

# Electrostatics and Electrodynamics

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## 1 Electric Fields

The electric force between two point charges are obtained by

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{R^2}$$

If the charges are similar in polarity, then  $\vec{F} > 0$ . Otherwise,  $\vec{F} < 0$ . Similarly, the electric field intensity can be obtained by placing a test charge  $Q_t$  at any point and obtaining the net force that is experienced by the charge at that point.

$$\vec{E} = \frac{\vec{F}_t}{Q_t}$$

### 1.1 Fields from Continuous Charge Distributions

The volume charge density is denoted by  $\rho_v$ . The amount of charge stored in in a small volume  $\Delta v$  is then

$$\Delta Q = \rho_v \Delta v$$

and  $\rho_v$  is mathematically defined as

$$\rho_v = \lim_{\Delta v \rightarrow 0} \frac{\Delta Q}{\Delta v}$$

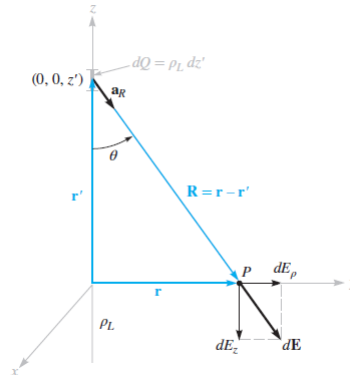
The total charge is then obtained by

$$Q = \int_{vol} \rho_v dv$$

### 1.2 Fields from Infinite Charge Distributions

#### 1.2.1 Field of a Line Charge

Consider a line charge with an infinite length extending from the negative z axis to the positive z axis. Varying  $\vec{a}_z$  does not change the experienced electric field since it has an infinite length. So does changing  $\vec{a}_\phi$  with a constant  $\vec{a}_\rho$ . The only thing that affects the electric field magnitude is  $\vec{a}_\rho$



Here, we select an arbitrary point along the y axis  $P(0, y, 0)$ .

Consider an infinitesimal charge  $dQ$  from a slice of the line charge with length  $z'$ . Now,

$$dE = \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz'}{\vec{R}^2}$$

where  $\rho_L$  is the linear charge density.

Note that here,  $\vec{R} = \vec{r} - \vec{r}'$  where  $\vec{r} = y\vec{a}_y = \rho\vec{a}_\rho$  and  $\vec{r}' = z'\vec{a}_z$ . Thus,

$$\begin{aligned} dE &= \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz' (\rho\vec{a}_\rho - z'\vec{a}_z)}{|\rho\vec{a}_\rho - z'\vec{a}_z|^3} \\ &= \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz' (\rho\vec{a}_\rho - z'\vec{a}_z)}{(\rho^2 + z'^2)^{\frac{3}{2}}} \end{aligned}$$

Since varying  $z'$  does not affect the electric field magnitude,

$$\begin{aligned} d\vec{E} &= \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\ \vec{E} &= \int_{-\infty}^{\infty} \frac{1}{4\pi\epsilon_0} \frac{\rho_L dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\ &= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\rho_L dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \end{aligned}$$

Using trigonometric substitution with  $z' = \rho \tan \theta$ ,

$$\begin{aligned} \vec{E} &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{[\rho^2(1 + \tan^2 \theta)]^{\frac{3}{2}}} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{[\rho^2(1 + \tan^2 \theta)]^{\frac{3}{2}}} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz' \rho\vec{a}_\rho}{\rho^3 \sec^3 \theta} \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int \frac{\rho\vec{a}_\rho}{\rho^3 \sec^3 \theta} \rho \sec^2 \theta d\theta \\ &= \frac{\rho_L}{4\pi\epsilon_0} \int \frac{\rho\vec{a}_\rho}{\rho^2 \sec \theta} d\theta \\ &= \frac{\rho_L}{4\pi\epsilon_0} \frac{\vec{a}_\rho}{\rho} \int \cos \theta d\theta \\ &= \frac{\rho_L}{4\pi\epsilon_0} \frac{\vec{a}_\rho}{\rho} \sin \theta \end{aligned}$$

And,

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0} \frac{\vec{a}_\rho}{\rho} \frac{z'}{\sqrt{z'^2 + \rho^2}} \Bigg|_{z=-\infty}^{\infty}$$

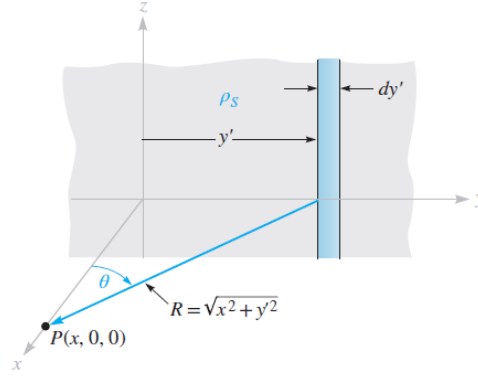
Finally,

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \vec{a}_\rho}$$

We can see that only the distance  $\rho$  affects the magnitude.

### 1.2.2 Field of a Sheet of Charge

Consider the sheet of charge placed at the y-z plane shown below.



We can use the electric field magnitude of a line charge to obtain the magnitude of this sheet.

We know that for a line charge,

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \vec{a}_\rho$$

Let  $\rho_S$  be the surface charge density and  $\rho_L = \rho_S dy'$  where  $dy'$  is the infinitesimal width of the strip. We select an arbitrary point along the x axis.

$$\begin{aligned} d\vec{E} &= \frac{\rho_S dy' \vec{a}_N}{2\pi\epsilon_0 \sqrt{x^2 + y'^2}} \cos \theta \\ &= \frac{\rho_S dy' \vec{a}_N}{2\pi\epsilon_0 \sqrt{x^2 + y'^2}} \frac{x}{\sqrt{x^2 + y'^2}} \\ \vec{E} &= \int_{-\infty}^{\infty} \frac{\rho_S \vec{a}_N}{2\pi\epsilon_0} \frac{x dy'}{x^2 + y'^2} \\ &= \frac{\rho_S \vec{a}_N}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x dy'}{x^2 + y'^2} \\ &= \frac{\rho_S \vec{a}_N}{2\pi\epsilon_0} \int_{-\infty}^{\infty} x \frac{dy'}{x^2 + y'^2} \\ &= \frac{\rho_S \vec{a}_N}{2\pi\epsilon_0} \tan^{-1} \left( \frac{y'}{x} \right) \Big|_{y'=-\infty}^{\infty} \end{aligned}$$

Finally,

$$\boxed{\vec{E} = \frac{\rho_S}{2\epsilon_0} \vec{a}_N}$$

This means that the electric field magnitude from a sheet of charge is constant. We can determine the sign depending on the normal vector from the sheet.

## 2 Flux Density, Gauss' Law, and Divergence