Group Assignment I
Advanced Econometrics

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QUESTION 1.

The goal is to show that the process $\{x_t\}_{t\in Z}$ is a stationary and ergodic (SE) sequence under certain conditions, using the following volatility model:

$$x_t = \mu + \lambda\sigma_t^2 + \sigma_t\epsilon_t$$

where $\epsilon_t \sim \text{TID}(\nu)$ and the volatility process σ_t^2 is given by the following recursive equation:

$$\sigma_t^2 = \omega + (\alpha + \delta\tanh(-\gamma x_{t-1}))\left(\frac{x_{t-1} - \mu - \lambda\sigma_{t-1}^2}{\sigma_{t-1}}\right)^2 + \beta\sigma_{t-1}^2$$

Step 1: Substituting x_{t-1}

We substitute $x_{t-1} = \mu + \lambda\sigma_{t-1}^2 + \sigma_{t-1}\epsilon_t$ into the volatility equation:

$$\sigma_t^2 = \omega + \left(\alpha + \delta\tanh\left(-\gamma(\mu + \lambda\sigma_{t-1}^2 + \sigma_{t-1}\epsilon_t)\right)\right)\epsilon_t^2 + \beta\sigma_{t-1}^2$$

Step 2: Proving A1 and A2 of Bougerol's Theorem

To prove condition A1, $\{\epsilon_t\}_{t\in Z}$ is an n_ϵ-variate SE sequence; Because the $\{\epsilon_t\}_{t\in Z}$ are TID distributed, which means i.i.d., they are an SE

sequence and thus condition A1 is fulfilled.
To prove Condition A2, we analyze the expectation $E[\log^+|\phi(\sigma_1^2, \epsilon_t)|]$. Substituting the recursive formula for σ_1^2 into the expression, we get:

$$\sigma_1^2 = \omega + (\alpha + \delta \tanh(-\gamma(\mu + \lambda\sigma_1^2 + \sigma_1\epsilon_t)))\epsilon_t^2 + \beta\sigma_1^2$$

Now, we want to show that $E[\log^+|\phi(\sigma_1^2, \epsilon_t)|]$ is finite. Substituting the expression for σ_1^2 into the expectation, we get:

$$E[\log^+|\phi(\sigma_1^2, \epsilon_t)|] = E\left[\log^+\left|\omega + (\alpha + \delta \tanh(-\gamma(\mu + \lambda\sigma_1^2 + \sigma_1\epsilon_t)))\epsilon_t^2 + \beta\sigma_1^2\right|\right]$$

Boundedness of $\tanh$ **and finiteness of expectation**

At this stage, we observe that $\tanh(x)$ is bounded for all real x by the range $(-1, 1)$. This means that:

$$\tanh(-\gamma(\mu + \lambda\sigma_1^2 + \sigma_1\epsilon_1)) \leq 1$$

Since the hyperbolic tangent function is bounded by 1, and therefore $< \infty$. Now we can check the remaining parameters if they are separately smaller than infinity.

$$E\left[\log^+\left|\omega + (\alpha + \delta \tanh(-\gamma(\mu + \lambda\sigma_1^2 + \sigma_1\epsilon_t)))\epsilon_t^2 + \beta\sigma_1^2\right|\right]$$

We can rewrite this equation as:

$$E \log^+\left|\omega + \left(\alpha + \delta \tanh(-\gamma(\mu + \lambda\sigma_1^2 + \sigma_1\epsilon_t))\right)\epsilon_t^2 + \beta\sigma_1^2\right|$$
$$\leq \left(|\omega| + \left(|\alpha| + |\delta|\left|\tanh(-\gamma(\mu + \lambda\sigma_1^2 + \sigma_1\epsilon_t)\right|\right)E|\epsilon_t^2| + |\beta|\sigma_1^2\right) < \infty$$

Boundedness of the Terms:

1. $|\omega|$ $|\alpha|$ $|\delta|$ are all constant and hence finite, $< \infty$.
2. $E[\epsilon_t^2]$ is finite because $\epsilon_t \sim TID(\nu)$ with $\nu > 2$, meaning $E[\epsilon_t^2] < \infty$.
3. $E[\sigma_1^2]$ is finite because σ_1^2 is finite as the initial condition.
4. $\tanh(-\gamma(\mu + \lambda\sigma_1^2 + \sigma_1\epsilon_t))$ is bounded because $\tanh(x) \leq 1$, implying this term is less than or equal to δ, and hence finite.

Thus, each term in the equation is finite, therefore we can conclude:

$$E\left[\log^{+}\left(|\omega+(\alpha+\delta\tanh(-\gamma(\mu+\lambda\sigma_1^2+\sigma_1\epsilon_1)))\epsilon_1^2+\beta\sigma_1^2|\right)\right]<\infty$$

This completes the proof for Condition A2 of Bougerol's Theorem, starting with the initialization at σ_1^2, and using the boundedness of tanh.

