

Electromagnetics

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1 Electric Fields

The electric force between two point charges are obtained by

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{R^2}$$

If the charges are similar in polarity, then $\vec{F} > 0$. Otherwise, $\vec{F} < 0$. Similarly, the electric field intensity can be obtained by placing a test charge Q_t at any point and obtaining the net force that is experienced by the charge at that point.

$$\vec{E} = \frac{\vec{F}_t}{Q_t}$$

1.1 Fields from Continuous Charge Distributions

The volume charge density is denoted by ρ_v . The amount of charge stored in a small volume Δv is then

$$\Delta Q = \rho_v \Delta v$$

and ρ_v is mathematically defined as

$$\rho_v = \lim_{\Delta v \rightarrow 0} \frac{\Delta Q}{\Delta v}$$

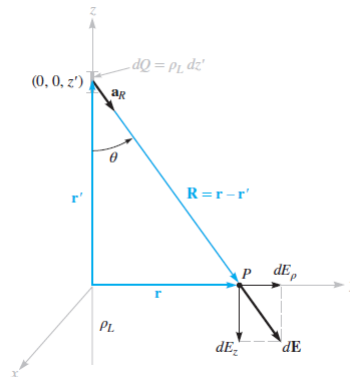
The total charge is then obtained by

$$Q = \int_{vol} \rho_v dv$$

1.2 Fields from Infinite Charge Distributions

1.2.1 Field of a Line Charge

Consider a line charge with an infinite length extending from the negative z axis to the positive z axis. Varying \vec{a}_z does not change the experienced electric field since it has an infinite length. So does changing \vec{a}_ϕ with a constant \vec{a}_ρ . The only thing that affects the electric field magnitude is \vec{a}_ρ



Here, we select an arbitrary point along the y axis $P(0, y, 0)$.

Consider an infinitesimal charge dQ from a slice of the line charge with length z' . Now,

$$dE = \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz'}{\vec{R}^2}$$

where ρ_L is the linear charge density.

Note that here, $\vec{R} = \vec{r} - \vec{r}'$ where $\vec{r} = y\vec{a}_y = \rho\vec{a}_\rho$ and $\vec{r}' = z'\vec{a}_z$. Thus,

$$\begin{aligned} dE &= \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz' (\rho\vec{a}_\rho - z'\vec{a}_z)}{|\rho\vec{a}_\rho - z'\vec{a}_z|^3} \\ &= \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz' (\rho\vec{a}_\rho - z'\vec{a}_z)}{(\rho^2 + z'^2)^{\frac{3}{2}}} \end{aligned}$$

Since varying z' does not affect the electric field magnitude,

$$\begin{aligned} d\vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\ \vec{E} &= \int_{-\infty}^{\infty} \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\ &= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho_L dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \end{aligned}$$

Using trigonometric substitution with $z' = \rho \tan \theta$,

$$\begin{aligned} \vec{E} &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{[\rho^2(1 + \tan^2 \theta)]^{\frac{3}{2}}} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{[\rho^2(1 + \tan^2 \theta)]^{\frac{3}{2}}} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{\rho^3 \sec^3 \theta} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int \frac{\rho\vec{a}_\rho}{\rho^3 \sec^3 \theta} \rho \sec^2 \theta d\theta \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int \frac{\rho\vec{a}_\rho}{\rho^2 \sec \theta} d\theta \\ &= \frac{\rho_L}{4\pi\epsilon_0} \frac{\vec{a}_\rho}{\rho} \int \cos \theta d\theta \\ &= \frac{\rho_L}{4\pi\epsilon_0} \frac{\vec{a}_\rho}{\rho} \sin \theta \end{aligned}$$

And,

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0} \frac{\vec{a}_\rho}{\rho} \frac{z'}{\sqrt{z'^2 + \rho^2}} \Bigg|_{z=-\infty}^{\infty}$$

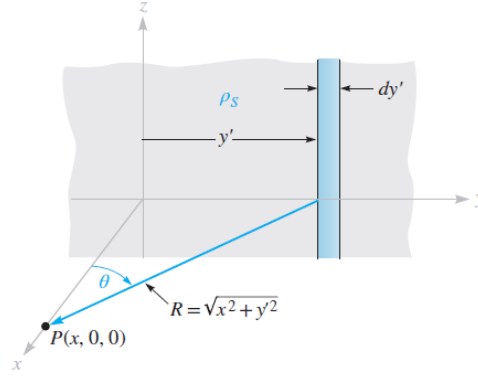
Finally,

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \vec{a}_\rho}$$

We can see that only the distance ρ affects the magnitude.

1.2.2 Field of a Sheet of Charge

Consider the sheet of charge placed at the y-z plane shown below.



We can use the electric field magnitude of a line charge to obtain the magnitude of this sheet.

We know that for a line charge,

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \vec{a}_\rho$$

Let ρ_S be the surface charge density and $\rho_L = \rho_S dy'$ where dy' is the infinitesimal width of the strip. We select an arbitrary point along the x axis.

$$\begin{aligned} d\vec{E} &= \frac{\rho_S dy' \vec{a}_N}{2\pi\epsilon_0 \sqrt{x^2 + y'^2}} \cos \theta \\ &= \frac{\rho_S dy' \vec{a}_N}{2\pi\epsilon_0 \sqrt{x^2 + y'^2}} \frac{x}{\sqrt{x^2 + y'^2}} \\ \vec{E} &= \int_{-\infty}^{\infty} \frac{\rho_S \vec{a}_N}{2\pi\epsilon_0} \frac{x dy'}{x^2 + y'^2} \\ &= \frac{\rho_S \vec{a}_N}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x dy'}{x^2 + y'^2} \\ &= \frac{\rho_S \vec{a}_N}{2\pi\epsilon_0} \int_{-\infty}^{\infty} x \frac{dy'}{x^2 + y'^2} \\ &= \frac{\rho_S \vec{a}_N}{2\pi\epsilon_0} \tan^{-1} \left(\frac{y'}{x} \right) \Big|_{y'=-\infty}^{\infty} \end{aligned}$$

Finally,

$$\boxed{\vec{E} = \frac{\rho_S}{2\epsilon_0} \vec{a}_N}$$

This means that the electric field magnitude from a sheet of charge is constant. We can determine the sign depending on the normal vector from the sheet.

2 Flux Density, Gauss' Law, and Divergence