Electrostatics and Electrodynamics

Emmanuel Estallo

August 3, 2021

1 Electric Fields

The electric force between two point charges are obtained by

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Q_1Q_2}{R^2}$$

If the charges are similar in polarity, then $\vec{F} > 0$. Otherwise, $\vec{F} < 0$. Similarly, the electric field intensity can be obtained by placing a test charge Q_t at any point and obtaining the net force that is experienced by the charge at that point.

$$\vec{E} = \frac{\vec{F_t}}{Q_t}$$

1.1 Fields from Continuous Charge Distributions

The volume charge density is denoted by ρ_v. The amount of charge stored in in a small volume Δv is then

$$\Delta Q = \rho_v \Delta v$$

and ρ_v is mathematically defined as

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$$

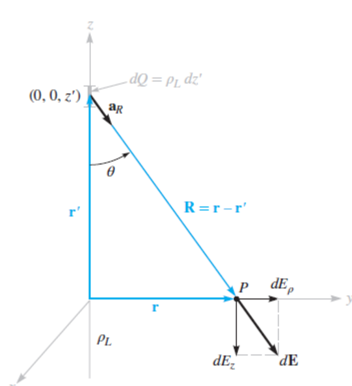
The total charge is then obtained by

$$Q = \int_{vol} \rho_v dv$$

1.2 Fields from Infinite Charge Distributions

1.2.1 Field of a Line Charge

Consider a line charge with an infinite length extending from the negative z axis to the positive z axis. Varying $\vec{a_z}$ does not change the experienced electric field since it has an infinite length. So does changing $\vec{a_\phi}$ with a constant $\vec{a_\rho}$. The only thing that affects the electric field magnitude is $\vec{a_\rho}$



Here, we select an arbitrary point along the y axis $P(0, y, 0)$.

Consider an infinitesimal charge dQ from a slice of the line charge with length z'. Now,

$$dE = \frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'}{\vec{R}^2}$$

where ρ_L is the linear charge density.

Note that here, $\vec{R} = \vec{r} - \vec{r'}$ where $\vec{r} = y\vec{a_y} = \rho\vec{a_\rho}$ and $\vec{r'} = z'\vec{a_z}$. Thus,

$$\begin{aligned}
dE &= \frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'(\rho\vec{a_\rho} - z'\vec{a_z})}{|(\rho\vec{a_\rho} - z'\vec{a_z})|^3} \\
&= \frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'(\rho\vec{a_\rho} - z'\vec{a_z})}{(\rho^2 + z'^2)^{\frac{3}{2}}}
\end{aligned}$$

Since varying z' does not affect the electric field magnitude,

$$\begin{aligned}
d\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\
\vec{E} &= \int_{-\infty}^{\infty}\frac{1}{4\pi\epsilon_0}\frac{\rho_L dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\
&= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\rho_L dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}}
\end{aligned}$$

Using trigonometric substitution with $z' = \rho\tan\theta$,

$$\begin{aligned}
\vec{E} &= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{(\rho^2 + z'^2)^{\frac{3}{2}}} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{[\rho^2(1 + \tan^2\theta)]^{\frac{3}{2}}} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{[\rho^2(1 + \tan^2\theta)]^{\frac{3}{2}}} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz'\rho\vec{a_\rho}}{\rho^3\sec^3\theta} \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int\frac{\rho\vec{a_\rho}}{\rho^3\sec^3\theta}\rho\sec^2\theta d\theta \\
&= \frac{\rho_L}{4\pi\epsilon_0}\int\frac{\rho\vec{a_\rho}}{\rho^2\sec\theta}d\theta \\
&= \frac{\rho_L}{4\pi\epsilon_0}\frac{\vec{a_\rho}}{\rho}\int\cos\theta d\theta \\
&= \frac{\rho_L}{4\pi\epsilon_0}\frac{\vec{a_\rho}}{\rho}\sin\theta
\end{aligned}$$

And,

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0}\frac{\vec{a_\rho}}{\rho}\frac{z'}{\sqrt{z'^2 + \rho^2}}\Bigg|_{z=-\infty}^{\infty}$$

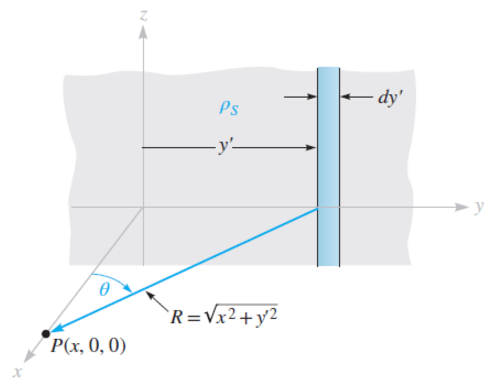
Finally,

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\vec{a_\rho}}$$

We can see that only the distance ρ affects the magnitude.

