Compressive sensing for multimode fiber optical imaging

Master project mathematics

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Practical information

- internship in the computational imaging group of Centrum Wiskunde & Informatica, in collaboration with the nanoscale imaging and metrology group of Advanced Research Center for Nanolithography
- supervision: dr. Svetlana Dubinkina (VU Amsterdam), prof.dr. Tristan van Leeuwen (CWI), dr. Lyubov Amitonova (ARCNL)
- interpret novel multimode fiber optical microscopy technique in the mathematical context of compressive sensing

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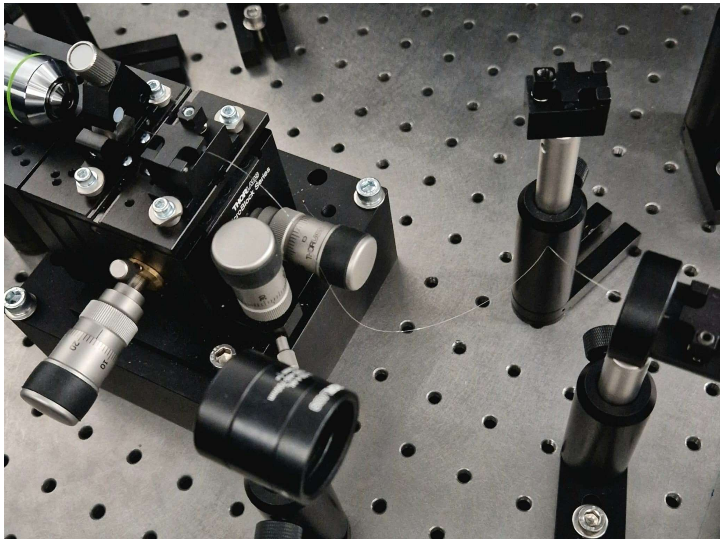
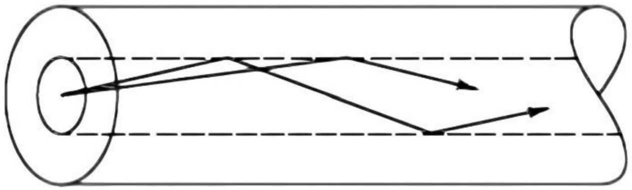
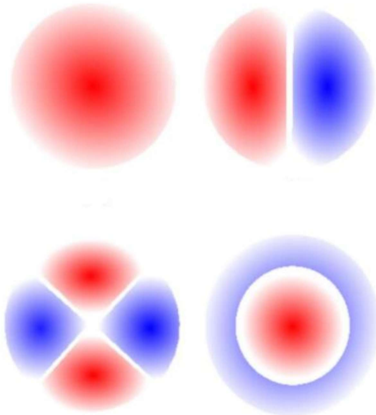
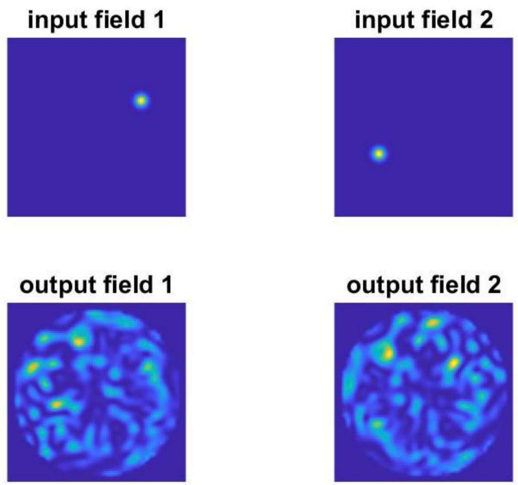
Multimode fiber

- glass fiber: transporter of light composed of core and cladding
- multiple modes $\rightsquigarrow$ light scattering $\rightsquigarrow$ speckle pattern
- $E_{\text{out}}(x, y) = \sum_{k=1}^{M} E_k e^{j\phi_k} \Psi_k(x, y)$



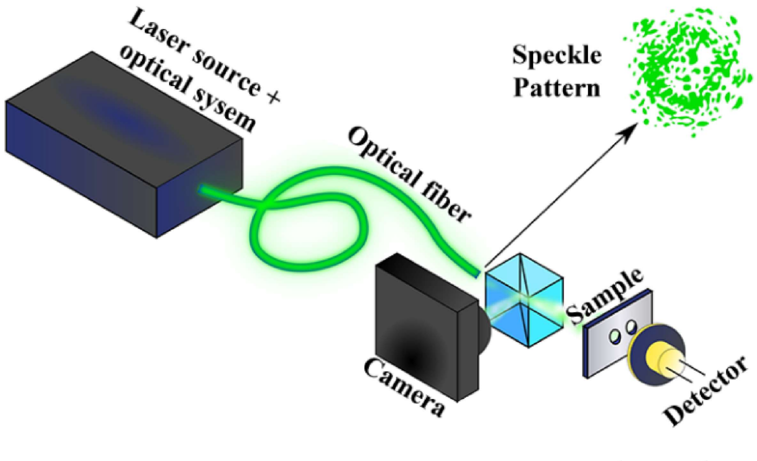


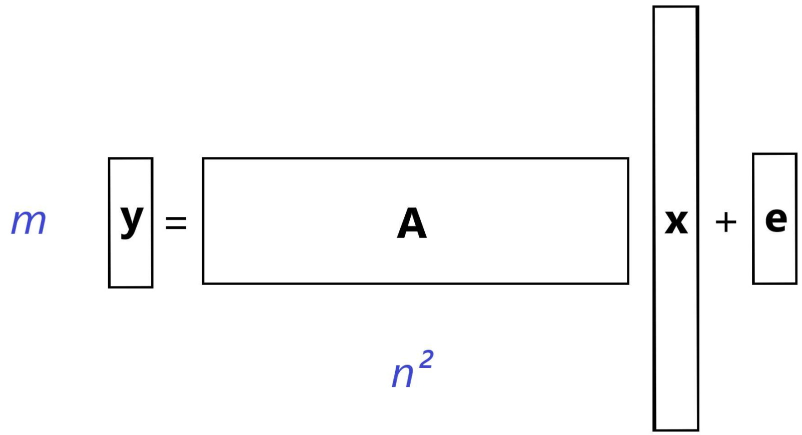




Multimode fiber compressive imaging (MMFCI)

