universität freiburg

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F(A) \arrow[rrr, "Ff"'] & & & F(X)
\end{tikzcd}$$

What you needa know about Yoneda

Emma Bach (she/her)

Seminar on Functional Programming and Logic, Summer Semester 2025

Motivation

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- "*Tell me your company, and I will tell you what you are.*"[1]
- The Yoneda Lemma is the result of applying this way of thinking to mathematical objects within the extremely general setting of *category theory*.
- As a result, a category $\mathbb{C}$ is often best understood by instead studying functors from that category into $\mathbb{S}et$.

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- If $\mathbb{C}(A, B)$ is a set, we call it the *homset* from A to B.
- For every category $\mathbb{C}$, there exists an *opposite category* $\mathbb{C}^{op}$, in which all morphisms are reversed.

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Natural Transformations

$$A \xrightarrow{f} B$$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\phi_A"] & F(B) \arrow[d, "\phi_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

- A structure-preserving map between functors.
  - Let $F, G : \mathbb{C} \to \mathbb{D}$ be functors.
  - A natural transformation ϕ is an indexed family of morphisms, such that for every object $A \in |\mathbb{C}|$, ϕ_A is a morphism from $F(A)$ to $G(A)$.
  - These morphisms satisfy the following *naturality condition*:

$$\forall f \in \mathbb{C}(A, B) : \phi_B \circ F(f) = G(f) \circ \phi_A$$

- Given two functors F and G, we write the collection of all natural transformation between them as $\mathrm{Nat}(F, G)$.

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- The naturality condition resembles an equality we saw a few weeks ago:

$$r_B \circ \mathtt{map}(a) = \mathtt{map}(a) \circ r_A$$

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- This is the free theorem we got for a parametrically polymorphic function `r :: [X] -> [X]` and an arbitrary function `a : A -> B`.
- This free theorem is equivalent to the statement that r is a natural transformation.

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- It turns out that parametrically polymorphic functions correspond exactly to natural transformations between endofunctors $\mathbb{S}et \to \mathbb{S}et$.

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- This category is known as a *functor category* and denoted $\mathbb{D}^{\mathbb{C}}$.
- A morphism $\phi \in \mathbb{D}^{\mathbb{C}}(F, G)$ is a natural transformation $F \to G$.

The Yoneda Embedding

- Remember that the goal is finding out everything about an object A through its relations to other objects.

$$\begin{tikzcd}
A \arrow[r, "f \,\in\, \mathbb{C}(A,B)"] \arrow[d, dotted, leftrightarrow, "\mathcal{Y}"'] & B \arrow[d, dotted, leftrightarrow, "\mathcal{Y}"] \\
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- Given $f \in \mathbb{C}(A, B)$, $\mathcal{Y}(f)$ has to be a morphism between $\mathbb{C}(A, -)$ and $\mathbb{C}(B, -)$ in the functor category $\mathbb{S}et^{\mathbb{C}}$.

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- Therefore, $\mathcal{Y}(f)$ has to be a natural transformation between $\mathbb{C}(A,-)$ and $\mathbb{C}(B,-)$

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  - Vice versa, if we know all natural transformations $\mathrm{Nat}(\mathbb{C}(A, -), F)$, we can construct the set $F(A)$.
- Note that this is the *covariant* version of the Yoneda lemma. The lemma is sometimes stated equivalently in terms of the *contravariant homfunctor* $\mathbb{C}(-, A)$.

Constructing the bijection

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- Remember that these functors are $\mathbb{C} \to \mathbb{S}et$.

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- Let $\phi \in \mathrm{Nat}(\mathbb{C}(A,-),F)$. Since ϕ is natural transformation, we have

$$F(f)\circ\phi_A = \phi_B \circ f\circ$$

- Remember that these functors are $\mathbb{C} \to \mathbb{S}et$.
- This means our morphisms are just regular set functions.

Constructing the bijection

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
id_A \arrow[r, "\mathbb{C}(A{,}f) = f\circ"] \arrow[d, "\phi_A"] & f \arrow[d, "\phi_B"] \\
u \in F(A) \arrow[r, "F(f)"] & \phi_B(f) = F(f)(u)
\end{tikzcd}$$

- If we apply these functions to the identity morphism id_A, we get:

$$\phi_B(f \circ id_A) = F(f)(\phi_A(id_A))$$

$$\phi_B(f) = F(f)(\phi_A(id_A)) := F(f)(u)$$

Cayley's Theorem

- Every group $(G, *, e)$ is isomorphic to a subgroup of the group of permutations of G.

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  - The other side sends a permutation f to the element $f(e)$
- The Yoneda lemma is often viewed as a generalization of Cayley's theorem.

Exercise 1 - Cayley's Theorem for Monoids

Use the Yoneda embedding to show that every monoid M is isomorphic to a monoid of functions $M \to M$.
Hint 1: The Yoneda embedding gives an isomorphism between objects and their homfunctors.
Hint 2: Two weeks ago we saw that every monoid M defines a category with a single object $*$ and a morphism m for each element $m \in M$.

Exercise 1 - Cayley's Theorem for Monoids

$$
\begin{tikzcd}
A \arrow[rr, "f \,\in\, \mathbb{C}(A{,}B)"] \arrow[d, leftrightarrow, dotted, "\mathcal{Y}"'] & & B \arrow[d, leftrightarrow, dotted, "\mathcal{Y}"] \\
\mathbb{C}(A, -) \arrow[rr, "\mathcal{Y}(f)"'] & & C(B, -)
\end{tikzcd}
$$

- The Yoneda embedding is an isomorphism mapping each object to its homfunctor.