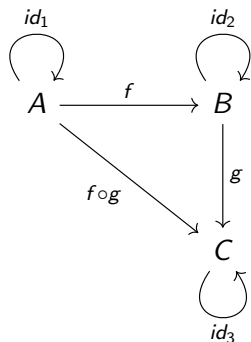
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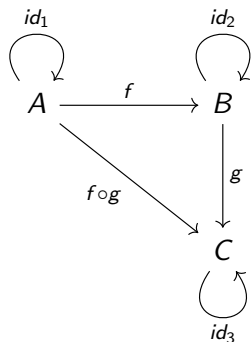
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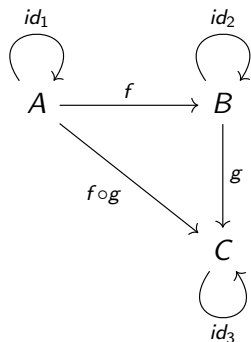
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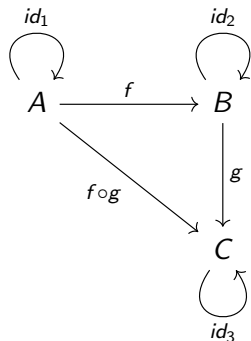
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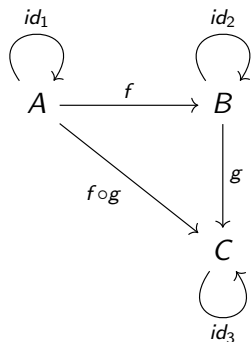
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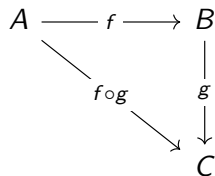
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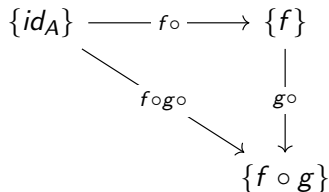


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- ▶ If $\mathbb{C}(A, B)$ is a set, we call it the homset from A to B .
- ▶ For every category \mathbb{C} , there exists an opposite category \mathbb{C}^{op} , in which all morphisms are reversed.

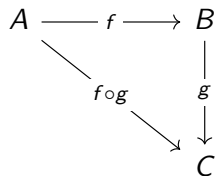
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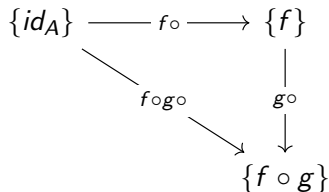
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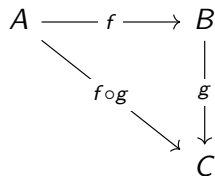
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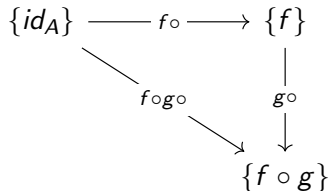
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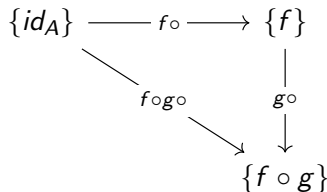
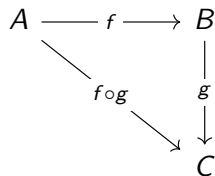
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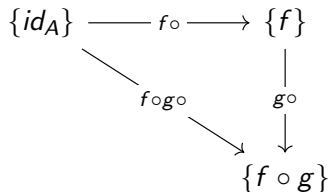
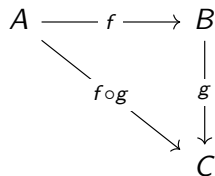


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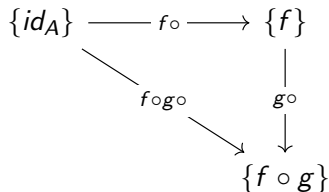
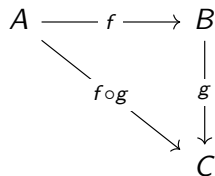
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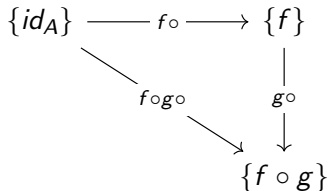
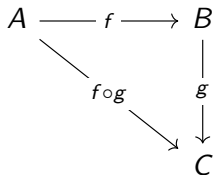
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Natural Transformations

- ▶ A structure-preserving map between functors.
 - ▶ Let $F, G : \mathbb{C} \rightarrow \mathbb{D}$ be functors.
 - ▶ A natural transformation ϕ is an indexed family of morphisms $\phi_A \in \mathbb{D}(F(A), G(A))$ from $F(A)$ to $G(A)$
 - ▶ These morphisms satisfy the following naturality condition:

$$\forall f \in \mathbb{C}(A, B) : \phi_B \circ F(f) = G(f) \circ \phi_A$$

- ▶ Given two functors F and G , we write the collection of all natural transformation between them as $\text{Nat}(F, G)$.

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- ▶ It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors $\mathbf{Set} \rightarrow \mathbf{Set}$.

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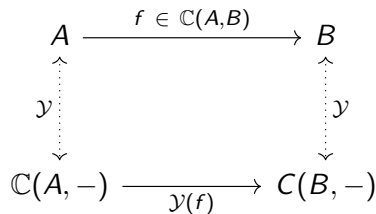
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- ▶ Vice versa, if we know all natural transformations $\text{Nat}(\mathbb{C}(A, -), F)$, we can construct the set $F(A)$.

Instances of the Yoneda Lemma

- ▶ Cayley's theorem in group theory
- ▶ Countless theorems in algebra, particularly in algebraic topology
- ▶ Proofs by indirect inequality: $b \preceq a$ iff. $\forall c : (a \preceq c) \implies (b \preceq c)$
- ▶ Profunctor optics in functional programming

Profunctor Optics