INTRODUCTION TO ANALYSIS I

MATH 279 – Lesson II

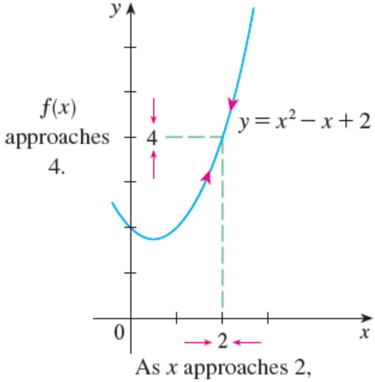
Prof. I. K. Dontwi

Department of Mathematics
Kwame Nkrumah University of Science and Technology

The limit of a function is a description of how a function behaves as the independent variable approaches a given value.

- Consider

$$f(x) = x^2 - x + 2 \quad \lim_{x\to 2}(x^2 - x + 2) = 4 \tag{1}$$



x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

Intuitive Definition of Limits

Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval contains a, expect possibly at a itself). Then we write

$$lim_{x\to a}f(x) = L \tag{2}$$

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a

- Use a numerical evidence to make a conjecture about the value of

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \tag{3}$$

- Use a numerical evidence to make a conjecture about the value of

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} \tag{4}$$

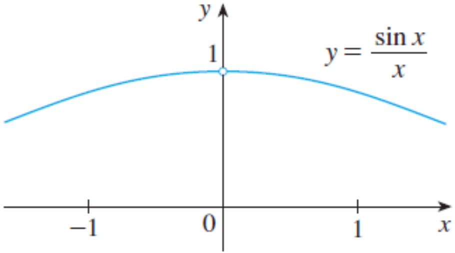
t	$\frac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	$0.162277\ldots$
± 0.5	$0.165525\ldots$
± 0.1	$0.166620\ldots$
± 0.05	$0.166655\ldots$
± 0.01	$0.166666\ldots$

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6} \tag{5}$$

Find the value of

$$\lim_{x \to 0} \frac{\sin x}{x} \tag{6}$$

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

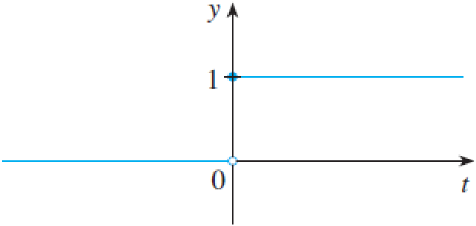


$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad (7)$$

Example

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases} \tag{8}$$

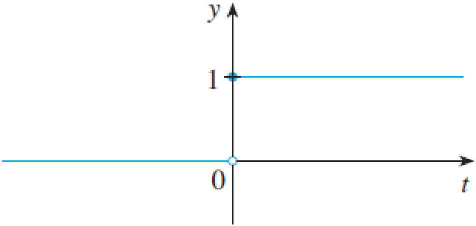


$$\lim_{t \to 0^-} H(t) = ? \qquad \lim_{t \to 0^+} H(t) = ? \tag{9}$$

Example

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases} \tag{10}$$



$$\lim_{t \to 0^-} H(t) = 0 \qquad \lim_{t \to 0^+} H(t) = 1 \tag{11}$$

Definition of One-Sided Limits

- The left-hand limit of $f(x)$ as x approaches a or limit of $f(x)$ as x approaches a from the left is denoted by

$$\lim_{x \to a^-} f(x) = L \tag{12}$$

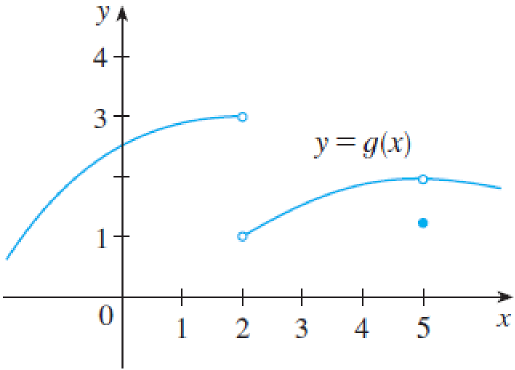
- The right-hand limit of $f(x)$ as x approaches a or limit of $f(x)$ as x approaches a from the right is denoted by

$$\lim_{x \to a^+} f(x) = L \tag{13}$$

Relationship One-side & Two-sided Limits

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L \tag{14}$$

Exercise



Evaluate the following

$$a.\ \lim_{x\to 2^-} g(x) = \quad b.\ \lim_{x\to 2^+} g(x) = \quad c.\ \lim_{x\to 2} g(x) = \tag{15}$$

