MA2860 Elementary Differential Equations Notes and Problems

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This text provides notes and problems that build on concepts in calculus for MA2860. It is open source and meant for students and instructors to use, adapt, and contribute. The computational component primarily consists of Sage code, and online symbolic calculators are provided for supplemental study.

University of Toledo, OH
September 2025

Contents

Preface

THIS TEXT PROVIDES NOTES AND PROBLEMS that build on concepts in calculus for *MA2860 Elementary Differential Equations* at the University of Toledo. The book is a companion to various textbooks including:

- Zill, Dennis G. *A First Course in Differential Equations with Modeling Applications*, 10th ed. Cengage Learning, 2012.
- Boyce, William E., DiPrima Richard C. *Elementary Differential Equations*, 8th ed. John Wiley & Sons Inc., 2004.

Although it can be used with other texts as well.

IN MAKING THIS STARTING RESOURCE OF LESSON PLANNING **open source**, I hope that each user (with code access) will contribute with notes, corrections, and exercises (with solutions). Users without code access may also email me any corrections. *Professors may email to request code access.* Each week has a portion of the assignment reserved for preparatory homework and another portion for further discussion or exam preparation.

KNOWING THAT ALL PEOPLE ARE MADE IN THE IMAGE OF GOD WITH A POTENTIAL FOR THE HIGHEST GOOD, I hope that all users of this material will work to make this work of highest quality for all educators.

The idea for this work came from my professor in Dynamical Systems during my grad school years, Dr. Marko Budišić at Clarkson University. He encouraged students to compile class notes into one document for easy review. All credit to him.

This text uses a Differential Equations workbook compiled by Clarkson University faculty as a starting point, although the content is mostly different: K. Black, D. White, G. Yao, J. Martin, and M. Budišić.

THE COMPUTATIONAL COMPONENT of this book primarily consists of SAGEMATH code. No prerequisite experience is needed, as problems can be completed by following links and editing code in the browser. Essential manual for SAGEMATH is the textbook:

- Bard, G.V. *Sage for Undergraduates*. American Mathematical Society, 2015. Available at no cost in PDF form: `http://www.gregorybard.com/Sage.html`.

ONLINE TOOLS for studying involve symbolic calculators at links: CAUTION: do not rely heavily on them while studying!

- `https://www.symbolab.com/solver/`
- `http://www.emathhelp.net/calculators/`

THE GOAL of this class is to teach the **principles** behind the calculations, so they can be built on and extended in future courses or practice, not convert people into human calculators. Therefore, emphasis is on algorithms, and memorization or **reliance on formulas and calculators is strongly discouraged**.

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Chapter 1: Introduction to Differential Equations

We begin our journey into the world of elementary differential equations by looking at some review.

Section 1.1: Review: Precalculus and Calculus

In this section, you will review the techniques from Calculus classes that you should absolutely know in and out: sketching functions, implicit derivatives, and three integration techniques: substitution, by-parts, and partial fractions.

a) Properties of exponents and logarithms For $a > 0$, $a \neq 1$, and integers m, n and $x, y > 0$. Then

- $a^m \cdot a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}, \quad (m > n)$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^0 = 1$
- $a^{-n} = \frac{1}{a^n}$
- $\ln 1 = 0$
- $\ln e = 1$
- $\ln xy = \ln x + \ln y$
- $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
- $\ln x^r) = r \cdot \ln x$
- $\ln(e^x) = x, e^{\ln(x)} = x$
- Should be familiar with graphs of the natural exponential and natural logarithm.

b) Trigonometric Properties

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

- Need to know the values of the trig functions for multiples of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
- Need to be familiar with polar coordinates: $x = r\cos(\theta), y = r\sin(\theta)$.

c) **Product rule:** If $f(x)$ and $g(x)$ are differentiable functions:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Example:

$$f(x) = x^2 \sin x$$

$$f'(x) = (2x)(\sin x) + (x^2)(\cos x) = 2x\sin x + x^2\cos x$$

d) **Chain rule:** If $y = f(g(x))$, then:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Example:

$$y = (3x^2 + 1)^5$$

$$\frac{dy}{dx} = 5(3x^2 + 1)^4 \cdot (6x) = 30x(3x^2 + 1)^4$$

e) **Integration by Substitution** If $u = g(x)$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Example:

$$\int 2x\cos(x^2)\,dx$$

Let $u = x^2$, so $du = 2x\,dx$:

$$\int 2x\cos(x^2)\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(x^2) + C$$

f) **Integration by Parts** From the product rule:

$$\int u\,dv = uv - \int v\,du$$

Example:

$$\int xe^x\,dx$$

Let $u = x \implies du = dx$ and $dv = e^x dx \implies v = e^x$:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = (x-1)e^x + C$$

g) **Partial Fraction Decomposition**: If the integrand is a rational function $\frac{P(x)}{Q(x)}$, decompose into simpler fractions.

