The Constrained Minimum Spanning Tree Problem

Extended Abstract by R. Ravi and M. X. Goemans

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Seminar zur Diskreten Optimierung, RWTH Aachen

May 27, 2021

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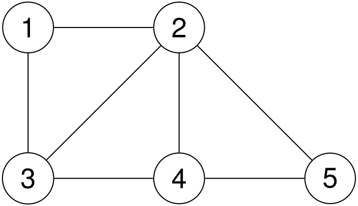

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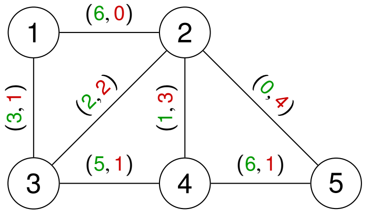

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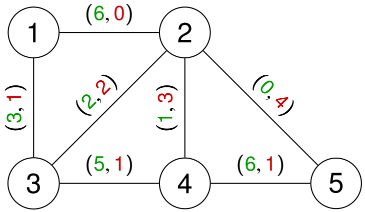

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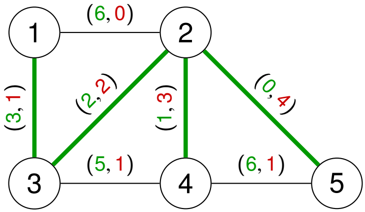

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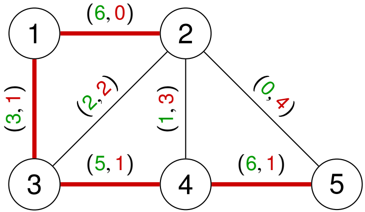

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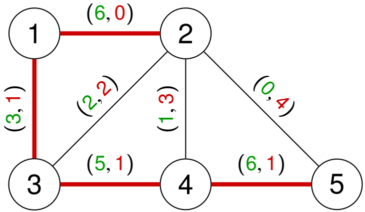

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- minimum total weight $w(T)$ and
- minimum total length $l(T)$.

$\Rightarrow$ Specify budget $L \in \mathbb{N}_{\geq 0}$ for the length and minimize the weight.

Exact Solution

Integer Program (CMST)

S is the set of incidence vectors of spanning trees of G and

$$\begin{aligned} W = \min \ & \sum_{e \in E} w_e x_e \\ \text{s.t.} \ & x \in S \\ & \sum_{e \in E} l_e x_e \leq L. \end{aligned}$$

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Theorem [RG96]

A (1,2)-approximation can be computed in polynomial runtime.

$\Rightarrow$ For fixed $\epsilon > 0$, we can even find $(1, 1+\epsilon)$-approximation.

Approximation Algorithm

1. Compute Lagrangian relaxation P_z of the IP

$$\ell(z) = \min \sum_{e \in E} (\underbrace{w_e + z l_e}_{= c_e}) x_e - zL$$

$$\text{s.t. } x \in S.$$

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2. Use Megiddo's algorithm to compute value z^* which maximizes

$$\mathcal{L} = \max_{z \geq 0} \ell(z)$$

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3. Compute sequence

$$T_{min} = T_0, T_1, \ldots, T_{i-1}, T_i, \ldots, T_k = T_{max}.$$

Pick first T_i such that $l(T_{i-1}) < L$ and $l(T_i) \geq L$, then

$$l(T_i) \leq L + l_{max} \leq 2L \quad \text{and} \quad w(T_i) \leq W.$$

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Lagrangian Relaxation

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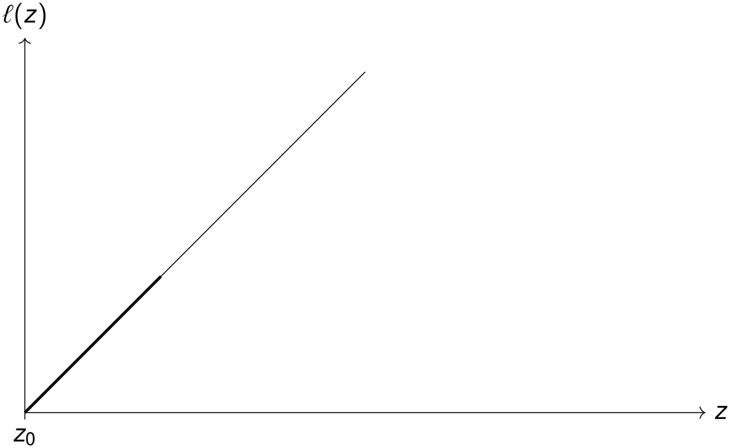
We obtain a set of *minimum spanning tree* problems for the cost function $c_z : E \to \mathbb{R}_{\geq 0}, e \mapsto w_e + z l_e$ and arbitrary $z \in \mathbb{R}_{\geq 0}$.

Lagrangian Relaxation



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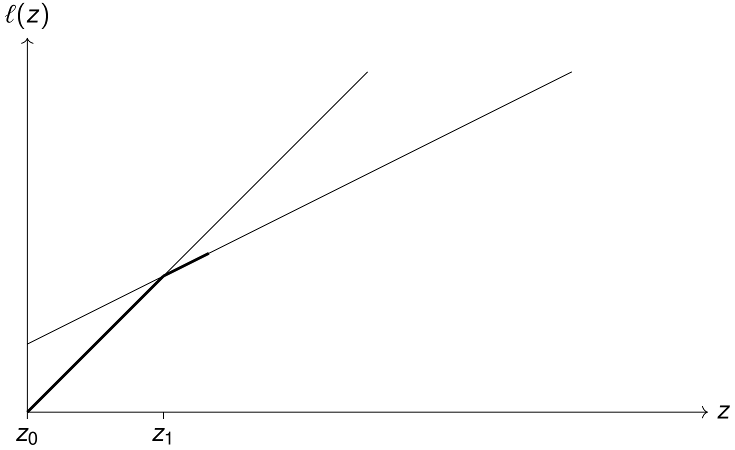