- inverse problem: reconstruct image of a sample from speckle pattern illuminations
  - store m flattened speckle patterns of n-by-n pixels in $\mathbf{A} \in \mathbb{R}^{m \times n^2}$ and intensities in $\mathbf{y} \in \mathbb{R}^m$
  - recover and reshape n-by-n pixels image $\mathbf{x} \in \mathbb{R}^{n^2}$
- underdetermined linear system $\left(m \ll n^2\right) \rightsquigarrow$ compressive sensing





Accomplishments and applications

- reconstructed 400-by-400 pixels image from 961 measurements
  - high resolution: overcomes diffraction limit 2.5 times
  - high speed: three orders of magnitude faster than state-of-the-art imaging, surpasses Nyquist rate
  - compact design: lensless
- applications: endoscopy, metrology

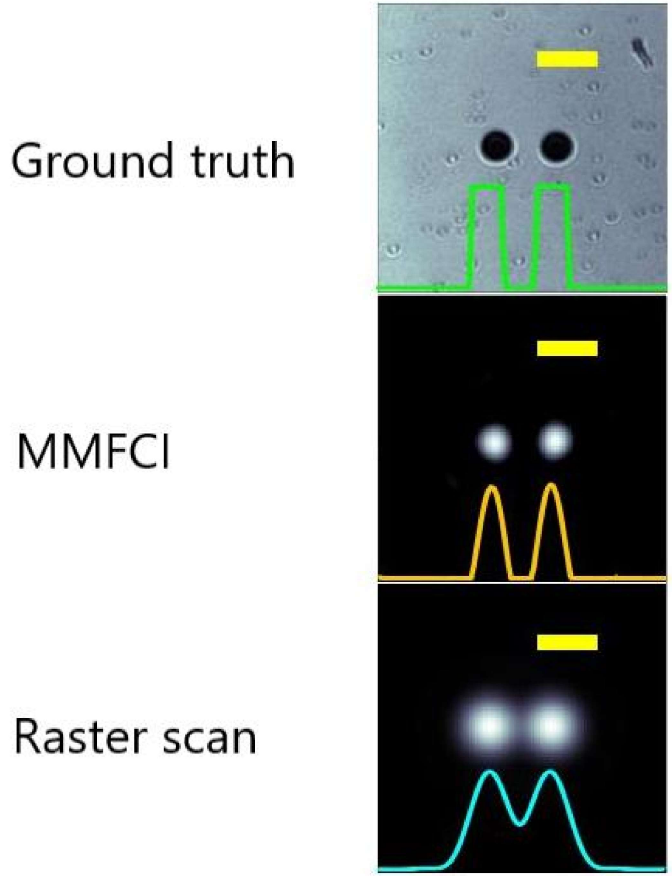


Figure: Abrashitova, 2022

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Problem statement

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

$$\mathbf{x} \in \mathbb{R}^N, \;\; \mathbf{A} \in \mathbb{R}^{m \times N}, \;\; \mathbf{y} \in \mathbb{R}^m, \;\;\; m \ll N$$

Well-posed inverse problem:

1. (*existence*) there is a solution
2. (*uniqueness*) the solution is unique
3. (*stability*) the solution depends continuously on the data

Compressive sensing: mathematical framework for stably retrieving unique solutions, assuming

- sparsity of $\mathbf{x}$
- incoherence of $\mathbf{A}$

Compressive sensing assumptions

Sparsity

* $\mathbf{x}$ is s-sparse if it has at most s nonzeros: $||\mathbf{x}||_0 := \#\{i : x_i \neq 0\} \leq s$
* collection of s-sparse signals: $\Sigma_s := \{\mathbf{x} : ||\mathbf{x}||_0 \leq s\}$
* sparsity in original domain

- uniqueness if $\mathbf{y} = \mathbf{A}\mathbf{x}$ is restricted to $\mathbf{x} \in \Sigma_s$ and every subcollection of $2s$ columns of $\mathbf{A}$ is linearly independent
- stability if all submatrices of $2s$ columns of $\mathbf{A}$ are well-conditioned

Mutual coherence

* $\mu(\mathbf{A}) := \max_{1 \leq i < j \leq N} \left| \left[\mathbf{A}^T \mathbf{A}\right]_{ij} \right|$ for $\mathbf{A}$ with unit-norm columns
* measures degree of correlation between the matrix's columns
* easy to compute, but large gaps with numerical results

Matrix properties

Restricted isometry property (RIP)

$\mathbf{A}$ satisfies the RIP of order s if there is a $\delta_s \in (0,1)$ such that

$$(1-\delta_s)||\mathbf{x}||_2^2 \leq ||\mathbf{Ax}||_2^2 \leq (1+\delta_s)||\mathbf{x}||_2^2 \;\; \forall \mathbf{x} \in \Sigma_s$$

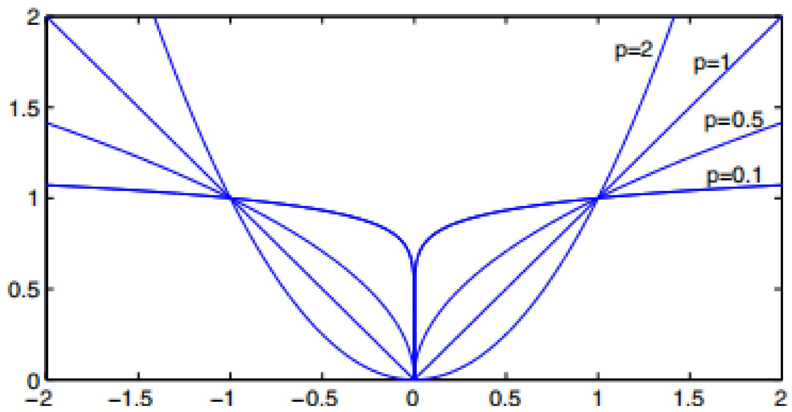
- the matrix approximately preserves distances between sparse vectors
- in general, computationally expensive to verify
  - proven for random matrices with i.i.d. entries following zero-centered distribution, e.g. standard Gaussian
  - deterministic matrices $\mathbf{X}$ are $(s,(s-1)\mu(\mathbf{X}))$-RIP

