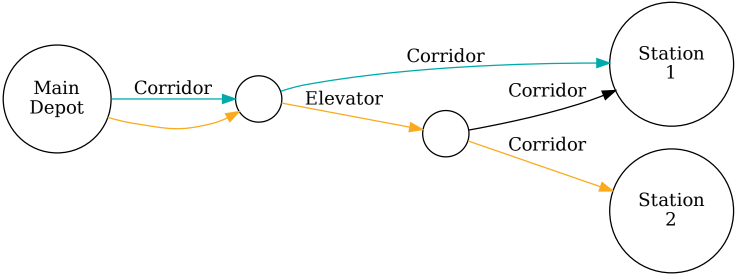
Generalized Temporally Repeated Flows for the Quickest Transshipment and Related Problems

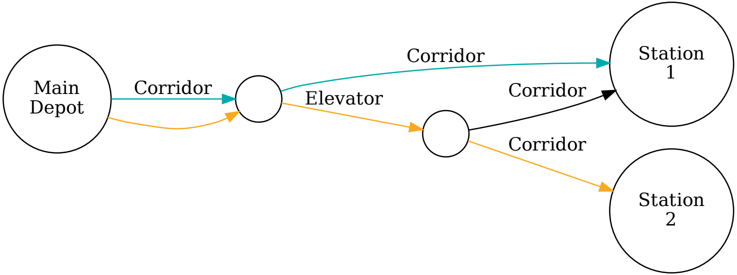
Emma Ahrens

Supervised by Prof. Dr. Christina Büsing

July 18, 2025



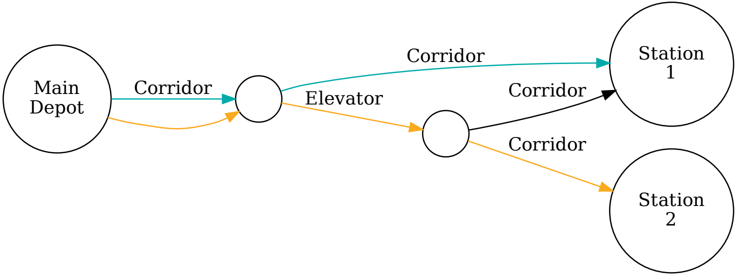
- Transportation of hospital beds during the day blocks elevators for time-critical processes



- Transportation of hospital beds during the day blocks elevators for time-critical processes

$\Rightarrow$ Distribution after the main working hours? What would be important?

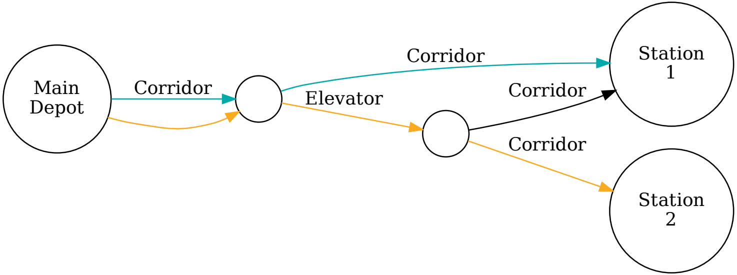
Motivation: Transportation of Beds in a Hospital



- Transportation of hospital beds during the day blocks elevators for time-critical processes

$\Rightarrow$ Distribution after the main working hours? What would be important?

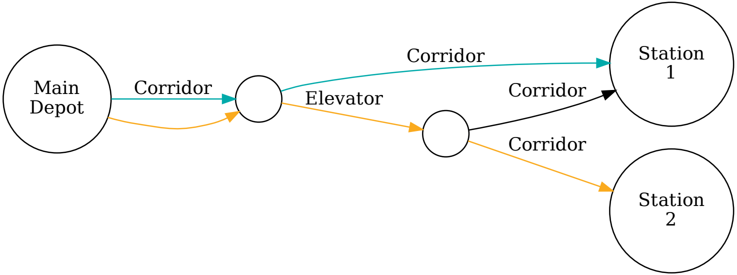
- As quick as possible to relieve the employees



- Transportation of hospital beds during the day blocks elevators for time-critical processes

⇒ Distribution after the main working hours? What would be important?

- As quick as possible to relieve the employees
- As easy to remember as possible to reduce confusion



- Transportation of hospital beds during the day blocks elevators for time-critical processes

⇒ Distribution after the main working hours? What would be important?

- As quick as possible to relieve the employees
- As easy to remember as possible to reduce confusion
- As few employees involved as possible

Contents

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Definition

For

- a directed graph $G = (V, A)$,
- a *capacity* function $u : A \to \mathbb{N}_0$,
- a *transit time* function $\tau : A \to \mathbb{N}_0$, and
- a *cost* function $c : A \to \mathbb{N}_0$,

(G,u), (G,u,c), (G,u,τ) and (G,u,τ,c) are *networks*.

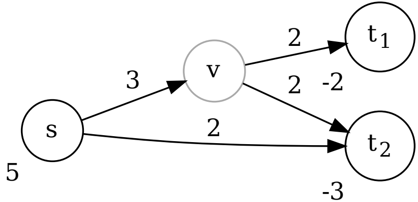

Figure: Network with labels u and b.

Definition

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(G, u), (G, u, c), (G, u, τ) and (G, u, τ, c) are *networks*.

Definition

A function $b : V \to \mathbb{Z}$ with $\sum_{v \in V} b(v) = 0$ is a *balance* function.

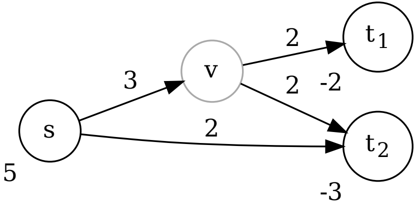


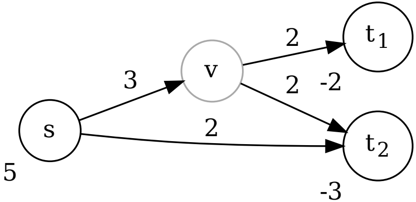
Figure: Network with labels u and b.

Definition

For

- a network (G, u), and
- a balance function b,

a *static b-flow* is a function $x : A \to \mathbb{R}_{\geq 0}$, which satisfies



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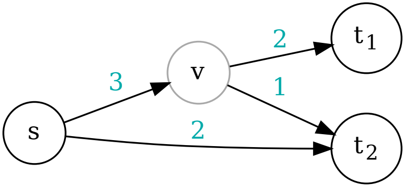

Figure: Flow represented by labels x on arcs.

Definition

For

- a network (G, u), and
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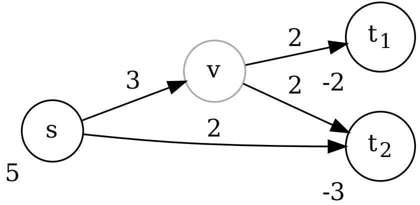
a *static b-flow* is a function $x : A \to \mathbb{R}_{\geq 0}$, which satisfies

1. the *capacity constraint* $0 \leq x(a) \leq u(a)$ for all $a \in A$, and



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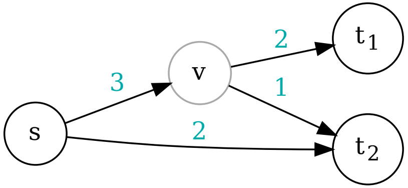

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Definition

For

- a network (G, u), and
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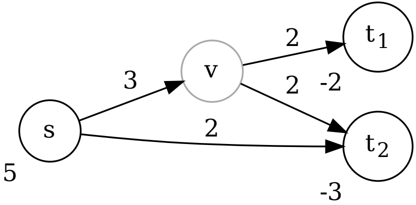
a *static* b*-flow* is a function $x : A \to \mathbb{R}_{\geq 0}$, which satisfies

2. the *flow conservation*

$$\sum_{a \in \delta^-(v)} x(a) - \sum_{a \in \delta^+(v)} x(a) = -b(v)$$

for all $v \in V$.