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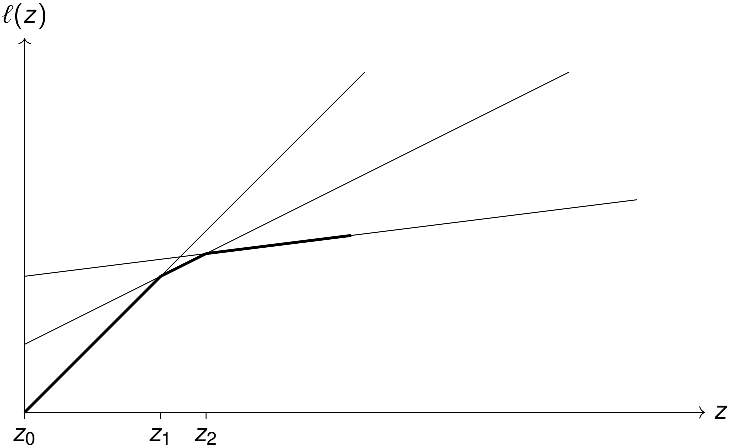

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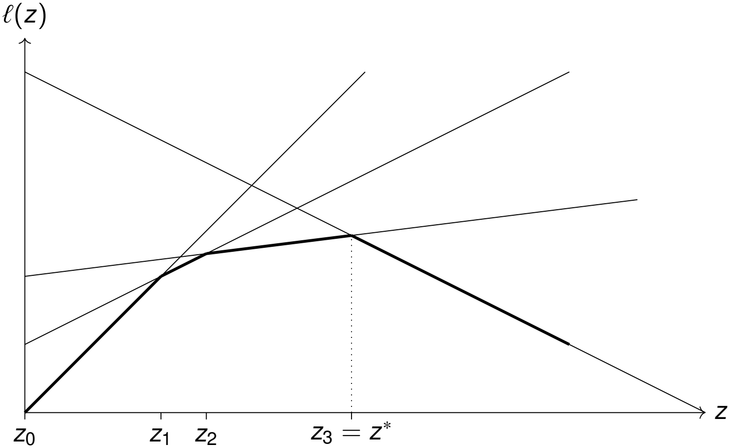

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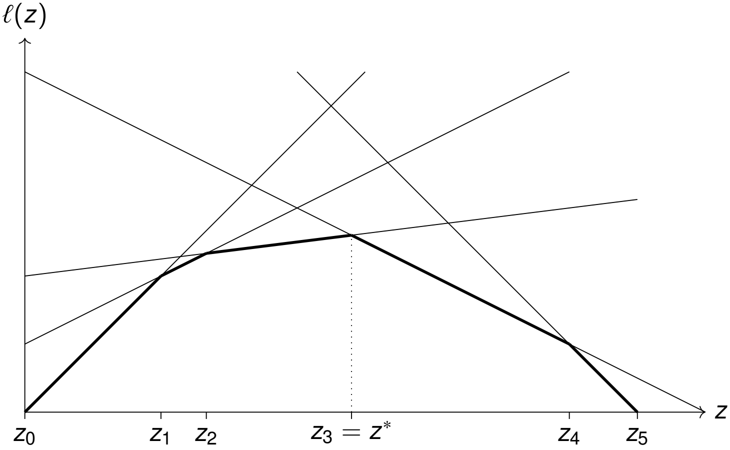

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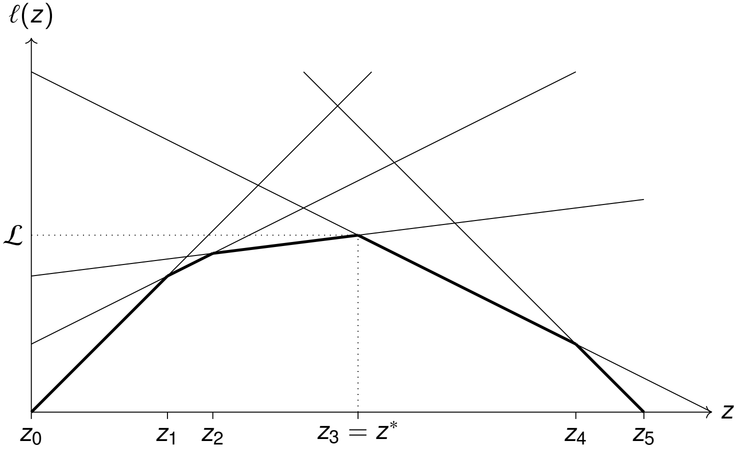

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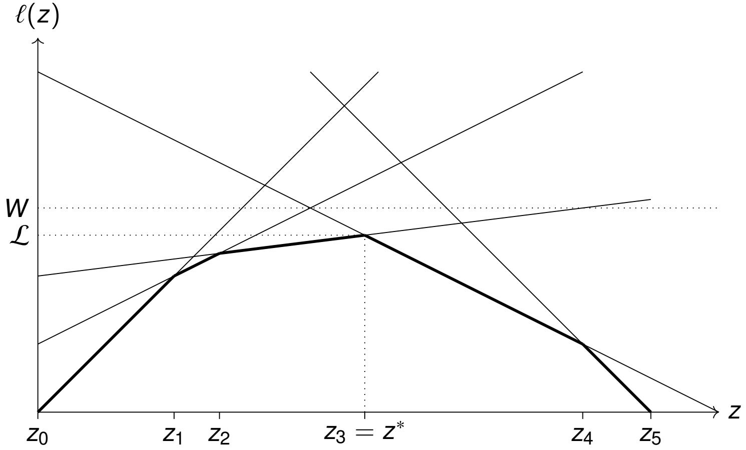

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Define $O_z \subseteq S$ as set of all MSTs for cost function c_z.

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Lemma

Let $T, T' \in O_z$. Then there exists a sequence

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in O_z such that T_i, T_{i+1} differ by a single edge swap, $i \in \{0, \ldots, k\}$.

