INTRODUCTION TO ANALYSIS I

MATH 279 – Lesson 0

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Outline

Lesson 0: Real Number Line

Abstract

- This lesson is designed to introduce students to the number line and its properties in relation to functions.

Objectives

By the end of the lesson participants should be able to:

1. understand various representation on the number line with points and intervals.
2. understand neighborhood points.
3. understand and prove statements using the principle of mathematical induction.

The Real Number Line

- This is a unique representation of real numbers by a set of points on a line.
- The origin corresponding to '0' divides the real number line into positive and negative numbers.

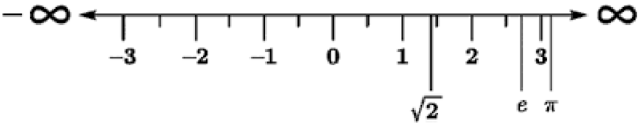

Figure: Geometric representation of the number line

Real Numbers

The following are sets of numbers;

- Natural Number; Positive whole numbers: $N = \{1, 2, 3, ...\}$
- Integers; Positive and negative whole numbers including 0. $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ..\}$
- Rational Numbers; Fractions expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
- Irrational Numbers; Numbers that cannot be expressed in the form $\frac{a}{b}$ i.e $\sqrt{2}$ and π
- The set of rational and irrational numbers are knows as **Real Numbers.** $N \subset Z \subset Q \subset R$

Real Number Operations

- $a + b$ and $ab \in R$- Closure
- $a + b = b + a$ - Commutative law of addition
- $a + (b + c) = (a + b) + c$ - Associative law of addition
- $a(b + c) = ab + ac$ - Distributive law
- $a + 0 = 0 + a = a$, $a \cdot 1 = 1 \cdot a = a$-0 and 1 are identity elements for addition and multiplication respectively.
- $-a$ - additive inverse.
- a^{-1} - multiplicative inverse.

Inequalities and Geometric Representations

- If $a - b$ is positive then a is greater than b: $a > b$.
- If $a - b$ is non-positive then a is less than or equal to$b$: $a \leq b$.
- If a, b, c are given real numbers then;

  1. Either $a > b$,$a = b$ or $a < b$ -Trichotomy.
  2. If $a > b$ and $b > c$ then $a > c$ -Transitivity.
  3. If $a > b$ then $a + c > b + c$.

Exponents, Roots and Logarithms

- A number to which another number is raised to the power is called its exponent.
  a exponent $b \implies a^b = a \times a \times ...a$ (b times)
- The following rules hold for exponents:

$$a^m \times a^n = a^{m+n} \tag{1}$$

$$(a^m)^n = a^{mn} \tag{2}$$

$$a^m \div a^n = a^{m-n} \tag{3}$$

$$(\frac{a}{b})^m = \frac{a^m}{b^m} \tag{4}$$

- If $a^m = N$ then $a = \sqrt[m]{N}$ and m is called the *mth* root of N.
- Generally,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \tag{5}$$

- If $a^m = N$, then m is called the logarithm of N to the base a. i.e

$$m = \log_a N \tag{6}$$

- The following rules hold for relations in logarithms:

$$\log_m AB = \log_m A + \log_m B \tag{7}$$

$$\log_m(\frac{A}{B}) = \log_m A - \log_m B \tag{8}$$

$$\log_m A^B = B \log_m A \tag{9}$$

- The log with base $e = 2.71828..$ is known as the natural logarithm, denoted by;

$$\ln(x) = \log_e x \tag{10}$$

Absolute Value of Real Numbers

- The absolute value of real number x is denoted by $|x|$.
- It is defined by x if $x > 0$, $-x$ if $x < 0$ and 0 if $x = 0$.

