INTRODUCTION TO ANALYSIS I

MATH 263 – Lesson V

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Least Upper & Greatest Lower Bounds

1. A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \qquad \text{for all} \quad n \geq 1$$

2. It is **bounded below** if there is a number m such that

$$m \leq a_n \qquad \text{for all} \quad n \geq 1$$

3. It is bounded above and below, then $\{a_n\}$ is a **bounded sequence**
4. A number M is called the **least upper bound** (l.u.b) of the sequence $\{a_n\}$ if $a_n \leq M \;\; n = 1, 2, 3, \ldots$ while at least one term is greater than $M - \epsilon$ for any $\epsilon > 0$
5. A number m is called the **greatest lower bound** (g.l.b) of the sequence $\{a_n\}$ if $a_n \geq m \;\; n = 1, 2, 3, \ldots$ while at least one term is greater than $M + \epsilon$ for any $\epsilon > 0$

- This theorem is a useful variant on themes of using the limits of other sequences that are similar to evaluate/analyze specified sequences.
- Here an unknown sequences is sandwiched between two convergent sequences, allowing us to conclude that the sequences converges. Thus the name **Squeeze Theorem**

Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c except at c itself. If $\lim\limits_{x \to c} g(x) = L = \lim\limits_{x \to c} h(x)$ then

$$\lim_{x \to c} f(x) = L \tag{1}$$

Proof: Squeeze Theorem

Let $\varepsilon > 0$, if we are able to find a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta \tag{2}$$

then we would have proved the theorem. Since $\lim_{x \to c} g(x) = L$, then $\exists \delta_1 > 0$ such that

$$|g(x) - L| < \varepsilon \qquad \text{for all} \qquad 0 < |x - c| < \delta_1$$

Thus

$$-\varepsilon < g(x) - L < \varepsilon \qquad \text{for all} \qquad 0 < |x - c| < \delta_1 \tag{3}$$

so

$$L - \varepsilon < g(x) < L + \varepsilon \qquad \text{for all} \qquad 0 < |x - c| < \delta_1 \tag{4}$$

Proof: Squeeze Theorem

Similarly some $\lim_{x \to c} h(x) = L$: using the definition of limits it means
$\exists! \delta_2 > 0$
such that

$$L - \varepsilon < h(x) < L + \varepsilon \qquad \text{for all} \qquad 0 < |x - c| < \delta_2 \tag{5}$$

Additionally, since $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, there exists some $\delta_3 > 0$ such that

$$g(x) \leq f(x) \leq h(x) \qquad \text{for all} \qquad 0 < |x - c| < \delta_3 \tag{6}$$

Now, we choose $\delta = min(\delta_1, \delta_2, \delta_3)$. Then by (4), (5) and (6),we have

$$L - \varepsilon < g(x) \leq f(x) \leq h(x) L + \varepsilon \qquad \text{for all} \qquad 0 < |x - c| < \delta$$

Proof: Squeeze Theorem

Therefore

$$-\varepsilon < f(x) - L < \varepsilon \qquad \text{for all} \qquad 0 < |x - c| < \delta \tag{7}$$

so

$$|f(x) - L| < \varepsilon \qquad \text{for all} \qquad 0 < |x - c| < \delta \tag{8}$$

Hence by definition of limits $\lim\limits_{x \to c} f(x) = L$

Limit Uniqueness Theorem

- This theorem helps us to establish the uniqueness of limits

Theorem

If $\lim_{n\to\infty} a_n$ exists, then it must be unique

Proof: Limit Uniqueness Theorem

We show in this proof that if $\lim_{n\to\infty} a_n = l_1$ and $\lim_{n\to\infty} a_n = l_2$ then $l_1 = l_2$
By hypothesis, for any $\varepsilon > 0$, we find N such that

$$|a_n - l_1| < \frac{1}{2}\varepsilon \qquad \text{when} \qquad n > N \tag{9}$$

Then

$$|l_1 - l_2| = |l_1 - a_n + a_n - l_2| \le |l_1 - a_n| + |a_n - l_2| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \tag{10}$$

$|l_1 - l_2|$ is less than any positive(however small) and so must be zero.
Thus $l_1 = l_2$