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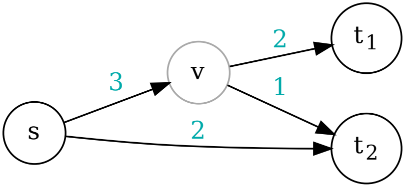

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Definition

For

- a network (G, u, τ),
- a time horizon $T \in \mathbb{N}_0$, and
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a b-*flow over time* is a family of functions $f_a : \mathbb{Z} \to \mathbb{R}_{\geq 0}$, $a \in A$, which satisfy

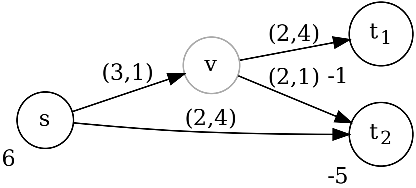

Figure: Network with labels representing (u, τ) and b.

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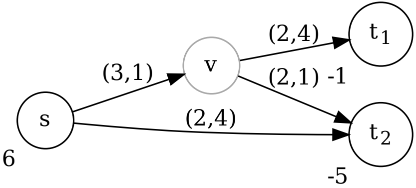

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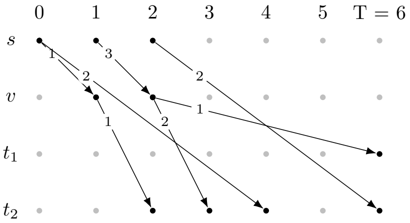

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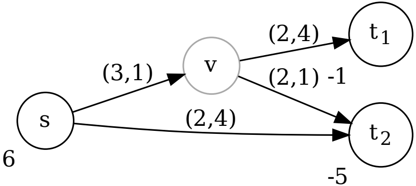
a *b-flow over time* is a family of functions $f_a : \mathbb{Z} \to \mathbb{R}_{\geq 0}$, $a \in A$, which satisfy

1. the *capacity constraint* $0 \leq f_a(\theta) \leq u(a)$ for all $a \in A$ and $\theta \in \{0, \ldots, T\}$,



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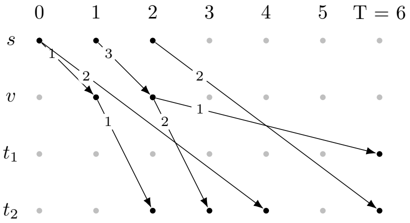

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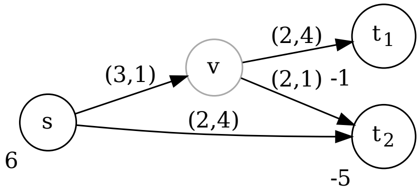
2. the *flow completion* $f_a(\theta) = 0$ for all $a \in A$ and $\theta > T - \tau_a$,



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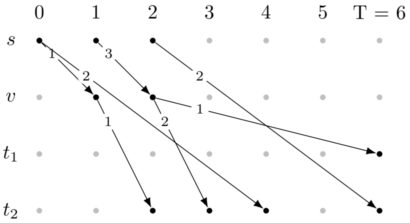

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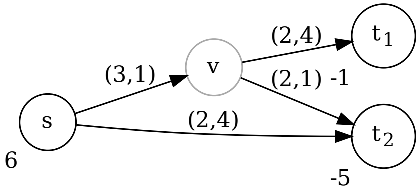
a *b-flow over time* is a family of functions $f_a : \mathbb{Z} \to \mathbb{R}_{\geq 0}$, $a \in A$, which satisfy

3. the *weak flow conservation* for all $v \in V \setminus \{s\}$ and $\theta \in \{0,\dots,T\}$

$$\sum_{a\in\delta^-(v)} \sum_{\xi=0}^{\theta-\tau_a} f_a(\xi) - \sum_{a\in\delta^+(v)} \sum_{\xi=0}^{\theta} f_a(\xi) \geq 0,$$



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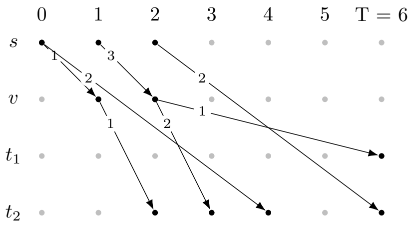

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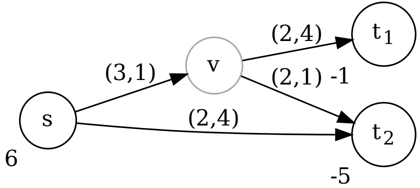
a *b-flow over time* is a family of functions $f_a : \mathbb{Z} \to \mathbb{R}_{\geq 0}$, $a \in A$, which satisfy

4. the *strict flow conservation* for all $v \in V$

$$\sum_{a \in \delta^-(v)} \sum_{\xi=0}^{T-\tau_a} f_a(\xi) - \sum_{a \in \delta^+(v)} \sum_{\xi=0}^{T} f_a(\xi) = -b(v),$$



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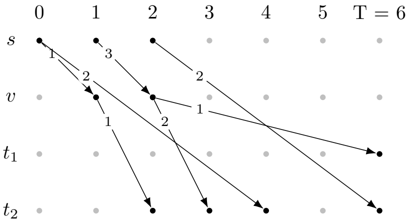

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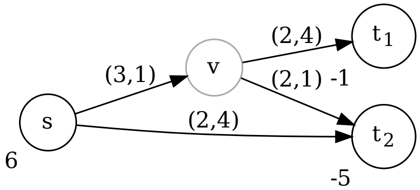
5. and $f_a(\theta) = 0$ for $\theta \notin \{0,\dots,T\}$.



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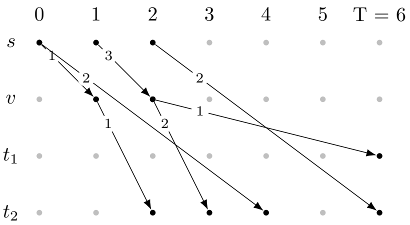

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Definition

For

- a network (G, u, τ),
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- a static flow x over (G, u) with path decomposition $y : P \to \mathbb{R}_{\geq 0}$ and $t := \max_{p \in P} \tau(p)$,

the associated *uniform flow* f is

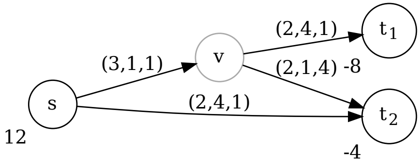

Figure: Network with labels representing (u, τ, c) and b.

