LAB REPORT: LAB 4

IMPLICIT SURFACES AND MODELING
TNM079 - MODELING AND ANIMATION

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Abstract

This report describes the fourth out of six labs in the course TNM079 - Modeling and Animation given at Linköping University. This report covers the subject of implicit surfaces and modeling, in which *Constructive Solid Geometry* operators together with discrete gradient and curvature are used to manipulate implicit surfaces. The results show that many objects of varying shapes can be created using these methods, and they are efficient when computing differentials. They are also useful since no self-intersections or holes will appear.

1 Introduction

The purpose of this lab is to show how implicit surfaces can be used in computer graphics and what advantages they have. An implicit surface is an indirect surface representation, in which the surface is defined by an equation which has to be solved in order to get the surface geometry. Compared to explicit surfaces in which the elements of the surface, such as triangles, edges and points, directly define the surface. The equation for an implicit surface maps all points in the current space onto a scalar. For an implicit surface, every point in space is classified as either inside, outside or directly on the implicit surface, see equation 1.

$$\begin{array}{ll} \text{Inside:} & f(\mathbf{x}) < C \\ \text{Outside:} & f(\mathbf{x}) > C \\ \text{On surface:} & f(\mathbf{x}) = C \end{array} \tag{1}$$

Here, the point $\mathbf{x}$ is classified with the iso-value C and a scalar function $f(x)$.

2 Assignments

The individual assignments that were performed in the lab are presented in this section. Each subsection below represents one lab assignment.

2.1 Implementing CSG operators

Constructive Solid Geometry, CSG, is a technique which constructs complex objects by combining primitive shapes, such as spheres and cubes, and a set of boolean operations. In this assignment the boolean operations union, intersection and difference are implemented. These operations are quite straightforward to implement on implicit surfaces, and to derive these three operators, Equations 2-4 have been implemented in the `GetValue`-functions in classes `Union`, `Intersection` and `Difference`. A and B in the equations denote two implicit surfaces.

$$\begin{aligned} \mathbf{Union}(A,B) &= A \cup B = \min(A,B) && (2)\\ \mathbf{Intersection}(A,B) &= A \cap B = \max(A,B) && (3)\\ \mathbf{Difference}(A,B) &= A - B = \max(A,-B) && (4) \end{aligned}$$

2.2 Implementing the quadric surface

A quadric surface is defined by a quadratic implicit function. The analytical expression for this function in a three-dimensional space is displayed in equation 5.

$$\begin{aligned} f(x,y,z) &= Ax^2 + 2Bxy + 2Cxz \\ &+ 2Dx + Ey^2 + 2Fyz \\ &+ 2Gy + Hz^2 + 2Iz \\ &+ J \end{aligned} \tag{5}$$

This function can also be written on matrix form, which is displayed in equation 6, in which **p** represents a point, and **Q** represents the matrix containing the constants A-J.

$$f(x,y,z) = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{p}^T\mathbf{Q}\mathbf{p} \tag{6}$$

The matrix **Q** can be used to create different quadric surfaces. This is done by changing the contstants in the matrix. In this assignment the following quadric surfaces were created:

$$Plane: \begin{cases} D = G = I = 0.5 \\ J = A = B = C = E = F = H = 0 \end{cases}$$

$$Cylinder: \begin{cases} A = E = 1 \\ J = -1 \\ B = C = D = F = G = H = I = 0 \end{cases}$$

$$Ellipsoid: \begin{cases} A = 1/a^2 \\ E = 1/b^2 \\ H = 1/c^2 \\ J = -1 \\ B = C = D = F = G = I = 0 \end{cases}$$

$$Cone: \begin{cases} A = E = 1 \\ H = -1 \\ B = C = D = F = G = I = J = 0 \end{cases}$$

$$Paraboloid: \begin{cases} A = E = 1 \\ I = 0.5 \\ B = C = D = F = G = H = J = 0 \end{cases}$$

$$Hyperboloid: \begin{cases} A = E = 1 \\ H = J = -1 \\ B = C = D = F = G = I = 0 \end{cases}$$