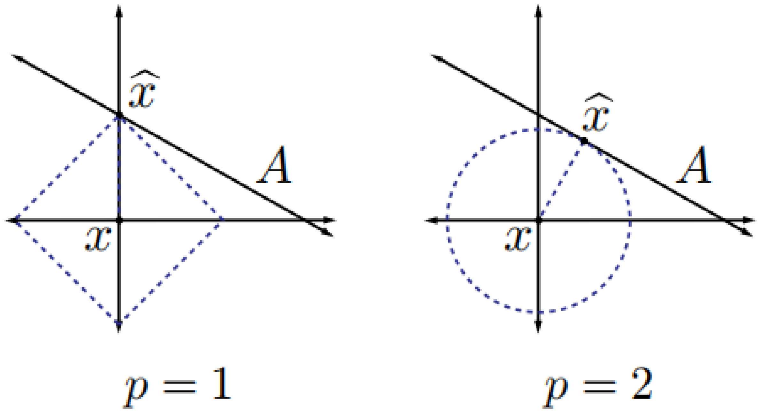
Sparse recovery

- minimization w.r.t. ℓ_1-norm: convex optimization and enforces sparsity in $\mathbf{x}$
- basis pursuit denoising (BPDN):

$$\min ||\mathbf{x}||_1 \;\; \textit{subject to} \;\; ||\mathbf{y} - \mathbf{A}\mathbf{x}||_2 \leq \eta$$

  - we typically set $\eta = 10^{-6}\left|\left|\mathbf{A}\mathbf{x}^{\dagger}\right|\right|_2$





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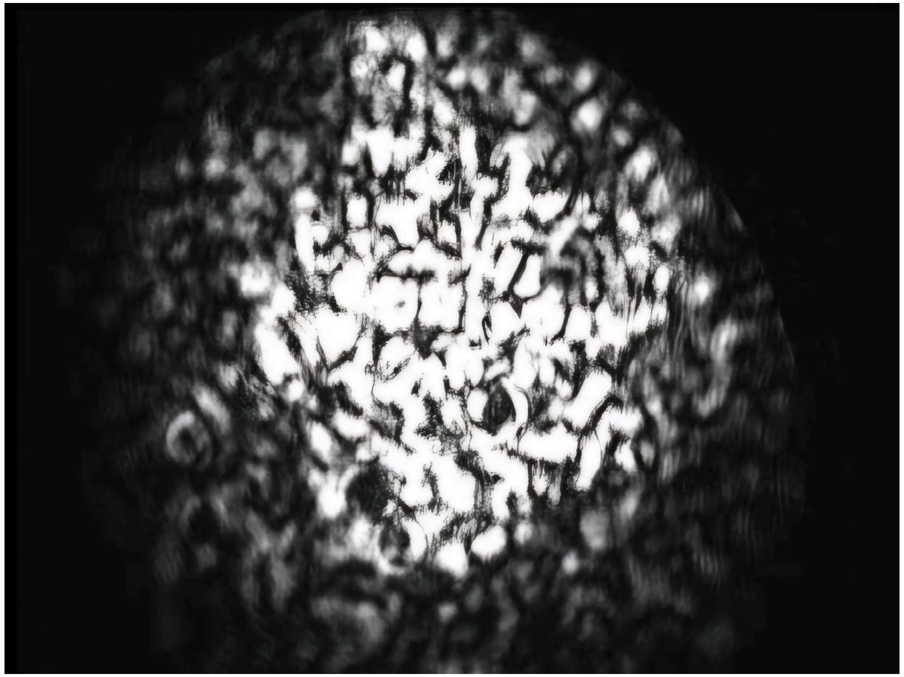
Question

To what extent do correlations present in speckle patterns affect reconstruction quality?



Model description

Devise a model that emulates correlations in speckle patterns

* start with random Gaussian baseline: $\mathbf{g} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$

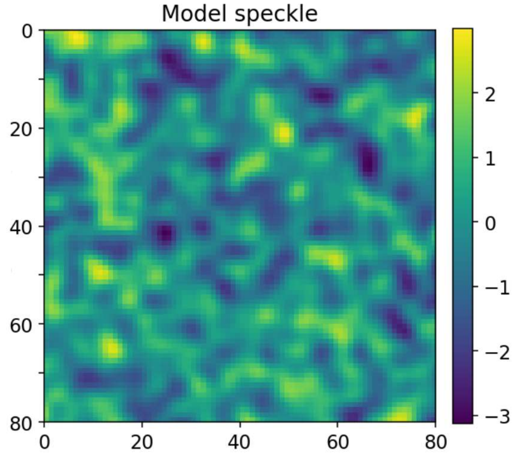
* apply smoothing operator: $\mathbf{g}\mathbf{S}_L \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma} = \mathbf{S}_L^T \mathbf{S}_L)$

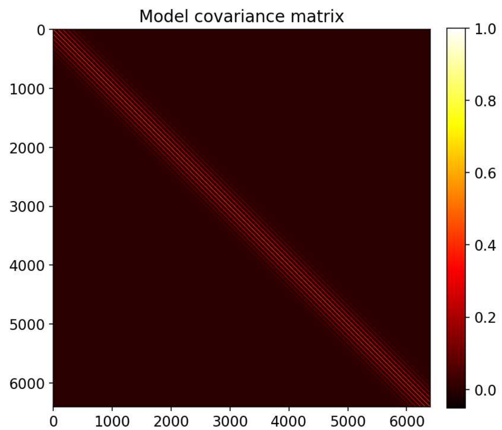
* choose $\mathbf{\Sigma}_{ij} := \exp\left(-\frac{d_{ij}^2}{L^2}\right)$