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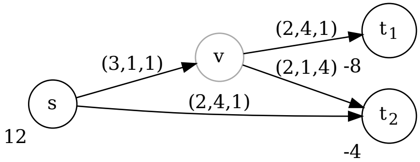

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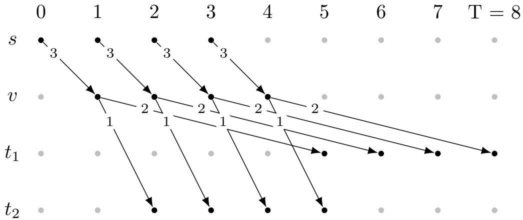

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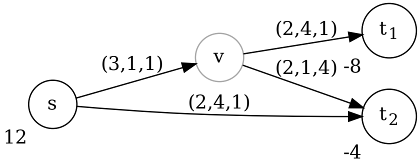
$$f_a(\theta) := \sum_{p\in P_a(\theta)} y(p) \qquad \forall a \in A,\ \theta \in \{0,\ldots,T\},$$

where
$P_a(\theta) := \{p \in P \mid a \in p,\ 0 \leq \theta - \tau(p_{[s,v]}) \leq T - t\}$
for $a = (v,w)$ and $p = (s,\ldots,t_i)$,
and $f_a(\theta) := 0$ for $\theta \notin \{0,\ldots,T\}$.



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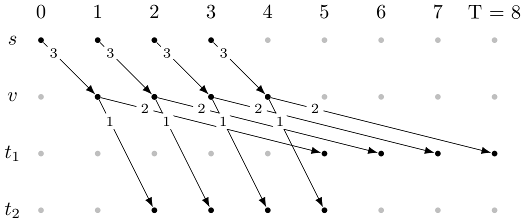

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the assoc. *temporally repeated flow* f is

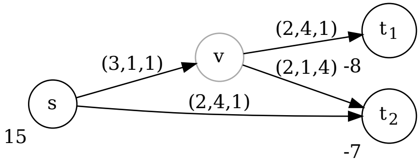

Figure: Network with labels representing (u,τ,c) and b.

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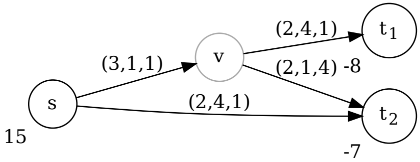

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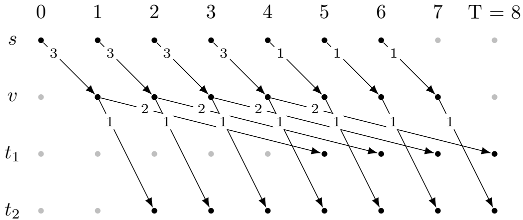

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Temporally Repeated Flows

Definition

For

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$$f_a(\theta) := \sum_{p \in P_a(\theta)} y(p) \qquad \forall a \in A,\ \theta \in \{0,\dots,T\},$$

where $P_a(\theta) := \{p \in P \mid a \in p,\ \tau(p_{[s,v]}) \leq \theta,$ $\tau(p_{[w,t]}) \leq T - \theta\}$
for $a = (v,w)$ and $p = (s,\dots,t_i)$,
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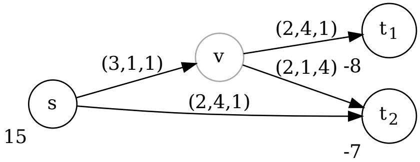

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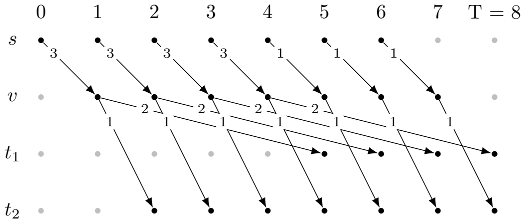

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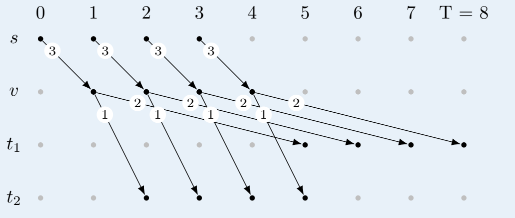
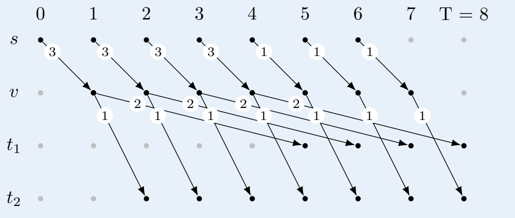
Uniform Flow



Temporally Repeated Flow



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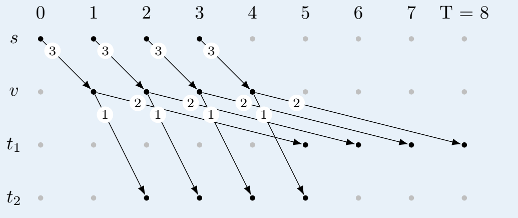
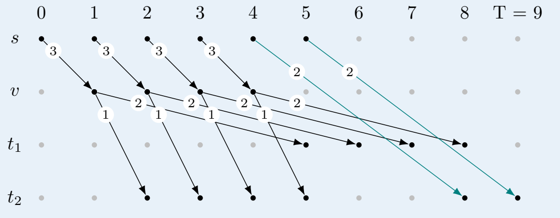
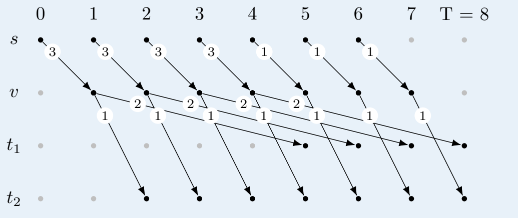


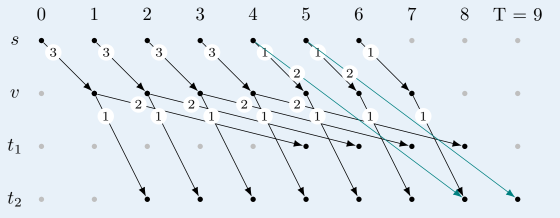
k-Uniform Flow for $k = 2$



Temporally Repeated Flow



k-Temporally Repeated Flow for $k = 2$



- Max Flow Problem

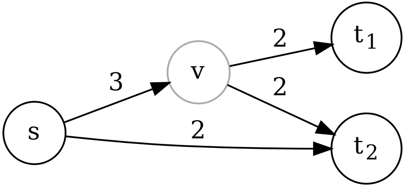


Figure: Network with labels representing u.

- Max Flow Problem
- Min Cost Flow Problem

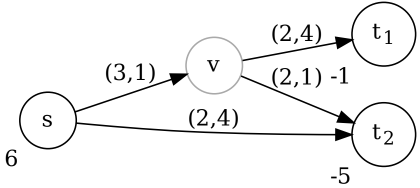


Figure: Network with labels representing (u,c) and b.

- Max Flow Problem
- Min Cost Flow Problem
- Max Flow over Time Problem

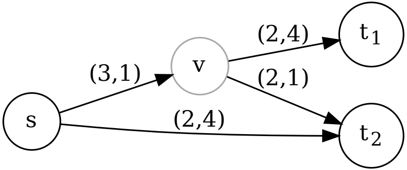


Figure: Network with labels representing (u, τ).

- Max Flow Problem
- Min Cost Flow Problem
- Max Flow over Time Problem
- Min Cost Flow over Time Problem

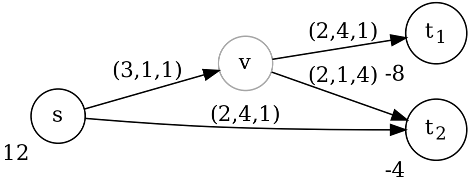


Figure: Network with labels representing (u, τ, c) and b.