After the quadric shapes were created, the normals were to be calculated for each shape. The normal for a quadric surface is defined in Equation 7. The differentiation can be applied directly to the quadric with the coefficient matrix $\mathbf{Q_{sub}}$, since the quadric surface is known analytically.

$$\begin{aligned} \nabla f(x, y, z) &= 2 \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ & & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \\ &= 2\mathbf{Q}_{sub}\mathbf{p} \end{aligned} \tag{7}$$

2.3 Implementing the discrete gradient operator for implicits

When computing gradients for an implicit object, the geometrical properties are of no concern. The implementation of the discrete gradient operator ∇ can be done using a central difference scheme, where its differentials can be determined using Equation 8.

$$\frac{\partial}{\partial x} = D_x \approx \frac{f(x_0+\epsilon) - f(x_0-\epsilon)}{2\epsilon}$$
$$\frac{\partial}{\partial y} = D_y \approx \frac{f(y_0+\epsilon) - f(y_0-\epsilon)}{2\epsilon} \tag{8}$$
$$\frac{\partial}{\partial z} = D_z \approx \frac{f(z_0+\epsilon) - f(z_0-\epsilon)}{2\epsilon}$$

Here, ϵ is the step size which will be set in the GUI. When choosing the value for ϵ, care must be taken in order to get accurate results. The gradient can then be formed using Equation 9.

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \tag{9}$$

2.4 Implementing the discrete curvature operator for implicits

The curvature κ for an implicit surface is calculated similarly to the gradient, but using second order differentials, see Equation 10.

$$\begin{aligned} \kappa &\approx \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &\textit{where} \\ \frac{\partial^2 f}{\partial x^2} &= D_{xx} \approx \frac{f(x_0+\epsilon) - 2f(x_0) + f(x_0-\epsilon)}{\epsilon^2} \\ \frac{\partial^2 f}{\partial y^2} &= D_{yy} \approx \frac{f(y_0+\epsilon) - 2f(y_0) + f(y_0-\epsilon)}{\epsilon^2} \\ \frac{\partial^2 f}{\partial z^2} &= D_{zz} \approx \frac{f(z_0+\epsilon) - 2f(z_0) + f(z_0-\epsilon)}{\epsilon^2} \end{aligned} \tag{10}$$

2.5 Implement super-elliptic blending

When utilizing the CSG-operators, presented in Equations 2-4, sharp edges within or between objects where the normal is undefined can appear. To avoid this, super-elliptic blending can be used. This method utilizes density functions to achieve smooth transitions, see Equation 11.

$$D_A(\mathbf{x}) = \begin{cases} > 1 & \text{if } \mathbf{x} \text{ is inside the surface} \\ = 1 & \text{if } \mathbf{x} \text{ is on the surface} \\ \in [0,1) & \text{if } \mathbf{x} \text{ is outside the surface} \end{cases} \tag{11}$$

In this equation, A represents an implicit surface and the density function is calculated according to Equation 12.

$$D_A(\mathbf{x}) = e^{-A(\mathbf{x})} \tag{12}$$

After the implicit surface has been transformed into a density function, the super elliptic blending can be performed by applying boolean operators, see Equations 13a-13c. The difference can be calculated by negating either A or B and thereafter performing an intersection. Equation 13c illustrates the difference when B is set to negative.

$$D_{A\cup B} = \left(D_A^p + D_B^p\right)^{1/p} \tag{13a}$$

$$D_{A\cap B} = \left(D_A^{-p} + D_B^{-p}\right)^{-1/p} \tag{13b}$$

$$D_{A-B} = \left(D_A^{-p} - D_B^{-p}\right)^{-1/p} \tag{13c}$$

The final implicit surface values are then obtained by inverting equation 12. In equations 14a-14c, the blended union, blended intersection and blended difference are shown.