Example:

$$\int \frac{1}{x^2 - 1}\,dx$$

Factor denominator:

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Multiply through:

$$1 = A(x+1) + B(x-1)$$

Setting $x = 1 \implies 1 = 2A \implies A = \frac{1}{2}$ Setting $x = -1 \implies 1 = -2B \implies B = -\frac{1}{2}$

So:

$$\frac{1}{x^2 - 1} = \frac{1}{2} \cdot \frac{1}{x-1} - \frac{1}{2} \cdot \frac{1}{x+1}$$

Now integrate:

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$
$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

In-Class Exercise

Exercise 1.1.1
Simplify the following expressions:

(a) $e^{4\ln|x+1|} =$

(b) $\exp\left[-\frac{1}{2}\ln(x^2)\right] =$

Exercise 1.1.2
Evaluate the following integrals using substition, integration by parts, and partial fractions, as appropriate.

(a) $\int_0^x xe^{x^2}dx$[1]

(b) $\int_1^2 x\sin(\pi x)dx$[2]

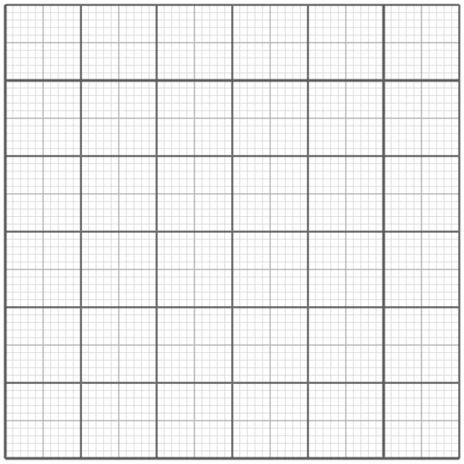
(c) $\int \frac{x}{x^2+x-6x}dx$

Exercise 1.1.3
Sketch the graph of the function $f(x) = (x^2-4x+3)e^{-2x}$ [3]

[1] Hint: Substitution

[2] Hint: Integration by parts

[3] Determine any minima/maxima, x-intercepts and y-intercepts, and asymptote $x \to +\infty$ for an accurate sketch.



Exercise 1.1.4
If $y(x) = e^x \sin(2x)$, evaluate the expression $y''(x) - 2y'(x) + 5y(x)$. Compute the required derivatives[4] first, and then substitute.

[4] We use y' to mean the derivative with respect to the argument, in this case dy/dx. When the function depends on time, say $x(t)$, then we sometimes additionally use the dot-notation $\dot{x} = dx/dt$.

Discussion Problems

Exercise 1.1.5
Compute the derivative $y'(x)$ if the following relation between y and x holds:

$$\ln|y^2 + 2| + x^3 \sin(2x) = C, \text{ a constant.}$$

Exercise 1.1.6
What is the domain of definition of each of the two functions:

(a) $\ln(2x + 1)$ (b) $\ln|2x + 1|$

Exercise 1.1.7
Split the following rational functions into partial fractions:

(a) $\frac{1}{x^2+4x+3}$ (b) $\frac{2x}{x^2+4x+3}$ (c) $\frac{x^2+4x}{x^2+4x+3}$

Exercise 1.1.8
A curve is defined by the implicit expression $\cos(2x) + \frac{1}{2}y^2 = 1$. Calculate **all** the points at which the tangent[5] to this curve is

(a) Vertical. (b) Horizontal.

[5] Hint: find the implicit derivative $y'(x)$
(a) $y'(x) \to \infty$, (b) $y'(x) \to \infty$

Exercise 1.1.9
Evaluate the integrals:[6]

(a) $\int \ln(x)dx$ (c) $\int \frac{x}{x+2}dx$ (e) $\int \frac{x^2}{x^2+x-6x}dx$
(b) $\int xe^x dx$ (d) $\int \frac{2x}{x^2-4}dx$ (f) $\int e^x \sin(\pi x)dx$

[6] Hint: (a)integration by parts, (b) substitution, (d) substitution/partial fraction, (f) by parts

Section 1.2: Basic Terminology & History

Basic Terminology

Definition 1.2.1. *A* differential equation *is an equation that contains one or more derivatives of an unknown function.*

A differential equation is an ordinary differential equation (ODE) if it involves an unknown function of only one independent variable. Example:

$$\frac{dy}{dt} = -ky.$$

A differential equation involving partial derivatives of a function of two or more independent variables is a partial differential equation. Example:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}.$$

Differential equations describe how one quantity changes in relation to other variables.