Step 3: Proving A3 - Derivative Condition

To satisfy condition A3 of Bougerol's Theorem, we need to show that:

$$E\left[\log^{+}\sup\left|\frac{\partial\sigma_t^2}{\partial\sigma_{t-1}^2}\right|\right]<0$$

This condition requires us to compute the derivative of σ_t^2 with respect to σ_{t-1}^2 and then analyze the supremum of this derivative. Specifically, we need to prove that the expected logarithm of the supremum of the derivative is less than zero.

Step 4: Calculating the derivative with respect to σ_{t-1}^2

We begin by calculating the derivative of σ_t^2 with respect to σ_{t-1}^2 based on the formula for σ_t^2:

$$\sigma_t^2=\omega+(\alpha+\delta\tanh(-\gamma(\mu+\lambda\sigma_{t-1}^2+\sigma_{t-1}\epsilon_t)))\epsilon_t^2+\beta\sigma_{t-1}^2$$

Taking the derivative of this equation with respect to σ_{t-1}^2, we get:

$$\frac{\partial\sigma_t^2}{\partial\sigma_{t-1}^2}=\beta+\delta\epsilon_t^2\left(1-\tanh^2\left(-\gamma(\mu+\lambda\sigma_{t-1}^2+\sigma_{t-1}\epsilon_t)\right)\right)\left(-\gamma\left(\lambda+\frac{\epsilon_t}{2\sqrt{\sigma_{t-1}^2}}\right)\right)$$

Step 5: Substituting the derivative into the supremum condition

Now, we substitute this derivative into the expression for the supremum of the logarithm, with respect to σ_t^2 of the derivative, which leads to the following:

$$E\left[\log^+ \sup \left|\beta + \delta\epsilon_t^2 \left(1 - \tanh^2 \left(-\gamma(\mu + \lambda\sigma_{t-1}^2 + \sigma_{t-1}\epsilon_t)\right)\right) \left(-\gamma \left(\lambda + \frac{\epsilon_t}{2\sqrt{\sigma_{t-1}^2}}\right)\right)\right|\right]$$

Step 6: Evaluating the Supremum Bounds

Supremum of $1-\tanh^2(x)$: Since $\tanh(x)$ is bounded by 1 for all real values of x, the supremum of $1 - \tanh^2(x)$ is given by:

$$\sup_{\sigma_t^2} \left(1 - \tanh^2 \left(-\gamma \left(\mu + \lambda\sigma_{t-1}^2 + \sigma_{t-1}\epsilon_t\right)\right)\right) = 1$$

Thus, in the supremum expression, $1 - \tanh^2(x)$ can be replaced by 1. This simplifies our expression to the following:

$$E\left[\log^+ \sup_{\sigma^2 \in R} \left|\beta + \delta\epsilon_t^2 \left(-\gamma \left(\lambda + \frac{\epsilon_t}{2\sqrt{\sigma_{t-1}^2}}\right)\right)\right|\right]$$

Supremum of $\frac{\epsilon_t}{2\sqrt{\sigma_{t-1}^2}}$: Next, we consider the supremum of $\frac{\epsilon_t}{2\sqrt{\sigma_{t-1}^2}}$. Since we want the supremum over a fraction with respect to σ_{t-1}^2, we want the denominator to be as small as possible with the smallest possible value for σ_{t-1}^2, so we maximize our value. Since $\sigma_{t-1}^2 \geq \omega$, and ω is the minimum value of σ_{t-1}^2, the supremum occurs when $\sigma_{t-1}^2 = \omega$. Since this is a fraction, to maximize the expression, we want to minimize the denominator. From the formula for σ_{t-1}^2, we know that the smallest possible value for σ_{t-1}^2 is ω. Therefore, the supremum with respect to σ_{t-1}^2 is given by:

$$\sup_{\sigma^2 \in R} \frac{\epsilon_t}{2\sqrt{\sigma_{t-1}^2}} = \frac{\epsilon_t}{2\sqrt{\omega}}$$

Substituting this into our expression, we get:

$$\frac{\partial\sigma_t^2}{\partial\sigma_{t-1}^2} \leq \beta + \delta\gamma\epsilon_t^2 \left(\lambda + \frac{\epsilon_t}{2\sqrt{\omega}}\right)$$

Step 7: Final Expectation and Conclusion

The final expectation we need to evaluate is the following:

$$E\left[\log^+\left|\beta+\delta\gamma\epsilon_t^2\left(\lambda+\frac{\epsilon_t}{2\sqrt{\omega}}\right)\right|\right]$$

For Condition A3 to hold, we need to prove that:

$$E\left[\log^+\sup_{\sigma^2\in R}\left|\frac{\partial\sigma_t^2}{\partial\sigma_{t-1}^2}\right|\right]<0$$

