Stability Analysis of a Minimal Within-Host TB Model

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1 Introduction

We consider a minimal model for within-host tuberculosis (TB) dynamics consisting of two interacting compartments: the TB burden $b(t)$, and the immune response $i(t)$. The system exhibits rich dynamics including bistability and tipping points between clearance and dominance. This document presents steady-state analysis, feasibility conditions, and local and global stability results.

2 Model Definition and Rescaling

We begin with the rescaled ODE system:

$$\frac{db}{dt} = \beta_b \, b \, [\eta_b(1-b) - i] \quad \text{and} \quad \frac{di}{dt} = \beta_i \, i \, [\eta_i(1-i) - b]$$

where $b(t)$ and $i(t)$ are non-dimensional variables representing TB and immune response (scaled 0–1), and the parameters are defined as:

- $\beta_b = \alpha_B K_I$, $\beta_i = \alpha_I K_B$ (feedback strength)
- $\eta_b = \frac{r_B}{\alpha_B K_I}$, $\eta_i = \frac{r_I}{\alpha_I K_B}$ (self-limiting growth factors)

3 Steady States

Setting the time derivatives to zero yields four equilibria:

- Trivial: $(b^*, i^*) = (0, 0)$
- TB-only: $(b^*, i^*) = (1, 0)$
- Immune-only: $(b^*, i^*) = (0, 1)$
- Coexistence equilibrium:

$$b^* = \frac{\eta_i(\eta_b - 1)}{\eta_i \eta_b - 1}, \qquad i^* = \frac{\eta_b(\eta_i - 1)}{\eta_i \eta_b - 1}$$

4 Existence and Feasibility Conditions

The coexistence equilibrium is biologically feasible (i.e., $b^* > 0$, $i^* > 0$) if either:

- $\eta_b > 1$ and $\eta_i > 1$
- $\eta_b < 1$ and $\eta_i < 1$

If one is above and one is below 1, then either b^* or i^* becomes negative, rendering the equilibrium nonphysical.

5 Local Stability Analysis

The Jacobian matrix evaluated at the coexistence point is:

$$J = \begin{bmatrix} -r_b b^* & -\beta_b b^* \\ -\beta_i i^* & -r_i i^* \end{bmatrix}$$

Stability requires:

- $\mathrm{tr}(J) < 0$
- $\det(J) > 0 \Rightarrow b^* i^* (r_b r_i - \beta_b \beta_i) > 0$

Thus, coexistence is locally stable if $r_b r_i > \beta_b \beta_i$ and $b^*, i^* > 0$.

6 Global Stability

Using the Bendixson–Dulac criterion with $B(b, i) = \frac{1}{bi}$ yields:

$$\frac{\partial(Bf)}{\partial b} + \frac{\partial(Bg)}{\partial i} = -\frac{r_b}{i} - \frac{r_i}{b} < 0$$

Hence, no periodic orbits exist and all bounded solutions tend to equilibria. This suggests global convergence to one of the steady states.

7 Lyapunov Stability

We construct a Lyapunov function for the coexistence equilibrium:

$$V(b, i) = b - b^* - b^* \log\left(\frac{b}{b^*}\right) + c\left(i - i^* - i^* \log\left(\frac{i}{i^*}\right)\right)$$

With $c = \beta_b/\beta_i$, the derivative along trajectories satisfies:

$$\dot{V} \leq -\beta_b(b - b^* + i - i^*)^2$$

indicating global asymptotic stability when $r_b \geq \beta_b$ and $r_i \geq \beta_i$.