Sum of Limits Theorem

Theorem

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then $\lim_{n\to\infty}(a_n + b_n) = A + B$

Proof: Sum of Limits Theorem

We must show that for any $\epsilon > 0$, we can find $N > 0$ such that

$$|(a_n + b_n) - (A + B)| < \varepsilon \qquad \text{for all} \qquad n > N \tag{11}$$

$$|(a_n + b_n) - (A + B)| = |(a_n - A) - (b_n - B)| \leq |a_n - A| + |b_n - B| \tag{12}$$

By hypothesis, given $\varepsilon > 0$ we can find N_1 and N_2 such that

$$|a_n - A| < \frac{1}{2}\varepsilon \qquad \text{for all} \qquad n > N_1 \tag{13}$$

$$|b_n - B| < \frac{1}{2}\varepsilon \qquad \text{for all} \qquad n > N_2 \tag{14}$$

From (12), (13), (14)

$$|a_n + b_n) - (A + B)| < \frac{1}{2}\epsilon + \frac{1}{2}\varepsilon = \varepsilon \tag{15}$$

where N is chosen as the larger of N_1 and N_2. Hence the proof.

- It is important to note here that the converse of this theorem is not true
- We can find examples of bounded sequences which have nonexistent limits

Theorem

Every Convergent Sequence is Bounded

Proof:

Given that $\lim_{n\to\infty} a_n = A$, we must show that there exists a positive number p such that $|a_n| < p$ for all n. Now

$$|a_n| = |a_n - A + A| < |a_n - A| + |A| \tag{16}$$

By hypothesis we can find N such that $|a_n - A| < \varepsilon$ for all $n > N$ that is

$$|a_n| < \varepsilon + |A| \qquad \text{for all} \qquad n > N \tag{17}$$

It follows that $|a_n| < p$ for all n if we choose p as the largest one of the numbers $a_1, a_2, a_3, ..., a_n,\ \varepsilon + |A|$

Proof: Limits Product Theorem

Theorem

If

$$\lim_{n \to \infty} a_n = A$$

and

$$\lim_{n \to \infty} b_n = B$$

, then

$$\lim_{n \to \infty} a_n b_n = AB$$

Proof: Limit Product Theorem

We start by having

$$
\begin{aligned}
|a_n b_n - AB| &= |a_n(b_n - B) + B(a_n - A)| && (18)\\
&\leq |a_n||(b_n - B)| + |B||(a_n - A)| && (19)\\
&\leq |P(b_N - B) + (|B| + 1)(a_n - A)| && (20)
\end{aligned}
$$

- But since $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$, given any $\epsilon > 0$ we can find N_1 and N_2 such that
  $|a_n - A| < \dfrac{\varepsilon}{2p}$ for all $n > N_1$, $|a_n - A| < \dfrac{\varepsilon}{2(|B|+1)}$ for all $n > N_2$
- Hence from (18) $|a_n b_n - AB| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ for all $n > N$,
- where N is the largest of N_1 and N_2. Hence the proof.

Proof: Limits Quotient Theorem

Theorem

Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences. Suppose further $b_n \neq 0$ or all n and that

$$\lim_{n\to\infty} b_n \neq 0$$

. Then

a. $\displaystyle \lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n}$ b. $\displaystyle \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

Proof: Limit Quotient Theorem

- We start with the proof of Part (a).
- Suppose that $\lim\limits_{a_n \to \infty} = A$ and $\lim\limits_{b_n \to \infty} = B$, we start the first part by showing that for any given $\varepsilon > 0$, we can find N such that

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|B - b_n|}{|B||b_n|} < \varepsilon \qquad \text{for all} \qquad n > N \tag{21}$$

- By hypothesis, given any $\varepsilon > 0$, we can find $N1$, such that $|b_n - B| < \frac{1}{2}B^2\varepsilon$ for all $n > N_1$. Also since $\lim\limits_{n \to \infty} b_n = B \neq 0$, we can find $|b_n| > \frac{1}{2}|B|$ for all $n > N_2$. Then if $N = max\{N_1, N_2\}$, we can write (21) as

$$|\frac{1}{b_n} - \frac{1}{B}| = \frac{|b_n - B|}{|B||b_n|} < \frac{\frac{1}{2}B^2\varepsilon}{|B|\frac{1}{2}|B|} = \varepsilon \qquad \text{for all} \qquad n > N$$

