Direction Cosine Matrix and Rotation Matrices

Vector

A vector is given properly in terms of it components within a given set of coordinate axes, e.g.,

$$\mathbf{c} = c_x\hat{\mathrm{i}} + c_y\hat{\mathrm{i}} + c_z\hat{\mathrm{i}} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}.$$

Magnitude of vector: the Dot-Product

The magnitude of a vector, **b**, is obtained by the dot-product of a vector with itself, i.e., (using a variety of usual notations)

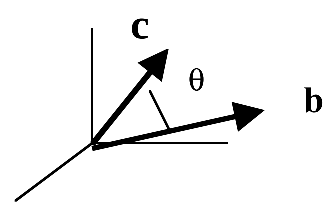
$$|\mathbf{b}|^2 = \mathbf{b} \bullet \mathbf{b} = \begin{pmatrix} b_x & b_y & b_z \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = b_i b_i = b_x^2 + b_y^2 + b_z^2 \quad ,$$

and, obviously, $|\mathbf{b}| = \sqrt{\mathbf{b} \bullet \mathbf{b}} = \sqrt{b_x^2 + b_y^2 + b_z^2}$.

What's the angle? Dot Product of Two Vectors

The cosine of the angle between two vectors is determined by the **dot product** of those two vectors. Namely, $\mathbf{b} \bullet \mathbf{c} = |b||c|\cos\theta$. Therefore,

$$\cos\theta = \frac{\mathbf{b} \bullet \mathbf{c}}{|b||c|} \quad .$$



Therefore, between a vector **n**=[hkl], say the normal to a crystal plane, and the **x**= [100] coordinate axes, the (direction) cosine is a_{Ni}= h / |**n**| |**x**| = h / |**n**|.

Examples

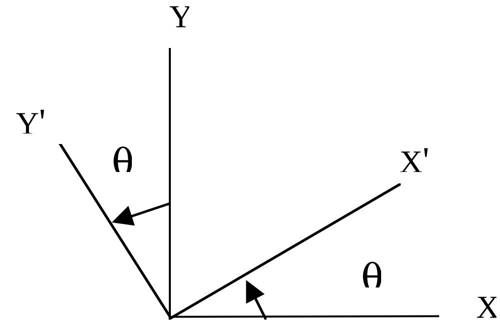
For **n**=[110] and **x**=[100], $\cos\theta = \frac{\hat{\mathrm{i}}\bullet\hat{\mathrm{i}}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, same for **n** and **y**=[010]. Zero for **z**=[001].

For **n**=[111] and **x**=[100], $\cos\theta = \frac{\hat{\mathrm{i}}\bullet\hat{\mathrm{i}}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, and same for **y**=[010] and **z**=[001].

Direction Cosine Matrix and Rotation Matrix

Coordinate Transforms: Rotation of Axes

Consider counterclockwise (positive) rotation, **R(θ)**, about z-axis (out of paper). The components of x' and y' in terms of the old x and y are given by the direction cosines, as above. (Think of x', or y', as just some vector.)



For coordinate transforms, $\underline{\underline{R}}(\theta) = \underline{\underline{a}}$, it is easiest to remember what the appropriate direction cosines, a_{ij}, are from the following aid:

Left index=new
Right index=old

		OLD AXES x	y	z
NEW AXES	x'	a_{11}	a_{12}	a_{13}
	y'	a_{21}	a_{22}	a_{23}
	z'	a_{31}	a_{32}	a_{33}

with , $a_{11}=a_{x'x}=\cos\theta_{x'x}$
$a_{12}=a_{x'y}=\cos\theta_{x'y}$
$a_{21}=a_{y'x}=\cos\theta_{y'x}$
$a_{22}=a_{y'y}=\cos\theta_{y'y}$

From the above picture, the angle (**from** y **to** x') $\theta_{x'y} = \pi/2 - \theta$, the angle (**from** x **to** y') $\theta_{y'x} = \pi/2 + \theta$, whereas the angles (**from** x **to** x' and y **to** y') $\theta_{x'x} = \theta_{x'x} = \theta$. A general vector, **v**, is given in terms of old vector components by the direction cosines of the angles between the two vectors $v'_i = a_{ij}\, v_j$.

