# Reduced form of climate-econ model

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## 1 Equations of the model

$$Y^0 = \frac{K}{\nu} \tag{1}$$

$$Y = (1 - \mathbf{D})(1 - A)Y^0 \tag{2}$$

$$L = \frac{Y^0}{a} \tag{3}$$

$$\dot{a} := \alpha\, a \tag{4}$$

$$\dot{N} := \beta N \tag{5}$$

$$\lambda := \frac{L}{N} \tag{6}$$

$$\dot{w} = w\left(\varphi(\lambda) + \gamma i\right) \tag{7}$$

$$i = \frac{\dot{p}}{p} = \eta(\xi\omega - 1) \tag{8}$$

$$\omega := \frac{wL}{pY} \tag{9}$$

$$\Pi := pY - wL - rD + p\Upsilon \tag{10}$$

$$\pi = \frac{\Pi}{pY} \tag{11}$$

$$d = \frac{D}{pY} \tag{12}$$

$$I = \kappa(\pi)Y \tag{13}$$

$$\dot{K} = I - \delta K \tag{14}$$

$$\dot{D} = pI - \Pi \tag{15}$$

$$E_{ind} = \sigma(1-n)Y^0 \tag{16}$$

$$\dot{E}_{land} = \delta_{E_{land}} E_{land} \tag{17}$$

$$\dot{\sigma} = g_\sigma \sigma \tag{18}$$
$$\dot{g_\sigma} = \delta_{g_\sigma} g_\sigma \tag{19}$$

$$\dot{p_{BS}} = \delta_{p_{BS}} p_{BS} \tag{20}$$

$$A = \frac{\sigma \, p_{BS}}{\theta} \, n^\theta \tag{21}$$

$$n = \min\left\{ \left( \frac{p_C}{(1-s_A)p_{BS}} \right)^{\frac{1}{\theta-1}} \, , \, 1 \right\} \tag{22}$$

$$\dot{T} = \frac{F - \frac{F_{dbl}}{S}T - \gamma^*(T - T_{LO})}{C} \tag{23}$$

$$\dot{T_{LO}} = \frac{\gamma^*(T - T_{LO})}{C_{LO}} \tag{24}$$

$$\begin{pmatrix} \dot{CO_2}^{AT} \\ \dot{CO_2}^{UP} \\ \dot{CO_2}^{LO} \end{pmatrix} = \begin{pmatrix} E_T \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\phi_{12} & \phi_{12} C_{UP}^{AT} & 0 \\ \phi_{12} & -\phi_{12} C_{UP}^{AT} - \phi_{23} & \phi_{23} C_{LO}^{UP} \\ 0 & \phi_{23} & -\phi_{23} C_{LO}^{UP} \end{pmatrix} \begin{pmatrix} CO_2^{AT} \\ CO_2^{UP} \\ CO_2^{LO} \end{pmatrix} \tag{25}$$

$$\mathbf{D} = 1 - \frac{1}{1 + \xi_1 T + \xi_2 T^2 + \xi_3 T^7} \tag{26}$$

# 2 Derivation

## 2.1 Growth of original output

Using (13) and (14), the growth rate of capital is

$$\frac{\dot{K}}{K} = \frac{I - \delta K}{K} = \frac{I}{K} - \delta = \frac{\kappa(\pi)}{\nu} - \delta \tag{27}$$

The growth rate of the original output $Y^0$ is then:

$$\frac{\dot{Y}^0}{Y^0} = \frac{\dot{K}}{\nu}\frac{\nu}{K} = \frac{I - \delta K}{K} = \frac{\kappa(\pi)Y}{K} - \delta = \frac{\kappa(\pi)}{\nu} - \delta \tag{28}$$

## 2.2 Growth of productive output

Since $Y = (1 - \mathbf{D})(1 - A)Y^0$,

$$\frac{\dot{Y}}{Y} = -\frac{\dot{\mathbf{D}}}{1 - \mathbf{D}} - \frac{\dot{A}}{1 - A} + \frac{\dot{Y}^0}{Y^0} \tag{29}$$

Derivative of $n$:

If $n < 1$, $n = \left(\frac{p_C}{(1-s_A)p_{BS}}\right)^{\frac{1}{\theta-1}}$

$$
\begin{aligned}
\dot{n} &= \left(\frac{1}{\theta-1}\right)\left(\frac{p_C}{(1-s_A)p_{BS}}\right)^{\frac{2-\theta}{\theta-1}}\left(\frac{\dot{p}_C p_{BS} + p_C \dot{p}_{BS}}{(1-s_A)p_{BS}^2}\right) \\
&= \left(\frac{1}{\theta-1}\right)\left(\frac{p_C}{(1-s_A)p_{BS}}\right)^{\frac{2-\theta}{\theta-1}}\left(\frac{\delta_C p_{BS} + p_C \delta_{p_{BS}} p_{BS}}{(1-s_A)p_{BS}^2}\right) \\
&= \left(\frac{1}{\theta-1}\right)\left(\frac{p_C}{(1-s_A)p_{BS}}\right)^{\frac{2-\theta}{\theta-1}}\left(\frac{\delta_C + p_C \delta_{p_{BS}}}{(1-s_A)p_{BS}}\right)
\end{aligned}
$$

If $n \geq 1$, $\dot{n} = 0$.