$$D_{A\cup B} = -log\left(\left(D_A^p + D_B^p\right)^{1/p}\right) \tag{14a}$$

$$D_{A\cap B} = -log\left(\left(D_A^{-p} + D_B^{-p}\right)^{-1/p}\right) \tag{14b}$$

$$D_{A-B} = -log\left(\left(D_A^{-p} - D_B^{-p}\right)^{-1/p}\right) \tag{14c}$$

3 Results

The important results for each of the assignments are presented in this section.

3.1 CSG operators

Figure 1 illustrates the different boolean operations that were used in this lab. To the left, two spheres are joined together by using the union operator. In the middle, the intersection between two spheres is shown. To the right the difference between two spheres is displayed.

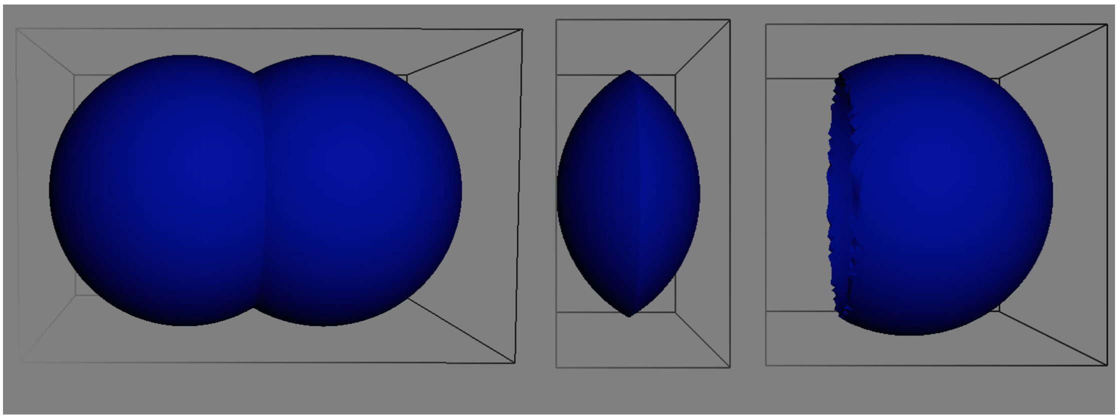

Figure 1: This image illustrates the different CSG operators that were implemented on two implicit spheres. From left to right: Union, Intersection and Difference. The Jet color map was used for the final shapes.

3.2 Quadric surfaces

Figure 2 illustrates the different quadric surfaces that were created. The constants used to create these shapes, i.e. the constants for the matrix **Q**, are presented in section 2.2.

