Name: Emma Furr
Perm: 4076063

Math 104A Final Project Report

Motivation:

In this project, we examined how much specific heat is required to raise the temperature of water (in ˚C), at some set temperature, by 1˚C. We used three methods with two data sets to approximate the specific heats at two distinct temperatures of water.

The question we plan to answer is which approximation method is the most effective (by comparing relative errors) in determining the specific heat needed to raise the temperature of water by one degree Celsius at a certain temperature?

The three methods we used to solve this problem were Neville's method, Lagrange Interpolation, and Natural Cubic Spline method. Each of these methods are used to find interpolating polynomials to approximate functions at specific nodes. Polynomials are functions with useful properties and their relatively simple form makes them an ideal candidate to use as approximations for more complex functions.

Problem Statement:

The problem we are focusing on is dealing with the temperature of water in a constant pressure environment, which approximates an Isobaric (i.e. constant pressure Cp) specific heat, and the goal is to find how much specific heat per unit of mass is needed to raise the temperature by 1˚C. Our unit for specific heat is J/(kg -˚C). We will test and evaluate the accuracy of our approximating polynomials by comparing already known specific heat values of 25˚C and 85˚C to our approximated results.

This solution is important because water covers about 70% of Earth's surface and has one of the highest specific heats on earth. Its high specific heat is very important because it means that water is able to absorb a lot of heat without a significant rise in its temperature. A relatively constant temperature without sudden increases or decreases is essential in sustaining life because most organisms require temperatures within a concise range for their survival. These approximations could be of great use for scientists who study water. This is important in the world today because of the rising problem of global warming. These methods could help scientists understand the direct implications of rising temperatures and how it affects our oceans.

Methods:

The problem will be solved with these three methods, Natural Cubic Spline, Lagrange polynomial interpolation, and Neville's method. These were the methods that we found most appropriate to use in approximating the amount of specific heat needed to raise the temperature of water at a certain temperature by 1˚C, when given a data set of temperatures (nodes) and the corresponding specific heats of those temperatures.

Natural cubic spline method is the most common piecewise polynomial approximation which uses cubic polynomials and interpolates a set of data points. A general cubic polynomial involves

Name: Emma Furr
Perm: 4076063

four constants, so there is sufficient flexibility in the cubic spline procedure to ensure that the interpolant not only is continuously differentiable on the interval, but also has a continuous second derivative. We used this method first, predicting that it will be the most accurate interpolation method.

Theorem 3.11 If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$, then f has a unique natural spline interpolant S on the nodes $x_0, x_1, \ldots, x_n$; that is, a spline interpolant that satisfies the natural boundary conditions $S''(a) = 0$ and $S''(b) = 0$.

Theorem 3.11 represents the natural cubic spline method. The spline is called a natural spline when the free boundary condition occurs, and the graph approximates the shape that long, flexible rod would take if it's forced to go through data points [(x0, f(x0)), (x1, f(x1)),...,(xn, f(xn))].

The Lagrange Polynomial is used for polynomial interpolation to find a linear combination of known functions to fit a set of data points to create a unique solution (polynomial) that fits the data to approximate it. We used this method second, predicting a less accurate interpolation method than Cubic Spline.

This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x), \tag{3.1}$$

where, for each $k = 0, 1, \ldots, n$,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \tag{3.2}$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}.$$

Theorem 3.2 defines the nth degree Lagrange Polynomial by using the formula above.

Neville's method can be applied when we need to interpolate a function f(x) at a given point x = p with increasingly higher order Lagrange interpolation polynomials. We used this method last predicting a similar outcome to that of Lagrange.

Theorem 3.5 Let f be defined at $x_0, x_1, \ldots, x_k$ and let x_j and x_i be two distinct numbers in this set. Then

$$P(x) = \frac{(x-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i-x_j)}$$

is the kth Lagrange polynomial that interpolates f at the $k+1$ points $x_0, x_1, \ldots, x_k$.

Theorem 3.5 shows how to recursively generate interpolating polynomials for approximating nodes, which is called Neville's method. The idea of the method is to use Lagrange polynomials of lower powers recursively in order to compute Lagrange polynomials of higher powers.

Name: Emma Furr
Perm: 4076063

The values we approximate from these three interpolating polynomial approximations will be used to compare to actual specific heat values that are known.

Results:

We approximated at two data points, 25˚C and 85˚C.

For water at temperature x = 25˚C, the actual specific heat is 4180 J/(kg - C˚). In this case, we used the data set (nodes), which is our water temperature in ˚C, of x = [20, 40, 50, 80, 90]. The actual specific heats to these nodes are f(x) = [4182, 4179, 4182, 4198, 4208]. In this case, for Natural Cubic Spline Method, our final approximation was 4180.542985 J/(kg - C˚) with relative error 1.299007 x 10^-4. For Lagrange Interpolation, our final approximation was 4179.441592 J/(kg - C˚) with relative error 1.335904 x 10^-4. For Neville's method, our final approximation was 4179.441592 J/(kg - C˚) with relative error 1.335904 x 10^-4.