Thus, according to Bougerol's Theorem, the process $\{\sigma_t^2\}_{t\in Z}$ is stationary and ergodic if the following condition is satisfied:

$$E\left[\log\left(|\gamma|\left(|\lambda\delta|\epsilon_t^2+\frac{|\delta||\epsilon_t^3|}{2\sqrt{\omega}}\right)+|\beta|\right)\right]<0$$

Proof that x_t is SE using Krengel's Theorem

Given the formula for x_t:

$$x_t=\mu+\lambda\sigma_t^2+\sigma_t\epsilon_t$$

We have already proven that σ_t^2 is stationary and ergodic (SE). Now, by Krengel's Theorem, we want to show that x_t is also stationary and ergodic. The process x_t depends on σ_t^2, which we know is SE, and ϵ_t, which is an i.i.d. sequence independent of σ_t^2.

where: - μ is a constant, - λ is a constant, - σ_t^2 is SE, - $\sigma_t\epsilon_t$ represents a multiplicative combination of σ_t, which is stationary and ergodic due to the properties of σ_t^2, and ϵ_t, which is independent and identically distributed (i.i.d).

Krengel's Theorem states that any measurable function of two **stationary ergodic (SE)** sequences will also be stationary and ergodic. Since x_t is a function of σ_t^2 (SE) and $\sigma_t\epsilon_t$ is (SE), xt is also stationary ergodic.

QUESTION 2.

The news impact curves shown in figure 1 are based on the GARCH-M-L model:

$$x_t = \mu + \lambda\sigma_t^2 + \sigma_t\epsilon_t$$

where $\epsilon_t \sim \mathrm{TID}(\nu)$ and the volatility process σ_t^2 is given by the following recursive equation:

$$\sigma_t^2 = \omega + (\alpha + \delta\tanh(-\gamma x_{t-1}))\left(\frac{x_{t-1} - \mu - \lambda\sigma_{t-1}^2}{\sigma_{t-1}}\right)^2 + \beta\sigma_{t-1}^2$$

For this question specifically we have adjusted the model by setting: $\mu = 0$, $\lambda = 0$, and $\alpha = 0.4$. Each graph in figure 1 shows the response of volatility to past returns with different values for $\delta(= 0, 0.1, 0.3, -0.3)$ and $\gamma = (0.01, 0.1, 1)$.