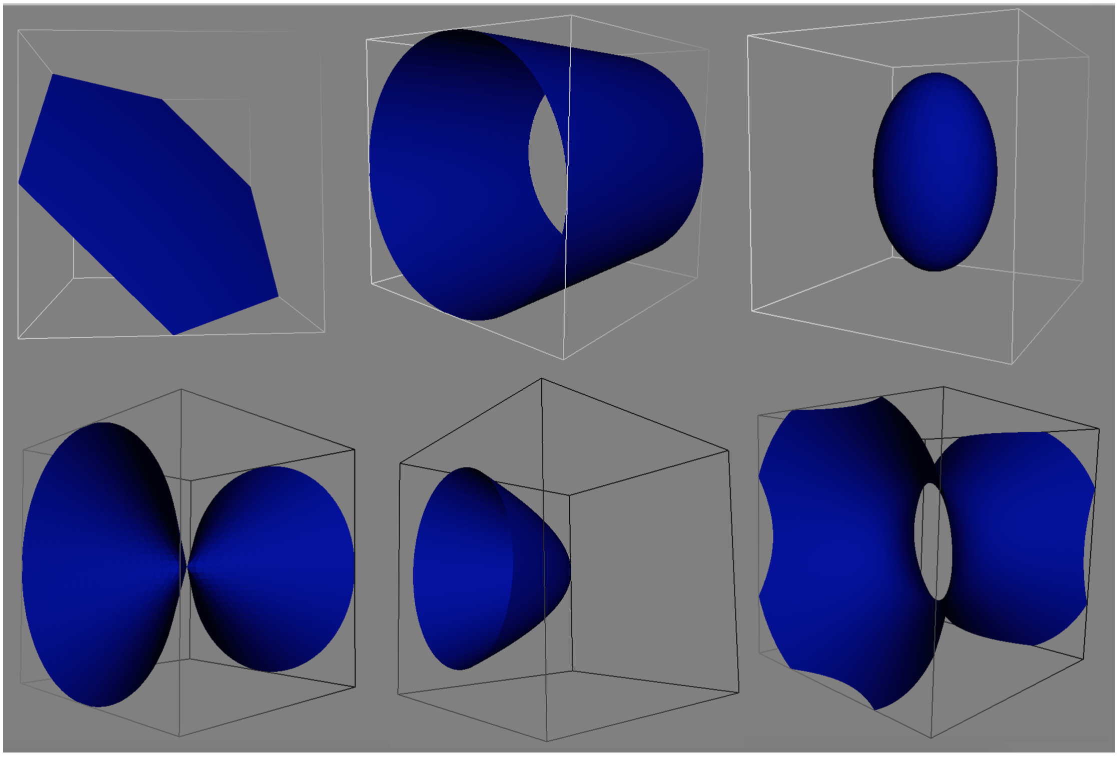

Figure 2: Quadric surfaces that were created by quadric implicit functions. The Jet color map has been used for all shapes in the image. From left to right with white bounding box: plane, cylinder and ellipsoid. From left to right with black bounding box: cone, paraboloid and hyperboloid.

3.3 Discrete gradient operator for implicits

The results for implementing the gradients are shown in Figure 3, which illustrates the gradients for an implicit sphere. When comparing different values for ϵ the results showed that when ϵ increases the gradient computation becomes inaccurate, which is visible in Figure 4 where the gradients are compared to the normals.

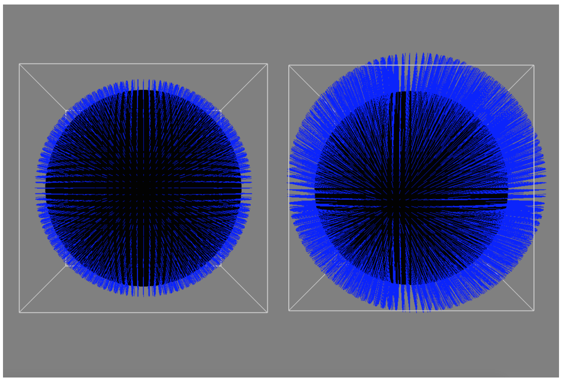

Figure 3: This figure illustrates the gradients for an implicit sphere. The gradients are illustrated with blue lines, and the spheres are colored with the color map Black-White. From left to right: Sphere with small value ϵ and sphere with large value ϵ.

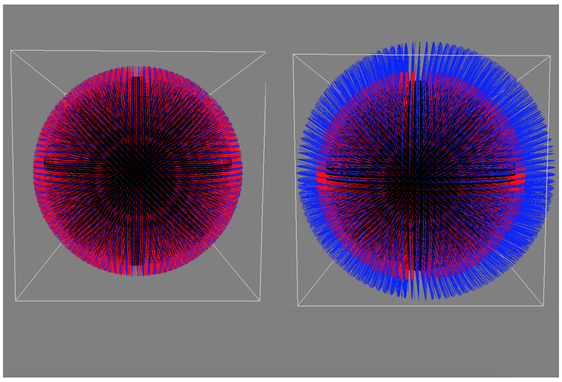

Figure 4: This figure illustrates the gradients and the normals for an implicit sphere. The gradients are illustrated with blue lines, the normals are illustrated with pink lines and the spheres are colored with the color map Black-White. From left to right: Sphere with small value ϵ and sphere with large value ϵ.

