INTRODUCTION TO ANALYSIS I

MATH 279 – Lesson III

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Outline

1. Precise definition of Limits

2. Continuity

3. Additional notes of computing limits

Lesson 3: Precise Definition of Limits

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, and we write

$$\lim_{x \to a} f(x) = L \tag{1}$$

- if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon \tag{2}$$

- $|x - a|$ is distance from x to a and $|f(x) - L|$ is distance from $f(x)$ to L we can say that
- "$\lim_{x \to a} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by requiring that the distance from x to a be sufficiently small (but not 0)."

Example

Prove that

$$\lim_{x \to 3} (4x - 5) = 7 \tag{3}$$

Example

- Preliminary analysis of the problem (guessing a value for δ). Let ϵ be a given positive number. We want to find a number δ such that

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |(4x-5)-7| < \epsilon \tag{4}$$

But $|(4x-5)-7| = |4x-12| = |4(x-3)| = 4|x-3|$. Therefore we want to δ such that

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad 4|x-3| < \epsilon \tag{5}$$

that is

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |x-3| < \frac{\epsilon}{4} \tag{6}$$

This implies that $\delta = \epsilon/4$

Example

- We can go on and show that δ works. Given $\epsilon > 0$, choose $\delta = \epsilon/4$. If n$0 < |x - 3| < \delta$ then

$$|(4x-5)-7| = |4x-12| = |4(x-3)| = 4|x-3| < 4\delta = 4\left(\frac{\epsilon}{4}\right) = \epsilon \quad (7)$$

- Thus

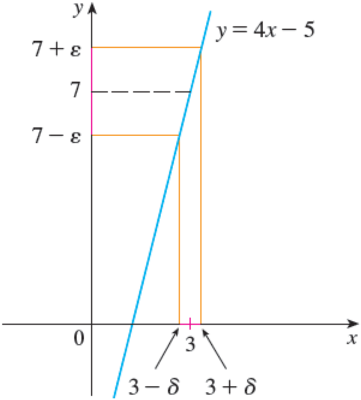
$$\text{if} \quad 0 < |x - 3| < \delta \quad \text{then} \quad |(4x - 5) - 7| < \epsilon \quad (8)$$

- Therefore, by the definition of a limit,

$$\lim_{x \to 3} (4x - 5) = 7 \quad (9)$$

Graphically

$$\lim_{x \to 3} (4x - 5) = 7 \tag{10}$$



Continuity

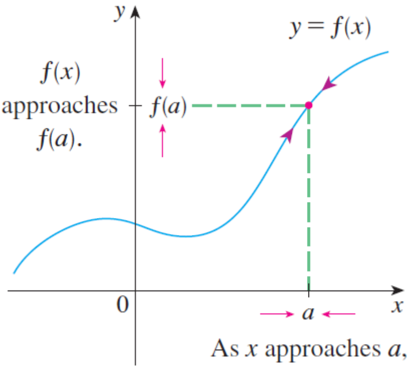
- Note that there are instances where the limit of a function as x approaches a is found by evaluating the function at a
- Such functions with this property are known as **continuous functions**

A function is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a) \tag{11}$$

- This implies that

1. $f(a)$ is defined (that is a is the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$



- The figure shown is an illustration of a continuous function f.
- f is continuous means that the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph
- Very importantly, there should be no gap in the curve

Continuity on an interval

Definition

- A function f is **continuous on an interval** if it is continuous at every number in the interval.
- If f is defined only on one side of an endpoint of the interval, we understand **continuous** at the endpoint to mean **continuous from the right or continuous from the left**

1. Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval [-1,1]

If $-1 < a < 1$, then we can have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(1-\sqrt{1-x^2}) \\
&= 1 - \lim_{x\to a}\sqrt{1-x^2} \\
&= 1 - \sqrt{\lim_{x\to a}(1-x^2)} \\
&= 1 - \sqrt{1-a^2} \\
&= f(a)
\end{aligned}$$

Based on the previous definition, f is continuous at a if $-1 < a < 1$. Similar calculations shows that

$$\lim x \to 1^+ f(x) = 1 = f(-1) \quad \text{and} \quad \lim x \to 1^- f(x) = 1 = f(1)$$

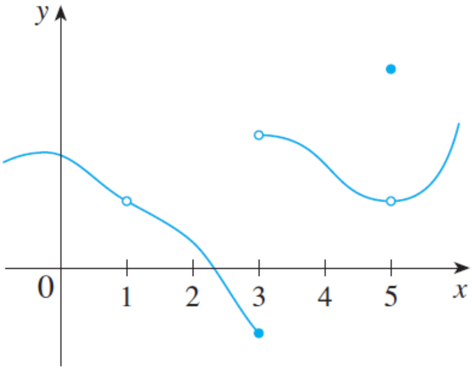
so f is continuous from the right at -1 and continuous from the left at 1. Therefore f is continuous on [-1,1]