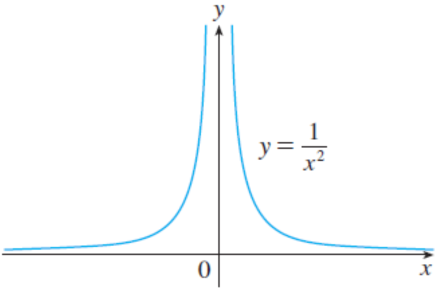
$$d.\ \lim_{x\to 5^-} g(x) = \quad e.\ \lim_{x\to 5^+} g(x) = \quad f.\ \lim_{x\to 5} g(x) = \tag{16}$$

Infinite Limits

Find

$$\lim_{x\to 0} \frac{1}{x^2} = ? \tag{17}$$

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000



$$\lim_{x\to 0} \frac{1}{x^2} = \infty \tag{18}$$

- This does not mean that ∞ is a number

Definition of an Infinite Limit

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty \tag{19}$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a

- the limit of $f(x)$ as x approaches a is infinity
- $f(x)$ becomes infinite as x approaches a
- $f(x)$ increases without bound as x approaches a

Vertical Asymptote

The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

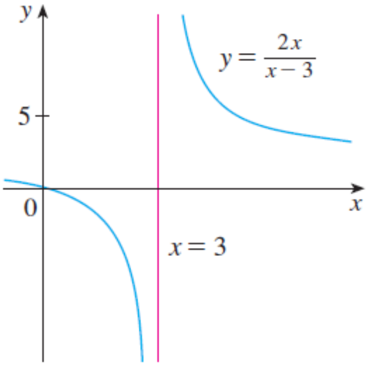
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a} f(x) = \infty \tag{20}$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a} f(x) = -\infty \tag{21}$$

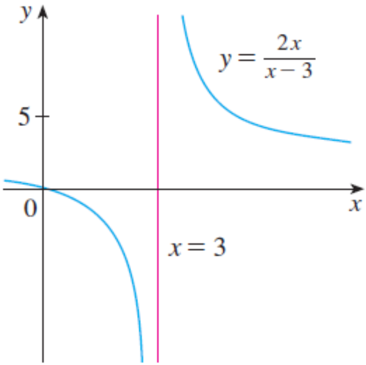
Example

Find

$$\lim_{x\to 3^+} \frac{2x}{x-3} \qquad \lim_{x\to 3^-} \frac{2x}{x-3} \tag{22}$$

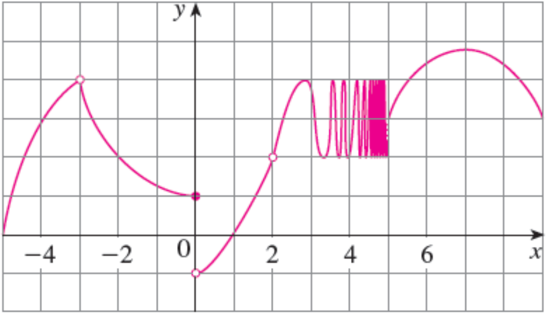


Example



$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty \qquad \lim_{x \to 3^-} \frac{2x}{x-3} - \infty \tag{23}$$

Exercise



$$(a) \lim_{x\to 0^-} g(x) = \quad (b) \lim_{x\to 0^+} g(x) = \quad (c) \lim_{x\to 0} g(x) = \tag{24}$$

$$(d) \lim_{x\to 2^-} g(x) = \quad (e) \lim_{x\to 2^+} g(x) = \quad (f) \lim_{x\to 2} g(x) = \tag{25}$$

$$(g)\ g(2) = \quad (h) \lim_{x\to 4} g(x) = \tag{26}$$