- Max Flow Problem
- Min Cost Flow Problem
- Max Flow over Time Problem
- Min Cost Flow over Time Problem
- Earliest Arrival Flow over Time Problem

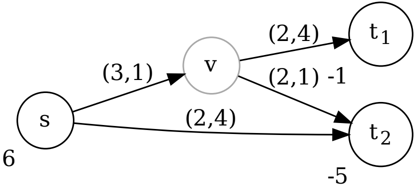


Figure: Network with labels representing (u, τ) and b.

- Max Flow Problem
- Min Cost Flow Problem
- Max Flow over Time Problem
- Min Cost Flow over Time Problem
- Earliest Arrival Flow over Time Problem
- Quickest Transshipment Problem

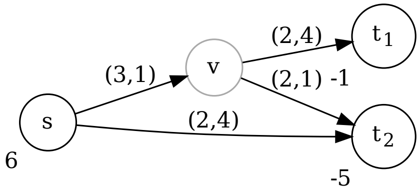


Figure: Network with labels representing (u, τ) and b.

- Max Flow Problem
- Min Cost Flow Problem
- Max Flow over Time Problem
- Min Cost Flow over Time Problem
- Earliest Arrival Flow over Time Problem
- Quickest Transshipment Problem
- Min Cost Uniform Flow Problem

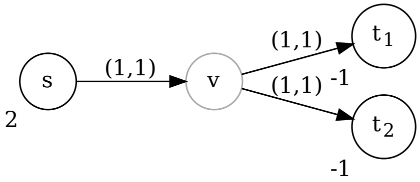


Figure: Network with labels representing (u, τ) and b.

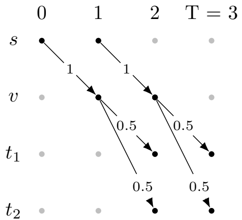


Figure: Labels represent loads on arcs.

- Max Flow Problem
- Min Cost Flow Problem
- Max Flow over Time Problem
- Min Cost Flow over Time Problem
- Earliest Arrival Flow over Time Problem
- Quickest Transshipment Problem
- Min Cost Uniform Flow Problem
- Quickest Transshipment Problem for Uniform Flows

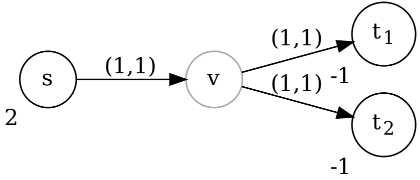


Figure: Network with labels representing (u, τ) and b.

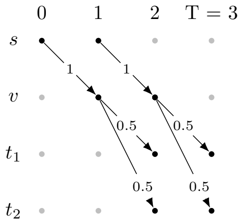


Figure: Labels represent loads on arcs.

Table of Contents

Definition

G is a tree, then (G, u, τ, c) is a *tree network*.

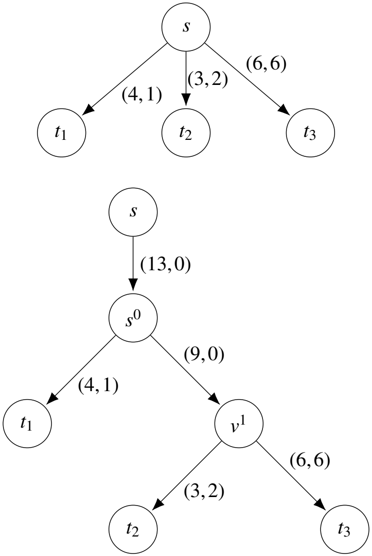


Figure: Networks with labels representing (u, τ).

Definition

G is a tree, then (G,u,τ,c) is a *tree network*.

Definition

Flow f on (G,u,τ,c) and f' on (G',u',τ',c').
Then $f \equiv f'$ if

- the number of sinks is equal,
- for each sink at each point in time the same number of units arrives with the same aggregated costs.

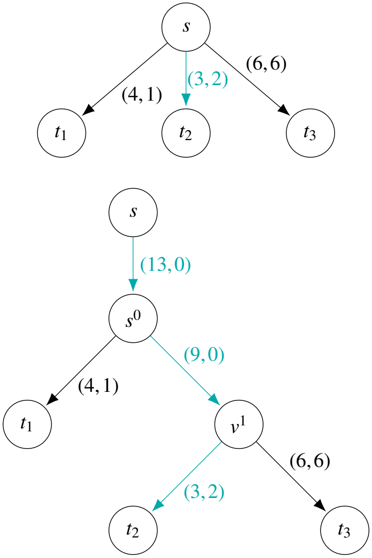


Figure: Networks with labels representing (u,τ).

Definition

Two tree networks (G,u,τ,c), (G',u',τ',c') are *equivalent*,

$$(G,u,\tau,c) \equiv (G',u',\tau',c'),$$

if for each flow f on (G,u,τ,c) there exists an equivalent flow f' on the other network (G',u',τ',c') and vice versa.

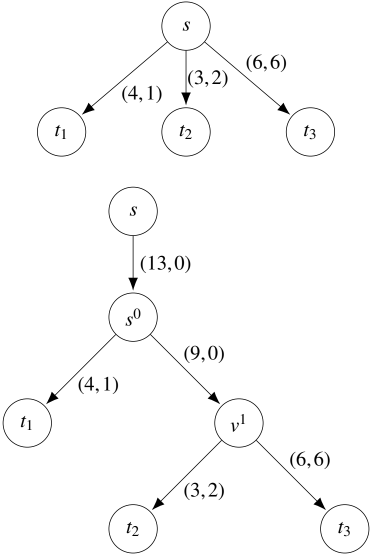


Figure: Networks with labels representing (u,τ).

Definition

Two tree networks (G,u,τ,c), (G',u',τ',c') are *equivalent*,

$$(G,u,\tau,c) \equiv (G',u',\tau',c'),$$

if for each flow f on (G,u,τ,c) there exists an equivalent flow f' on the other network (G',u',τ',c') and vice versa.

$\Rightarrow$ For all mentioned network problems: Given two equivalent networks, all optimal solutions have the *same objective value*. The optimal solutions have equivalent counterparts.

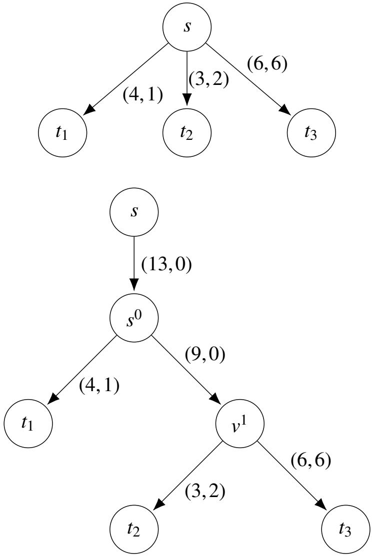


Figure: Networks with labels representing (u,τ).

Definition

Tree network (G,u,τ,c) and node $v \in V$. We define operation $\rho_1((G,u,\tau,c),v)$:
It maps to (G,u,τ,c) if v is

- a leaf,
- has at least two children, or
- is the root.

Otherwise, it maps to (G',u',τ',c') which resembles the network (G,u,τ,c) with the changes in the picture.

