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 \downarrow \Phi_A & & \downarrow \Phi_X \\
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What you needa know about Yoneda

Emma Bach (she/her)

Seminar on Functional Programming and Logic, Summer Semester 2025

Motivation

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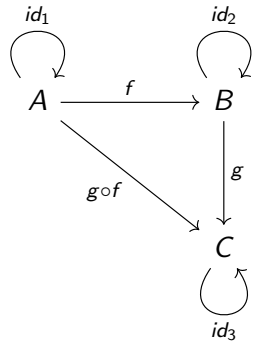
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- ▶ “*Tell me your company, and I will tell you what you are.*”¹
- ▶ The Yoneda Lemma is the result of applying this way of thinking to mathematical objects within the extremely general setting of *category theory*.
- ▶ As a result, a category \mathbb{C} is often best understood by instead studying functors from that category into \mathbf{Set} .

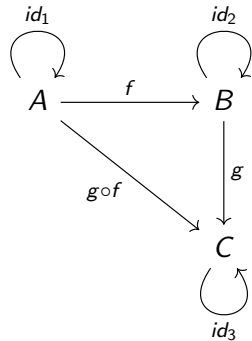
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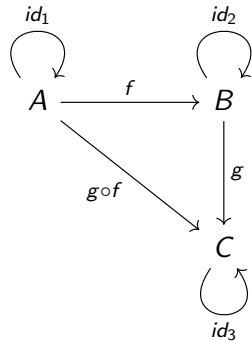
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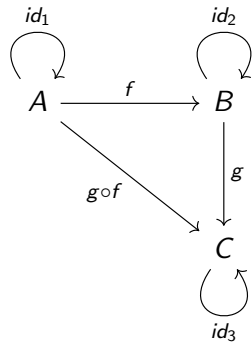
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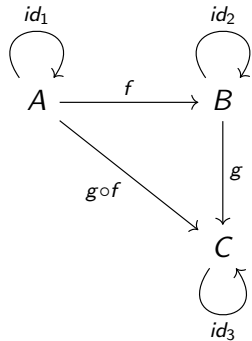
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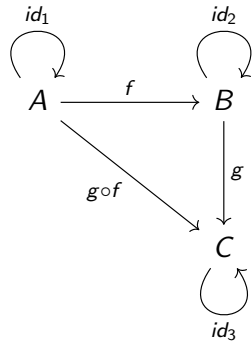
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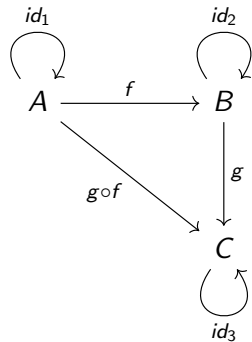
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- ▶ If $\mathbb{C}(A, B)$ is a set, we call it the *homset* from A to B .
- ▶ For every category \mathbb{C} , there exists an *opposite category* \mathbb{C}^{op} , in which all morphisms are reversed.

Functors

A *functor* $F : \mathbb{C} \rightarrow \mathbb{D}$ is a structure-preserving map between two categories:

$$\begin{array}{ccc} A & \xrightarrow{f \in \mathbb{C}(A,B)} & B \\ \vdots F \downarrow & & \downarrow F \vdots \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array}$$

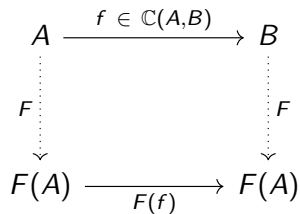
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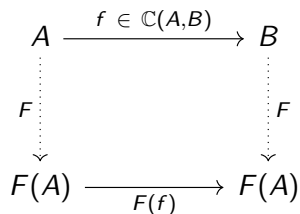
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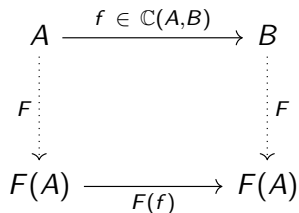
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- ▶ $F(g \circ f) = F(g) \circ F(f)$

Functors from a category into itself are known as *endofunctors*.

Natural Transformations

- Structure-preserving maps between functors.

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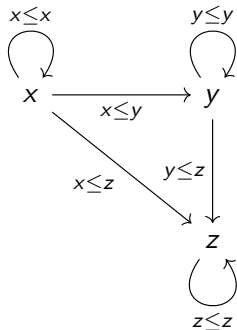
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- ▶ Given two functors F and G , we write the collection of all natural transformation between them as $\text{Nat}(F, G)$.

Exercise 1 : Order Categories

- a) Let \leq be a reflexive, transitive order (a *preorder*) on a set M . Show that if we define objects by $|\mathbb{P}re(M, \leq)| = M$ and morphisms by $f_{x \leq y} \in \mathbb{P}re(x, y) \Leftrightarrow x \leq y$, then $\mathbb{P}re(M, \leq)$ forms a category.
- b) Let $F : \mathbb{M} \rightarrow \mathbb{M}$ be an endofunctor on \mathbb{M} . Show that F defines a monotonic function $M \rightarrow M$, i.e. $\forall x, y : x \leq y \implies F(x) \leq F(y)$.
- c) Let $F, G : M \rightarrow M$ be monotonic functions. Let ϕ be a natural transformation $F \rightarrow G$. Show that $\forall x \in M : F(x) \leq G(x)$.

Exercise 1 : Order Categories, Solution a)



- ▶ \leq is reflexive, so we have
 $\forall x : x \leq x \implies \exists id_{x \leq x} \in \mathbb{P}re(x, x).$
- ▶ Because of transitivity, for every pair of morphisms $f_{x \leq y}$ and $g_{y \leq z}$, we have a composed morphism $(g \circ f)_{x \leq z}$.
- ▶ Since our morphisms are just witnesses of an ordering, they don't care about the order of function application, so composition is associative.