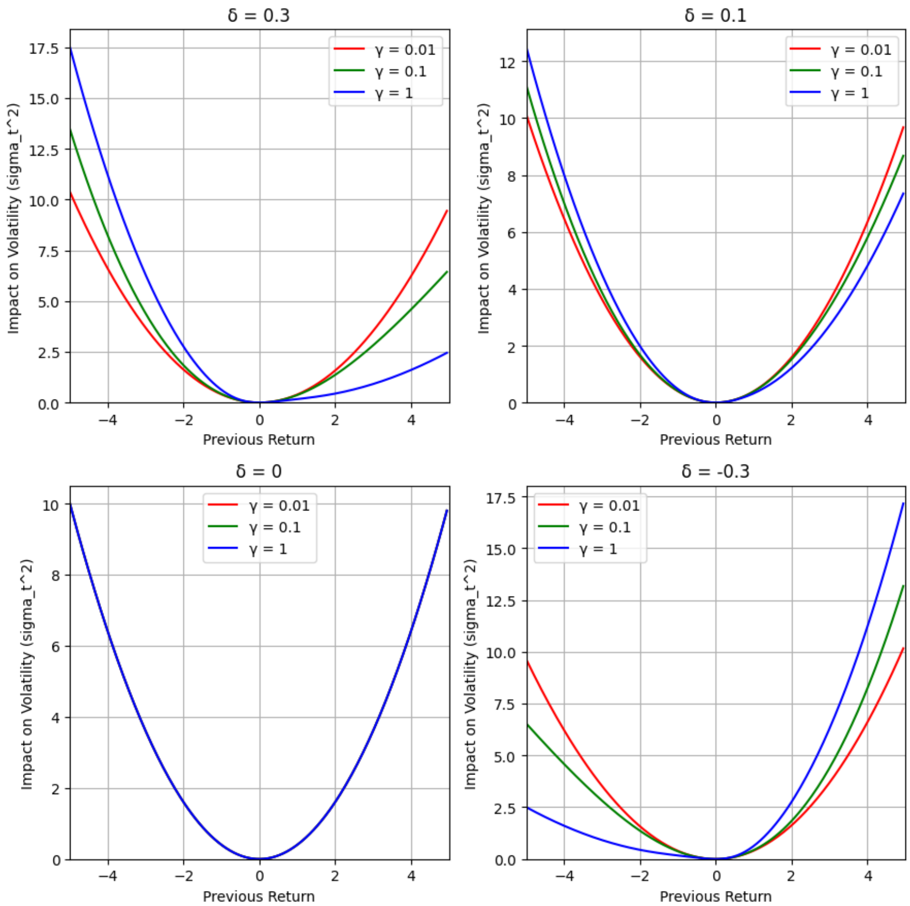

Figure 1: News Impact Curves for Different Values of Delta and Gamma

Each graph in figure 1 shows the response of volatility to past returns, with the x-axis representing the previous return x_{t-1} and the y-axis representing the impact on the volatility σ_t^2. The figure is a 2x2 grid where each plot corresponds to a specific δ value: Top-left: $\delta = 0.3$, Bottom-left: $\delta = 0$, Top-right: $\delta = 0.1$, Bottom-right: $\delta = -0.3$. Additionally, each plot contains three curves for $\gamma \in \{0.01 \text{ (red)}, 0.1 \text{ (green)}, 1 \text{ (blue)}\}$.

In the upper left plot, we observe that when δ has a high positive magnitude, which represents the magnitude of the smooth transition function, positive returns produce less volatility compared to negative returns. This results in an asymmetric curve, implying that negative returns generate higher volatility then positive returns. This phenomenon is known as the leverage effect. As δ approaches zero, this asymmetry diminishes, and the effect of the smooth transition/leverage effect decreases as well.

When $\delta = 0$, the function tanh is removed from the model, which means that there is no smooth transition between different volatility regimes. Furthermore, no γ values are present because γ depends on the tanh function. This is evident in the lower left graph, where a symmetric blue hyperbolic parabola is visible. In this case, there is no leverage effect or transition, meaning that the volatility responds equally to both positive and negative returns.

In the bottom right graph, it can be seen that when δ becomes negative, we observe that the news impact curve is mirrored compared to the positive value with the same magnitude. Positive returns now produce higher volatility compared to negative returns in exactly the same magnitudes as when $\delta = 0.3$.

In addition, we observe that as γ increases, the leverage effect and the asymmetry become more pronounced. However, when γ approaches infinity, the impact curve remains unchanged because the tanh function is bounded by 1.

QUESTION 3.

Table 1. Descriptive Statistics of the Stocks: PFE, JNJ, MRK and AAPL

Stock	Count	Mean	Median	Std Dev	Skew	Kurtosis	Min	Max
PFE	3270	0.041	0.000	1.368	0.223	2.284	-0.077	0.109
JNJ	3270	0.046	0.036	1.076	-0.150	6.353	-0.100	0.080
MRK	3270	0.057	0.046	1.306	0.083	3.625	-0.099	0.104
AAPL	3270	0.107	0.089	1.783	-0.076	2.350	-0.129	0.120

Note: Table 1 presents summary statistics for four major stocks: Pfizer (PFE), Johnson & Johnson (JNJ), Merck & Co (MRK), and Apple (AAPL). The statistics include mean, median, standard deviation, skewness, kurtosis, and the range (minimum and maximum) of returns.

As can be seen in Table 1, Apple has the highest mean return (0.107) and volatility (1.783 Std Dev). Johnson & Johnson shows the most negative skewness (-0.150) and highest kurtosis (6.353), indicating more extreme returns. Pfizer and Merck & Co have moderate returns and volatility, with skewness and kurtosis values closer to normal.

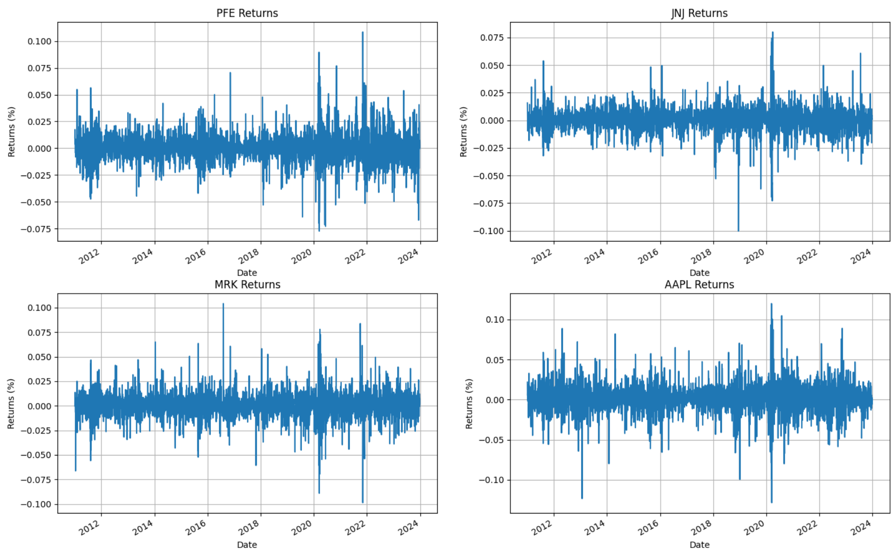

Figure 2: Stock Returns

Note: Figure 2 is a 2x2 grid where in each plot the returns of a specific stock is shown over time. On the left side the stocks Pfizer (top) and Merck & Co (bottom) are plotted and on the right side the stocks Johnson & Johnson (top) and Apple (bottom) are shown. The y-axis represents the returns in percentages and the x-axis corresponds to the dates ranging from 2012 until 2024.

