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Lesson 1

1. (Equivalent definition of continuity): A map f is continuous if and only if the preimage of any closed set is closed.

Solution

Let $f : X \to Y$ be a map between topological spaces.

Assume f is continuous,

Let $C \subseteq Y$ be a closed set. Then $Y \setminus C$ is open. Since f is continuous, the preimage $f^{-1}(Y \setminus C)$ is open.
By properties of set preimages,

$$f^{-1}(Y \setminus C) = X \setminus f^{-1}(C).$$

Since the complement $X \setminus f^{-1}(C)$ is open, it follows that $f^{-1}(C)$ is closed.

Assume that for every closed set $C \subseteq Y$, $f^{-1}(C)$ is closed,

Let $U \subseteq Y$ be an open set. Then $Y \setminus U$ is closed. By assumption, the preimage $f^{-1}(Y \setminus U)$ is closed.
By properties of set preimages,

$$f^{-1}(Y \setminus U) = X \setminus f^{-1}(U).$$

Since the complement $X \setminus f^{-1}(U)$ is closed, $f^{-1}(U)$ is open.
This shows that f is continuous.

Since we have shown the implication holds in both directions, the statement, f is continuous if and only if the preimage of any closed set is closed is true.

2. Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in B(x, r)$. Show that

$$B\left(y, r - \|x - y\|\right) \subseteq B(x, r).$$

Solution

Lets consider an open ball,

$$B(x, r) = \{z \in \mathbb{R}^n : \|x - z\| < r\}.$$

Now, let $z \in B(y, r - \|x - y\|)$, where,

$$B(y, r - \|x - y\|) = \{z \in \mathbb{R}^n : \|y - z\| < r - \|x - y\|\}.$$

We check $\|x - z\|$
Using the triangle inequality,

$$\begin{aligned}
&\|x - z\| \leq \|x - y\| + \|y - z\| \\
\text{Substituting} \quad &\|y - z\| \text{ from the definition of } B(y, r - \|x - y\|), \\
&\|x - z\| < \|x - y\| + (r - \|x - y\|) \\
\text{Simplifying,} \quad &\|x - z\| < \|x - y\| + r - \|x - y\| \\
\text{Thus,} \quad &\|x - z\| < r
\end{aligned}$$

which implies $z \in B(x, r)$
Since,

$$z \in B(y, r - \|x - y\|) \quad \textit{and} \quad y \in B(x, r)$$

we conclude that,

$$B(y, r - \|x - y\|) \subset B(x, r).$$

3. Show that the open balls $B(x, r)$ of $\mathbb{R}^n$ are open sets (with respect to the Euclidean topology).

Solution

Let $y \in B(x, r)$. By definition of the open ball, $\|y - x\| < r$

$$\text{Define} \quad \delta = r - \|y - x\|$$

so that $\delta > 0$
Now consider the open ball,

$$B(y, \delta) = \{z \in \mathbb{R}^n : \|z - y\| < \delta\}.$$

For any $z \in B(y, \delta)$, we have,

$$\|z - x\| \leq \|z - y\| + \|y - x\| \quad \text{(by the triangle inequality)}$$

Since,

$$\|z - y\| < \delta = r - \|y - x\|$$

it follows that,

$$\|z - x\| < (r - \|y - x\|) + \|y - x\|$$

$$\|z - x\| < r$$

Thus, $z \in B(x, r)$, and we have shown that

$$B(y, \delta) \subseteq B(x, r).$$

For every $y \in B(x, r)$, there exists an open ball $B(y, \delta)$ centered at y that is contained within $B(x, r)$. This demonstrates that $B(x, r)$ is an open set.

4. Let $x, y \in \mathbb{R}^n$, and $r = \|x - y\|$. Show that

$$B\left(\frac{x+y}{2}, \frac{r}{2}\right) \subseteq B(x,r) \cap B(y,r)$$

Solution

$$\text{For} \quad B\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset B(x,r) \cap B(y,r),$$

it must be true that,

$$B\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset B(x,r) \quad \text{and} \quad B\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset B(y,r)$$

Proof for $B\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset B(x,r)$

Let $z \in B\left(\frac{x+y}{2}, \frac{r}{2}\right)$, where,

$$B\left(\frac{x+y}{2}, \frac{r}{2}\right) = \left\{z \in \mathbb{R}^n : \left\|z - \frac{x+y}{2}\right\| \leq \frac{r}{2}\right\}$$