$|5| = 5 \qquad |-5| = 5 \qquad |0| = 0$

- The following rules hold;

$$|ab| = |a||b| \tag{11}$$

$$|a+b| \leq |a| + |b| \tag{12}$$

$$|a-b| \geq |a| - |b| \tag{13}$$

Points, Intervals and Representation

- A set points or real numbers located on the real axis is called a one-dimensional point set.
- Closed interval: all set of points $a \leq x \leq b \implies [a, b]$
- Half open/closed sets $a < x \leq b \implies (a, b]$ or $a \leq x < b \implies [a, b)$
- Open interval: $a < x < b \implies (a, b)$.
- x- represents a set of points known as a variable and a and b are fixed points (constants)
- $|x| < 2 \implies -2 < x < 2$ or $x \in (-2, 2)$
- $x < a$ can be expressed as $-\infty < x < a$.
- Real numbers are represented by $-\infty < x < \infty$.

Neighborhood of Points

- The neighborhood of a point a refers to all set of points such that $|x - a| < \delta$ for $\delta > 0$

- Specifically this is called the δ - neighborhood of the point a

- The set of all points x such that $0 < |x - a| < \delta$ in which $x = a$ is excluded is called the deleted delta-neighborhood of a - open ball about a.

Mathematical Induction

- This is an important concept for proving statements/relations involving positive integers.

- The principle of mathematical induction;

Given infinitely many statements $S_1, S_2, S_3, ..$ and suppose that;

1. S_1 is true.

2. If S_n is true then S_{n+1} must also be true, then all the statements must be true.

Steps In Proving Mathematical Induction.

- Let $P(n)$ be some propositional function involving an integer n. E.g
  1. $$P(n) = n(n+3) \tag{14}$$
     is an even integer.
  2. $$P(n) = 1 + 3 + \ldots + (2n-1) = n^2 \tag{15}$$
- The following are steps to prove $P(n)$.
  1. Basis: Prove $P(1)$ is true.
  2. Hypothesis: Assume $P(k)$ is true for any arbitrary $k \geq 1$.
  3. Induction: Establish $P(k+1)$

  NB:Steps 2 and 3 are known as induction steps.

Example

Prove by induction that;

$$1 + 3 + ... + (2n - 1) = n^2 \tag{16}$$

1. Basis: Prove that when $n = 1$, the proposition $P(1)$ is true. i.e;

$$\begin{aligned} P(n) &= 1 + 3 + ... + (2n - 1) = n^2 \\ P(1) &= 1 = 1^2 = 1 \end{aligned} \tag{17}$$

2. Hypothesis: We shall assume that $P(k)$ is true for some arbitrary k.

$$P(k) = 1 + 3 + ... + (2k - 1) = k^2 \tag{18}$$

Example

- Inductive Step: Using the hypothesis, we prove that $P(k+1)$ is true.

$$
\begin{aligned}
P(k) &= 1+3+....+(2k-1) = k^2 \\
P(k+1) &= 1+3+....+(2k-1)+(2(k+1)-1) = (k+1)^2 \\
P(k+1) &= P(k)+(2(k+1)-1) = (k+1)^2 \\
P(k+1) &= k^2+2k+1 = (k+1)^2
\end{aligned}
\tag{19}
$$

$\therefore P(k+1)$ is true.

Prove by induction that;

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+...+n^2 = \frac{n(n+1)(2n+1)}{6} \tag{20}$$

is true

Basis and Hypothesis

Basis;

$$
\begin{aligned}
P(n) &= \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \\
P(1) &= 1^2 = \frac{1(1+1)(2(1)+1)}{1} = \frac{6}{6} = 1
\end{aligned}
\tag{21}
$$

Hypothesis; Answer that $P(k)$ is true.

$$
P(k) = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \tag{22}
$$

Inductive Step

By the hypothesis, we have

$$
\begin{aligned}
P(k) =& 1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \\
P(k+1) =& 1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \\
P(k+1) =& \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \\
P(k+1) =& \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \\
P(k+1) =& (k+1)\frac{(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \\
P(k+1) =& \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}
\tag{23}
$$

$\therefore P(k+1)$ is true

Exercise

Prove the following statements by mathematical induction;

1. $$1 + 2 + ... + n = \frac{n(n+1)}{2} \tag{24}$$

2. $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \tag{25}$$

3. $$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3} \tag{26}$$