In Figure 2, all stocks show volatility clustering, with spikes in returns around 2020 and 2022, likely due to market turbulence, due to the outbreak of Covid-19. AAPL exhibits the highest volatility, consistent with its higher standard deviation in Table 1, while JNJ shows a relatively lower volatility with more stable returns.

QUESTION 4.

Formulas

For simplicity, the models are described as M1 (GARCH), M2 (GARCH_M), and M3 (GARCH_M_L). Here a loglikelihood function and subsequent optimization function where used to optimize the parameters of the 3 given models.

The loglikelihood of the $GARCH_{ML}$ is given by the following formula;

$$\log f(x_t|\sigma_t^2) = \Gamma\left(\frac{\nu+1}{2}\right) - \Gamma\left(\frac{\nu}{2}\right) - \frac{1}{2}\log(\nu\pi) - \frac{1}{2}\log(\sigma_t) - \frac{\nu+1}{2}\log\left(1+\frac{\epsilon_t^2}{\nu}\right) - 0.001\gamma^2$$

For the optimization we used the common BFGS models, with appropriate bounds for all parameters. Furthermore, negative values for the variances are avoided, and the loglikelihood is penalized by 0,001*γ^2. This only affects tje model that has a value greater than 0 for gamma (M3).

The results and GARCH models interpretation

For the formula descriptions of the model please refer to Q5. The results are presented in Table 2. The best model is the one with the lowest AIC and BIC and the highest value for the Log-Likelihood.

BIC penalizes model complexity, and model 3 is also additionally penalized due to the penaziling factor of 0,001*γ^2. According to the Log Likelihood, AIC AND BIC model 2 performs best on AAPL and JNJ, but model 3 performs best on MRK and PFE. Overall there is a reason to believe that the relative model complexity of M3 is therefore not the determining factor and still M3 performs relatively well (best in some cases).

Table 2. Model Parameters, Log-Likelihood, AIC, and BIC for Different Stocks

Stock	**AAPL**	**JNJ**	**MRK**	**PFE**
M1 Parameters	$\mu = 0.15339$ $\omega = 0.03795$ $\alpha = 0.09028$ $\beta = 0.87364$ $\nu = 4.14587$	$\mu = 0.07239$ $\omega = 0.00000$ $\alpha = 0.02547$ $\beta = 0.91872$ $\nu = 4.38521$	$\mu = 0.06578$ $\omega = 0.01420$ $\alpha = 0.03899$ $\beta = 0.90628$ $\nu = 4.49554$	$\mu = 0.05899$ $\omega = 0.00000$ $\alpha = 0.04273$ $\beta = 0.91540$ $\nu = 4.64902$
Log-Likelihood	-4660.87	-3281.39	-3891.82	-3770.87
AIC	9331.74	6572.78	7793.64	7551.73
BIC	9360.86	6601.90	7822.76	7580.85
M2 Parameters	$\mu = 0.12012$ $\lambda = 0.02526$ $\omega = 0.99990$ $\alpha = 0.07715$ $\beta = 0.82972$ $\nu = 4.51961$	$\mu = 0.06161$ $\lambda = 0.00000$ $\omega = 0.99990$ $\alpha = 0.02255$ $\beta = 0.90828$ $\nu = 4.57018$	$\mu = 0.05575$ $\lambda = 0.02307$ $\omega = 0.08306$ $\alpha = 0.03096$ $\beta = 0.90593$ $\nu = 4.17096$	$\mu = 0.06583$ $\lambda = 0.04683$ $\omega = 0.06085$ $\alpha = 0.04681$ $\beta = 0.84792$ $\nu = 4.16476$
Log-Likelihood	-4626.91	-3269.87	-3891.49	-3785.90
AIC	9265.83	6551.74	7794.98	7583.81
BIC	9300.77	6586.69	7829.93	7618.75
M3 Parameters	$\mu = 0.10958$ $\lambda = 0.01990$ $\omega = 0.01224$ $\alpha = 0.07311$ $\beta = 0.91539$ $\delta = 0.07062$ $\gamma = 0.44022$ $\nu = 4.39757$	$\mu = 0.03014$ $\lambda = 0.06965$ $\omega = 0.00257$ $\alpha = 0.02555$ $\beta = 0.91600$ $\delta = 0.01723$ $\gamma = 1.21563$ $\nu = 4.54827$	$\mu = -0.05664$ $\lambda = 0.12895$ $\omega = 0.01948$ $\alpha = 0.03997$ $\beta = 0.90559$ $\delta = 0.03802$ $\gamma = 6.18130$ $\nu = 4.68513$	$\mu = -0.00255$ $\lambda = 0.09483$ $\omega = 0.00361$ $\alpha = 0.05616$ $\beta = 0.89085$ $\delta = 0.09916$ $\gamma = 0.10438$ $\nu = 4.15533$
Log-Likelihood	-4631.07	-3270.44	-3868.02	-3766.00
AIC	9278.03	6556.89	7752.04	7548.01
BIC	9324.63	6603.48	7798.64	7594.60