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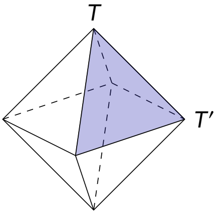

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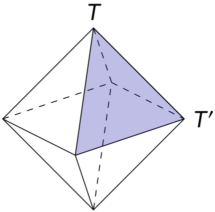

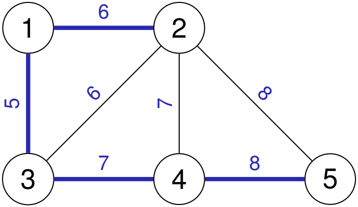

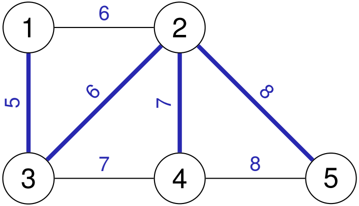

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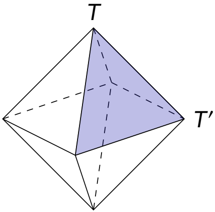

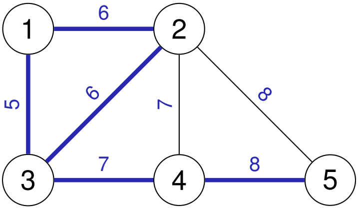

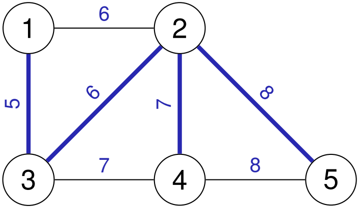

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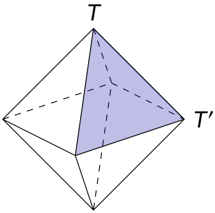

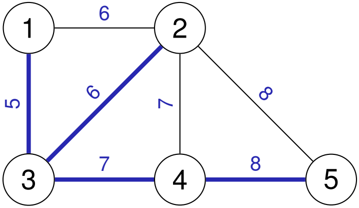

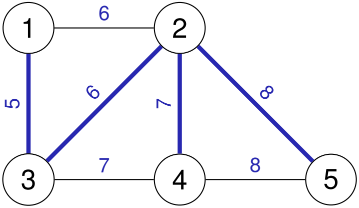

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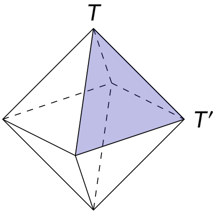

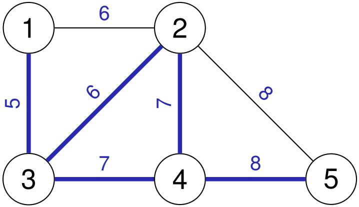

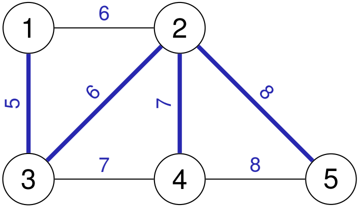

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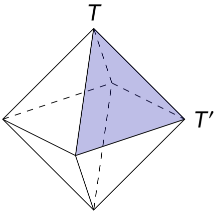

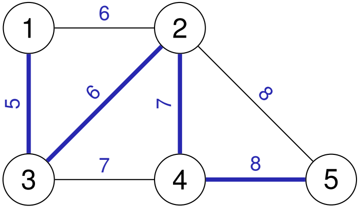

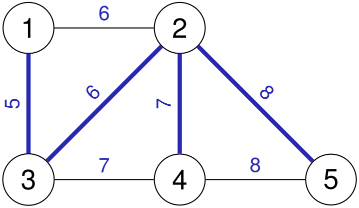

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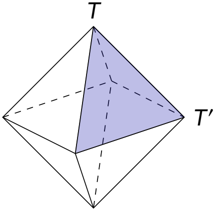

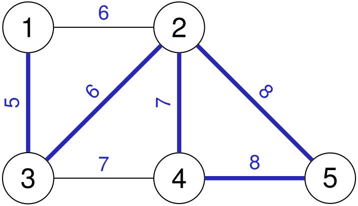

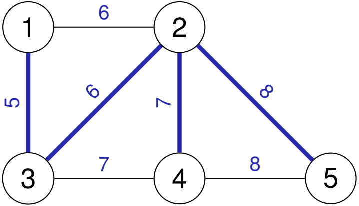

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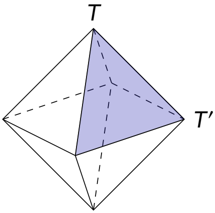

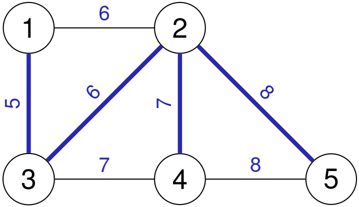

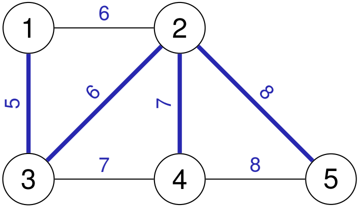

Existence of $(1, 2)$-Approximation

Theorem

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

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and $w(T) \leq \mathcal{L} \leq W \iff l(T) \geq L$.

Existence of $(1, 2)$-Approximation

Theorem

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq L + l_{\max}.$$

Proof

Let $T \in O_{z^*}$. Then

$$\begin{aligned} w(T) &= w(T) + z^*(l(T) - L) - z^*(l(T) - L) \\ &= \ell(z^*) - z^*(l(T) - L) = \mathcal{L} - z^*(\underbrace{l(T) - L}_{\geq 0}) \end{aligned}$$

and $w(T) \leq \mathcal{L} \leq W \iff l(T) \geq L$.