If $z \in B(x,r)$, then,

$$B(x,r) = \{z \in \mathbb{R}^n : \|z - x\| \leq r\}$$

Using the triangle inequality,

$$\|z - x\| \leq \left\|z - \frac{x+y}{2}\right\| + \left\|\frac{x+y}{2} - x\right\|$$

$$\|z - x\| \leq \left\|z - \frac{x+y}{2}\right\| + \left\|\frac{x+y}{2} - \frac{2x}{2}\right\|$$

Thus,

$$\|z - x\| \leq \left\|z - \frac{x+y}{2}\right\| + \frac{1}{2}\|y - x\|$$

Since $\|y - x\| = r$ and $\left\|z - \frac{x+y}{2}\right\| < \frac{r}{2}$,

$$\|z - x\| < \frac{r}{2} + \frac{r}{2}$$

$$\|z - x\| < r$$

Therefore, $z \in B(x,r)$

Proof for $B\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset B(y,r)$

Let $z \in B\left(\frac{x+y}{2}, \frac{r}{2}\right)$
Then

$$B(y,r) = \{z \in \mathbb{R}^n : \|z - y\| \leq r\}$$

Using the triangle inequality,

$$\|z - y\| \leq \left\|z - \frac{x+y}{2}\right\| + \left\|\frac{x+y}{2} - y\right\|$$

Simplifying,

$$\|z - y\| \leq \left\|z - \frac{x+y}{2}\right\| + \frac{1}{2}\|x - y\|.$$

Since $\|x - y\| = r$ and $\left\|z - \frac{x+y}{2}\right\| < \frac{r}{2}$:

$$\|z - y\| < \frac{r}{2} + \frac{r}{2}$$

$$\|z - y\| < r$$

Therefore, $z \in B(y,r)$.

Conclusion

Since $z \in B\left(\frac{x+y}{2}, \frac{r}{2}\right)$ implies $z \in B(x,r)$ and $z \in B(y,r)$, we conclude that,

$$B\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset B(x,r) \cap B(y,r).$$

5. Show that the set of rational numbers is not an open subset of $\mathbb{R}$

Let $\mathbb{Q}$ be the set of rational numbers. Suppose, for contradiction, that $\mathbb{Q}$ is an open subset of $\mathbb{R}$.

Proof by Contradiction

If $\mathbb{Q}$ is open, then for any $q \in \mathbb{Q}$, there exists $\epsilon > 0$ such that the open interval $(q-\epsilon, q+\epsilon) \subseteq \mathbb{Q}$.
However, it is a known property of the real numbers that between any two real numbers, there exists an irrational number.
Therefore, for any such $\epsilon > 0$, the interval $(q - \epsilon, q + \epsilon)$ will contain at least one irrational number.
This contradicts the assumption that $(q - \epsilon, q + \epsilon) \subseteq \mathbb{Q}$, since the presence of an irrational number implies that the interval cannot be fully contained within $\mathbb{Q}$

Thus, our assumption that $\mathbb{Q}$ is an open subset of $\mathbb{R}$ leads to a contradiction.
Therefore, $\mathbb{Q}$ is not an open subset of $\mathbb{R}$.

6. Show that a map is continuous if and only if the preimage of closed sets are closed sets

Solution

Let $f : X \to Y$ be a map between topological spaces.

Assume f is continuous,

Let $C \subseteq Y$ be a closed set. Then $Y \setminus C$ is open. Since f is continuous, the preimage $f^{-1}(Y \setminus C)$ is open.
By properties of set preimages,

$$f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$$

Since the complement $X \setminus f^{-1}(C)$ is open, it follows that $f^{-1}(C)$ is closed.

Assume that for every closed set $C \subseteq Y$, $f^{-1}(C)$ is closed,

Let $U \subseteq Y$ be an open set. Then $Y \setminus U$ is closed. By assumption, the preimage $f^{-1}(Y \setminus U)$ is closed.
By properties of set preimages,

$$f^{-1}(Y \setminus U) = X \setminus f^{-1}(U).$$

Since the complement $X \setminus f^{-1}(U)$ is closed, $f^{-1}(U)$ is open.
This shows that f is continuous.

Since we have shown the implication holds in both directions, the statement, f is continuous if and only if the preimage of any closed set is closed is true.