3.4 Discrete curvature operator for implicits

In Figure 5 the results of the curvature operator for a paraboloid is illustrated. When comparing different values for ϵ, the results showed that the curvature gets inaccurate as ϵ increases, see Figure 6.

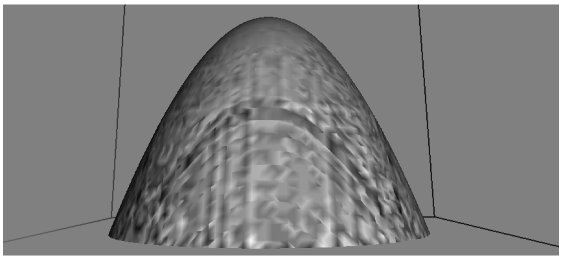

Figure 5: This figure illustrates the curvature for a Paraboloid. The Black-White color map was used.

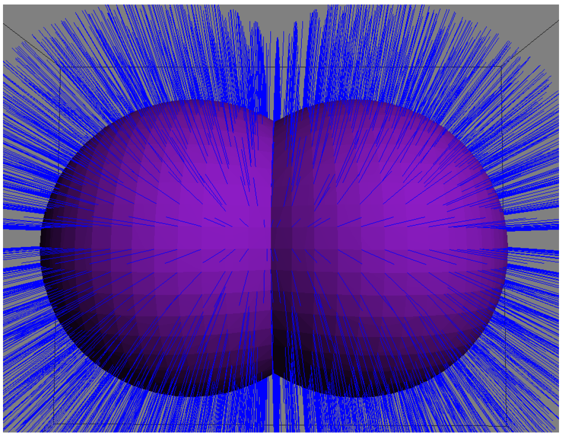

Figure 6: This figure illustrates the impact ϵ can have on the normals of a mesh.

3.5 Super-elliptic blending

In Figure 7, blended union, intersection and difference are shown. In the image the curvature for the shapes is also shown.

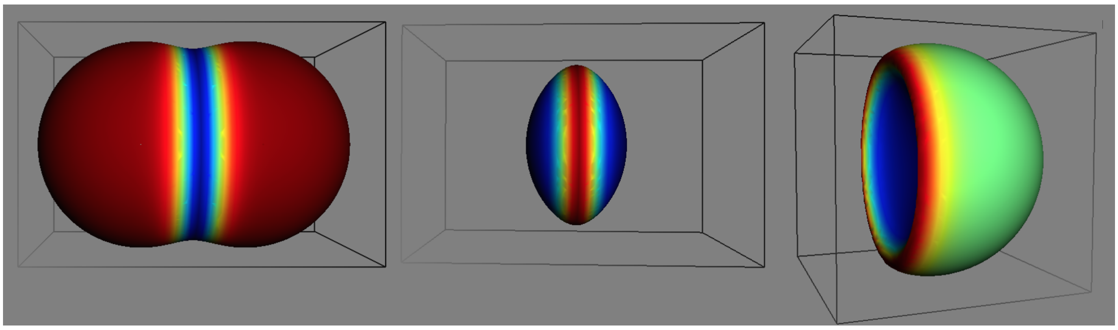

Figure 7: This figure illustrates the curvature for implicit spheres which has been blended, with $p = 6$. From left to right: blended union, intersection, difference. The color map Jet was used for all shapes.

When comparing different values for p, in Equations 14a through 14c, the results were that the blending becomes less smooth with higher value for p. This is illustrated in Figure 8 when different values for p have been tested when performing blended union on two spheres.