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

Existence of $(1, 2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

Existence of $(1, 2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

- There exists $T_{\leq} \in O_{z^*}$ s.t. $l(T_{\leq}) \leq L$:

Existence of $(1,2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

- There exists $T_{\leq} \in O_{z^*}$ s.t. $l(T_{\leq}) \leq L$:
  Choose $\epsilon > 0$ with $O_{z^*+\epsilon} \subseteq O_{z^*}$ (without proof) and $T_{\leq} \in O_{z^*+\epsilon}$:

Existence of $(1,2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

- There exists $T_\leq \in O_{z^*}$ s.t. $l(T_\leq) \leq L$:
  Choose $\epsilon > 0$ with $O_{z^*+\epsilon} \subseteq O_{z^*}$ (without proof) and $T_\leq \in O_{z^*+\epsilon}$:

$$\ell(z^* + \epsilon) = c_{z^*+\epsilon}(T_\leq) - (z^* + \epsilon)L \leq c_{z^*}(T_\leq) - z^* L = \ell(z^*)$$

Existence of $(1, 2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

- There exists $T_{\leq} \in O_{z^*}$ s.t. $l(T_{\leq}) \leq L$:
  Choose $\epsilon > 0$ with $O_{z^*+\epsilon} \subseteq O_{z^*}$ (without proof) and $T_{\leq} \in O_{z^*+\epsilon}$:

$$\ell(z^* + \epsilon) = c_{z^*+\epsilon}(T_{\leq}) - (z^* + \epsilon)L \leq c_{z^*}(T_{\leq}) - z^* L = \ell(z^*)$$
$$\Leftrightarrow w(T_{\leq}) + (z^* + \epsilon)(l(T_{\leq}) - L) \leq w(T_{\leq}) + z^*(l(T_{\leq}) - L)$$

Existence of $(1,2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

- There exists $T_{\leq} \in O_{z^*}$ s.t. $l(T_{\leq}) \leq L$:
  Choose $\epsilon > 0$ with $O_{z^*+\epsilon} \subseteq O_{z^*}$ (without proof) and $T_{\leq} \in O_{z^*+\epsilon}$:

$$\begin{aligned} \ell(z^* + \epsilon) = c_{z^*+\epsilon}(T_{\leq}) - (z^* + \epsilon)L \leq c_{z^*}(T_{\leq}) - z^* L = \ell(z^*) \\ \Leftrightarrow w(T_{\leq}) + (z^* + \epsilon)(l(T_{\leq}) - L) \leq w(T_{\leq}) + z^*(l(T_{\leq}) - L) \\ \Leftrightarrow \epsilon(l(T_{\leq}) - L) \leq 0 \Leftrightarrow l(T_{\leq}) \leq L. \end{aligned}$$

Existence of $(1,2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

- There exists $T_{\leq} \in O_{z^*}$ s.t. $l(T_{\leq}) \leq L$:
  Choose $\epsilon > 0$ with $O_{z^*+\epsilon} \subseteq O_{z^*}$ (without proof) and $T_{\leq} \in O_{z^*+\epsilon}$:

$$\begin{aligned}\ell(z^* + \epsilon) = c_{z^*+\epsilon}(T_{\leq}) - (z^* + \epsilon)L \leq c_{z^*}(T_{\leq}) - z^* L = \ell(z^*)\\ \Leftrightarrow w(T_{\leq}) + (z^* + \epsilon)(l(T_{\leq}) - L) \leq w(T_{\leq}) + z^*(l(T_{\leq}) - L)\\ \Leftrightarrow \epsilon(l(T_{\leq}) - L) \leq 0 \Leftrightarrow l(T_{\leq}) \leq L.\end{aligned}$$

- There exists $T_{\geq} \in O_{z^*}$ s.t. $l(T_{\geq}) \geq L$: Analogous proof.

Existence of $(1,2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

- There exists $T_\leq \in O_{z^*}$ s.t. $l(T_\leq) \leq L$:
  Choose $\epsilon > 0$ with $O_{z^*+\epsilon} \subseteq O_{z^*}$ (without proof) and $T_\leq \in O_{z^*+\epsilon}$:

$$\begin{aligned} \ell(z^* + \epsilon) = c_{z^*+\epsilon}(T_\leq) - (z^* + \epsilon)L \leq c_{z^*}(T_\leq) - z^*L = \ell(z^*) \\ \Leftrightarrow w(T_\leq) + (z^* + \epsilon)(l(T_\leq) - L) \leq w(T_\leq) + z^*(l(T_\leq) - L) \\ \Leftrightarrow \epsilon(l(T_\leq) - L) \leq 0 \Leftrightarrow l(T_\leq) \leq L. \end{aligned}$$

- There exists $T_\geq \in O_{z^*}$ s.t. $l(T_\geq) \geq L$: Analogous proof.
- Lemma: There exists sequence $T_\leq = T_0, T_1, \ldots, T_k, T_{k+1} = T_\geq$ in O_{z^*}.

Existence of $(1,2)$-Approximation

Proof (continued)

$\Rightarrow$ Show that there exists $T \in O_{z^*}$ with $L \leq l(T) \leq L + l_{\max}$.

- There exists $T_{\leq} \in O_{z^*}$ s.t. $l(T_{\leq}) \leq L$:
  Choose $\epsilon > 0$ with $O_{z^*+\epsilon} \subseteq O_{z^*}$ (without proof) and $T_{\leq} \in O_{z^*+\epsilon}$:

$$\ell(z^* + \epsilon) = c_{z^*+\epsilon}(T_{\leq}) - (z^* + \epsilon)L \leq c_{z^*}(T_{\leq}) - z^*L = \ell(z^*)$$
$$\Leftrightarrow w(T_{\leq}) + (z^* + \epsilon)(l(T_{\leq}) - L) \leq w(T_{\leq}) + z^*(l(T_{\leq}) - L)$$
$$\Leftrightarrow \epsilon(l(T_{\leq}) - L) \leq 0 \Leftrightarrow l(T_{\leq}) \leq L.$$

- There exists $T_{\geq} \in O_{z^*}$ s.t. $l(T_{\geq}) \geq L$: Analogous proof.
- Lemma: There exists sequence $T_{\leq} = T_0, T_1, \ldots, T_k, T_{k+1} = T_{\geq}$ in O_{z^*}.

$\Rightarrow$ Show that there exists element T_{i+1} such that

$$L \leq l(T_{i+1}) \leq L + l_{\max}.$$

Existence of $(1, 2)$-Approximation

Theorem

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

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Existence of $(1, 2)$-Approximation

Theorem

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq L + l_{\max}.$$

Proof (continued)

We have $l(T_{\leq}) \leq L$, $T_{\leq} = T_0, T_1, \ldots, T_k, T_{k+1} = T_{\geq}$ and $l(T_{\geq}) \geq L$.