Exercise 1 : Order Categories, Solution b)

$$\begin{array}{ccc} x & \xrightarrow{x \leq y} & y \\ \downarrow F & & \downarrow F \\ F(x) & \xrightarrow{F(x) \leq F(y)} & F(y) \end{array}$$

- By the definition of functors, F must take each morphism $f : x \leq y$ to a morphism $F(f)_{F(x) \leq F(y)}$.

Exercise 1 : Order Categories, Solution c)

$$\begin{array}{c} F(x) \\ \downarrow \phi_x \\ G(x) \end{array}$$

- By the definition of natural transformations, for every object x , ϕ_x is a morphism $F(x) \rightarrow G(x)$. If such a morphism exists, we have $F(x) \leq G(x)$.

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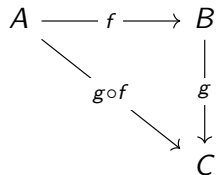
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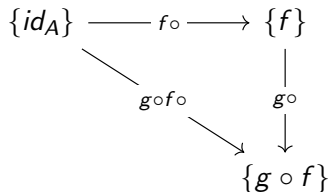
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- ▶ So our free theorem is a proof that any parametrically polymorphic function r is a natural transformation!
- ▶ It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors $\mathcal{Set} \rightarrow \mathcal{Set}$.

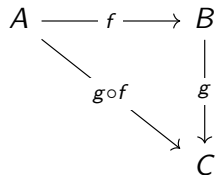
Homfunctors



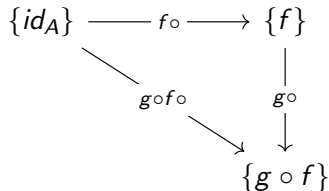
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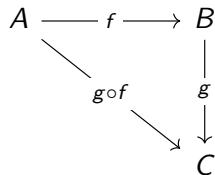
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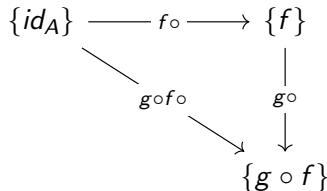
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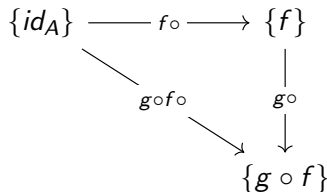
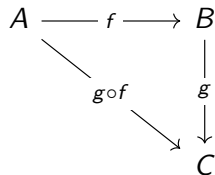
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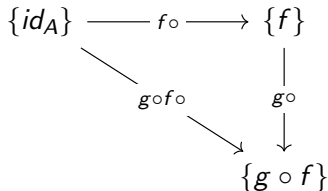
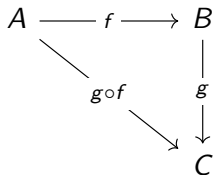


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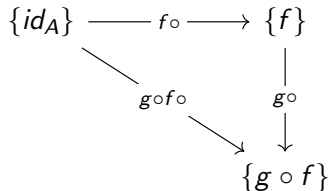
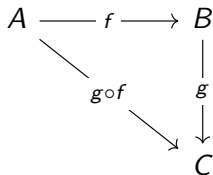
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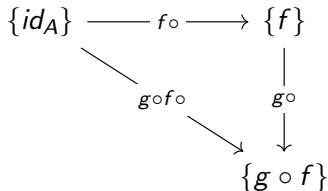
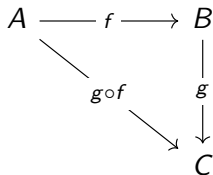
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- ▶ A morphism $\phi \in \mathbb{D}^{\mathbb{C}}(F, G)$ is a natural transformation $F \rightarrow G$.

The Yoneda Embedding

- Remember that the goal is finding out everything about an object A through its relations to other objects.

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- ▶ Note that this is the *covariant* version of the Yoneda lemma. The lemma is sometimes stated equivalently in terms of the *contravariant homfunctor* $\mathbb{C}(-, A)$.

Constructing the bijection

$$A \xrightarrow{f} B$$

► Let $\phi \in \text{Nat}(\mathbb{C}(A, -), F)$. Since ϕ is natural transformation, we have

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- ▶ Remember that these functors are $\mathbb{C} \rightarrow \text{Set}$.
- ▶ This means our morphisms are just regular set functions.

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$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \\ id_A & \xrightarrow{\mathbb{C}(A,f)=f \circ} & f \\ \downarrow \phi_A & & \downarrow \phi_B \\ u \in F(A) & \xrightarrow{F(f)} & \phi_B(f) = F(f)(u) \end{array}$$

- If we apply these functions to the identity morphism id_A , we get:

$$\phi_B(f \circ id_A) = F(f)(\phi_A(id_A))$$

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- ▶ The Yoneda lemma is often viewed as a generalization of Cayley's theorem.

Exercise 2 - Cayley's Theorem for Monoids

Use the Yoneda embedding to show that every monoid M is isomorphic to a monoid of functions $M \rightarrow M$.

Hint 1: The Yoneda embedding gives an isomorphism between objects and their homfunctors.

Hint 2: Two weeks ago we saw that every monoid M defines a category \mathbb{M} with a single object $*$ and a morphism m for each element $m \in M$, where we define morphism composition to be the monoid operation.

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- ▶ Thus, the Yoneda embedding on M is an isomorphism between monoid objects and a set of functions $M \rightarrow M$. These functions form a monoid under composition.

