universität freiburg

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[rr, "\mathrm{Hom}(A{,}f)"] \arrow[d, "\Phi_A"'] & & \mathrm{Hom}(A,X) \arrow[d, "\Phi_X"] \\
F(A) \arrow[rr, "Ff"'] & & F(X)
\end{tikzcd}$$

$$\begin{tikzcd}
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u \arrow[r, mapsto] & (Ff)u = \Phi_X(f)
\end{tikzcd}$$

What you needa know about Yoneda

Emma Bach (she/her)
Seminar on Functional Programming and Logic, Summer Semester 2025

Motivation

- A common sentiment in many cultures is the idea that things are defined by how they interact with their surroundings.

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- A common sentiment in many cultures is the idea that things are defined by how they interact with their surroundings.
- "*Tell me your company, and I will tell you what you are.*"[1]
- The Yoneda lemma is the result of applying this way of thinking to mathematical objects in category theory.
- As a result, a category $\mathbb{C}$ is often best understood by instead studying functors from that category into $\mathbb{S}et$.

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Categories

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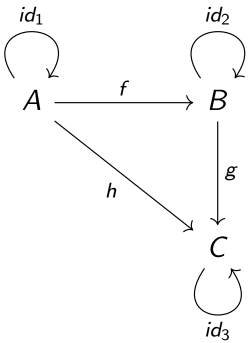

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- For every category $\mathbb{C}$, the *opposite category* $\mathbb{C}^{op}$ is a category.
- For every pair of categories $\mathbb{C}$, $\mathbb{D}$, the *product category* $\mathbb{C} \times \mathbb{D}$ is a category.

Functors

$$\begin{tikzcd}
A \arrow[r, "f \in \mathbb{C}(A{,}B)"] \arrow[d, dotted, "F"'] & B \arrow[d, dotted, "F"] \\
F(A) \arrow[r, "F(f)"'] & F(A)
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- $F(g \circ f) = F(g) \circ F(f)$

Functors from a category into itself are known as *endofunctors*.

Natural Transformations

$$A \xrightarrow{f} B$$

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- Given two functors F and G, we write the collection of all natural transformation between them as $\mathrm{Nat}(F, G)$.

Exercise 1 : Order Categories

a) Let $\leq$ be a reflexive, transitive order (a *preorder*) on a set M. Show that if we define objects by $|\mathbb{P}re(M, \leq)| = M$ and morphisms by $\exists! f_{x \leq y} \in \mathbb{P}re(x, y) \Leftrightarrow x \leq y$, then $\mathbb{P}re(M, \leq)$ forms a category.

b) Let $F : \mathbb{M} \to \mathbb{M}$ be an endofunctor on $\mathbb{M}$. Show that F defines a monotonic function $M \to M$, i.e. $\forall x, y : x \leq y \implies F(x) \leq F(y)$.

c) Let $F, G : M \to M$ be monotonic functions. Let ϕ be a natural transformation $F \to G$. Show that $\forall x \in M : F(x) \leq G(x)$.

Exercise 1 : Order Categories, Solution a)

$$
\begin{tikzcd}
x \arrow[loop above, "x\leq x"] \arrow[r, "x\leq y"'] \arrow[dr, "x\leq z"'] & y \arrow[loop above, "y\leq y"] \arrow[d, "y\leq z"'] \\
 & z \arrow[loop below, "z\leq z"]
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$$

- $\leq$ is reflexive, so we have $\forall x : x \leq x \implies \exists id_{x\leq x} \in \mathbb{P}re(x, x)$.
- Because of transitivity, for every pair of morphisms $f_{x\leq y}$ and $g_{y\leq z}$, we have a composed morphism $(g \circ f)_{x\leq z}$.
- Since our morphisms are just witnesses of an ordering, they dont care about the order of function application, so composition is associative.

Exercise 1 : Order Categories, Solution b)

$$\begin{tikzcd} x \arrow[r, "x\leq y"] \arrow[d, "F"] & y \arrow[d, "F"] \\ F(x) \arrow[r, "F(x)\leq F(y)"] & F(y) \end{tikzcd}$$

- By the definition of functors, F must take each morphism $f_{x\leq y} \in \mathbb{P}re(x, y)$ to a morphism $F(f)_{F(x)\leq F(y)} \in \mathbb{P}re(F(x), F(y))$.

Exercise 1 : Order Categories, Solution c)

$$\begin{tikzcd} F(x) \arrow[d, "\phi_x \ :\ F(x) \leq G(x)"] \\ G(x) \end{tikzcd}$$

- By the definition of natural transformations, for every object x, ϕ_x is a morphism $F(x) \to G(x)$. If such a morphism exists, we have $F(x) \leq G(x)$.