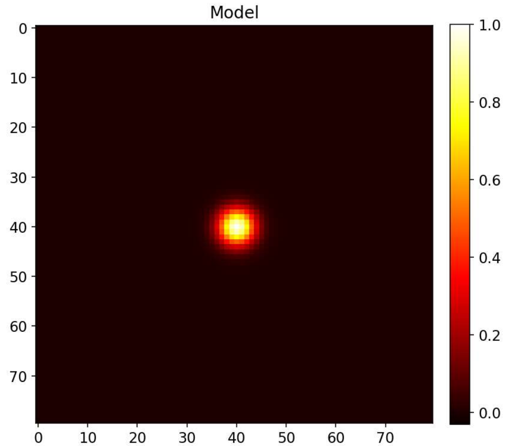
with $d_{ij} = ||\mathbf{p}_i - \mathbf{p}_j||_2$

and $L = \sqrt{L_x + L_y}$

p_0 (0,0)	p_1 (0,1)	p_2 (0,2)	p_3 (0,3)	p_4 (0,4)
p_5 (1,0)	p_6 (1,1)	p_7 (1,2)	p_8 (1,3)	p_9 (1,4)
p_{10} (2,0)	p_{11} (2,1)	p_{12} (2,2)	p_{13} (2,3)	p_{14} (2,4)
p_{15} (3,0)	p_{16} (3,1)	p_{17} (3,2)	p_{18} (3,3)	p_{19} (3,4)
p_{20} (4,0)	p_{21} (4,1)	p_{22} (4,2)	p_{23} (4,3)	p_{24} (4,4)







Theoretical upper bound

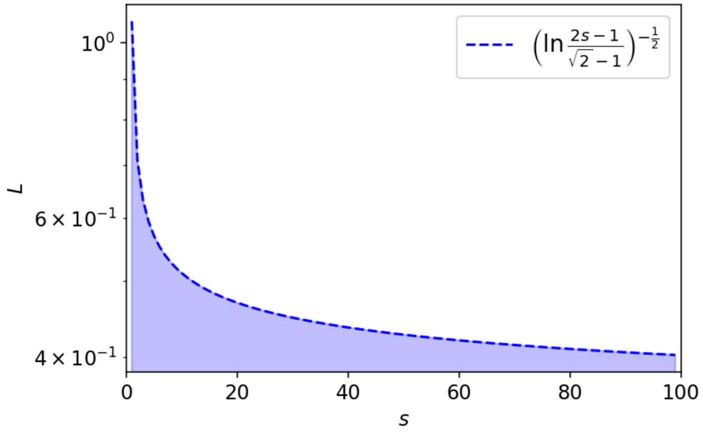
Derive a theoretical upper bound on the smoothing length

Reconstruction guarantee (Candès, 2008)

If $\mathbf{A}$ satisfies the RIP of order $2s$ with $\delta_{2s} < \sqrt{2} - 1$ and $\mathbf{y} = \mathbf{Ax}$, then the solution $\hat{\mathbf{x}}$ to BPDN obeys

$$||\hat{\mathbf{x}} - \mathbf{x}||_2 \leq 4 \frac{\sqrt{1 + \delta_{2s}}}{1 - (1 + \sqrt{2})\delta_{2s}} \eta$$

- $\mathbf{GS}_L$ satisfies the RIP if $\mathbf{S}_L$ has the RIP
- $\mathbf{S}_L$ is $\left(2s, (2s-1)\mu(\mathbf{S}_L)\right)$-RIP
- $\mu(\mathbf{S}_L) = \exp\left(-L^{-2}\right)$
- $L < \left(\ln \frac{2s-1}{\sqrt{2}-1}\right)^{-\frac{1}{2}}$



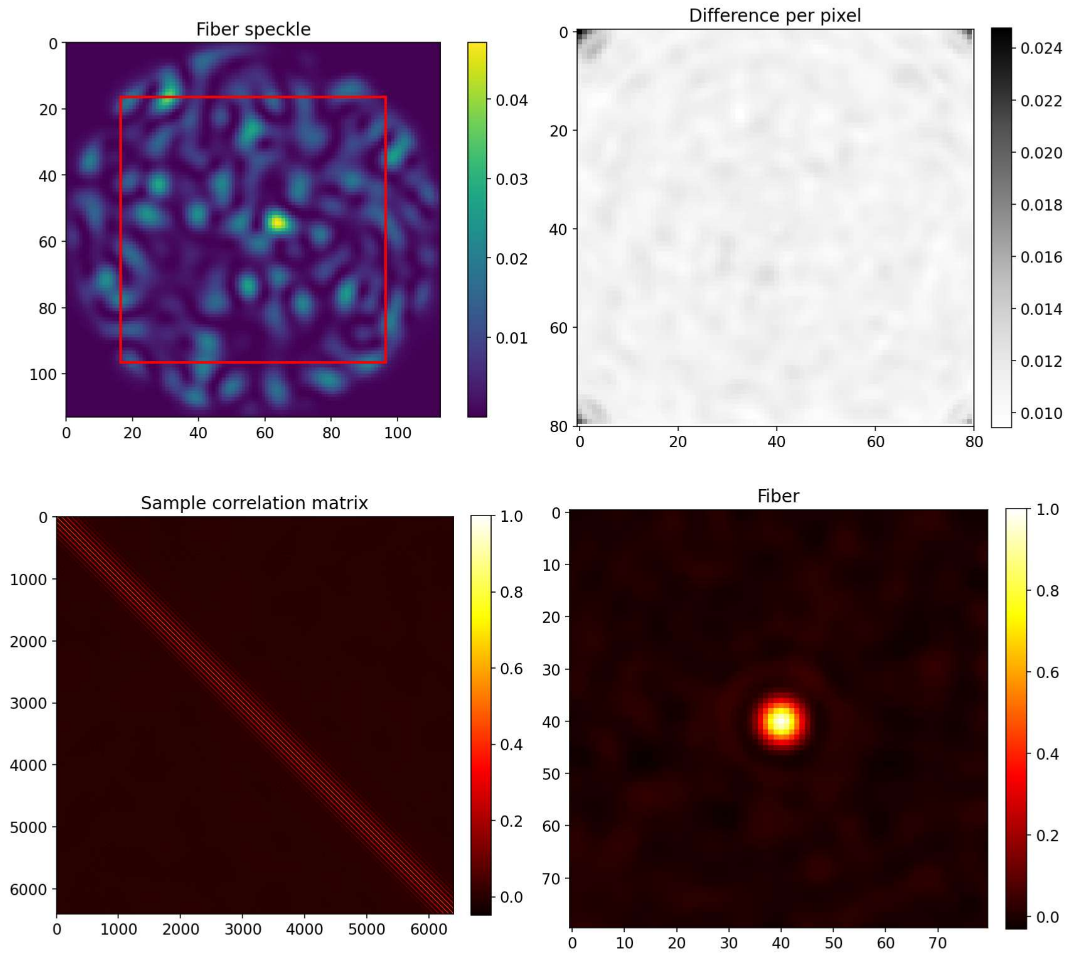
Fiber-simulated speckle patterns

Compare the correlation model with speckle patterns simulated from multimode fibers