Lesson 2

1. Show that the following spaces are homeomorphic:

(i) Let $a \neq b$. Show that the interval $[0,1]$ is homeomorphic to $[a,b]$.

Proof:
Define the function $f:[0,1] \to [a,b]$ by

$$f(x) = (b-a)x + a.$$

Continuity: f is a linear function and thus continuous.

Bijective: Since $a \neq b$, f has a non-zero slope and is injective. The range of f is $[a,b]$ because $f(0) = a$ and $f(1) = b$. Thus f is bijective.

Inverse Continuous: The inverse is

$$f^{-1}(x) = \frac{x-a}{b-a},$$

which is also a linear function and thus continuous.

Since f is bijective and both f and f^{-1} are continuous, f is a homeomorphism, and therefore

$$[0,1] \cong [a,b]$$

.

(ii) The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is homeomorphic to the interval $(0,1)$.

Proof:

To show that two spaces are homeomorphic, we need to find a bijective continuous function between them that has a continuous inverse.

Define a function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (0,1)$ as follows:

$$f(x) = \frac{1}{2}\left(\frac{x}{\frac{\pi}{2}} + 1\right) = \frac{x}{\pi} + \frac{1}{2}.$$

This function is a linear transformation that maps the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to the interval $(0,1)$.

The function $f(x) = \frac{x}{\pi} + \frac{1}{2}$ is a linear function. Linear functions are continuous everywhere, and thus f is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Injective (One-to-one): Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then:

$$\frac{x_1}{\pi} + \frac{1}{2} = \frac{x_2}{\pi} + \frac{1}{2}.$$

Simplifying:

$$\frac{x_1}{\pi} = \frac{x_2}{\pi} \implies x_1 = x_2.$$

Therefore, f is injective.

Surjective (Onto): For any $y \in (0, 1)$, we need to show that there exists an $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f(x) = y$. Let's solve for x in the equation $y = \frac{x}{\pi} + \frac{1}{2}$:

$$y - \frac{1}{2} = \frac{x}{\pi} \implies x = \pi\left(y - \frac{1}{2}\right).$$

Since $0 < y < 1$, we have $-\frac{1}{2} < y - \frac{1}{2} < \frac{1}{2}$, and multiplying by π, we get:

$$-\frac{\pi}{2} < \pi\left(y - \frac{1}{2}\right) < \frac{\pi}{2}.$$

Thus, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and f is surjective.

Since f is both injective and surjective, it is a bijection.

We already solved for x in terms of y in the surjectivity argument. The inverse function $f^{-1} : (0, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is given by:

$$f^{-1}(y) = \pi\left(y - \frac{1}{2}\right) = \pi y - \frac{\pi}{2}.$$

The inverse function is a linear function, therefore f^{-1} is continuous on $(0, 1)$.

Since:

- $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (0, 1)$ is a bijection,
- f is continuous, and
- f^{-1} is continuous,

the function f is a homeomorphism. Therefore, the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is homeomorphic to the interval $(0, 1)$, denoted as:

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cong (0, 1).$$

(iii) The interval $(-1, 1)$ is homeomorphic to $\mathbb{R}$.

Proof:
Define the function $f : (-1, 1) \to \mathbb{R}$ as

$$f(x) = \tan\left(\frac{\pi x}{2}\right).$$

Continuity: $\tan\left(\frac{\pi x}{2}\right)$ is continuous for $x \in (-1, 1)$.

Bijective: The range of $\tan(x)$ is all real numbers, and when restricted to the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$, the tan function is bijective. The map $\frac{\pi x}{2}$ maps $(-1, 1)$ bijectively to $(-\frac{\pi}{2}, \frac{\pi}{2})$, so $f(x)$ is also bijective.

Inverse Continuous: The inverse is

$$f^{-1}(y) = \frac{2}{\pi} \arctan(y),$$

which is continuous.

Since f is bijective and both f and f^{-1} are continuous, f is a homomorphism, and therefore

$$(-1, 1) \cong \mathbb{R}$$

.

2. Show that the function $f : \mathbb{R}^n \to [0, \infty)$ defined by

$$f(x) = \|x\|$$

is continuous

Solution

Given an arbitrary $x_0 \in \mathbb{R}^n$ and an arbitrary $\epsilon > 0$.

we need to find $\delta > 0$ such that if $\|x - x_0\| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Now let $\delta = \epsilon$.

