SVD Computation. Example: $W = \begin{bmatrix} 4 & 0 \\ 3 & -5 \\ 0 & 2 \end{bmatrix}$

Recall SVD $= U \Sigma V^T$

$$W^T W = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 \times 4 + 3 \times 3 + 0 \times 0 & 4 \times 0 + 3 \times -5 + 0 \times 2 \\ 0 \times 4 + -5 \times 3 + 2 \times 0 & 0 \times 0 + -5 \times -5 + 2 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -15 \\ -15 & 29 \end{bmatrix}$$

$$WW^T = \begin{bmatrix} 4 & 0 \\ 3 & -5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 \\ 0 & -5 & 2 \end{bmatrix} \qquad 2\times 3 \quad 3\times 2$$

$$\begin{bmatrix} 4\times4+0\times0 & 4\times3+0\times-5 & 4\times0+0\times2 \\ 3\times4+-5\times0 & 3\times3+-5\times-5 & 3\times0+-5\times2 \\ 0\times4+2\times0 & 0\times3+2\times-5 & 0\times0+2\times2 \end{bmatrix}$$

$$\begin{bmatrix} 16 & 12 & 0 \\ 12 & 34 & -10 \\ 0 & -10 & 4 \end{bmatrix} \mathbb{R}^{3\times3}$$

Eigenvalue W^TW

~~Eigenvalue~~ $\cancel{\lambda} = \det(\lambda I - W^TW)$ $\quad$ I for 2x2 matrix

$$\lambda I = \cancel{\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \; \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 25 & 15 \\ 15 & 25 \end{bmatrix}}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 25 & -15 \\ -15 & 29 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-25 & 15 \\ 15 & \lambda-29 \end{bmatrix}$$

$$\det \begin{bmatrix} \lambda - 25 & 15 \\ 15 & \lambda - 29 \end{bmatrix}$$

$$((\lambda-25)\times(\lambda-29)) + (15 \times 15)$$

$$\cancel{\lambda^2 - 29\lambda - 25\lambda + 725 + 225} \quad \lambda^2 - 29\lambda - 25\lambda + 725 - 225$$

$$\lambda^2 - 54\lambda + 500$$

$$\lambda^2 \cancel{- 54\lambda + 950}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$

$b = -54$

$c = \cancel{950}\ 500$

$$\frac{-(-54) \pm \sqrt{54^2 - 4(1)(500)}}{2(1)}$$

$$= \frac{54 \pm \sqrt{2916 - 2000}}{2}$$

$$= \frac{54 \pm \sqrt{916}}{2} = 42.14,\ 11.86$$

Eigenvalue for $LoLo^T$ = $\det(\lambda I - LoLo^T)$ =

Identity matrix of $\mathbb{R}^{3\times3}$ = $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$\lambda = 0, 42.14, 11.86$

$$-\begin{bmatrix} \lambda-16 & -12 & 0 \\ -12 & \lambda-34 & 10 \\ 0 & 10 & \lambda-4 \end{bmatrix}$$

Singular Values, $\sigma_1 = \sqrt{42.14} = 6.49$ $\sigma_2 = \sqrt{11.86} = 3.44$

Compute V on W^TW = $(W^TW - \lambda I)V = 0$

$$\begin{bmatrix} 25-42.14 & -15 \\ -15 & 29-42.14 \end{bmatrix} = \begin{bmatrix} 17.14 & -15 \\ -15 & -13.14 \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$$

$-17.14V_{11} + 15V_{12} = 0$ — (i)

$-15V_{11} - 13.14V_{12} = 0$ — (ii)

from eqn 1

$$V_{11} = \frac{-15}{17.14} V_{12} = -0.875 V_{12}$$

$-15(-0.875) - 13.14 V_{12} = 0$

$V_{12} = 1$

$V_{11} = -0.875$

$$\begin{bmatrix} -0.875 \\ 1 \end{bmatrix}$$

$$\|V_1\| = \sqrt{-0.875^2 + 1^2} = 1.33$$

$$1/1.33 \begin{bmatrix} -0.875 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.657 \\ 0.751 \end{bmatrix}$$

$$\begin{bmatrix} 25-11.86 & -15 \\ -15 & 29-11.86 \end{bmatrix} = \begin{bmatrix} 13.14 & -15 \\ -15 & 17.14 \end{bmatrix}$$

$$\|V_2\| = \begin{bmatrix} 0.752 \\ 0.659 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.657 & 0.752 \\ 0.751 & 0.659 \end{bmatrix}$$

U = do same as V where $WW^T = U$

$$U = \begin{bmatrix} 0.576 & 0.638 \\ 0.817 & -0.374 \\ 0.029 & 0.677 \end{bmatrix}$$

Σ is the singular value in a diagonal matrix

$$\begin{bmatrix} 6.49 & 0 \\ 0 & 3.44 \\ 0 & 0 \end{bmatrix}$$

Recall: $W = U \Sigma V^T$

But for Low rank Adaptation

$$W_{low-rank} = \sigma_1 U_1 V_1^T$$

$$6.49 \times \begin{bmatrix} 0.576 \\ 0.817 \\ 0.029 \end{bmatrix} \times \begin{bmatrix} -0.657 & 0.751 \end{bmatrix}$$

$$= \begin{bmatrix} 6.49 \times 0.576 \times 0.657 & 6.49 \times 0.576 \times 0.751 \\ 6.49 \times 0.817 \times -0.657 & 6.49 \times 0.817 \times 0.751 \\ 6.49 \times 0.029 \times -0.657 & 6.49 \times 0.029 \times 0.751 \end{bmatrix}$$

$$= \begin{bmatrix} -2.39 & 2.74 \\ -3.47 & 3.96 \\ -0.13 & 0.15 \end{bmatrix}$$ for Rank 1

Rank 2: $3.44 \times \begin{bmatrix} 0.638 \\ -0.374 \\ 0.677 \end{bmatrix} \times \begin{bmatrix} 0.752 & 0.657 \end{bmatrix}$

$$= \begin{bmatrix} 3.44 \times 0.638 \times 0.752 & 3.44 \times 0.638 \times 0.657 \\ 3.44 \times 0.374 \times 0.752 & 3.44 \times 0.374 \times 0.657 \\ 3.44 \times 0.677 \times 0.752 & 3.44 \times 0.677 \times 0.657 \end{bmatrix}$$

$$\begin{bmatrix} 1.65 & 1.49 \\ -0.97 & -0.85 \\ 1.75 & 1.53 \end{bmatrix}$$

full matrix rank

$$\text{low rank} = \sigma_1 U_1 V_1^T + \sigma_2 U_2 V_2^T$$

$$= \begin{bmatrix} -2.39 & 2.74 \\ -3.47 & 3.96 \\ 0.13 & 0.15 \end{bmatrix} + \begin{bmatrix} 1.65 & 1.49 \\ -0.97 & -0.85 \\ 1.75 & 1.53 \end{bmatrix}$$