Discontinuity

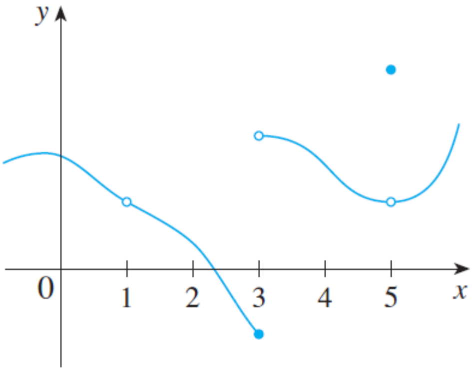
- A function that is not continuous at a point a has a **discontinuity at** a
- Physical phenomena are usually continuous e.g. velocity of a vehicle varying continuously with time
- The Heaviside function is discontinuous at 0 because $\lim_{x\to 0} H(t)$ does not exist

Example

1. At which numbers of f is there discontinuity



Example



1. At $a = 1$; there is a break. $f(1)$ is undefined
2. At $a = 3$; $f(3)$ is defined but the $\lim_{x\to 3} f(x)$ does not exist
3. At $a = 5$; $f(5)$ is defined and the $\lim_{x\to 5} f(x)$ exists but they are not equal

Exercise

Where are each of the following functions discontinuous?

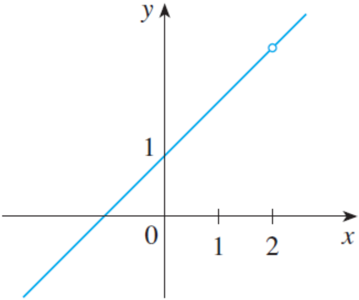
$$(a) \qquad f(x) = \frac{x^2 - x - 2}{x - 2} \tag{12}$$

$$(c) \qquad f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \tag{13}$$

$$(c) \qquad f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases} \tag{14}$$

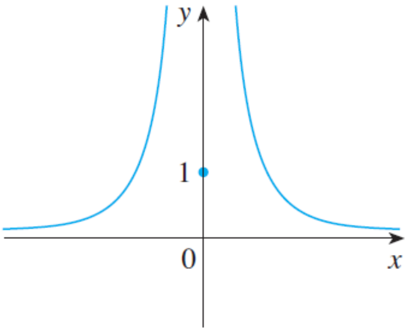
Removable Discontinuity

$$(a) \qquad f(x) = \frac{x^2 - x - 2}{x - 2} \tag{15}$$



Notice that $f(2)$ is not defined, so f is discontinuous at 2.

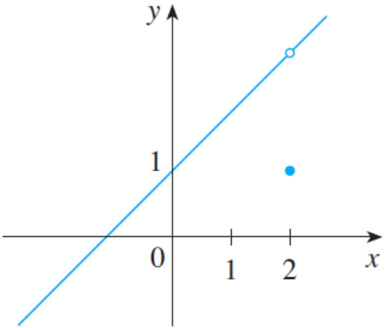
$$(c) \qquad f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \tag{16}$$



Notice that $f(0)$ is defined, but $\lim_{x\to 0} f(x)$ does not exist

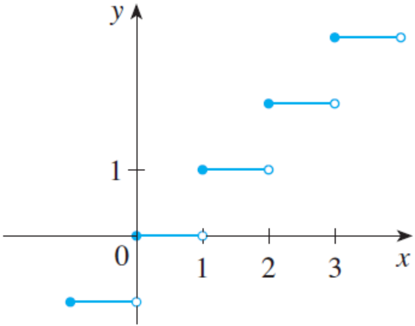
Removable Discontinuity

$$(c) \qquad f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases} \tag{17}$$



Notice that $\lim_{x \to 2} \neq f(2)$

Jump Discontinuity



Exercises

1. Prove the state using the ϵ, δ definition of a limit

$$\lim_{x \to 1} \frac{2+4x}{3} = 2 \tag{18}$$

$$\lim_{x \to 4} \frac{x^2-2x-8}{x-4} = 6 \tag{19}$$

$$\lim_{x \to 10} \left(3 - \frac{4}{5}x\right) = -5 \tag{20}$$

Additional Notes on Limits

1. Limit Laws
2. Direct Substitution Property

Limit Laws

Suppose that c is a constant and

$$\lim_{x \to a} f(x) \quad \lim_{x \to a} g(x)$$

1. $$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} f(x)$$

2. $$\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$$

3. $$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

4. $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if} \quad \lim_{x \to a} g(x) \neq 0$$

Direct Substitution Property

If a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a) \tag{21}$$

- Functions with the Direct Substitution Property are called continuous at a.
- However, not all limits can be evaluated by direct substitution, as the following example shows

Find

$$lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Find

$$lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

- We cannot find the limit by substituting $x = 1$ because $f(x)$ is undefined
- We factor the numerator as a difference of two squares

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1}$$

$$\begin{aligned} lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} \\ &= lim_{x \to 1} (x+1) \\ &= 1 + 1 = 2 \end{aligned}$$