- Hence the proof

Proof: Limit Quotient Theorem

- We then consider the Part (b).
- From the first Part (a) and the theorem on the product of limits, we can write

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} (a_n \cdot \frac{1}{b_n}) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = A \cdot \frac{1}{B} = \frac{A}{B} \qquad (22)$$

- Hence the proof

Monotonic Sequence Theorem

Theorem

Every bounded, monotonic sequence is convergent

Proof: Monotonic Sequence Theorem

- Suppose $\{a_n\}$ is an increasing sequence
- Since $\{a_n\}$ is bounded $\Rightarrow S = \{a_n | n \geq 1\}$ has an upper bound
- By the **Completeness Axiom** S has an upper bound, i.e.

$$a_N > L - \epsilon \quad \text{for some integer } N \tag{23}$$

- Since $\{a_n\}$ is increasing $\Rightarrow a_n \geq a_N$ for every $n > N$. Thus

$$a_n > L - \epsilon \quad \Rightarrow \quad 0 \leq L - a_n < \epsilon \tag{24}$$

- Since $a_n \leq L$ then

$$|L - a_n| < \epsilon \quad \text{whenever } n > N \tag{25}$$

Further Theorems on Monotone Sequences

Suppose that $\{a_n\}$ is a monotonic sequence. Then $\{a_n\}$ is convergent if and only if $\{a_n\}$ is bounded. More specifically

1. If $\{a_n\}$ is non-decreasing then either $\{a_n\}$ is bounded and converges to $\sup\{a_n\}$ or else $\{a_n\}$ is unbounded and $a_n \to \infty$
2. $\{a_n\}$ is non-increasing then either $\{a_n\}$ is bounded and converges to $\inf\{a_n\}$ or else $\{a_n\}$ is unbounded and $a_n \to -\infty$

Proof:

- If the sequence is unbounded, then it diverges, this applies to all sequences and not only monotonic sequences.

- We then have to show that for any bounded sequence $\{a_n\}$, the limit is $\sup\{a_n\}$ in the case where the sequence is non decreasing.
- In the case where the sequence is non-increasing, the it is $\inf\{a_n\}$. We proceed to prove for each of these cases.
- To apply this theorem, suppose a sequence $\{a_n\}$ is given a_n, we recognize it as increasing(or may be just decreasing), then to establish that it converges, we need to show that it is bounded above. That is,. we need to find just one number M with

$$a_n \leq M \qquad \text{for all n}$$

.

Proof:

- Let $\{a_n\}$ be assumed to be non-decreasing and bounded, and let

$$L = sup\{a_n\}$$

- Then $a_n \leq L$ for all n and if $\beta < L$ there must be some term a_m say with $a_m > \beta$. Let $\varepsilon > 0$. We know that there is an m so that

$$a_n \geq a_m > L - \varepsilon$$

  for all $n \geq m$ But we already know that every term $a_n \leq L$

- Combining these together, we have

$$L - \varepsilon < a_n \leq L < L + \varepsilon$$

$$\Rightarrow |a_n - L| < \varepsilon \tag{26}$$

- for all $n \geq m$. By definition then $a_n \to L$ as required.

Example

- Let us show that the sequence

$$a_n = \frac{1}{\sqrt{n}} \qquad \textit{converges}$$

• It is evident that this is a decreasing sequence

• Then we have to find a lower bound.

• Zero(0) is a good choice because all the terms are positive. Therefore zero is a lower bound.

• Consequently, the sequence must converge.

