# Seminar Computeralgebra 2018

## Geck

- Introduction of
  - algebraic sets in affine space
  - affine algebras and algebraic groups
- Section 1.1
  - basic definitions
  - version of Hilbert's Nullstellensatz for hypersurfaces
- Section 1.2
  - Groebner bases and applications
  - define dimension of an algebraic set by Hilbert polynomial of an ideal

### Kapitel 1.1

- k is a field and k[X_{1}, . . . , X_{n}] a polynomial ring with X_{i} independent- commutative ring A with 1 is noetherian ring if every ideal of A is generated by finite number of elements
- Hilbert's Basis Theorem: If A is noetherian and X an indeterminate over A, then A[X] is also noetherian. If k is a field, then k[X_{1}, . . . , X_{n}] is noetherian.
  - $x \in k^n \Rightarrow \exists \varepsilon_x : k[X_1, ..., X_n] \rightarrow k$ (Einsetzungshomomorphismus)
  - f is uniquely determined by $\dot{f}$ if |k| = infinity
- V(S) ist die Menge aller gemeinsamen Nullstellen aller Polynome aus S
  - genannt algebraisches Set
  - V(S) = V(I), wenn I = das von S generierte Ideal ist
  - I(V) sind alle Polynome, für die alle $x \in V$ Nullstellen sind (genannt *vanishing ideal*)
  - S ist Teilmenge, algebraisches Set wird gefunden, daraus wird wieder eine Teilmenge des Polynomrings kreiert (diese ist größer/gleich S und ein Ideal)
  - A[V] := k[X_{1}, . . . , X_{n}]/I(V) heißt affine Algebra von V
  - Schnitte von algebraischen Sets sind wieder ein algebraisches Set, unendliche Vereinigungen jedoch nicht
- Zariski topology
  - beliebige Schnittmengen und endliche Vereinigungen von algebraischen Sets sind algebraische Sets, genauso wie $\varnothing$ and $k^n$
  - $X \subseteq k^n$ is open, if $k^n \setminus X$ is closed (algebraic)
  - $V \subseteq k^n$ (not necessarily algebraic) then $\bar{V} = V(I(V))$ is its closure in the Zariski topology ⇒ V(I(V) = V if V is algebraic set, hence $I : \{V \subseteq k^n | V algebraic\} \rightarrow \{I \subseteq k[X_1, ..., X_n] | I ideal\}$ is injective
  - I is not surjective: f polynomial and non-constant, then $H_f := V(f) \subseteq k^n$ is called hypersurface (n = 2 ⇒ plane curve), it may happen that

$H_f = \varnothing$, but only if k is algebraically closed and n $\geq 2$

- Hilbert's Nullstellensatz for hypersurfaces: k algebraically closed field, $f \in k[X_1, ..., X_n]$ non-constant and $\varnothing \neq H_f \subseteq k^n$ the corresponding hypersurface. $f = f_1^{n_1} \cdots f_r^{n_r}$ with $f_1, ..., f_r$ irreducible and pairwise coprime, then $H_f = H_{f_1} \cup \cdots \cup H_{f_r}$ and $I(H_f) = (f_1 \cdots f_r)$.
- Topological space $Z \neq \varnothing$ is reducible, if $Z = Z_1 \cup Z_2$ where $Z_1, Z_2$ are non-empty closed subsets with $Z_i \neq Z$
  - $Y \subseteq Z$ is irreducible if Y is irreducible with induced topology
  - $\varnothing$ is not considered irreducible
- Z is noetherian topological space if every chain of closed sets $Z_1 \supseteq Z_2 \supseteq \cdots$ in Z becomes stationary
- $Z \neq \varnothing$ is noetherian topological space, then there are only finitely many maximal closed irreducible subsets in Z; they are called the irreducible components of Z. If these are $Z_1, ..., Z_r$, we have $Z = Z_1 \cup \cdots \cup Z_r$.
- $\varnothing \neq V \subseteq k^n$ is algebraic set
  - V is noetherian with respect to the Zariski topology. Thus $V = V_1 \cup \cdots \cup V_r$ where $V_i$ are maximal closed irreducible subsets of V.
  - V is irreducible (in Zariski topology) $\Leftrightarrow$ I(V) is prime ideal.
- Principal open subsets: V non-empty algebraic set, $f \in k[X_1, ..., X_n]$ such that $f \notin I(V)$ and $V_f := \{v \in V | f(v) \neq 0\} \subseteq V$.Then $V_f$ is non-empty open set in V and called *principal open set*. Any open set in V is a finite union of principal open sets.
  - Indeen let U $\subseteq V$ be open. Then V minus U is closed and so there exists an algebraic set W such that $VminusU = W \cap V$. Now, by Hilbert's Basis Theorem, $W \cap V$ is defined by a finite collection of polynomials, $f_1, \ldots, f_r$. Then we have $U = V_{f_1} \cup \cdots \cup V_{f_r}$.
  - Principle open sets can be regarded as algebraic sets in their own right.

**Kapitel 1.2**

**Thoughts**

- Algebraische Geometrie: Menge von Nullstellen einer Menge von Polynomen
  - Algebraisches Set: K Körper, S Menge von Polynomen in n Variablen, V(S) Menge von gemeinsamen Nullstellen und V(S) $\subset K^n$
  - Algebraische Gruppe: Gruppe bzgl. Polynomfunktionen und algebraisches Set
  - Historisch: Analytischer Kontext und Lie Gruppen
- Algebraisch abgeschlossener Körper K: Jedes Polynom mit Koeffizienten aus K hat eine Nullstelle in K, z.B. $\mathbb{C}$
- Es gibt Garben, Halme und Keime! :)
- Definition of "algebraic" (see e.g. Example 1.1.3(c))
- "maximal closed" (Proposition 1.1.11)

- Principal open subsets: Warum sind das Subsets? Das stimmt gar nicht…
- Lemma 1.1.15

## CLOS

### Kapitel 1

### Kapitel 2

### Kapitel 9