Existence of $(1, 2)$-Approximation

Theorem

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq L + l_{\max}.$$

Proof (continued)

We have $l(T_{\leq}) \leq L$, $T_{\leq} = T_0, T_1, \ldots, T_k, T_{k+1} = T_{\geq}$ and $l(T_{\geq}) \geq L$.
If $l(T_i) \leq L$ and $l(T_{i+1}) \geq L$ and $T_{i+1} = T_i - e + e'$ for $e, e' \in E$, then

$$l(T_{i+1}) = l(T_i - e + e') = l(T_i) - l_e + l_{e'} \leq L + l_{\max}.$$

$\square$

Existence of $(1, 2)$-Approximation

Corollary

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq 2L.$$

Existence of $(1, 2)$-Approximation

Corollary

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq 2L.$$

$\Rightarrow$ Such a spanning tree $T \in O_{z^*}$ is a $(1, 2)$-approximation.

Existence of $(1, 2)$-Approximation

Corollary

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq 2L.$$

$\Rightarrow$ Such a spanning tree $T \in O_{z^*}$ is a $(1, 2)$-approximation.
For a fixed $\epsilon > 0$, $\epsilon \in \mathbb{R}_{\geq 0}$, restrict edges to subset

$$E' := \{e \in E \mid l_e \leq \epsilon L\} \subseteq E.$$

Existence of $(1, 2)$-Approximation

Corollary

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

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$\Rightarrow$ Such a spanning tree $T \in O_{z^*}$ is a $(1, 2)$-approximation.
For a fixed $\epsilon > 0$, $\epsilon \in \mathbb{R}_{\geq 0}$, restrict edges to subset

$$E' := \{e \in E \mid l_e \leq \epsilon L\} \subseteq E.$$

Corollary

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq (1 + \epsilon)L.$$

Existence of $(1, 2)$-Approximation

Corollary

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq 2L.$$

$\Rightarrow$ Such a spanning tree $T \in O_{z^*}$ is a $(1, 2)$-approximation.
For a fixed $\epsilon > 0$, $\epsilon \in \mathbb{R}_{\geq 0}$, restrict edges to subset

$$E' := \{e \in E \mid l_e \leq \epsilon L\} \subseteq E.$$

Corollary

Let $z^* \in \mathbb{R}_{\geq 0}$ be optimal s.t. $\mathcal{L} = \ell(z^*)$. There exists $T \in O_{z^*}$ with

$$w(T) \leq W \quad \text{and} \quad l(T) \leq (1 + \epsilon)L.$$

$\Rightarrow$ We can find a $(1, 1 + \epsilon)$-approximation.

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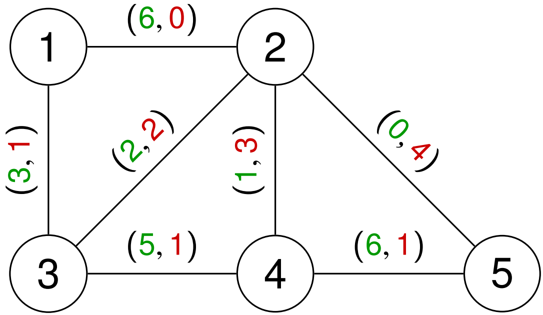
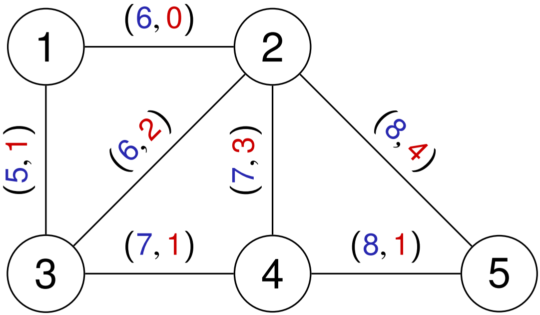
Optimum Spanning Tree with Minimum Length

$L = 8 \qquad z = 2 \qquad c_z = w_e + z l_e$



Optimum Spanning Tree with Minimum Length

$L = 8 \qquad z = 2 \qquad c_z = w_e + z l_e$



Optimum Spanning Tree with Minimum Length

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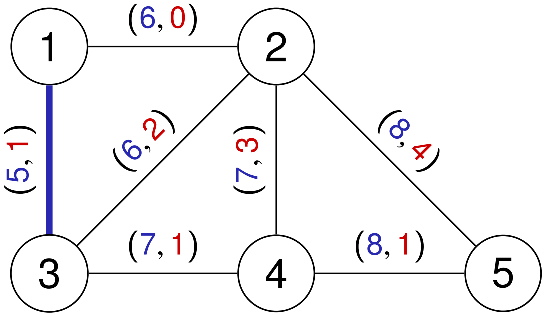
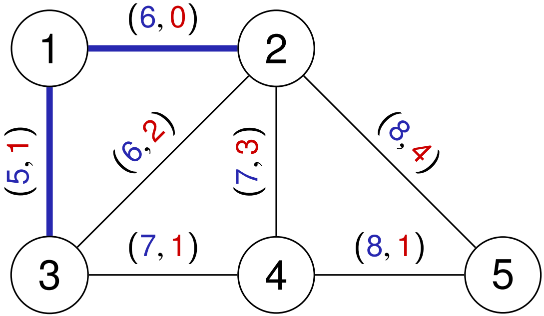


$(5, 1)$

Optimum Spanning Tree with Minimum Length

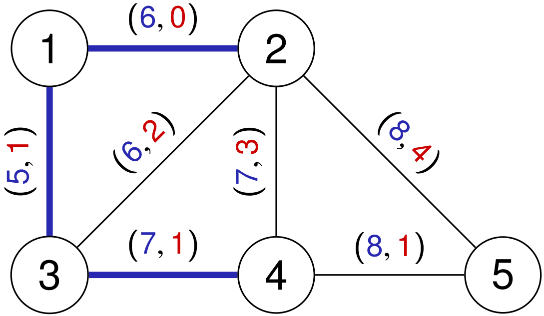
$L = 8 \qquad z = 2 \qquad c_z = w_e + z l_e$