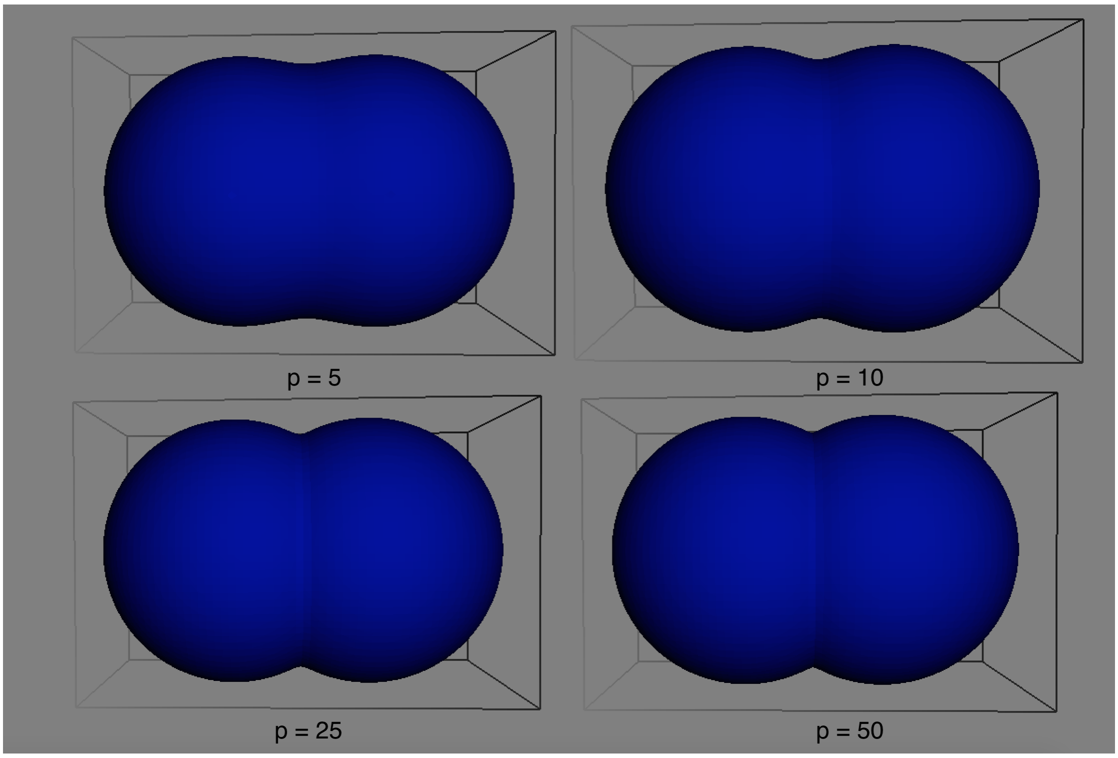

Figure 8: This figure illustrates blended union on two spheres, with different values for p. From left to right, top to bottom: $p = 5$, $p = 10$, $p = 25$ and $p = 50$. The color map used for all shapes is Jet.

4 Conclusion

The results show that many different objects can be created by using implicit functions and CSG operators. These objects can range from simple objects to more complex since the computations are easy to perform with implicit surfaces. Implicit surfaces also have the advantage that no holes or self-intersections will appear.

Equation 8 and **??** illustrate that the D_x and D_{xx}, respectively, are both influenced by the value of ϵ. When choosing ϵ for both discrete gradient and curvature a small value should be used. This is illustrated in Figure 6, in which a large value for ϵ has been used, and therefore the normals close to the cusp don not behave as they should. This is due to the fact that the sampling distance will cover both sides of the cusp, and the sampled normal values will negate each other.

The results illustrate that the assignments for this lab have been successfully carried out.

5 Lab partner and grade

This lab was completed in collaboration with Martin Gråd, margr484@student.liu.se. Since I have completed all the assignments marked with (*) and (**), I should qualify for grade 5.