The coordinate rotation, **R(θ),** from "OLD" to "NEW" is the direction cosine matrix:

$$R(\theta) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} .$$

Example: 180^0 z-axis i.e., x'= –x y'= –y z'=z

$$R(\theta) = \begin{pmatrix} \cos 180 & \cos 90 & \cos 90 \\ \cos 270 & \cos 180 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For general rotations, the angles are determine by picture above or equation on pg.1.

Direction Cosine Matrix and Rotation Matrix

Similarly, the reverse rotation, $\mathbf{R}^{-1}(\theta)$, from "NEW" to "OLD" is a **new** cosine matrix:

Left index=old
Right index=new

		New AXES x'	y'	z'
OLD AXES	x	$\tilde{a}_{11}$	$\tilde{a}_{12}$	$\tilde{a}_{13}$
	y	$\tilde{a}_{21}$	$\tilde{a}_{22}$	$\tilde{a}_{23}$
	z	$\tilde{a}_{31}$	$\tilde{a}_{32}$	$\tilde{a}_{33}$

with , $\tilde{a}_{11}=\tilde{a}_{xx'}=\cos\theta_{xx'}$
$\tilde{a}_{12}=\tilde{a}_{xy'}=\cos\theta_{xy'}$

$\tilde{a}_{21}=\tilde{a}_{yx'}=\cos\theta_{yx'}$
$\tilde{a}_{22}=\tilde{a}_{yy'}=\cos\theta_{yy'}$

From the picture, notice the sign changes! Compared to before, the angle (**from** y' **to** x) $\theta_{xy'}=\pi/2+\theta$, the angle (**from** x' **to** y) $\theta_{yx'}=\pi/2-\theta$, whereas the angles (**from** x **to** x' and y **to** y') $\theta_{xx'}=\theta_{xx'}=-\theta$. Note that $\mathbf{R}^{-1}(\theta)$ (to go from "NEW" to "OLD") is the transpose of $\mathbf{R}(\theta)$, where the rows (across) and columns (down) exchange places.

$$R^{-1}(\theta)=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}=R^{T}(\theta)$$

Note that $\mathbf{R}(\theta)\mathbf{R}^{-1}(\theta)=\mathbf{1}$. In other words, two rotations, one counterclockwise by θ and the other clockwise by θ, you are back to same "OLD" axes! So, in fact, you know everything in terms of the original a's.

In 2-D, we have

$$R(\theta)=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} \cos\theta & \cos(\frac{\pi}{2}+\theta) \\ \cos(\frac{\pi}{2}-\theta) & \cos\theta \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

We immediately see that $RR^{-1}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since

$\sin^2\theta+\cos^2\theta=1$. And, we have actually returned to old coordinate system, x'=x and y'=y.

Blast from past: Coordinate Transforms from Matrix Methods

From all this, and since $\underline{\mathbf{v}}' = \underline{\underline{\mathbf{R}}}\,\underline{\mathbf{v}}$, we can also reproduce our general coordinate transformation equations. Consider matrix equation $Mv = w$ and rotation, R, of axes:

$$RMv = Rw \rightarrow RM\underbrace{R^{-1}R}_{R^{-1}R=1}v = Rw \rightarrow M'\,v' = w' \quad \text{matrix eq. in new axes}$$

Vectors

Therefore, in new axes, $v'=Rv$ which is $v'_i = a_{ij}\, v_j$ with the labels put on the vector and rotation matrix elements.

Matrices

Similarly, in new axes, $M'=RM\,R^{-1}$, or, with labels, $M'_{ij} = a_{ik}a_{jl}M_{kl}$. Just like stress tensors transform from breaking forces and areas into vector components, or from Mohr's Circle. This is how you did in linear algebra!

Inverse Transformations (NEW to OLD)

The inverse transform arises because $M= R^{-1}\, M'\, R$, or, with labels, $M_{ij} = a_{ki}a_{lj}M'_{kl}$. Indices arise due to how matrices multiply. Recall matrix multiplication does not commute, i.e. $AB \neq BA$ when multiplying matrices.

*Thus, already we know what to expect for transformation equations of symmetric 2nd ranked tensors, like stress and strain. For higher-order tensors, this is not straightforward to think about. But, there is an easier way.