$(5, 1) \prec_{\min} (6, 0) \prec_{\min} (6, 2)$

Optimum Spanning Tree with Minimum Length

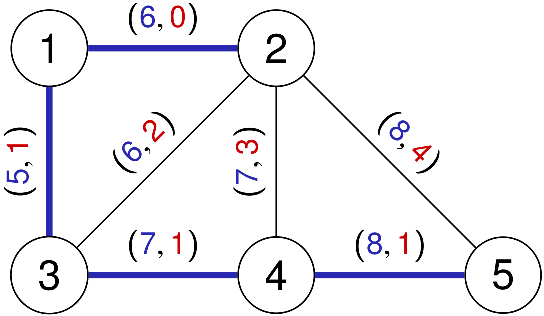
$L = 8 \qquad z = 2 \qquad c_z = w_e + z l_e$



$(5, 1) \prec_{\min} (6, 0) \prec_{\min} (6, 2) \prec_{\min} (7, 1) \prec_{\min} (7, 3)$

Optimum Spanning Tree with Minimum Length

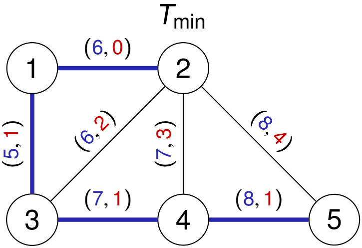
$L = 8 \qquad z = 2 \qquad c_z = w_e + z l_e$

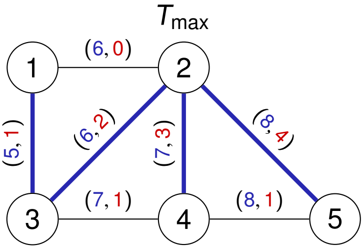


$(5,1) \prec_{\min} (6,0) \prec_{\min} (6,2) \prec_{\min} (7,1) \prec_{\min} (7,3) \prec_{\min} (8,1) \prec_{\min} (8,4)$

Lexicographic Order

$L = 8 \qquad z = 2$



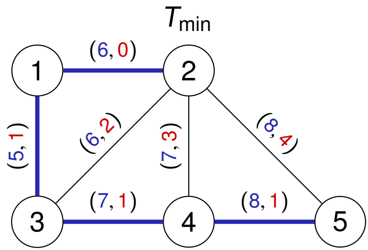


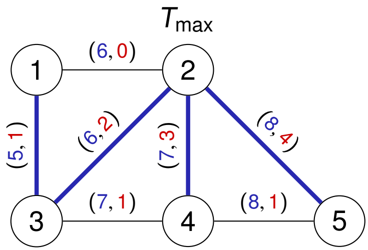
$(6, 0) \prec_{\min} (6, 2)$

$(6, 2) \prec_{\max} (6, 0)$

Lexicographic Order

$L = 8 \qquad z = 2$





$(6, 0) \prec_{\min} (6, 2)$

$(6, 2) \prec_{\max} (6, 0)$

$(c, l) \prec_{\min} (c', l')$
if $c < c'$
or $c = c'$ and $l < l'$

$(c, l) \prec_{\max} (c', l')$
if $c < c'$
or $c = c'$ and $l > l'$

Check Lagrange Multiplier z

Listing 1: check(z).

```
Input: Graph G, z ∈ ℝ≥0 and
   cost functions c_z(e), l_e : E → ℝ≥0
Output: Two spanning trees T_min and T_max
   and a valuation of z with respect to z*

1. Calculate tuples (c_z(e), l_e) for e ∈ E

2. Sort tuples based on order ≺_min
3. Find MST T_min via Prim's algorithm
4. Sort tuples based on order ≺_max
5. Find MST T_max via Prim's algorithm

6. If l(T_min) > L, then z < z*,
   if l(T_max) < L, then z > z*,
   otherwise z is an acceptable value
```

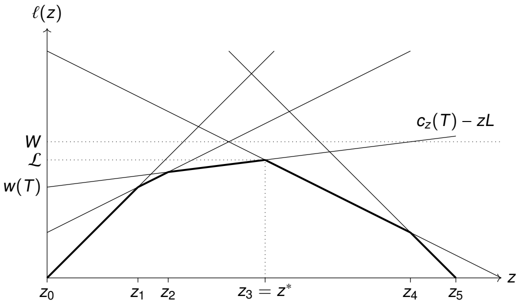
Check Lagrange Multiplier z

Listing 2: check(z).

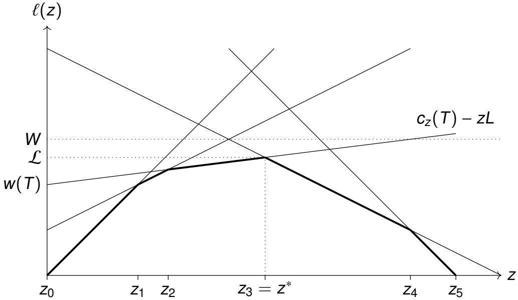
```
Input: Graph G, z ∈ ℝ≥0 and
   cost functions c_z(e), l_e : E → ℝ≥0
Output: Two spanning trees T_min and T_max
   and a valuation of z with respect to z*

1. Calculate tuples (c_z(e), l_e) for e ∈ E

2. Sort tuples based on order ≺_min
3. Find MST T_min via Prim's algorithm
4. Sort tuples based on order ≺_max
5. Find MST T_max via Prim's algorithm

6. If l(T_min) > L, then z < z*,
   if l(T_max) < L, then z > z*,
   otherwise z is an acceptable value
```

⇒ Runtime for $n = |V|$, $m = |E|$ is $O(m + n\log(n))$ [CLRS01].

