HOMEWORK 4 FOR MATH 2860 (ELEMENTARY DIFFERENTIAL EQUATIONS)

E. A. ATINDAMA, PHD MATHEMATICS

Due at 10:00 EST in class

Question Q1. Solve the Bernoulli equation

$$y' = y(xy^3 - 1), \quad y(1) = 1.$$

Solution Rearranging the given Bernoulli equation, we have

$$y' + y = xy^4.$$

This is the standard form of a Bernoulli equation $y' + P(x)y = Q(x)y^n$, with $n = 4$.
Make the substitution

$$v = y^{1-n} = y^{-3} \Rightarrow v' = -3y^{-4}y' \;\Rightarrow\; y^{-4}y' = -\frac{1}{3}v'$$

So that the differential equation becomes linear in v given by

$$-\frac{1}{3}v' + v = x.$$

We can solve using integration factor to obtain

$$e^{-3x}v = xe^{-3x} - \frac{1}{3}e^{-3x} + C \quad \Rightarrow \quad v = x - \frac{1}{3} + Ce^{3x}.$$

Substitute back $v = y^{-3}$ to obtain $y^{-3} = x - \frac{1}{3} + Ce^{3x}$.
Substitute the initial condition $y(1) = 1$ into the solution to find C.

$$1^{-3} = 1 - \frac{1}{3} + Ce^{3(1)} \quad \Rightarrow \quad 1 = \frac{2}{3} + Ce^{3} \quad \Rightarrow \quad C = \frac{1}{3e^3}$$

.
The solution to the Bernoulli equation is:

$$y^{-3} = x - \frac{1}{3} + \frac{1}{3e^3}e^{3x} \quad \text{or} \quad \mathbf{y} = \left(\mathbf{x} - \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{e^{3x-3}}}{\mathbf{3}}\right)^{\mathbf{-1/3}}$$

Question Q2. Determine whether the differential equation

$$-y\,dx + (x + \sqrt{x+y})\,dy = 0$$

is homogeneous. If it is, solve it. If not, state the reason why it is not.

(E. A. Atindama) . Email addresses: emmanuel.atindama@utoledo.edu
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Solution A differential equation of the form $M(x,y)\,dx+N(x,y)\,dy=0$ is homogeneous if both M and N are homogeneous functions of the same degree. A function $f(x,y)$ is homogeneous of degree n if for all $t>0$,

$$f(tx,ty)=t^n f(x,y).$$

In our case, we have

$$M(x,y)=-y \quad \text{and} \quad N(x,y)=x+\sqrt{x+y}.$$

$$M(tx,ty)=-ty=t^1(-y)=t^1M(x,y), \quad \text{so} M(x,y) \text{ is homogeneous of degree 1..}$$

$$N(tx,ty)=tx+\sqrt{tx+ty}=tx+\sqrt{t(x+y)}=tx+t^{1/2}\sqrt{x+y},$$

which is not a single term multiplied by a power of t. Therefore, N is not homogeneous. Since M and N are not both homogeneous functions of the same degree, the differential equation is not homogeneous.

Question Q3. Determine whether the differential equation

$$-y\,dx+(x+\sqrt{xy})\,dy=0$$

is homogeneous. If it is, solve it. If not, state the reason why it is not.

Solution A differential equation of the form $M(x,y)\,dx+N(x,y)\,dy=0$ is homogeneous if both M and N are homogeneous functions of the same degree. A function $f(x,y)$ is homogeneous of degree n if for all $t>0$,

$$f(tx,ty)=t^n f(x,y).$$

In our case, we have

$$M(x,y)=-y \quad \text{and} \quad N(x,y)=x+\sqrt{xy}.$$

$$M(tx,ty)=-ty=t^1(-y)=t^1M(x,y), \quad \text{so} M(x,y) \text{ is homogeneous of degree 1..}$$

$$N(tx,ty)=tx+\sqrt{tx\cdot ty}=tx+\sqrt{t^2(xy)}=tx+t\sqrt{x+y},$$

Hence, N is homogeneous of degree 1. So the differential equation is homogeneous.

Make the substitution

$$y=vx \Rightarrow dy=v\,dx+x\,dv.$$

$$\begin{aligned}
-y\,dx+(x+\sqrt{xy})\,dy&=0\\
-(vx)\,dx+(x+\sqrt{x(vx)})(v\,dx+x\,dv)&=0\\
-vx\,dx+(x+x\sqrt{v})(v\,dx+x\,dv)&=0\\
-vx\,dx+x(1+\sqrt{v})(v\,dx+x\,dv)&=0\\
-v\,dx+(1+\sqrt{v})(v\,dx+x\,dv)&=0\\
-v\,dx+v(1+\sqrt{v})\,dx+x(1+\sqrt{v})\,dv&=0\\
(v(1+\sqrt{v})-v)\,dx+x(1+\sqrt{v})\,dv&=0\\
(v+v^{3/2}-v)\,dx+x(1+\sqrt{v})\,dv&=0\\
v^{3/2}\,dx+x(1+\sqrt{v})\,dv&=0
\end{aligned}$$

which is separable.

Thus,

$$\begin{aligned}
\frac{1}{x}\,dx&=-\frac{1+\sqrt{v}}{v^{3/2}}\,dv\\
\int\frac{1}{x}\,dx&=-\int\left(\frac{1}{v^{3/2}}+\frac{\sqrt{v}}{v^{3/2}}\right)dv\\
\ln|x|&=-\int\left(v^{-3/2}+\frac{1}{v}\right)dv\\
\ln|x|&=\frac{2}{\sqrt{v}}-\ln|v|+C
\end{aligned}$$

$y=vx$, which means $v=\frac{y}{x}$

Hence the implicit solution is

$$-\frac{2\sqrt{x}}{\sqrt{y}}+\ln|y|=C$$

or

$$\ln|y|=\frac{2\sqrt{x}}{\sqrt{y}}+C.$$