* fix mode profiles $\{\Psi_k\}_{k=1}^{197}$, simulate 10^4 patterns

* compute sample correlation matrix $\bar{\boldsymbol{\Sigma}}$ and avg. difference $\sqrt{\frac{1}{n^2}\sum_{j=0}^{n^2-1}\left(\boldsymbol{\Sigma}_{ij}-\bar{\boldsymbol{\Sigma}}_{ij}\right)^2}$ for each pixel i

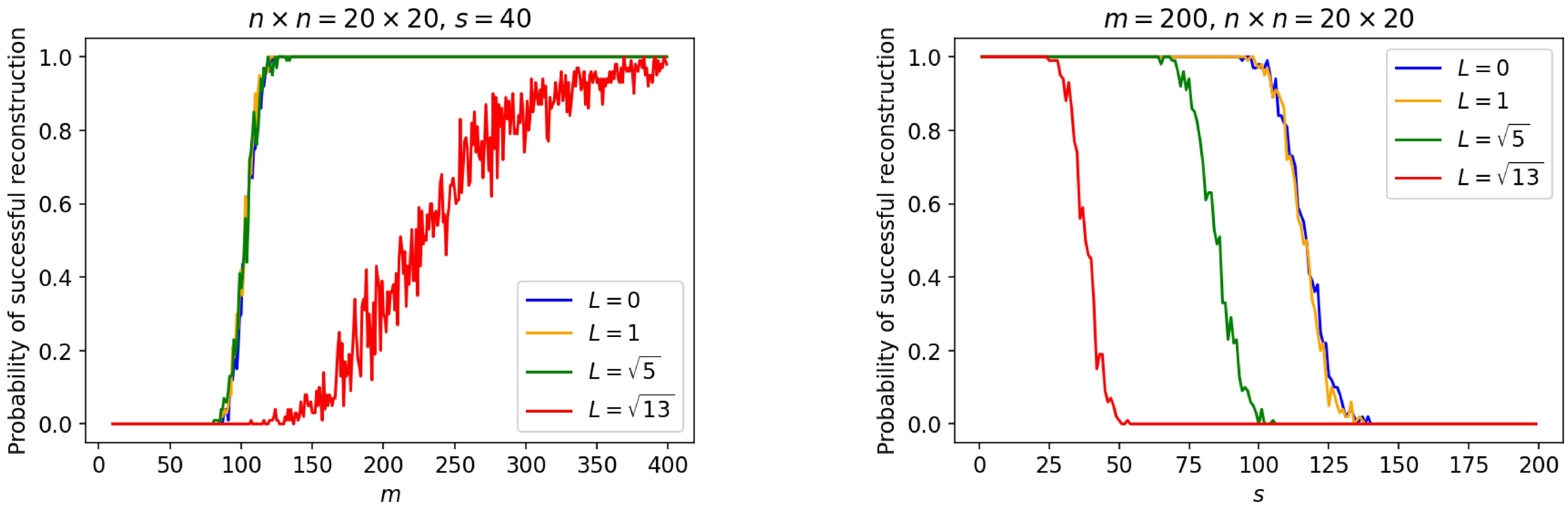
* $\min_L \left|\left|\boldsymbol{\Sigma}-\bar{\boldsymbol{\Sigma}}\right|\right|_F \rightsquigarrow$ $L \approx \sqrt{13}$ for $n^2 = 80^2$



Numerical experiment

Assess the effect of smoothing through a numerical experiment

- downscale: effect of $L \approx 1$ in 20×20 patterns
- for every m, s, L, generate 100 problem sets $\mathbf{y} = \mathbf{GS}_L\mathbf{x}$ and count number of successful BPDN reconstructions $\hat{\mathbf{x}}$
- conclusion: correlation effect is negligible as long as speckle size is sufficiently small



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Question

Does the probability distribution of speckle patterns' light intensity influence reconstruction quality?