Subsequences

- Consider the sequence

$$a_n = \{1, -1, 2, -2, 3, -3, 4, -4, 5, -5\} \tag{27}$$

This appear to have two sequences

$$a_{n1} = \{1, 2, 3, 4, 5, \} \qquad \text{and} \qquad a_{n2} = \{-1, -2, -3, -4 - 5\} \tag{28}$$

- The latter two sequences a_{n1} and a_{n2} in (28) are known as subsequences of the main sequence a_n in (27).
- The formal mathematical definition of subsequence are given as follows :
  Let $a_1, a_2, a_3, a_4, ...$ be any sequence. Then the subsequence of this sequence is

$$a_{n1}, a_{n2}, a_{n3}, a_4, ... \tag{3}$$

where $n_1 < n_2 < n_3 < n_4$ is an increasing sequence of natural numbers.

Monotonic Subsequences

Theorem

- Every sequence contains a monotonic subsequence

Proof: Monotonic Subsequences

- We start by constructing a non-increasing subsequence. We call the m^{th} element x_m of the sequence $\{x_n\}$ a turn-back point if all the later elements are less than or equal to it, symbolically $x_m \geq x_n$ for all $n > m$.
- If these is an infinite sequence of turn-back points $a_{m1}, a_{m2}, a_{m3}, a_{m4}, \ldots$ then we have found our non-increasing subsequence since

$$a_{m1} \geq a_{m2} \geq a_{m3} \geq a_{m4}, \geq \cdots$$

- This would not be possible if there are only finitely many turn-points.
- Let us suppose that x_m is the last turn-back points so that any element x_n for $n > M$ is not a turn-back point.

Proof: Monotonic Subsequences

- Since if it is not, there must be an element further in the sequence greater than it, in symbols $x_m > x_n$ for some $m > n$.

- Thus we can choose $x_{m1} > x_{M+1}$ with $m > M + 1$ then $x_{m2} > x_{m1}$ with $m_2 > m_1$ and then $x_{m3} > x_{m2}$ with $m_3 > m_2$

- So we can obtain an increasing subsequence

$$x_{M+1} < x_{m1} < x_{m2} < x_{m3} < x_{m4} < \cdots \tag{29}$$

Key Notes

- Based on the monotone convergence criterion, we were able to show that a sequence can be proved to converge without knowing its limit.
- Apart from this criterion there are other criteria that can lead us to the same conclusion.
- The following are provisions for the presentation of another criteria.
- Suppose that a sequence $\{x_n\}$ converges to x. Then for $\varepsilon > 0$, $\exists N$ such that $|x_n - x_m| < \frac{\varepsilon}{2}$ for all $n, m > N$ we have

$$|x_n - x_m| = |x_n - x + x - x_m| < |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

- The conclusion therefore is as follows :
  If a sequence x_n converges then it satisfies the **Cauchy' Criterion** :
  for $\varepsilon > 0$, there exists N such that $|x_n - x_m| < \varepsilon$ for all $n, m > N$

Key Notes

- If a sequence converges then the elements of the sequence get close to the limit as n increases.
- In case of a sequence satisfying Cauchy's Criterion, the elements get close to each other as m, n increases.
- **Note also** that a sequence satisfying Cauchy's Criterion is a bounded sequence with some additional property
- Moreover, it seems to intuitively converge.
- We will show that a sequence satisfying Cauchy's Criterion does converge.

Nested Interval Theorem

Theorem

- For each n, let $I_n = [a_n, b_n]$ be a (non-empty) bounded interval of real numbers such that

$$I_1 \supset I_2 \supset I_3 \supset \ldots \supset I_n \supset I_{n+1} \ldots$$

and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n$ contains only one point.

Proof: Nested Interval Theorem

- We start by defining $[a_n, b_n]$, $n = 1, 2, 3, \cdots$ as nested interval.
- Then by this definition, $a_{n+1} \geq a_n$, b_{n+1}, $b_{n+1} \leq b_n$, $n = 1, 2, 3, \ldots$ and lim $\lim_{n\to\infty}(a_n - b_n) = 0$
- Then $a_1 \leq a_n \leq b_n \leq b_1$ and the sequences $\{a_n\}$ and $\{b_n\}$ are bounded with $\{a_n\}$ being monotonic increasing and $\{b_n\}$ being monotonic decreasing which converges to a and b respectively.
- To show that $a = b$ and thus prove the required result, we note that

$$b - a = (b - b_n) + (b_n - a_n) + (a_n - a) \tag{30}$$

$$|b - a| \leq |b - b_n| + |b_n - a_n| + |a_n - a| \tag{31}$$

- Now given $\varepsilon > 0$, we can find N such that for all $n > N$

$$|b - b_n| < \frac{\varepsilon}{3}, \quad |b_n - a_n| < \frac{\varepsilon}{3}, \quad |a_n - a| < \frac{\varepsilon}{3} \tag{32}$$

- Therefore from (30), (31), (32), $|b - a| < \varepsilon$. Since ε is any positive number, we must have interval theorem.