Derivative of $A$, $A = \frac{\sigma\, p_{BS}}{\theta}\, n^{\theta}$:

$$
\begin{aligned}
\dot{A} &= \frac{1}{\theta}\left(\dot{\sigma} p_{BS} n^{\theta} + \sigma \dot{p}_{BS} n^{\theta} + \sigma p_{BS} \theta n^{\theta-1}\dot{n}\right) \\
&= \frac{1}{\theta}\left(g_{\sigma}\sigma p_{BS} n^{\theta} + \sigma \delta_{p_{BS}} p_{BS} n^{\theta} + \sigma p_{BS} \theta n^{\theta-1}\dot{n}\right) \\
&= \frac{\sigma p_{BS} n^{\theta}}{\theta}\left(g_{\sigma} + \delta_{p_{BS}} + \theta n^{-1}\dot{n}\right) \\
&= A\left(g_{\sigma} + \delta_{p_{BS}} + \theta\frac{\dot{n}}{n}\right) \\
&= A\left(g_{\sigma} + \delta_{p_{BS}} + \left(\frac{\theta}{\theta-1}\right)\left(\frac{p_C}{(1-s_A)p_{BS}}\right)^{-1}\left(\frac{\delta_C + p_C\delta_{p_{BS}}}{(1-s_A)p_{BS}}\right)\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\dot{A}}{1-A} &= \frac{A}{1-A}\left(g_{\sigma} + \delta_{p_{BS}} + \left(\frac{\theta}{\theta-1}\right)\left(\frac{(1-s_A)p_{BS}}{p_C}\right)\left(\frac{\delta_C + p_C\delta_{p_{BS}}}{(1-s_A)p_{BS}}\right)\right) \\
&= \frac{\sigma p_{BS} n^{\theta}}{\theta - \sigma p_{BS} n^{\theta}}\left(g_{\sigma} + \delta_{p_{BS}} + \left(\frac{\theta}{\theta-1}\right)\left(\frac{(1-s_A)p_{BS}}{p_C}\right)\left(\frac{\delta_C + p_C\delta_{p_{BS}}}{(1-s_A)p_{BS}}\right)\right)
\end{aligned}
$$

Derivative of the damage function, $\mathbf{D} = 1 - \frac{1}{1+\xi_1 T + \xi_2 T^2 + \xi_3 T^7}$:

$$
\dot{\mathbf{D}} = \frac{d\mathbf{D}}{dT}\frac{dT}{dt} = \left(\frac{\xi_1 + 2\xi_2 T + 7\xi_3 T^6}{(1+\xi_1 T + \xi_2 T^2 + \xi_3 T^7)^2}\right)\dot{T} \tag{30}
$$

Therefore

$$
\frac{\dot{\mathbf{D}}}{1-\mathbf{D}} = \left(\frac{\xi_1 + 2\xi_2 T + 7\xi_3 T^6}{1+\xi_1 T + \xi_2 T^2 + \xi_3 T^7}\right)\dot{T}
$$

### 2.3 Growth rate of employment and the workforce

Using (3), (4), and (28), the growth rate of employment is

$$\frac{\dot{L}}{L} = \frac{\dot{Y^0}}{Y^0} - \frac{\dot{a}}{a} = \frac{\kappa(\pi)}{\nu} - \delta - \alpha \tag{31}$$

This means that the growth rate of the employment rate is

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{L}}{L} - \frac{\dot{N}}{N} = \frac{\kappa(\pi)}{\nu} - \delta - \alpha - \beta \tag{32}$$

using (5), (6), and (31).

### 2.4 Growth rate of the wage share

Since $\omega = \frac{wL}{pY}$ by (9), (7), we have :

$$\begin{aligned}
\frac{\dot{\omega}}{\omega} &= \frac{\dot{w}}{w} + \frac{\dot{L}}{L} - \frac{\dot{p}}{p} - \frac{\dot{Y}}{Y} \\
&= \varphi(\lambda) + \gamma i + \frac{\dot{Y^0}}{Y^0} - \alpha - i + \frac{\dot{\mathbf{D}}}{1-\mathbf{D}} + \frac{\dot{A}}{1-A} - \frac{\dot{Y}^0}{Y^0} \\
&= \varphi(\lambda) + (\gamma - 1)i - \alpha + \frac{\dot{\mathbf{D}}}{1-\mathbf{D}} + \frac{\dot{A}}{1-A}
\end{aligned}$$