Lemma

For tree network (G,u,τ,c) and any node $v \in V$, it is $\rho_1((G,u,\tau,c),v) \equiv (G,u,\tau,c)$.

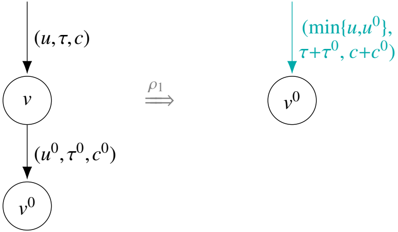


Figure: The operation ρ_1.

Definition

Tree network (G,u,τ,c) and node $v \in V$. We define operation $\rho_2((G,u,\tau,c),v)$:
It maps to (G,u,τ,c) if v has

- at most two children.

Otherwise, it maps to (G',u',τ',c') which resembles the network (G,u,τ,c) with the changes in the picture.

Lemma

For tree network (G,u,τ,c) and any node $v \in V$, it is $\rho_2((G,u,\tau,c),v) \equiv (G,u,\tau,c)$.

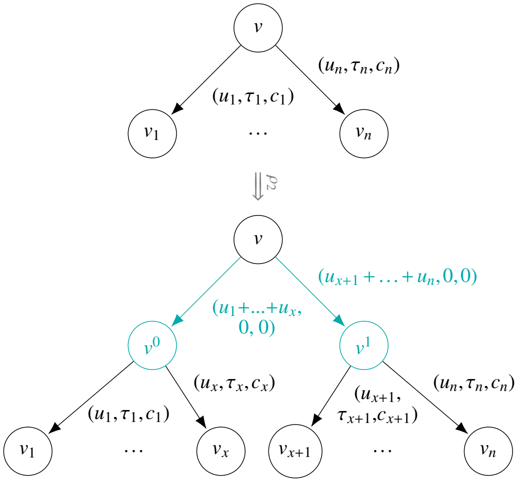


Figure: The operation ρ_2.

Definition

Tree network (G,u,τ,c) and node $v \in V$. We define operation $\rho_3((G,u,\tau,c),v)$:
It maps to (G,u,τ,c) if v has

- at most one child.

Otherwise, it maps to (G',u',τ',c') which resembles the network (G,u,τ,c) with the changes in the picture.

Lemma

For tree network (G,u,τ,c) and any node $v \in V$, it is $\rho_3((G,u,\tau,c),v) \equiv (G,u,\tau,c)$.

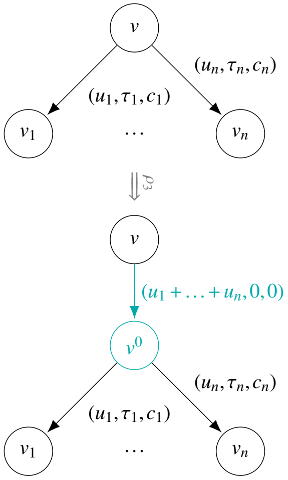


Figure: The operation ρ_3.

Definition

An *almost-binary tree* is a tree G with

- the root node $r \in V$ has at most one child $v \in V$, and
- the subtree of v is a complete binary tree (if v exists).

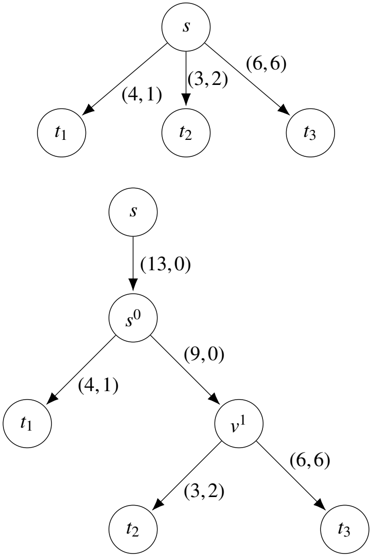


Figure: Networks with labels representing (u,τ).

Definition

An *almost-binary tree* is a tree G with

- the root node $r \in V$ has at most one child $v \in V$, and
- the subtree of v is a complete binary tree (if v exists).

Theorem

For each tree network (G, u, τ, c), there exists an equivalent tree network where the underlying graph is an almost-binary tree.

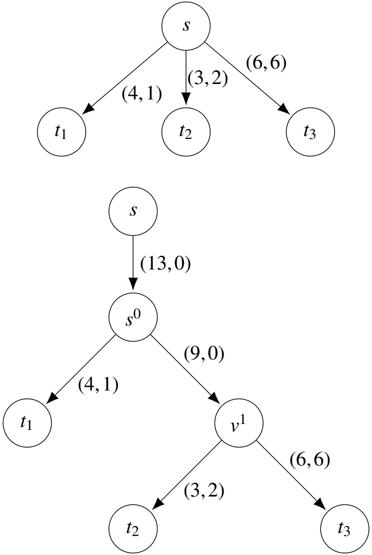


Figure: Networks with labels representing (u, τ).

Theorem

For each tree network (G, u, τ, c), there exists an equivalent tree network where the underlying graph is an almost-binary tree.

Theorem

Let (G, u, τ, c) be tree network, where G has degree $k \in \mathbb{N}_0$ and $|G| = n$. Then, an equivalent binary tree network can be computed in $\mathcal{O}(n \cdot k)$ steps.

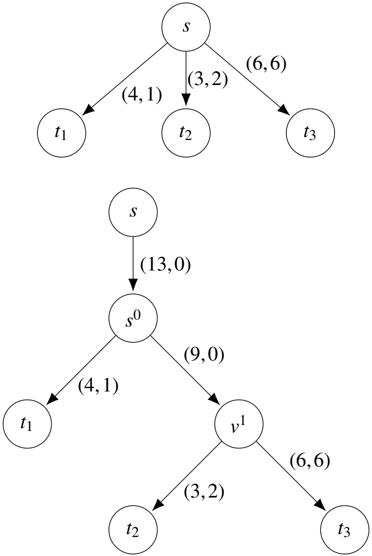
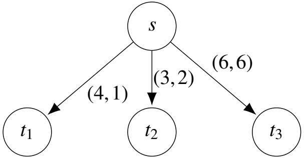
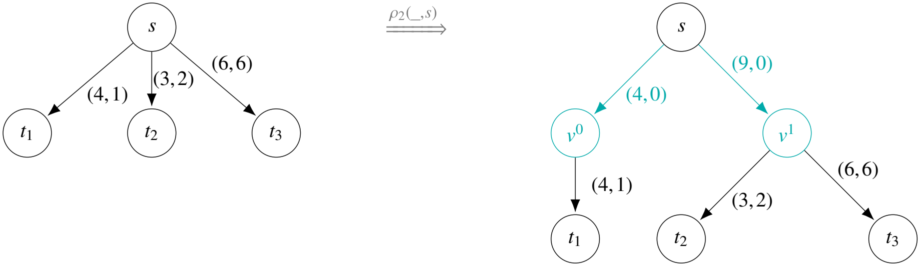


Figure: Networks with labels representing (u, τ).