Note: Table 2 provides the estimated model parameters, log-likelihood, AIC, and BIC for four different stocks: Apple (AAPL), Johnson Johnson (JNJ), Merck Co (MRK), and Pfizer (PFE). The parameters are estimated for three different models (M1, M2, M3). For each stock, the table reports the values of μ, ω, α, β, and ν, which represent the key parameters of the GARCH model. The table also provides the log-likelihood values, as well as the AIC and BIC criteria, to evaluate the relative performance of the models.

QUESTION 5.

The news-impact curves

Formulas

The generated plots show the estimated news-impact curves for three alternative GARCH models for three different stocks (JNJ, MRK, and PFE). Each news-impact curve shows how volatility reacts to previous returns based on different GARCH models.

GARCH M-L

The GARCH M-L model introduces an additional complexity, possibly including mean and leverage effects, which is shown in the green curve. The estimated news impact curve formula for GARCH M-L is the following.

$$\sigma_t^2 = (\alpha + \delta \cdot \tanh(-\gamma \cdot x_{t-1})) \left(\frac{x_{t-1} - \mu - \lambda \cdot \sigma_{t-1}^2}{\sigma_{t-1}} \right)^2$$

GARCH-M

The GARCH-M model shows a slightly different impact on volatility, where it captures the mean effect. The estimated news impact curve formula for GARCH-M is the following.

$$\sigma_t^2 = \alpha \cdot \left(\frac{x_{t-1} - \mu - \lambda \cdot \sigma_{t-1}^2}{\sigma_{t-1}} \right)^2$$

GARCH

The standard GARCH model shows a symmetric relationship between past returns and volatility. Negative returns increase volatility more than positive returns. The estimated news impact curve formula for GARCH is the following.

$$\sigma_t^2 = \omega + \alpha \cdot x_{t-1}^2$$

The plots

The Figure 3. presents the estimated news-impact-curves for the three alternative models for each of the 3 stocks.

X-axis represents the previous period returns. Negative values represent negative returns and positive values represent positive returns. Y-axis represents the impact of the previous return on the conditional volatility at time t, which is the volatility forecasted for the current period. The higher the value on the Y-axis, the greater the impact of the return on volatility. This is measured as the increase in volatility due to the shock.

The GARCH models interpretation

GARCH-M-L

For GARCH-M-L, the model introduces both the mean effect and a leverage effect. In this case negative returns increase volatility more than positive returns. This is reflected in the asymmetry of the curve. For negative returns, the curve tends to rise more steep than for positive returns.

GARCH-M

For GARCH-M, the model includes a mean effect. This model slightly changes the impact of past returns on volatility in comparison to simpler standard GARCH.In the plots, there is a similar U-shape, but the curve becomes steeper. Therefore, higher returns have a somewhat differentiated impact on future volatility.

GARCH

For standard GARCH, the relationship between previous returns and volatility is symmetric. Both negative and positive returns of the same magnitude have the same impact on future volatility. It is indicated by the U-shape. There no leverage effect.

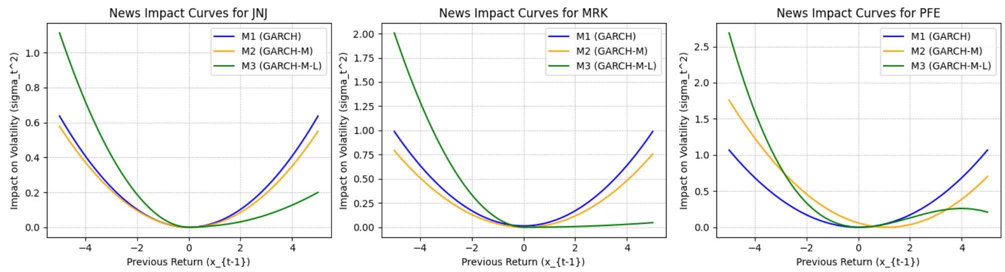

Figure 3: The Estimated News Impact Curves for JNJ, MRK and PFE Stocks.