Assume $\|x - x_0\| < \delta$
Then, using the reverse triangle inequality,

$$|f(x) - f(x_0)| = \big|\|x\| - \|x_0\|\big| \leq \|x - x_0\|$$

Since $\|x - x_0\| < \delta$,
we have,

$$|f(x) - f(x_0)| \leq \|x - x_0\| < \delta$$

Substituting $\delta = \epsilon$, it follows that,

$$|f(x) - f(x_0)| < \epsilon.$$

For all $\epsilon > 0$, there exists $\delta > 0$ (specifically $\delta = \epsilon$)
such that if $\|x - x_0\| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
Thus, $f(x) = \|x\|$ is continuous on $\mathbb{R}^n$.

3. Consider the sets

$$D = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}, \quad S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}.$$

(a) Let $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$. Show that $f : D \setminus S^1 \to \mathbb{R}^2$ defined by

$$f(x) = \frac{x}{1 - \|x\|}$$

is a homeomorphism.

Solution

Let $x, y \in D \setminus S^1$. Suppose

$$f(x) = f(y) \implies \frac{x}{1 - \|x\|} = \frac{y}{1 - \|y\|}.$$

Cross-multiplying gives,

$$x(1 - \|y\|) = y(1 - \|x\|)$$

which simplifies to

$$x - x\|y\| = y - y\|x\| \implies x = y.$$

Thus,

$$f \quad \text{is injective}$$

.
Also letting $y \in \mathbb{R}^2$,
we solve,

$$y = \frac{x}{1 - \|x\|}.$$

Taking the norm, we get

$$\|y\| = \frac{\|x\|}{1 - \|x\|} \implies \|x\| = \frac{\|y\|}{1 + \|y\|}.$$

Since

$$\|x\| < 1, \quad x \in D \setminus S^1$$

$$\text{Thus, } f \quad \text{is surjective.}$$

Let $\epsilon > 0$ and $x_0 \in D \setminus S^1$.
Choose

$$\delta = \frac{\epsilon}{(1 - \|x\|)(1 - \|x_0\|)}$$

.
For $\|x - x_0\| < \delta$, we need to show

$$\|f(x) - f(x_0)\| < \epsilon.$$

We have

$$\|f(x) - f(x_0)\| = \left\| \frac{x}{1 - \|x\|} - \frac{x_0}{1 - \|x_0\|} \right\|.$$

Using the reverse triangle inequality,

$$\|f(x) - f(x_0)\| \leq \frac{1}{(1 - \|x\|)(1 - \|x_0\|)} \left(\|x - x_0\| + \|x\|\|x_0\| \left\| \frac{x_0}{\|x_0\|} - \frac{x}{\|x\|} \right\| \right).$$

Since $\|x - x_0\| < \delta$, we conclude $\|f(x) - f(x_0)\| < \epsilon$.

Thus, f is continuous.

Let $g : \mathbb{R}^2 \to D \setminus S^1$, where,

$$g(y) = \frac{y}{1 + \|y\|}.$$

Then for $y \in \mathbb{R}^2$,

$$f(g(y)) = f\left(\frac{y}{1 + \|y\|}\right) = \frac{\frac{y}{1 + \|y\|}}{1 - \left\|\frac{y}{1 + \|y\|}\right\|} = y.$$

Similarly, for $x \in D \setminus S^1$,

$$g(f(x)) = g\left(\frac{x}{1 - \|x\|}\right) = \frac{\frac{x}{1 - \|x\|}}{1 + \left\|\frac{x}{1 - \|x\|}\right\|} = x.$$

Thus, $f : D \setminus S^1 \to \mathbb{R}^2$ is a homeomorphism.

(b) Show that the map $g : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ defined by

$$g(x) = \frac{1}{1 - x_3}(x_1, x_2)$$

is a homeomorphism.

Solution

The map is defined for all points $(x_1, x_2, x_3) \in S^2 \setminus \{(0,0,1)\}$.

The components x_1, x_2, and x_3 of the input vector x are continuous functions of x.
The function $1 - x_3$ is also a continuous function of x.
Since $1 - x_3 \neq 0$, the function $1/(1 - x_3)$ is continuous.
The map $g(x)$ is a combination of continuous functions, so $g(x)$ is continuous.