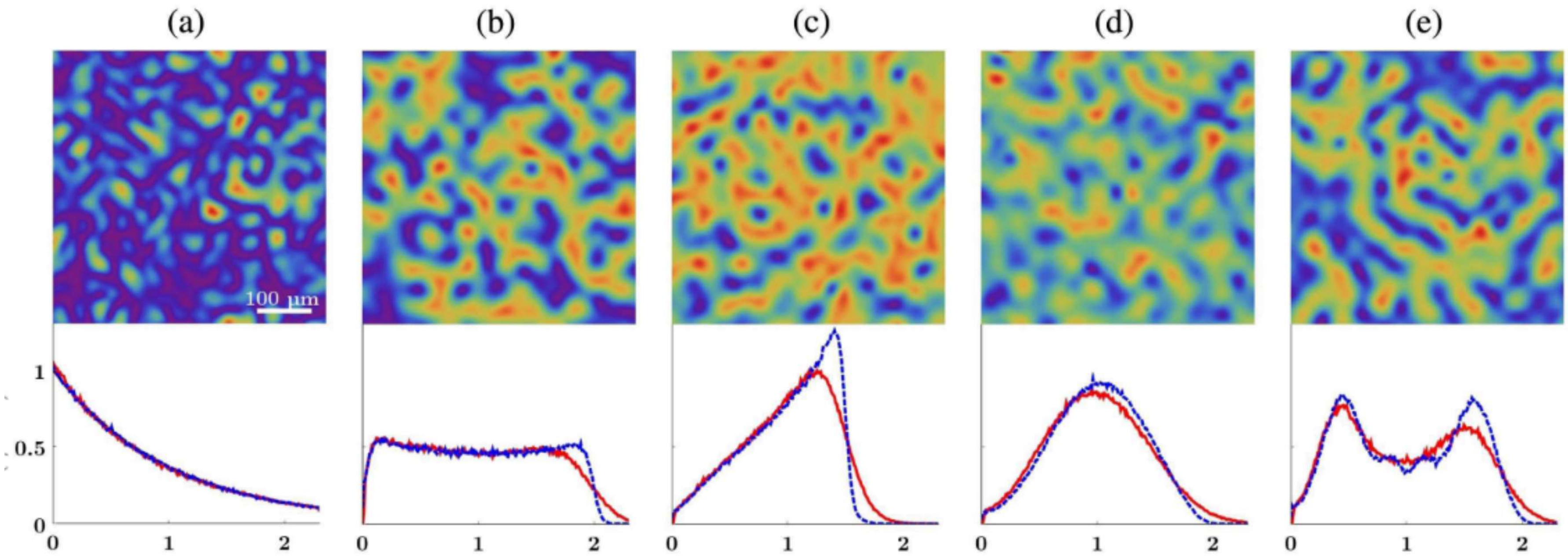

Figure: Bender, 2018

Model description

Extend the smoothing model to follow different distributions

Inversion method (Derflinger, 2010)

Realizations r of a random variable R with target distribution function F_R can be generated by applying F_R^{-1} to realizations u of $U \sim \mathrm{Unif}[0,1]$

$\left(F_R^{-1} \circ \Phi\right)(\mathbf{gS}_L)$ has smoothing length L and distribution R

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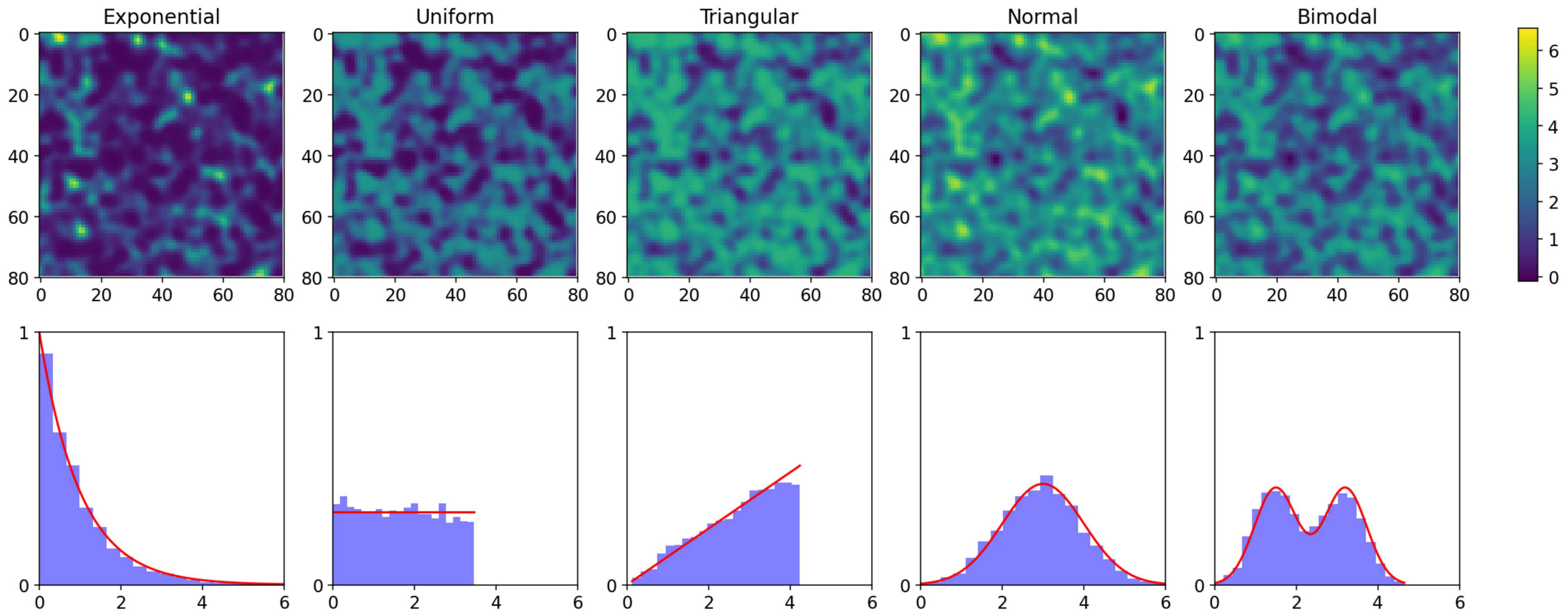
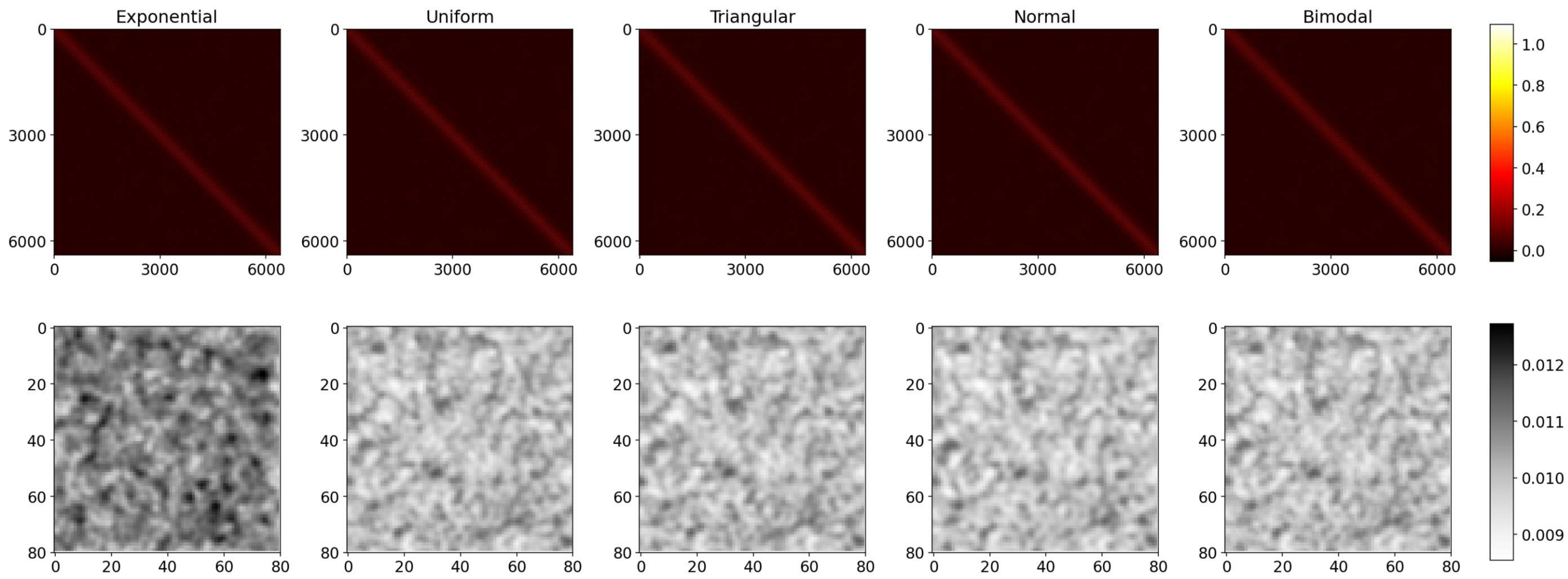
$$\left(F_R^{-1} \circ \Phi\right)(\mathbf{gS}_L) \text{ has smoothing length } L \text{ and distribution } R$$