### 2.5 Growth rate of debt

Since $d = D/(pY)$,

$$\begin{aligned}
\dot{d} &= \frac{d\left(\frac{D}{pY}\right)}{dt} \\
&= \frac{\dot{D}}{pY} - d\left(\frac{\dot{p}}{p} + \frac{\dot{Y}}{Y}\right) \\
&= \frac{pI - \Pi}{pY} - d\left(i + \frac{\dot{Y}}{Y}\right) \\
&= \kappa(\pi) - \pi - d\left(i - \frac{\dot{\mathbf{D}}}{1-\mathbf{D}} - \frac{\dot{A}}{1-A} + \frac{\kappa(\pi)}{\nu} - \delta\right)
\end{aligned}$$

## 3 Possible differential system

$$\dot{\lambda} = \lambda \left( \frac{\kappa(\pi)}{\nu} - \delta - \alpha - \beta \right) \tag{33}$$

$$\dot{\omega} = \omega \left( \varphi(\lambda) + (\gamma - 1)i - \alpha + \frac{\dot{\mathbf{D}}}{1 - \mathbf{D}} + \frac{\dot{A}}{1 - A} \right) \tag{34}$$

$$\dot{d} = \kappa(\pi) - \pi - d \left( i - \frac{\dot{\mathbf{D}}}{1 - \mathbf{D}} - \frac{\dot{A}}{1 - A} + \frac{\kappa(\pi)}{\nu} - \delta \right) \tag{35}$$

$$\dot{p} = p\eta(\xi\omega - 1) \tag{36}$$

$$\dot{T} = \frac{F - \frac{F_{dbl}}{S} T - \gamma^*(T - T_{LO})}{C} \tag{37}$$

$$\dot{T_{LO}} = \frac{\gamma^*(T - T_{LO})}{C_{LO}} \tag{38}$$

$$\dot{\sigma} = g_\sigma \sigma \tag{39}$$

$$\dot{g_\sigma} = \delta_{g_\sigma} g_\sigma \tag{40}$$

$$\dot{E}_{land} = \delta_{E_{land}} \tag{41}$$

$$\dot{p_{BS}} = \delta_{p_{BS}} p_{BS} \tag{42}$$

$$\dot{p}_C = \delta_C \tag{43}$$

$$\begin{pmatrix} \dot{\mathrm{CO}_2}^{AT} \\ \dot{\mathrm{CO}_2}^{UP} \\ \dot{\mathrm{CO}_2}^{LO} \end{pmatrix} = \begin{pmatrix} E_T \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\phi_{12} & \phi_{12} C_{UP}^{AT} & 0 \\ \phi_{12} & -\phi_{12} C_{UP}^{AT} - \phi_{23} & \phi_{23} C_{LO}^{UP} \\ 0 & \phi_{23} & -\phi_{23} C_{LO}^{UP} \end{pmatrix} \begin{pmatrix} \mathrm{CO}_2^{AT} \\ \mathrm{CO}_2^{UP} \\ \mathrm{CO}_2^{LO} \end{pmatrix} \tag{44}$$

where the following are left as auxiliary equations:

$$\pi = 1 - \omega - rd \tag{45}$$

$$\frac{\dot{\mathbf{D}}}{1 - \mathbf{D}} = \left( \frac{\xi_1 + 2\xi_2 T + 7\xi_3 T^6}{1 + \xi_1 T + \xi_2 T^2 + \xi_3 T^7} \right) \dot{T} \tag{46}$$

$$\frac{\dot{A}}{1 - A} = \frac{\sigma p_{BS} n^\theta}{\theta - \sigma p_{BS} n^\theta} \left( g_\sigma + \delta_{p_{BS}} + \left( \frac{\theta}{\theta - 1} \right) \left( \frac{(1 - s_A) p_{BS}}{p_C} \right) \left( \frac{\delta_C + p_C \delta_{p_{BS}}}{(1 - s_A) p_{BS}} \right) \right) \tag{47}$$

$$n = \min \left\{ \left( \frac{p_C}{(1 - s_A) p_{BS}} \right)^{\frac{1}{\theta - 1}}, 1 \right\} \tag{48}$$

$$F_{ind} = \frac{F_{dbl}}{\log(2)} \log \left( \frac{CO_2^{AT}}{C_{AT_{preind}}} \right) + F_{exo} \tag{49}$$

$$E_T = E_{land} + \sigma(1 - n) K / \nu \tag{50}$$

**Problem:** Emissions depend on output ($E_T = E_{land} + \sigma(1-n)Y^0$), and emissions are needed to feed into the climate cycle in equation (25). Therefore the stock value of output or capital would still be needed to code the model.