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- The naturality condition resembles an equality we saw a few weeks ago:

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- This free theorem is equivalent to the statement that r is a natural transformation.

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- So our free theorem is a proof that any parametrically polymorphic function `r` is a natural transformation!
- It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors $\mathbb{S}et \to \mathbb{S}et$.

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- A morphism $\phi \in \mathbb{D}^{\mathbb{C}}(F, G)$ is a natural transformation $F \to G$.

The Yoneda Embedding

$$
\begin{tikzcd}
 & C & & D & & & \\
B & & & & E & & \\
 & & A \arrow[ul] \arrow[ur] \arrow[ull] \arrow[urr] \arrow[dll] \arrow[drr] \arrow[ddl] \arrow[ddr] & & & \Longleftrightarrow & \mathbb{C}(A, -) \\
I & & & & F & & \\
 & H & & G & & &
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- Is it actually possible to construct all of the necessary natural transformations?

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  - So we can construct the Yoneda embedding $\mathcal{Y}$ from the set $F(A)$.
  - Vice versa, if we know all natural transformations $\mathrm{Nat}(\mathbb{C}(A, -), F)$, we can construct the set $F(A)$.

Constructing the bijection

$$
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\mathbb{C}(A, A) \arrow[r, "\mathbb{C}(A,f) = f\circ"] \arrow[d, "\phi_A"] & \mathbb{C}(A, B) \arrow[d, "\phi_B"] \\
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- Let $\phi \in \mathrm{Nat}(\mathbb{C}(A, -), F)$. Since ϕ is natural transformation, we have

$$F(f) \circ \phi_A = \phi_B \circ f \circ$$

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- This means our morphisms are just regular set functions.

Constructing the bijection

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
id_A \arrow[r, "\mathbb{C}(A,f)=f\circ"] \arrow[d, "\phi_A"] & f \arrow[d, "\phi_B"] \\
u \in F(A) \arrow[r, "F(f)"] & \phi_B(f) = F(f)(u)
\end{tikzcd}$$

- If we apply these functions to the identity morphism id_A, we get:

$$\phi_B(f \circ id_A) = F(f)(\phi_A(id_A))$$

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Application : Cayley's Theorem

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  - The other side sends a permutation f to the element $f(e)$
- The Yoneda lemma is often viewed as a generalization of Cayley's theorem.

Exercise 2 - Cayley's Theorem for Monoids

Use the Yoneda embedding to show that every monoid M is isomorphic to a monoid of functions $M \to M$.
Hint 1: The Yoneda embedding gives an isomorphism between objects and their homfunctors.
Hint 2: Two weeks ago we saw that every monoid M defines a category $\mathbb{M}$ with a single object $*$ and a morphism m for each element $m \in M$, where we define morphism composition to be the monoid operation.

Exercise 2 - Cayley's Theorem for Monoids

- The Yoneda embedding is an isomorphism mapping each object to its homfunctor.

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- Thus, the Yoneda embedding on M is an isomorphism between monoid objects and a set of functions $M \to M$. These functions form a monoid under composition.

$\square$

Profunctor Optics

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- *Profunctor optics* are neat and flexible representations of optics as individual polymorphic function.
- In particular, profunctor optics make composition of optics trivial.
- Equivalence between optics and their profunctor representations comes down to the Yoneda lemma.

Adapters

```haskell
data Adapter a b s t = Adapter { from :: s -> a, to :: b -> t }
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- This lets us view the category $\mathbb{H}ask^{op} \times \mathbb{H}ask$ as the category $\mathbb{A}da$ of adapters.

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- We define the category $\mathbb{P}rof$ of Profunctors on Haskell types to be the functor category $\mathbb{S}et^{(\mathbb{H}ask^{op} \times \mathbb{H}ask)} = \mathbb{S}et^{\mathbb{A}da}$

Functor Application as a Functor

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, dotted, "-(A)"'] & G \arrow[d, dotted, "-(A)"] \\ F(A) \arrow[r, "\eta_A"'] & G(A) \end{tikzcd}$$

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- We write $-(A)$ for this functor.

Profunctor Adapters

- The profunctor representation of an adapter is given by:

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type AdapterP a b s t =
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- Specifically, the homsets in $\mathbb{A}daP$ are:

$$\mathbb{A}daP((A,B),(S,T)) = \mathbb{S}et^{\mathbb{P}rof}(-(A,B), -(S,T))$$

Equivalence of the representations

$$\mathbb{S}et^{\mathbb{P}rof}(-(A,B),-(S,T)) \overset{?}{\simeq} \mathbb{A}da((A,B),(S,T))$$

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- Equivalence of adapters and profunctor adapters can be shown by applying the Yoneda embedding twice.
- Similar techniques can be used to show the equivalence of any optic and its profunctor representation.