Common variance and preservation of smoothing length

- let each target distribution have unit variance
- compute for each distribution
  * sample covariance matrix $\bar{\mathbf{\Sigma}}$
  * pixel-wise average differences with $\mathbf{\Sigma}$



Numerical experiment

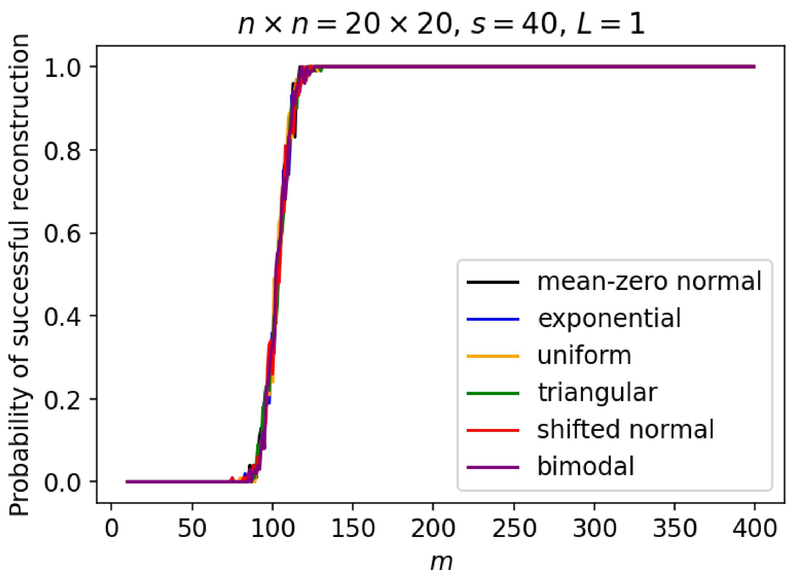
Assess whether using a customized intensity distribution improves performance

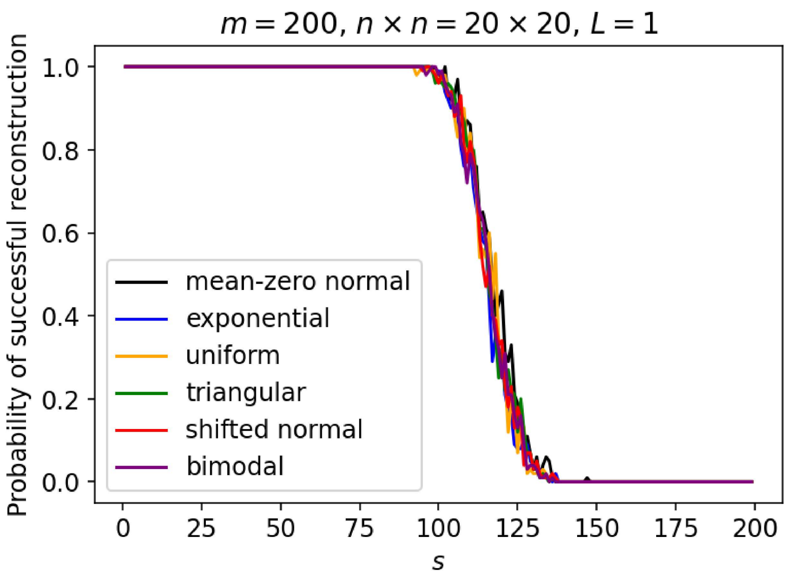
- for every m, s, L, R, generate 100 problem sets $\mathbf{y} = \left(F_R^{-1} \circ \Phi\right) (\mathbf{GS}_L)\mathbf{x}$ and count number of successful BPDN reconstructions $\hat{\mathbf{x}}$

Numerical experiment

Assess whether using a customized intensity distribution improves performance

- for every m, s, L, R, generate 100 problem sets $\mathbf{y} = \left(F_R^{-1} \circ \Phi\right) (\mathbf{G}\mathbf{S}_L)\mathbf{x}$ and count number of successful BPDN reconstructions $\hat{\mathbf{x}}$
- conclusion: all distributions perform equally well in recovering arbitrary sparse signals





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Are there reconstruction schemes other than conventional BPDN suited for the optical experiment?