Suppose $g(x) = g(y)$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then:

$$\frac{1}{1 - x_3}(x_1, x_2) = \frac{1}{1 - y_3}(y_1, y_2)$$

This implies:

$$\frac{x_1}{1 - x_3} = \frac{y_1}{1 - y_3}, \quad \frac{x_2}{1 - x_3} = \frac{y_2}{1 - y_3}$$

We also know that

$$x_1^2 + x_2^2 + x_3^2 = 1$$

and

$$y_1^2 + y_2^2 + y_3^2 = 1$$

. Let $(u, v) = \dfrac{(x_1, x_2)}{1 - x_3}$.
Solving for x_1, x_2, and x_3 in terms of u and v, we get:

$$x_3 = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}, \quad x_1 = \frac{2u}{u^2 + v^2 + 1}, \quad x_2 = \frac{2v}{u^2 + v^2 + 1}$$

Since these expressions uniquely determine x, g is injective.

Using the inverse map,

$$g^{-1}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right)$$

We can verify that $x_1^2 + x_2^2 + x_3^2 = 1$ and $x_3 \neq 1$.
Hence, g is surjective.

Since the components of g^{-1} are rational functions of u and v with a nonzero denominator, they are continuous. Therefore, g^{-1} is continuous.

Since g is continuous, bijective and g^{-1} is continuous.

$$\therefore g \text{ is a homeomorphism.}$$

Lesson 3

1. Definition: Let X be a topological space, and $x, y \in X$. A path between the points x and y is a continuous function $\eta : [0,1] \to X$ such that $\eta(0) = x$ and $\eta(1) = y$. Show that the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is path-connected for $n \geq 1$

Solution

Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ for $n \geq 1$, and let $x, y \in S^n$ be arbitrary points.

Case 1: $x \neq -y$

Define a path $\eta : [0,1] \to S^n$ by

$$\eta(t) = \frac{(1-t)x + ty}{\|(1-t)x + ty\|}$$

The numerator

$$(1-t)x + ty$$

and the denominator

$$\|(1-t)x + ty\|$$

are continuous functions of t. Since $x \neq -y$, the denominator is never zero.
Therefore, $\eta(t)$ is continuous.

Path Conditions
- $\eta(0) = \dfrac{(1-0)x + 0y}{\|(1-0)x + 0y\|} = \dfrac{x}{\|x\|} = x.$
- $\eta(1) = \dfrac{(1-1)x + 1y}{\|(1-1)x + 1y\|} = \dfrac{y}{\|y\|} = y.$

Thus, the path η connects x to y.

Case 2: $x = -y$

Since $n \geq 1$, there exists a point $z \in S^n$ such that $z \neq x$ and $z \neq y$.
Define a combined path $\eta : [0,1] \to S^n$ by

$$\eta(t) = \begin{cases} \eta_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \eta_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where η_1 is the path from x to z and η_2 is the path from z to y, as in Case 1.

Since both η_1 and η_2 are continuous, and $\eta_1(1) = \eta_2(0)$, the path η is continuous.

- $\eta(0) = \eta_1(0) = x.$
- $\eta(1) = \eta_2(1) = y.$

Thus, the combined path η connects x to y.

In either case, we have shown that a continuous path exists from x to y.

$$\text{Therefore, } S^n \text{ is path-connected for } n \geq 1$$

2. Intermediate Value Theorem: Let X be a connected topological space and let $f : X \to X$ be a continuous function. If $a < b$ are points in $\mathbb{R}$ such that $a = f(x)$ and $b = f(y)$ for some $x, y \in X$, then for each $c \in (a, b)$ there exists $z \in X$ such that $c = f(z)$. Apply the Intermediate Value Theorem to answer the following: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) \cdot f(f(x)) = 2$ for all $x \in \mathbb{R}$ and $f(3) = 10$. Find the value of $f(5)$

Solution

Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x) \cdot f(f(x)) = 2, \quad \forall x \in \mathbb{R},$$

and $f(3) = 10$.
From the given equation, substituting $x = 3$:

$$f(3) \cdot f(f(3)) = 2.$$

Let $f(3) = 10$. Then:

$$10 \cdot f(10) = 2 \quad \Rightarrow \quad f(10) = \frac{1}{5}.$$

This implies:

$$f(10) < f(s) < f(3), \quad \text{where } s \in \mathbb{R}.$$

Thus,

$$\frac{1}{5} < f(s) < 10.$$

using the Intermediate Value Theorem,
there must exist a $z \in \mathbb{R}$ such that,

$$f(z) = 5$$

and

$$z \in (\frac{1}{5}, 10)$$

.
Substituting z into the function,

$$f(z) \cdot f(f(z)) = 2$$

$$5 \cdot f(5) = 2$$

$$\Rightarrow f(5) = \frac{2}{5}$$

3. Let $f : S^n \to \mathbb{R}$ be a continuous function. Show that there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Here, if $x = (x_1, x_2, \ldots, x_{n+1})$, then $-x = (-x_1, -x_2, \ldots, -x_{n+1})$