Note: Figure 3 shows the estimated news impact curves for three different models (M1, M2, M3) applied to the stocks Johnson Johnson (JNJ), Merck Co (MRK), and Pfizer (PFE). The x-axis represents the previous return (x_{t-1}), while the y-axis represents the impact on volatility (σ_t^2). Each curve corresponds to one of the three models (GARCH-M, GARCH-M-L1, GARCH-M-L2), showing the relationship between past returns and the volatility generated by each model.

The filtered volatilities

The Figure 4. presents filtered volatilities obtained from the three alternative models for each of the 3 stocks over the entire sample.

The description

The x-axis represents the time in days, ranging from 0 to around 3000. This is the total time period over which the stock returns and volatilities are observed and filtered. The y-axis shows the magnitude of the real returns (gray) and the filtered volatilities (blue) for each stock. The scale differs slightly for each stock. Vertical dash line represents end of in-sample (2500 observations).

Plots visualize how the volatility in stock returns changes over time, with each stock (JNJ, MRK, and PFE) exhibiting different patterns of volatility. The filtered volatility curve smooths out the raw returns and highlights significant periods of market risk.

The interpretation

The blue line represents the filtered volatility, which measures the standard deviation of the returns over time, acting as an indicator of risk. The filtered volatility smooths the noisy return data, highlighting periods of increased or decreased market risk. Peaks in the blue line correspond to periods of heightened volatility in the stock prices.

JNJ stock

For JNJ stock, the returns of fluctuate mostly within a moderate range, with several noticeable spikes and dips. The fluctuations indicate variability in daily returns. There are visible periods of heightened volatility, particularly towards the end of the sample.

The blue filtered volatility line stays relatively low in the earlier period but experiences a significant increase toward the middle of the time frame. This suggests an increase in market risk around this point. Toward the end of the in-sample period, the volatility increases further, reflecting a period of market instability. The volatility spikes correspond to moments of extreme returns, indicating times when market conditions for JNJ were highly volatile.

After the dashed vertical line, volatility continues to rise slightly, indicating that the forecasted out-of-sample period exhibits sustained risk.

MRK stock

For MRK, there are some spikes, implying that MRK experienced some sudden large changes in stock prices during the observed period.

The filtered volatility line remains relatively stable for most of the time period, with a moderate rise in volatility during certain periods around 1700 and 2300 observations. The volatility increases towards the end, similar to JNJ, but the peaks are less sharp, indicating that while there is volatility, it does not reach extreme levels. Therefore, the stock shows steady behavior, but like JNJ, there are increasing volatility spikes toward the end, suggesting rising market risk.

After the dashed line, the model shows an increase in volatility. The post-sample period exhibits relatively steady volatility with huge high peak around 2700 observation, indicating that the model forecasts a continuation of moderate risk levels.

PFE stock

The returns for PFE show similar characteristics to JNJ, with large fluctuations, especially towards the end of the time period. The magnitude of spikes and drops in returns is notable, indicating periods where PFE's stock experienced rapid price changes.

The volatility line follows a similar pattern to JNJ and MRK but with a slightly more gradual increase in volatility over the time period. The volatility is relatively low at the beginning, increases mid-period, and peaks toward the end of the sample. The pattern suggests PFE was relatively stable for the first half of the time period but became riskier overtime.

The out-of-sample period shows a continuation of increased volatility.

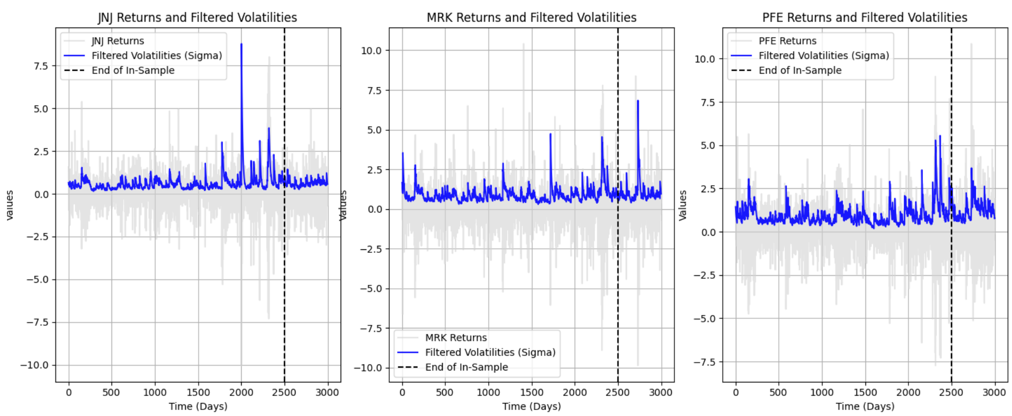

Figure 4: The Filtered Volatilities for JNJ, MRK and PFE Stocks.