Theorem

- Every bounded sequence in $\mathbb{R}$ has a convergent subsequence

Proof: Bolzano-Weierstrass Theorem

- We prove this in two parts.
- First we show that the intersection of nested intervals converge to a single point and secondly show that a subsequence converges to that same point.
- Let $\{w_n\}$ be a bounded sequence. Then, there exists an interval $[a_1, b_1]$ such that $a_1 \leq w_n \leq b_1$ for all n.
- This means that

$$\left[a_1, \frac{a_1 + b_1}{2}\right] \qquad \text{or} \qquad \left[\frac{a_1 + b_1}{2}, b_1\right] \tag{4}$$

contains infinitely many terms of $\{w_n\}$. That is, there exists infinitely many n in I, such that $\{w_n\}$ is in $\left[a_1, \frac{a_1 + b_1}{2}\right]$ or there exists infinitely many n in I, such that $\{w_n\}$ is in $\left[\frac{a_1 + b_1}{2}, b_1\right]$.

Proof: Bolzano-Weierstrass Theorem

- If $\left[a_1, \dfrac{a_1 + b_1}{2}\right]$ contains infinitely many terms of $\{w_n\}$ let $[a_2, b_2] = \left[a_1, \dfrac{a_1 + b_1}{2}\right]$. Otherwise let $[a_2, b_2] = \left[\dfrac{a_1 + b_1}{2}, b_1\right]$.
- We continue to show that either $\left[a_2, \dfrac{a_2 + b_2}{2}\right]$ or $\left[\dfrac{a_2 + b_2}{2}, b_2\right]$ contains infinitely many terms of $\{w_n\}$. If $\left[a_2, \dfrac{a_2 + b_2}{2}\right]$ contains infinitely many terms of $\{w_n\}$, let $[a_3, b_3] = \left[a_2, \dfrac{a_2 + b_2}{2}\right]$.
- Otherwise, let $[a_3, b_3] = \left[\dfrac{a_2 + b_2}{2}, b_2\right]$.

Proof: Bolzano-Weierstrass Theorem

- By mathematical induction, we can continue this construction and obtain a sequence of interval $\{[a_n, b_n]\}$ such that for each n

1. $[a_n, b_n]$ contains infinitely many terms of $\{w_n\}$
2. $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
3. $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$

- Based on the nested interval theorem, it implies that the intersection of the intervals $[a_n, b_n]$ is a single point w.

Proof: Bolzano-Weierstrass Theorem

- We will now proceed to construct a subsequence of $\{w_n\}$ which will converge to w.
- Since $[a_1, b_1]$ contains infinitely many terms of $\{w_n\}$, there exists k_1 in I such that w_{k1} is in $[a_1, b_1]$.
- Since $[a_2, b_2]$ contains infinitely many terms of $\{w_n\}$, there exists k_2 in I, $k_2 > k_1$, such that w_{k2} is in $[a_2, b_2]$.
- Since $[a_3, b_3]$ contains infinitely many terms of $\{w_n\}$, there exits k_3 in I, $k_3 > k_2$, such that $\{w_{k3}\}$ is in $[a_3, b_3]$.
- Continuing this process by induction, we obtain a sequence $\{w_{kn}\}$ such that w_{kn} is in $[a_n, b_n]$ for each n.
- The sequence $\{w_{kn}\}$ is a subsequence of $\{w_n\}$ since $k_{n+1} > k_n$ for each n. Since $a_n \to w, \quad b_n \to w$ and $a_n \leq w_n \leq b_n$ for each n,
- The squeeze theorem implies that $w_{kn} \to w$ as $n \to \infty$.