Equivalence Classes of Trees



Equivalence Classes of Trees



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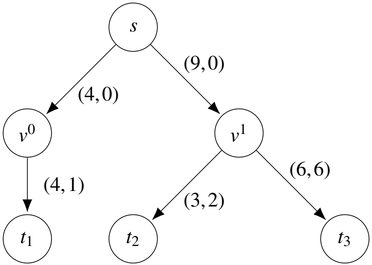

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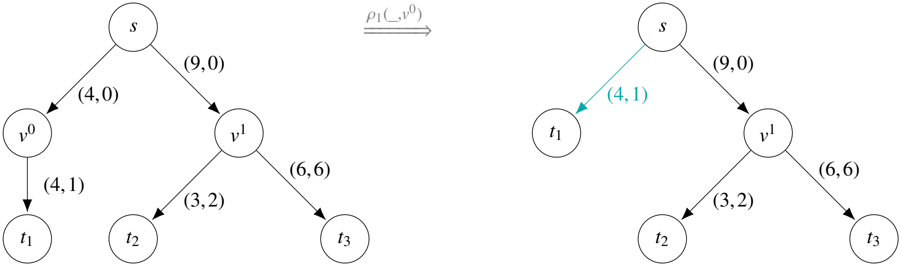
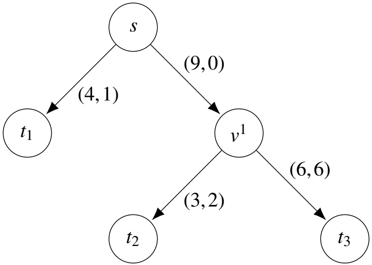
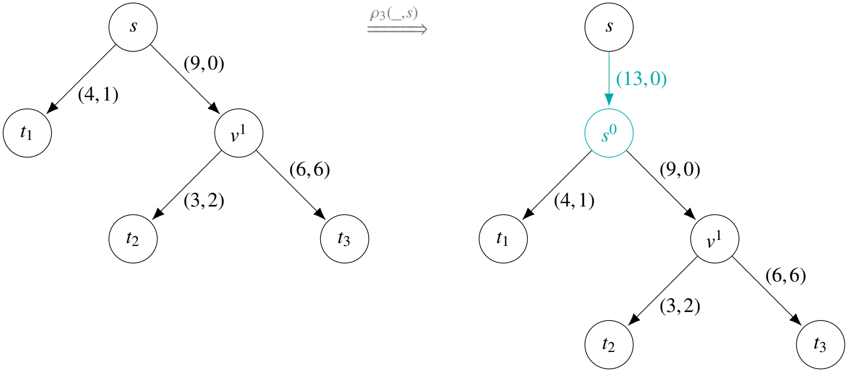



s
$(9,0)$
$(4,1)$
t_1
v^1
$(6,6)$
$(3,2)$
t_2
t_3

Equivalence Classes of Trees



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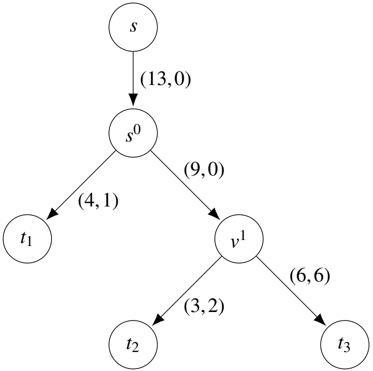

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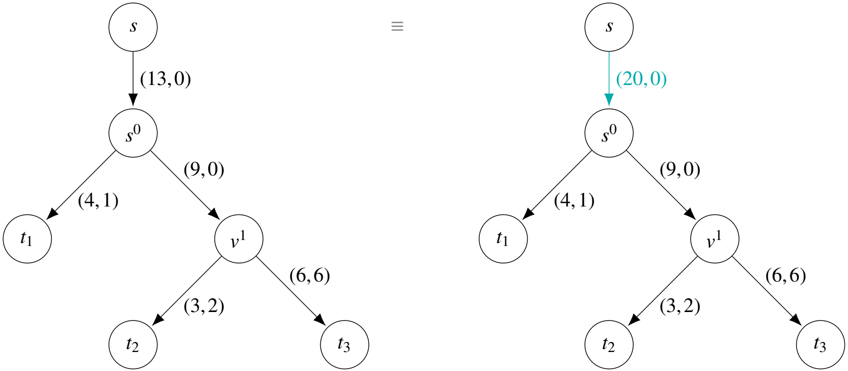

Definition

Given a tree network (G, u, τ), and a balance function b, find an *integer* k*-uniform flow* with arbitrary $k \leq h$ which

- satisfies the balances and
- has minimal overall time horizon $T \in N$.

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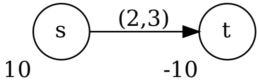

Figure: Network with labels representing (u,τ) and b.

Definition

Given a tree network (G,u,τ), and a balance function b, find an *integer k-uniform flow* with arbitrary $k \leq h$ which

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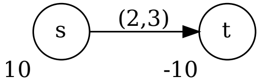


Figure: Network with labels representing (u,τ) and b.

Lemma

For an almost-binary tree network (G,u,τ,c) with one leaf and balance function b. Then, the minimal time horizon for a 1-uniform flow satisfying b is

$$T := \left\lceil \frac{b}{u} \right\rceil + \tau - 1.$$

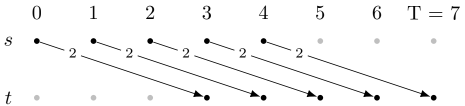
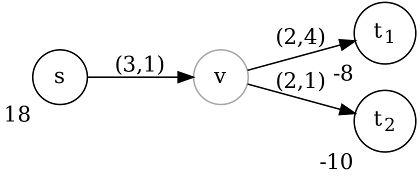


Figure: Labels represent loads on arcs.



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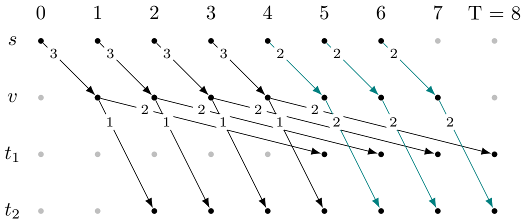

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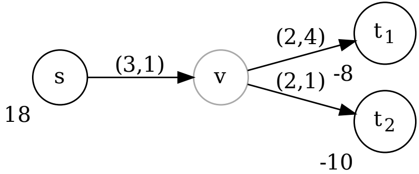

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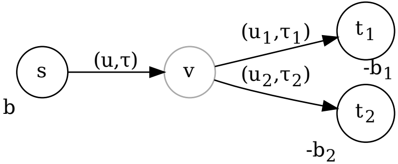
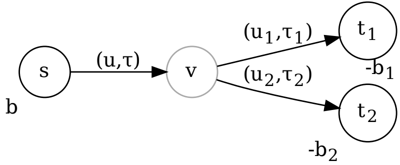


Figure: Network with labels representing (u, τ) and b.