Naïve Approach

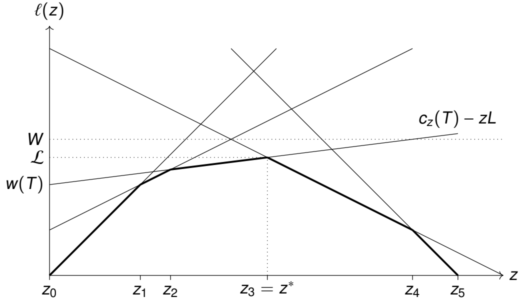


Naïve Approach



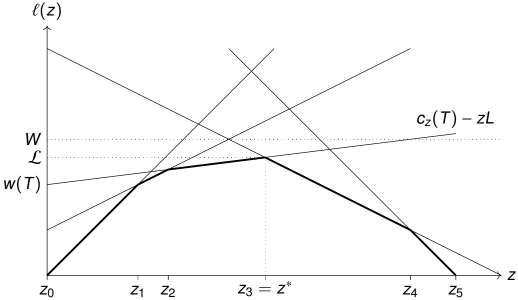
- $Z := \{z \in \mathbb{R}_{\geq 0} \mid \exists e, e' \in E \text{ s.t. } c_z(e) = c_z(e')\}$ with $|Z| \approx m^2$

Naïve Approach



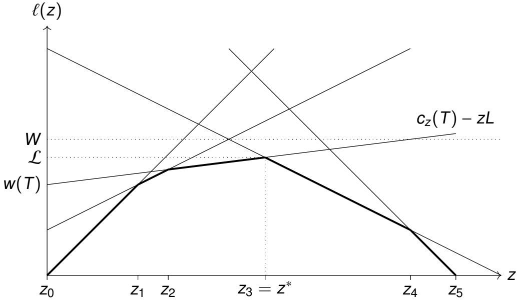
- $Z := \{z \in \mathbb{R}_{\geq 0} \mid \exists e, e' \in E \text{ s.t. } c_z(e) = c_z(e')\}$ with $|Z| \approx m^2$
- Sort in $O(m^2 \log(m^2)) = O(m^2 \log(m))$

Naïve Approach



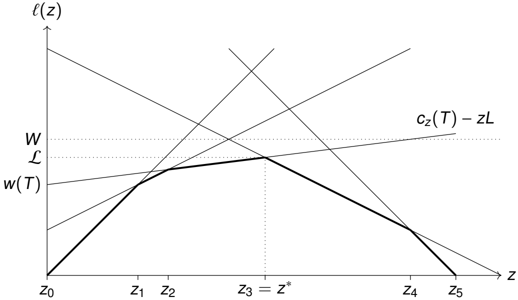
- $Z := \{z \in \mathbb{R}_{\geq 0} \mid \exists e, e' \in E \text{ s.t. } c_z(e) = c_z(e')\}$ with $|Z| \approx m^2$
- Sort in $O(m^2 \log(m^2)) = O(m^2 \log(m))$
- Binary search with $\texttt{check(z)}$ in $O(\log(m^2)(m + n\log(n))) = O(\log(m)(m + n\log(n)))$

Naïve Approach



- $Z := \{z \in \mathbb{R}_{\geq 0} \mid \exists e, e' \in E \text{ s.t. } c_z(e) = c_z(e')\}$ with $|Z| \approx m^2$
- Sort in $O(m^2 \log(m^2)) = O(m^2 \log(m))$
- Binary search with $\texttt{check}(z)$ in $O(\log(m^2)(m + n\log(n))) = O(\log(m)(m + n\log(n)))$
- Compute sequence in $O(n\log(n))$ [ST83]

Naïve Approach



- $Z := \{z \in \mathbb{R}_{\geq 0} \mid \exists e, e' \in E \text{ s.t. } c_z(e) = c_z(e')\}$ with $|Z| \approx m^2$
- Sort in $O(m^2 \log(m^2)) = O(m^2 \log(m))$
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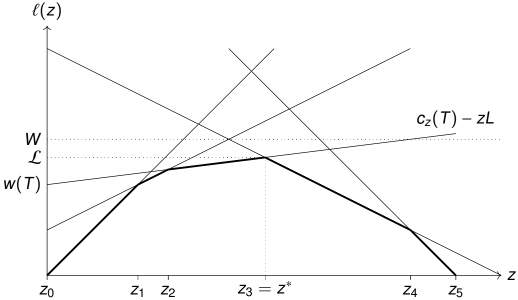
- Compute sequence in $O(n\log(n))$ [ST83]

$\Rightarrow$ Runtime $O(m^2 \log(m) + \log(m)(m + n\log(n)) + n\log(n))$

Naïve Approach

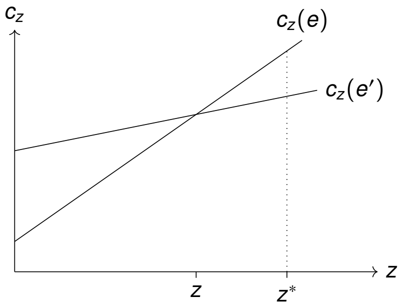


- $Z := \{z \in \mathbb{R}_{\geq 0} \mid \exists e, e' \in E \text{ s.t. } c_z(e) = c_z(e')\}$ with $|Z| \approx m^2$
- Sort in $O(m^2 \log(m^2)) = O(m^2 \log(m))$
- Binary search with $\texttt{check}(\texttt{z})$ in $O(\log(m^2)(m + n\log(n))) = O(\log(m)(m + n\log(n)))$
- Compute sequence in $O(n\log(n))$ [ST83]

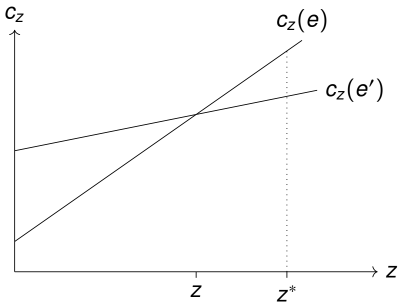
$\Rightarrow$ Runtime $O(m^2\log(m) + \log(m)(m + n\log(n)) + n\log(n))$

$$= O(m^2\log(m) + n\log(m)\log(n))$$

Compare Edges



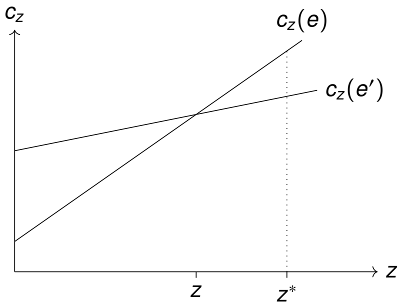
Compare Edges



Let $l_e > l_{e'}$. If the algorithm $\texttt{check}(z)$ returns

1. $z < z^*$, then $c_{z^*}(e) > c_{z^*}(e')$,
2. $z > z^*$, then $c_{z^*}(e) < c_{z^*}(e')$, and
3. z is accepted, then $c_{z^*}(e) = c_{z^*}(e')$.