Note: Figure 4 shows the returns and filtered volatilities for the stocks Johnson Johnson (JNJ), Merck Co (MRK), and Pfizer (PFE). The x-axis represents the time (in days), while the y-axis represents the values of returns and volatilities. The blue line indicates the filtered volatilities (in terms of sigma), and the gray line represents the stock returns. The dashed vertical line marks the end of the in-sample period.

QUESTION 6.

Simulated VaR estimation

The approach

To simulate the 1-, 5- and 20-step ahead VaR for M1, M2 and M3 models in a non-parametric way, the general idea is to avoid assuming a specific distribution for the innovations and instead rely on simulations.

The following approach was applied:

- Calculate the conditional variance at time $t = 0$, which corresponds to the last observed volatility from the in-sample data.
- Use the suitable GARCH model equation to calculate the conditional variance at time $t = 1$, using the past returns and volatilities.
- For each step, draw innovations from $T(\hat{\theta})$, simulating the return for that time step x_{t1}.
- Repeat the process iteratively to generate returns and update the conditional variance for each time step up to $t = 20$.
- After simulating returns for 1, 5, and 20 days, compute the compound returns over these periods. Use the simulated returns to calculate VaR at the 1%, 5%, and 10% confidence levels for each horizon.

The Results

The results of Value at Risk (VaR) simulation are shown Table 3.

There is 81 VaR estimates for JNJ, MRK, and PFE stocks across three models (M1, M2, M3) for different horizons (1-step, 5-step, 20-step) and confidence levels (1%, 5%, 10%).

General Observations

Generally speaking, across all three stocks, as the forecast horizon increases (1 to 20 steps ahead), the VaR values become more negative. This indicates a greater potential for loss as the time horizon extends, suggesting increasing uncertainty and risk over longer periods. Furthermore, for each model and stock, the VaR estimates at lower confidence levels (1%) are more negative compared to the 5%

and 10% levels. This is expected, as the 1% VaR represents a more extreme tail event, that is the higher risk.

JNJ

GARCH (M1) consistently shows lower VaR values compared to GARCH-M (M2) and GARCH M-L (M3) across all horizons, implying slightly lower risk according to the GARCH model.

The VaR for M3 is generally more negative, especially in the longer horizon (20-step), suggesting more risk when accounting for GARCH-M-L.

MRK

Similar to JNJ, M1 has slightly less negative VaR values, particularly for the 20-step horizon, compared to M2 and M3. M3 shows the most negative values for 20 steps at the 1% confidence level. It means that MRK is more likely to exhibit higher losses over longer periods when considering leverage effects.

PFE

PFE exhibits the largest VaR values across all models and horizons, especially in M3 for the 20-step horizon, indicating that PFE is expected to have the highest risk compared to JNJ and MRK.

The summary of the results

JNJ appears to be the least risky stock across all models and horizons, particularly in the shorter-term forecasts.

MRK shows moderate risk, with GARCH-M-L indicating slightly higher VaR values over longer periods, signaling increased risk.

PFE exhibits the highest potential for loss due to more negative VaR, especially under longer forecast horizons and GARCH-M-L.

Table 3. Updated Value at Risk (VaR) Estimates for JNJ, MRK, and PFE Stocks

Stock	**Model**	**1 step**			**5 steps**			**20 steps**		
		1%	**5%**	**10%**	**1%**	**5%**	**10%**	**1%**	**5%**	**10%**
JNJ	**GARCH**	-2.39	-1.41	-1.01	-5.10	-3.04	-2.15	-10.32	-5.71	-3.87
	GARCH-M	-2.24	-1.25	-0.89	-4.92	-2.86	-1.99	-9.69	-5.74	-3.80
	GARCH-M-L	-2.31	-1.40	-1.02	-5.80	-3.24	-2.31	-12.50	-7.00	-4.86
MRK	**GARCH**	-2.77	-1.58	-1.14	-5.99	-3.47	-2.47	-11.79	-6.99	-4.92
	GARCH-M	-2.77	-1.61	-1.13	-6.22	-3.77	-2.70	-12.80	-7.79	-5.47
	GARCH-M-L	-2.62	-1.55	-1.15	-6.92	-4.08	-2.97	-16.10	-10.05	-7.37
PFE	**GARCH**	-4.35	-2.56	-1.82	-8.61	-5.41	-3.96	-15.10	-9.31	-6.85
	GARCH-M	-4.24	-2.43	-1.70	-8.13	-5.02	-3.67	-14.11	-8.79	-6.30
	GARCH-M-L	-4.00	-2.31	-1.67	-9.19	-5.40	-3.94	-18.43	-10.89	-7.75

Updated VaR estimates for Johnson Johnson (JNJ), Merck Co (MRK), and Pfizer (PFE) stocks using GARCH, GARCH-M, and GARCH-M-L models. The estimates are for 1, 5, and 20 steps at three confidence levels: 1%, 5%, and 10%. Values represent the maximum expected loss (in percentage terms) over the specified time horizon.