Lemma

We set $\mathbf{u}_i := \min\{u, u_i\}$, $\boldsymbol{\tau}_i := \tau + \tau_i$.
The quickest 1*- or* 2*-uniform flow has time horizon*

$$\begin{aligned} T := \min\big(\, &\{\, f(\mathbf{u}_1, \boldsymbol{\tau}_1, b_1, \mathbf{u}_2, \boldsymbol{\tau}_2, b_2) \,\} \\ &\cup \{\, g(\mathbf{u}_1, \boldsymbol{\tau}_1, b_1, \mathbf{u}_2, \boldsymbol{\tau}_2, b_2, x_1, x_2) \mid \\ &\quad 1 \le x_1 \le \mathbf{u}_1,\ 1 \le x_2 \le \mathbf{u}_2,\ x_1 + x_2 \le u \,\} \\ &\cup \{\, h(\boldsymbol{\tau}_1, b_1, \boldsymbol{\tau}_2, b_2, x_1, x_2, y_1, y_2, d) \mid \\ &\quad 1 \le x_1 \le \mathbf{u}_1,\ 1 \le y_1 \le \mathbf{u}_1,\ 1 \le x_2 \le \mathbf{u}_2, \\ &\quad 1 \le y_2 \le \mathbf{u}_2,\ x_1 + x_2 \le u,\ y_1 + y_2 \le u, \\ &\quad d < \min\left\{\left\lceil \tfrac{b_1}{x_1} \right\rceil, \left\lceil \tfrac{b_2}{x_2} \right\rceil\right\},\ \left\lceil \tfrac{b_1 - d \cdot x_1}{y_1} \right\rceil = \left\lceil \tfrac{b_2 - d \cdot x_2}{y_2} \right\rceil \,\}\,\big). \end{aligned}$$



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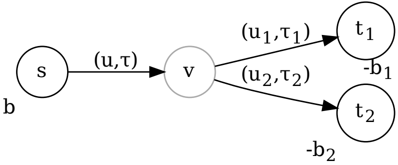
$$
\begin{aligned}
T := \min\big(&\{ f(\mathbf{u}_1, \boldsymbol{\tau}_1, b_1, \mathbf{u}_2, \boldsymbol{\tau}_2, b_2) \} \\
&\cup \{ g(\mathbf{u}_1, \boldsymbol{\tau}_1, b_1, \mathbf{u}_2, \boldsymbol{\tau}_2, b_2, x_1, x_2) \mid \\
&\quad 1 \le x_1 \le \mathbf{u}_1,\ 1 \le x_2 \le \mathbf{u}_2,\ x_1 + x_2 \le u \} \\
&\cup \{ h(\boldsymbol{\tau}_1, b_1, \boldsymbol{\tau}_2, b_2, x_1, x_2, y_1, y_2, d) \mid \\
&\quad 1 \le x_1 \le \mathbf{u}_1,\ 1 \le y_1 \le \mathbf{u}_1,\ 1 \le x_2 \le \mathbf{u}_2, \\
&\quad 1 \le y_2 \le \mathbf{u}_2,\ x_1 + x_2 \le u,\ y_1 + y_2 \le u, \\
&\quad d < \min\left\{\left\lceil \tfrac{b_1}{x_1} \right\rceil, \left\lceil \tfrac{b_2}{x_2} \right\rceil\right\},\ \left\lceil \tfrac{b_1 - d \cdot x_1}{y_1} \right\rceil = \left\lceil \tfrac{b_2 - d \cdot x_2}{y_2} \right\rceil \} \big).
\end{aligned}
$$

It is $f : \mathbb{N}_0^6 \to \mathbb{N}_0, (\mathbf{u}_1, \boldsymbol{\tau}_1, b_1, \mathbf{u}_2, \boldsymbol{\tau}_2, b_2) \mapsto$

$$
\begin{cases}
\left\lceil \frac{b_1}{\mathbf{u}_1} \right\rceil + \left\lceil \frac{b_2}{\mathbf{u}_2} \right\rceil + \max\left\{\boldsymbol{\tau}_2, \boldsymbol{\tau}_1 - \left\lceil \frac{b_2}{\mathbf{u}_2} \right\rceil\right\} - 1, & \boldsymbol{\tau}_1 \ge \boldsymbol{\tau}_2, \\
f(\mathbf{u}_2, \boldsymbol{\tau}_2, b_2, \mathbf{u}_1, \boldsymbol{\tau}_1, b_1), & \textit{otherwise.}
\end{cases}
$$



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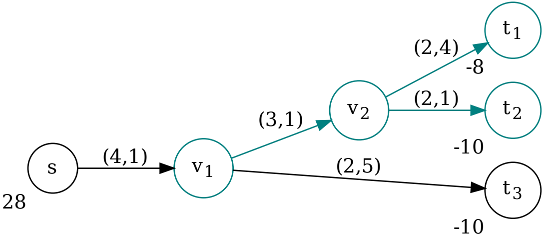

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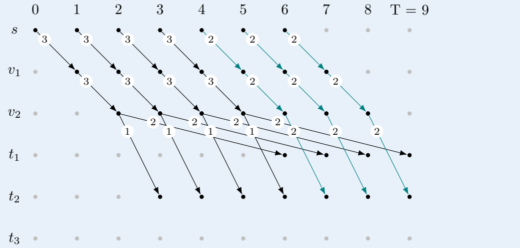
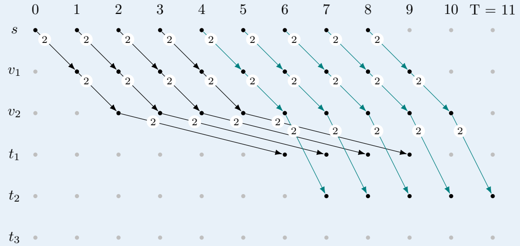
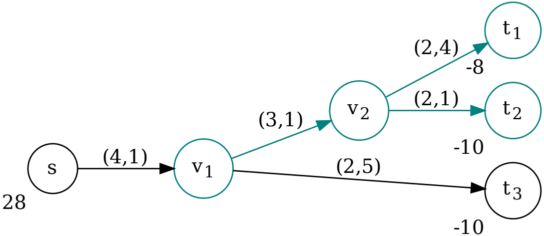
Optimal Solution for Subgraph



Non-optimal Solution





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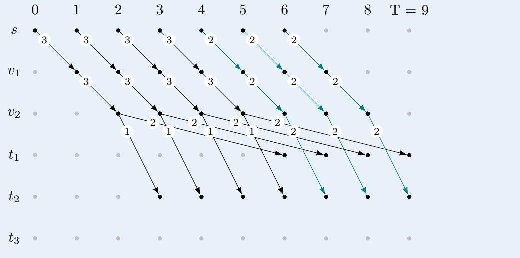
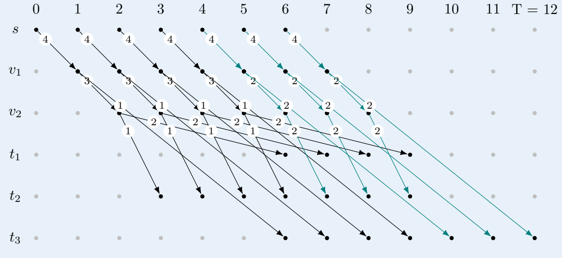
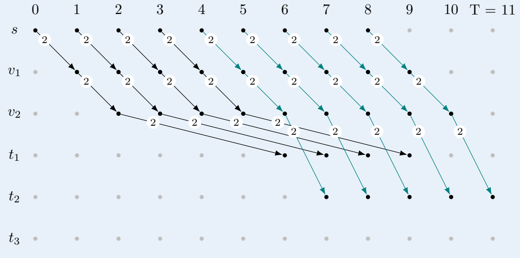
Optimal Solution for Subgraph



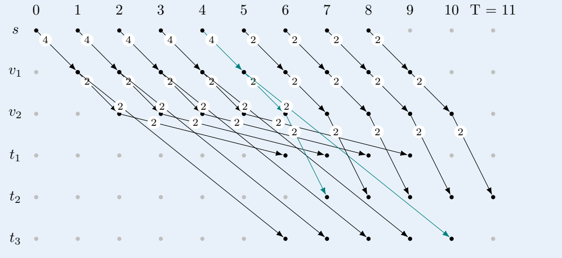
Extension of Optimal Solution



Non-optimal Solution



Extension of Non-optimal Solution



1-Uniform Flow with Minimal Time Horizon for Tree Network

$$\min \quad d \quad \text{s.t.} \quad \sum_{i \in I_a} -b_i \leq d \cdot u(a), \quad a \in A, I_a := \{i \in \{1,\dots,h\} \mid a \in p_i\}, \quad d \in \mathbb{N}_0.$$

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$\Rightarrow$ Linear algorithm transforms an optimal solution of the *linear relaxation* into an optimal *load-consistent flow*.