Table 1

Temperature (°C)	Specific Heat (J/kg-C)
20.00	4182.00
40.00	4179.00
50.00	4182.00
80.00	4198.00
90.00	4208.00

Results from Table 1 for x = 25°C

Method	Approximation (J/kg-C)	Relative Error
Cubic Spline	4180.542985	1.299007 * 10^-4
Lagrange	4179.441592	1.335904 * 10^-4
Neville's Method	4179.441592	1.335904 * 10^-4

For water at temperature x = 85˚C, the actual specific heat is 4203 J/(kg - C˚). In this case, we used the data set (nodes), which is our water temperature in ˚C, of x = [10, 30, 50, 70, 90]. The actual specific heats to these nodes are f(x) = [4192, 4178, 4182, 4191, 4208]. In this case, for Natural Cubic Spline Method, our final approximation was 4203.257565 J/(kg - C˚) with relative error 6.128123 x 10^-5. For Lagrange Interpolation, our final approximation was 4202.234375 J/(kg - C˚) with relative error 1.821616 x 10^-4. For Neville's method, our final approximation was 4202.234375 J/(kg - C˚) with relative error 1.821616 x 10^-4.

Table 2

Temperature (°C)	Specific Heat (J/kg-C)
10.00	4192.00
30.00	4178.00
50.00	4182.00
70.00	4191.00
90.00	4208.00

Results from Table 2 for x = 85°C

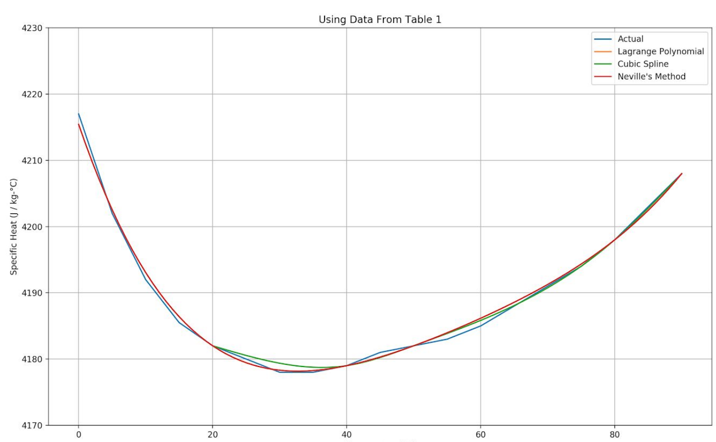
Method	Approximation (J/kg-C)	Relative Error
Cubic Spline	4203.257565	6.128123 * 10^-5
Lagrange	4202.234375	1.821616 * 10^-4
Neville's Method	4202.234375	1.821616 * 10^-4

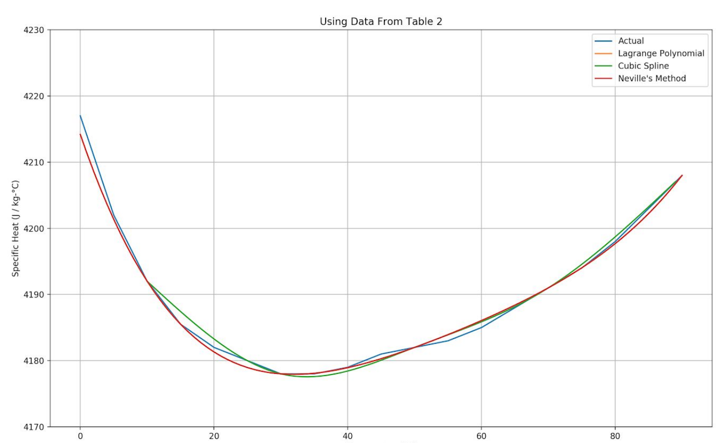
In theory, Lagrange interpolation and Neville's method are different methods computing the values of the same function, so the interpolated value resulting from each of the two methods is about the same. The Lagrange interpolation can be susceptible to strong variations if the data points are not strong enough. A cubic spline may differ from the true function if the derivatives on the endpoints are not constrained. If the function to interpolate is not smooth enough, which is possible when there are not enough data points, or nodes, a cubic spline method is preferable to the lagrange interpolation polynomial. Lagrange interpolation is slower than Neville's method when the number of points is more than three. Thus, the Natural Cubic Spline method should be the best approximation method.

Some interesting findings in our results were that we found that Natural Cubic Spline was the most accurate method when approximating our data, with small relative error. It is the most common and accurate piecewise polynomial approximation, thus the results follow. We, also, acquired the same results from the Lagrange polynomial interpolation and Neville's method. This is because we used fourth degree interpolating polynomials for both cases, so the results came out the same in both cases. Each of these methods gave approximations close to the actual specific heat values of 25˚C and 85˚C, with low relative errors.

The approximation of the specific heat of water is important because the high specific heat of water helps stabilize ocean temperatures, which creates a sustainable environment for marine life. Also, because of water's high specific heat, the water that covers most of Earth's surface keeps temperature fluctuations on land and in water within boundaries that allow life.

Below are two graphs showing the function of temperatures to their actual specific heat, the lagrange polynomial, neville's method, and cubic spline method for water temperatures of 25˚C and 85˚C. It is evident that Cubic Spline was the best approximation method, as its polynomial follows closest to the actual function. Lagrange polynomial and Neville's method have the same polynomial, so their functions are the same, and follow close to the actual specific heats.





Name: Emma Furr
Perm: 4076063

Group Members:
Kelcie Opp 3754850
Amit Yativ 6657977
Shawden Soltani 4446191
Kameron Quackenbush 8868416
Niko Kalpakian 4416723