Compare Edges



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2. $z > z^*$, then $c_{z^*}(e) < c_{z^*}(e')$, and
3. z is accepted, then $c_{z^*}(e) = c_{z^*}(e')$.

$\Rightarrow$ Sort edges in E by cost c_{z^*} without knowing z^*.

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Preparata's Algorithm for Parallel Sorting

m elements

3	4	8	2	6	1	$\cdots$	4	1	9	2	3	5

Preparata's Algorithm for Parallel Sorting

$m\log(m)$ processors

$\frac{m}{m\log(m)}$ elements

m elements

3	4	8	2	6	1	$\cdots$	4	1	9	2	3	5

Preparata's Algorithm for Parallel Sorting

$m \log(m)$ processors

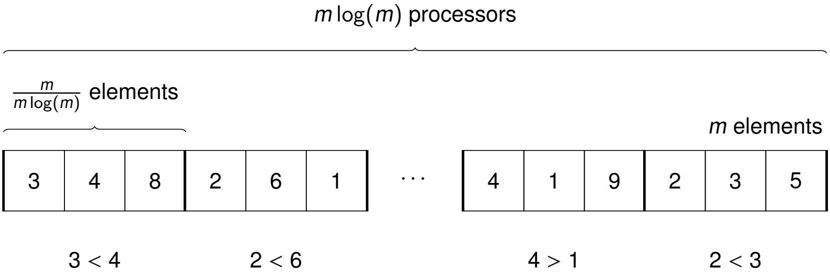
$\frac{m}{m \log(m)}$ elements

m elements

3	4	8	2	6	1	...	4	1	9	2	3	5

3 ? 4   2 ? 6   4 ? 1   2 ? 3

Preparata's Algorithm for Parallel Sorting



Preparata's Algorithm for Parallel Sorting

$m\log(m)$ processors

$\frac{m}{m\log(m)}$ elements

m elements

3	4	8	2	6	1	$\cdots$	4	1	9	2	3	5

$3 < 4$ $\quad$ $2 < 6$ $\quad$ $4 > 1$ $\quad$ $2 < 3$

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

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Preparata's Algorithm for Parallel Sorting

$$\overbrace{\underbrace{\boxed{3}\,\boxed{4}\,\boxed{8}}_{}^{\frac{m}{m\log(m)} \text{ elements}}\,\boxed{2}\,\boxed{6}\,\boxed{1}\ \cdots\ \boxed{4}\,\boxed{1}\,\boxed{9}\,\boxed{2}\,\boxed{3}\,\boxed{5}}^{m\log(m) \text{ processors}}$$

m elements

3	4	8	2	6	1	⋯	4	1	9	2	3	5

$3 < 4$ $\quad 2 < 6$ $\quad 4 > 1$ $\quad 2 < 3$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$\log(m)$ steps $\quad \Rightarrow m \cdot \log(m)^2$ serial steps

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Megiddo's Algorithm

$m\log(m)$ processors

$\overbrace{e_1 \quad e_2 \quad e_3}^{\frac{m}{m\log(m)} \text{ elements}}$

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

Megiddo's Algorithm

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

Megiddo's Algorithm

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

e_1 ? e_2 $\quad$ e_4 ? e_5 $\quad$ e_{m-5} ? e_{m-4} $\quad$ e_{m-2} ? e_{m-1}

Megiddo's Algorithm

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

e_1 ? e_2 e_4 ? e_5 e_{m-5} ? e_{m-4} e_{m-2} ? e_{m-1}

$z_{1,2}$ $z_{4,5}$ $z_{m-5,m-4}$ $z_{m-2,m-1}$

Megiddo's Algorithm

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

$e_1 \ ? \ e_2$ $\quad$ $e_4 \ ? \ e_5$ $\quad$ $e_{m-5} \ ? \ e_{m-4}$ $\quad$ $e_{m-2} \ ? \ e_{m-1}$

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sort breakpoints

Megiddo's Algorithm

m elements

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Megiddo's Algorithm

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

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$$z_{m-5,m-4} \leq z_{1,2} \leq \cdots \leq z_{4,5} \leq z_{m-2,m-1}$$

$$\uparrow$$

$$z^*$$

Megiddo's Algorithm

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

$e_1 \ ? \ e_2$ $\quad$ $e_4 \ ? \ e_5$ $\quad$ $e_{m-5} \ ? \ e_{m-4}$ $\quad$ $e_{m-2} \ ? \ e_{m-1}$

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m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

$e_1 \; ? \; e_2$ $\quad$ $e_4 \; ? \; e_5$ $\quad$ $e_{m-5} \; ? \; e_{m-4}$ $\quad$ $e_{m-2} \; ? \; e_{m-1}$

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$e_1 < e_2$ $\quad$ $e_4 > e_5$ $\quad$ $e_{m-5} > e_{m-4}$ $\quad$ $e_{m-2} < e_{m-1}$

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z^*

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Since $z^* \in Z$:

- $\Rightarrow$ Eventually we compare $e, e' \in E$ with $z_{e,e'} = z^*$.
- $\Rightarrow$ $\mathtt{check}(z_{e,e'})$ returns $z_{e,e'}$ is optimal, and T_{min} and T_{max}.
- $\Rightarrow$ Swapping edges, we obtain spanning tree $T \subseteq E$ such that T is a $(1, 2)$-approximation.

Runtime

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

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sort breakpoints

- Sort breakpoints in $O(m\log(m)\cdot\log(m\log(m))) = O(m\log(m)^2)$

Runtime

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

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Runtime

m elements

e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

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$$O(m\log(m)^3 + \log(m)^2(m+n\log(n)))$$

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- Compute sequence in $O(n \log(n))$ [ST83]

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e_1	e_2	e_3	e_4	e_5	e_6	$\cdots$	e_{m-5}	e_{m-4}	e_{m-3}	e_{m-2}	e_{m-1}	e_m

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- Compute sequence in $O(n\log(n))$ [ST83]

$\Rightarrow$ Total runtime is $O(m\log(m)^3 + \log(m)^2(m + n\log(n)) + n\log(n)) =$

$$O(m\log(m)^3 + n\log(m)^2\log(n)))$$

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Conclusion

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Thank you for your attention!

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