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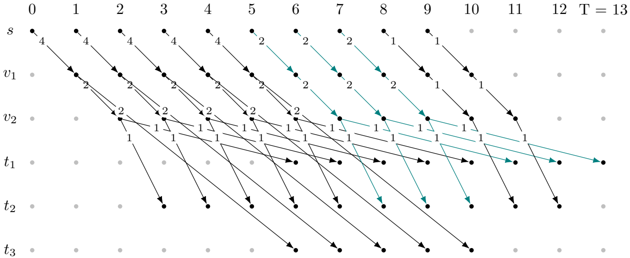


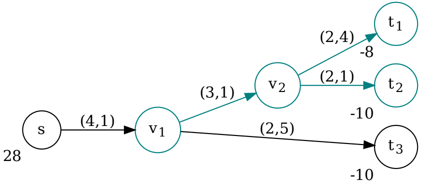


Table of Contents

Definition

Given a network (G, u, τ, c), source and sink $s, t \in V$, and a time horizon $T \in \mathbb{N}_0$, find an *integer temporally repeated flow* with

- maximal flow and
- minimized maximal costs over all points in time.

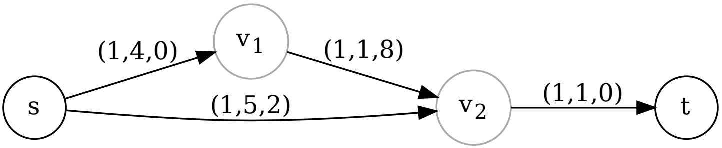


Figure: Network with labels representing (u, τ, c).

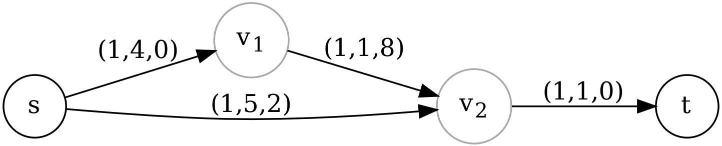
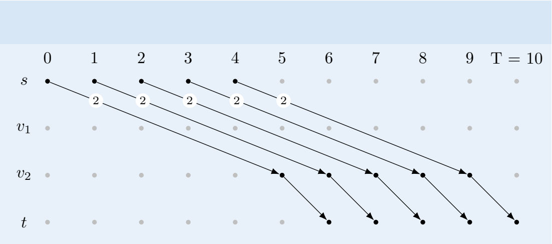
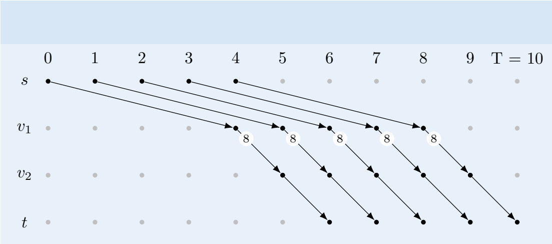


Figure: Network with labels representing (u, τ, c).





- Both flows have *total value* 5 and *time horizon* 10.
- The left flow has *total costs* 10, whereas the right flow has *total costs* 40.

Cost Minimization at each Point in Time

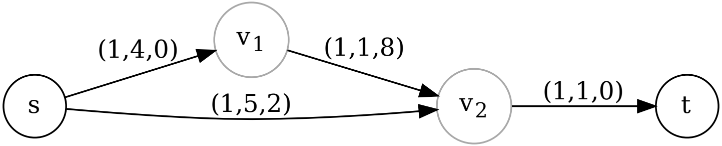
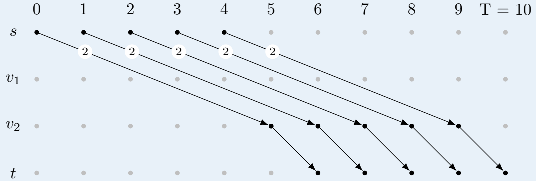
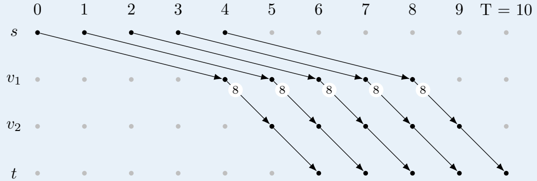


Figure: Network with labels representing (u, τ, c).

Maximal costs are 10 at time 4.



Maximal costs are 8 at time 4.



- Both flows have *total value* 5 and *time horizon* 10.
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Listing: Ford-Fulkerson Algorithm for Maximal (s,t)-Flows over Time

```
Input: Network (G,u,τ),
   source and sink s,t ∈ V, and
   time horizon T
Output: Temporally repeated flow f
   with time horizon T
   with maximal flow

Calculate static (s,t)-flow x
   that maximizes
   T · value(x) − ∑_{a∈A} τ(a) · x(a)
Calculate path decomposition
Construct temporally repeated flow f
   with time horizon T
Return f
```

Listing: Maximal (s,t)-Flows with Minimal Costs over all Points in Time

```
Input: Network (G,u,τ,c),
   source and sink s,t ∈ V, and
   time horizon T
Output: Temporally repeated flow f
   with time horizon T
   with maximal flow and minimized
   maximal costs at each time point

Set d : A → ℕ_0, a ↦ M · τ(a) + c(a) · τ(a)
Calculate static (s,t)-flow x
   that maximizes
   M · T · value(x) − ∑_{a∈A} d(a) · x(a)
Calculate path decomposition
Construct temporally repeated flow f
   with time horizon T
Return f
```

$$
\begin{aligned}
T \cdot \mathrm{value}(x) - \sum_{a \in A} \tau(a) \cdot x(a) \quad &\leftrightarrow \quad M \cdot T \cdot \mathrm{value}(x) - \sum_{a \in A} d(a) \cdot x(a) \\
&\Leftrightarrow \quad M \cdot T \cdot \mathrm{value}(x) - \sum_{a \in A} \big(M \cdot \tau(a) + c(a) \cdot \tau(a) \big) \cdot x(a) \\
&\Leftrightarrow \quad M \cdot \underbrace{\Big(T \cdot \mathrm{value}(x) - \sum_{a \in A} \tau(a) \cdot x(a) \Big)}_{\text{Maximize first.}} - \underbrace{\sum_{a \in A} c(a) \cdot \tau(a) \cdot x(a)}_{\text{Minimize second.}}
\end{aligned}
$$

Conclusion

Achievements

- Several types of flows over time

Future Work

Achievements

- Several types of flows over time
- Observation of tree networks and notion of equivalence

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Future Work

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- Analyze QTP on differently structured graphs
- Explore whether the solutions of subtrees for almost-binary tree networks can be restricted to a very small (of constant size?) set of relevant solutions

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