1.2.2 Field of a Sheet of Charge

Consider the sheet of charge placed at the y-z plane shown below.



We can use the electric field magnitude of a line charge to obtain the magnitude of this sheet.

We know that for a line charge,

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\vec{a_\rho}$$

Let ρ_S be the surface charge density and $\rho_L = \rho_S dy'$ where dy' is the infinitesimal width of the strip. We select an arbitrary point along the x axis.

$$\begin{aligned}
d\vec{E} &= \frac{\rho_S dy' \vec{a_N}}{2\pi\epsilon_0 \sqrt{x^2+y'^2}} \cos\theta \\
&= \frac{\rho_S dy' \vec{a_N}}{2\pi\epsilon_0 \sqrt{x^2+y'^2}} \frac{x}{\sqrt{x^2+y'^2}} \\
\vec{E} &= \int_{-\infty}^{\infty} \frac{\rho_S \vec{a_N}}{2\pi\epsilon_0} \frac{x dy'}{x^2+y'^2} \\
&= \frac{\rho_S \vec{a_N}}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x dy'}{x^2+y'^2} \\
&= \frac{\rho_S \vec{a_N}}{2\pi\epsilon_0} \int_{-\infty}^{\infty} x \frac{dy'}{x^2+y'^2} \\
&= \frac{\rho_S \vec{a_N}}{2\pi\epsilon_0} \tan^{-1}\left(\frac{y'}{x}\right)\Bigg|_{y'=-\infty}^{\infty}
\end{aligned}$$

Finally,

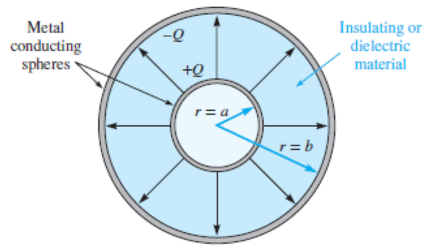
$$\boxed{\vec{E} = \frac{\rho_S}{2\epsilon_0}\vec{a_N}}$$

This means that the electric field magnitude from a sheet of charge is constant. We can determine the sign depending on the normal vector from the sheet.

2 Flux Density, Gauss' Law, and Divergence

2.1 Electric Flux Density

Denote the electric flux density $\vec{D}$. The direction of $\vec{D}$ at a point is the direction of the flux lines at that point. The magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area. Consider the figure below,



For the sphere,

$$D\Big|_{r=a} = \frac{Q}{4\pi a^2}\vec{a_r}\ (inner\ sphere)$$
$$D\Big|_{r=b} = \frac{Q}{4\pi b^2}\vec{a_r}\ (outer\ sphere)$$

at any point inside the sphere where $a \le r \le b$,

$$D = \frac{Q}{4\pi r^2}\vec{a_r}$$

For a free space,

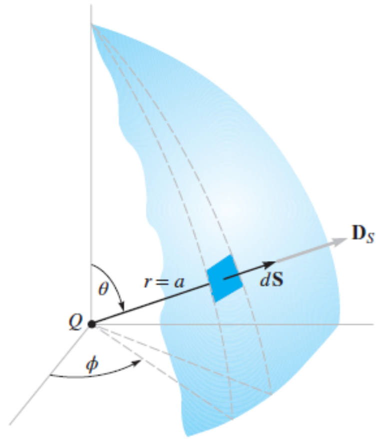
$$D = \epsilon_0 E$$
$$\boxed{D = \int_{vol} \frac{\rho_v dv}{4\pi R^2}\vec{a_R}}$$

2.2 Gauss' Law

The electric flux passing through any imaginary spherical surface lying between two conducting spheres is equal to the charge enclosed within that imaginary surface.

$$\Psi = \oint_S D_S \cdot dS = Q$$
$$\oint_S D_S \cdot dS = \int_{vol} \rho_v dv$$

Consider the figure below.



Here, the differential area dS is obtained by multiplying the length and width of the infinitesimal surface.

$$
\begin{aligned}
L &= r d\theta \\
W &= r \sin\theta d\phi \\
\vec{dS} &= r^2 \sin\theta d\theta d\phi \; \vec{a_R}
\end{aligned}
$$

We obtain the integrand

$$
\begin{aligned}
D_S \cdot dS &= \frac{Q}{4\pi a^2} \; \vec{a_R} \cdot a^2 \sin\theta d\theta d\phi \; \vec{a_R} \\
&= \frac{Q}{4\pi} \sin\theta d\theta d\phi
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\Psi &= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta d\theta d\phi \\
&= \int_0^{2\pi} \frac{Q}{2\pi} d\phi \\
&= Q
\end{aligned}
$$