Electromagnetics

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1 Electric Fields

The electric force between two point charges are obtained by

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{R^2}$$

If the charges are similar in polarity, then $\vec{F} > 0$. Otherwise, $\vec{F} < 0$. Similarly, the electric field intensity can be obtained by placing a test charge Q_t at any point and obtaining the net force that is experienced by the charge at that point.

$$\vec{E} = \frac{\vec{F_t}}{Q_t}$$

1.1 Fields from Continuous Charge Distributions

The volume charge density is denoted by ρ_v. The amount of charge stored in in a small volume Δv is then

$$\Delta Q = \rho_v \Delta v$$

and ρ_v is mathematically defined as

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$$

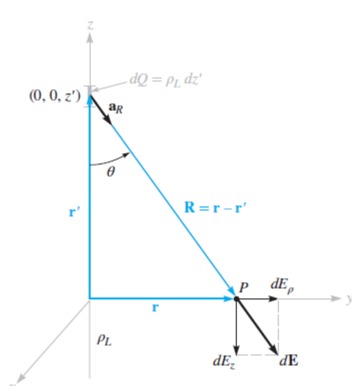
The total charge is then obtained by

$$Q = \int_{vol} \rho_v dv$$

1.2 Fields from Infinite Charge Distributions

1.2.1 Field of a Line Charge

Consider a line charge with an infinite length extending from the negative z axis to the positive z axis. Varying $\vec{a_z}$ does not change the experienced electric field since it has an infinite length. So does changing $\vec{a_\phi}$ with a constant $\vec{a_\rho}$. The only thing that affects the electric field magnitude is $\vec{a_\rho}$



Here, we select an arbitrary point along the y axis $P(0, y, 0)$.

Consider an infinitesimal charge dQ from a slice of the line charge with length z'. Now,

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'}{\vec{R}^2}$$

where ρ_L is the linear charge density.

Note that here, $\vec{R} = \vec{r} - \vec{r'}$ where $\vec{r} = y\vec{a_y} = \rho\vec{a_\rho}$ and $\vec{r'} = z'\vec{a_z}$. Thus,

$$\begin{aligned}
dE &= \frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'(\rho\vec{a_\rho} - z'\vec{a_z})}{|(\rho\vec{a_\rho} - z'\vec{a_z})|^3} \\
&= \frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'(\rho\vec{a_\rho} - z'\vec{a_z})}{(\rho^2 + z'^2)^{\frac{3}{2}}}
\end{aligned}$$

Since varying z' does not affect the electric field magnitude,

$$\begin{aligned}
d\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\
\vec{E} &= \int_{-\infty}^{\infty}\frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\
&= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\rho_L dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}}
\end{aligned}$$

Using trigonometric substitution with $z' = \rho\tan\theta$,

$$\begin{aligned}
\vec{E} &= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{[\rho^2(1+\tan^2\theta)]^{\frac{3}{2}}} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{[\rho^2(1+\tan^2\theta)]^{\frac{3}{2}}} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{\rho^3\sec^3\theta} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int\frac{\rho\vec{a_\rho}}{\rho^3\sec^3\theta}\rho\sec^2\theta d\theta \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int\frac{\rho\vec{a_\rho}}{\rho^2\sec\theta}d\theta \\
&= \frac{\rho_L}{4\pi\epsilon_0}\frac{\vec{a_\rho}}{\rho}\int\cos\theta d\theta \\
&= \frac{\rho_L}{4\pi\epsilon_0}\frac{\vec{a_\rho}}{\rho}\sin\theta
\end{aligned}$$

And,

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0}\frac{\vec{a_\rho}}{\rho}\frac{z'}{\sqrt{z'^2+\rho^2}}\Bigg|_{z=-\infty}^{\infty}$$

Finally,

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}}$$

We can see that