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Sparse recovery with nonnegative least-squares (NNLS) (Foucart, 2013)

If $\mathbf{x}$ is nonnegative, and $\mathbf{A}$ satisfies the ℓ_2-robust nullspace property (ℓ_2-robust NSP) and $\mathcal{M}^+$-criterion, one can resort to NNLS

$$\min_{\mathbf{x}\geq\mathbf{0}} \; ||\mathbf{y} - \mathbf{A}\mathbf{x}||_2$$

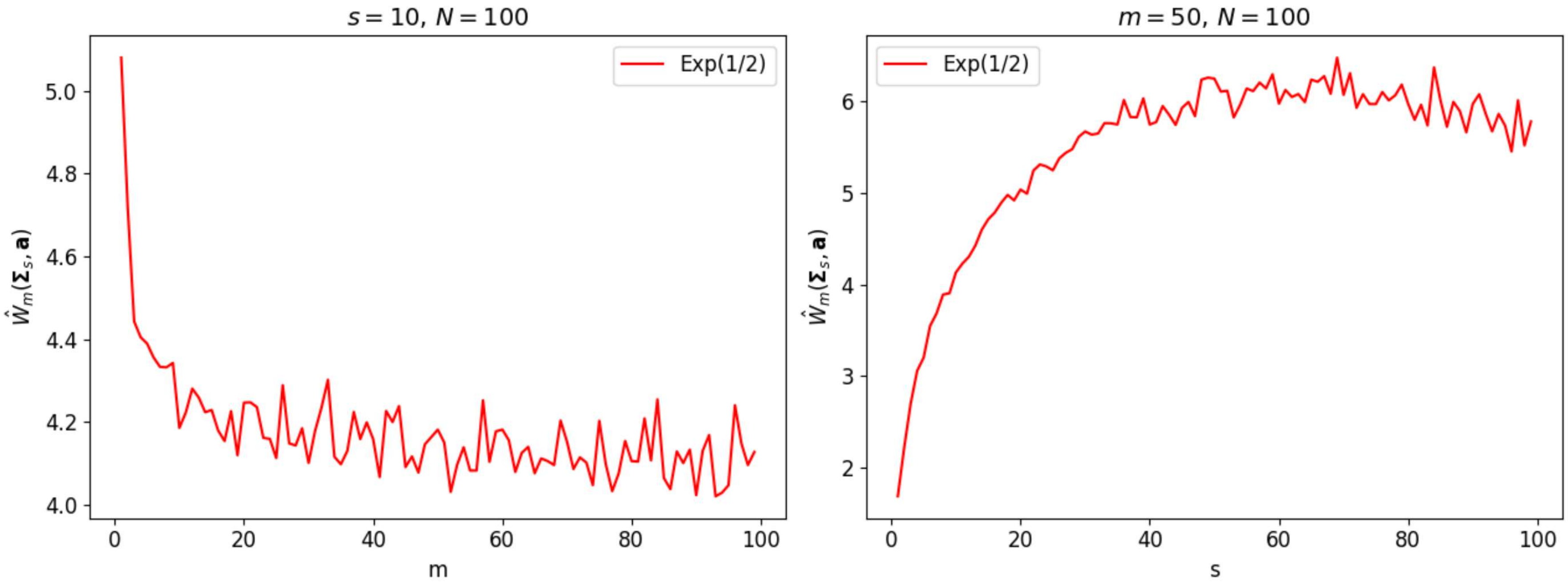
- NNLS requires no a-priori noise bound
- we indicate that $\mathbf{A} \in \mathbb{R}^{m\times N}$ composed of i.i.d. entries $a_{ij} \sim \text{Exp}\left(\lambda = 1/\langle I\rangle\right)$ with $\langle I\rangle$ representing the avg. light intensity satisfy the ℓ_2-robust NSP and $\mathcal{M}^+$-criterion

ℓ_2-robust nullspace property

ℓ_2-robust NSP (Foucart, 2013)

$\mathbf{A} \in \mathbb{R}^{m \times N}$ satisfies the ℓ_2-robust NSP of order s if for all $S \subset \{1, \ldots, N\}$ with $|S| \leq s$ and $\mathbf{v} \in \mathbb{R}^N$, $||\mathbf{v}_S||_2 \leq \frac{\rho}{\sqrt{s}}||\mathbf{v}_{\bar{S}}||_1 + \tau||\mathbf{A}\mathbf{v}||_2$

* suitable for nonnegative matrices instead of RIP
* showed general idea behind ℓ_2-robust NSP
* upper bound on $\mathbb{E}\left(\sup_{\mathbf{u}\in\Sigma_s^2}\left\langle \frac{1}{\sqrt{m}}\sum_{k=1}^m \epsilon_k \mathbf{a}_k, \mathbf{u}\right\rangle\right)$



$\mathcal{M}^+$-criterion

- $\exists \mathbf{t} \in \mathbb{R}^m$ such that $\mathbf{w} := \mathbf{A}^T \mathbf{t}$ is strictly positive $\rightsquigarrow$ $\mathbf{A} \in \mathcal{M}^+$
- pick $\mathbf{t}$ such that $\mathbb{E}(\mathbf{w}) = \mathbf{1}_N$, and show that $|w_i - 1| < \frac{1}{2}$ with high probability for all $i = 1, \ldots, N$

$\mathcal{M}^+$-criterion for random exponential matrices

Suppose that $\mathbf{A} \in \mathbb{R}^{m \times N}$ is a random exponential matrix with parameter $\lambda = 1/\langle I \rangle > 0$, then $\mathbf{A}$ fulfills the $\mathcal{M}^+$-criterion with probability $\geq 1 - 2N \exp(-em/(4e^2 + 4e + 16))$

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- general idea behind ℓ_2-robust NSP ✓
- proof of $\mathcal{M}^+$-criterion ✓
- conclusion: optical experiment is likely to benefit from NNLS for samples that are sparse in the original domain

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Findings

Correlation effect:

- as long as the fiber excites sufficiently many modes, the effect of correlations is negligible

Intensity distribution:

- different distributions perform equally well in recovering arbitrary sparse signals

Reconstruction scheme:

- one can resort to NNLS for recovering samples that are sparse in original domain

Other:

- Pylops' SPGL1 is sensitive to correlations and large values in the measurement matrix
- pyMMF has a bug for square fiber mode computation

Future research

- formally prove ℓ_2-robust NSP for random exponential matrices
- examine how different intensity distributions perform in reconstructing specific 2D samples
- study the effect of noise (in relation to NNLS)
- use of machine learning to predict speckle patterns
- nonsparse image recovery for metrology

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