Solution

Let $f : S^n \to \mathbb{R}$ be a continuous function. Define $g : S^n \to \mathbb{R}$ by

$$g(x) = f(x) - f(-x)$$

Since f is continuous and $-x$ is a continuous function of x, the composition $f(-x)$ is continuous. Since subtraction preserves continuity, g is continuous.

For any $x \in S^n$, we have,

$$g(-x) = f(-x) - f(-(-x)) = f(-x) - f(x) = -(f(x) - f(-x)) = -g(x)$$

Thus, $g(-x) = -g(x)$, so g is an odd function with respect to the antipodal map.

Assume a contradiction that there is no point $x \in S^n$ such that $g(x) = 0$.
Therefore, for all $x \in S^n$, $g(x)$ is either always positive or always negative.

- If $g(x)$ is always positive, then from the antipodal property, $g(-x)$ is always negative.
- If $g(x)$ is always negative, then $g(-x)$ is always positive.

Since S^n is a path-connected space and g is continuous, the image $g(S^n)$ must be a path-connected interval.

Since the image contains both positive and negative values, there must be a value in the image where $g(x) = 0$, which is a contradiction to our assumption that $g(x) \neq 0$ for all $x \in S^n$.

Therefore, there exists some $x \in S^n$ such that $g(x) = 0$, which implies

$$f(x) - f(-x) = 0$$

Thus

$$f(x) = f(-x)$$

Therefore, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.

4. Show that the intervals $[0,1)$ and $(0,1)$ are not homeomorphic

Solution

Let $I_1 = [0,1)$ and $I_2 = (0,1)$ denote the two intervals.

- Let $y \in I_2$. Then $I_2 \setminus \{y\}$ is disconnected, with two connected components: $(0,y)$ and $(y,1)$.
- Let $x \in (0,1)$. Then $I_1 \setminus \{x\}$ is disconnected.
- $I_1 \setminus \{0\}$ is connected (the space $(0,1)$).

Assume there exists a homeomorphism $h : I_1 \to I_2$

Let $z = h(0)$. Since h is a homeomorphism, it maps connected components to connected components.
Therefore, $I_1 \setminus \{0\}$ should be homeomorphic to $I_2 \setminus \{z\}$.

However, since removing 0 from I_1 results in a connected space, and removing z from I_2 results in a disconnected space, this leads to a contradiction.

Since h cannot preserve the connectedness properties, no homeomorphism exists between $[0,1)$ and $(0,1)$.

Therefore, the interval $[0,1)$ is not homeomorphic to $(0,1)$.

5. Show that the closed interval $[0, 1]$ is not homeomorphic to a cross

Solution

Let $I = [0, 1]$ denote the closed interval and let X denote the cross.

- If $x \in (0, 1)$, then $I \setminus \{x\}$ has two connected components: $(0, x)$ and $(x, 1)$.
- If $x = 0$ or $x = 1$, then $I \setminus \{x\}$ has one connected component, either $(0, 1]$ or $[0, 1)$.
- Let p denote the intersection point of the cross X. Then $X \setminus \{p\}$ has four connected components.

Suppose there exists a homeomorphism $h : I \to X$.
Let $p \in X$ be the intersection point.
Let $q = h^{-1}(p) \in I$.
Since h is a homeomorphism, it maps connected components of $I \setminus \{q\}$ to connected components of $X \setminus \{p\}$.

- $I \setminus \{q\}$ has either 1 or 2 connected components, depending on whether q is an endpoint or not, respectively.
- $X \setminus \{p\}$ has four connected components.

Therefore, h cannot be a homeomorphism because it doesn't preserve the number of connected components when a single point is removed.

Therefore, no homeomorphism $h : I \to X$ can exist, so the closed interval $[0, 1]$ is not homeomorphic to the cross X.