For a standard algebraic equation, the solution is a number, but for a differential equation, the solution is a function.

- $y(x)' = y + x^2 + 3$ is a differential equation. The unknown function is $y$depending on the variable x and the derivative of y is involved in the equation.
- $\cos x = -1$ is not a differential equation. There is no unknown function.
- $y = x^2 - 2x + 1$ is not a differential equation since there is no derivative involved.
- $y'(x) = y$ is a differential equation, with unknown function y with its derivative depending on the variable x.
- $\frac{d}{dx}(\tan x) = \sec^2 x$ is not a differential equation since there is no unknown function

Definition 1.2.2. *The* order *of a differential equation is the highest derivative that appears in it.*

- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ is a second-order ODE.
- $y' + 3x^2 = 1$ is a first-order ODE.
- $y'' - x^2y' + y = \cos x$ is a second-order ODE.
- $y^4 - xy' + y^2 = 3$ is a fourth-order ODE.

Definition 1.2.3 (Linearity). *An ODE is* linear *if the dependent variable y and its derivatives $y', y'', \ldots, y^n$ all appear only to the first power and not multiplied together. Example:*

$$a_2(x)y'' + a_1(x)y' + P(x)y = Q(x).$$

An ODE is nonlinear *if it contains products, powers, or nonlinear functions of the dependent variable or its derivatives. Example:*

$$\frac{dy}{dx} = y^2 + \sin(y).$$

Definition 1.2.4. *A differential equation is homogeneous if all terms depend on the unknown function or its derivatives. If there is a term that does not, it is nonhomogeneous.*

- ***Homogeneous***: $\frac{dy}{dx} + p(x)y = 0$
- ***Nonhomogeneous***: $\frac{dy}{dx} + p(x)y = q(x)$

Definition 1.2.5. *An* Initial Value Problem (IVP) *of an $n-th$ order differential equation is the differential equation together with an initial condition of y and the first $n-1$ derivatives specified at a single point. Example:*

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0.$$

Definition 1.2.6. *A* Boundary Value Problem (BVP): *is a differential equation together with conditions specified at two or more points. Example:*

$$\frac{d^2y}{dx^2} + y = 0, \quad y(0) = 0, \; y(\pi) = 0.$$

Definition 1.2.7. *A* solution *of a differential equation is any function that satisfies the differential equation and its initial conditions on some open interval.*

Note that solutions should be accompanied by intervals, since they can provide important information about the solution.[7] The solution is general if no initial condition is provided.

[7] If y is a solution to the differential equation, differentiate to obtain y', then substitute y and y' into the differential equation to verify.

Example 1.2.8.

$$\frac{dy}{dx} = 2x \quad \Rightarrow \quad y = x^2 + C$$

is the general solution. If $y(0) = 3$, then $y = x^2 + 3$ is the particular solution.

Exercise 1.2.1

Determine whether the given function is a solution of the differential equation.

a) Determine whether $y = e^{2x} + 1$ is a solution of

$$\frac{dy}{dx} - 2y = -2.$$

b) Determine whether $y = \sin(x) + \cos(x)$ is a solution of

$$\frac{d^2y}{dx^2} + y = 0.$$

c) Determine whether $y = x^2$ is a solution of

$$\frac{dy}{dx} - 2x = 0.$$

d) Determine whether $y = e^{2x}$ is a solution of

$$\frac{dy}{dx} - 2y = 0.$$

e) Is $x = e^{4t}$ a solution to

$$x''' - 12x'' + 48x' - 64x = 0?$$

f) Is $y = \sin t$ a solution to

$$\left(\frac{dy}{dt}\right)^2 = 1 - y^2?$$

g) Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.

h) Determine whether $y = x^{-\frac{3}{2}}, y = x^{-\frac{1}{2}}, y = -9x^{-\frac{3}{2}}$, and $y = 7x^{-\frac{1}{2}} - 9x^{-\frac{3}{2}}$ are solutions of the differential equation $4x^2y'' + 12xy' + 3y = 0$, for $x > 0$.

Historical Background

The study of ODEs is closely tied to the development of calculus and mathematical physics:

- *17th Century (Foundations):* The birth of differential equations coincides with the invention of calculus by *Isaac Newton* and *Gottfried Wilhelm Leibniz* in the late 1600s. Newton formulated equations of motion, such as

  $$F = ma \quad \Rightarrow \quad m\frac{d^2x}{dt^2} = F(x,t),$$

  which is a second-order ODE.

  The Bernoulli brothers also developed methods to solve differential equations to extend the range of their applications

- *18th Century (Development):* Mathematicians like *Leonhard Euler*, a student of Johann Bernoulli and *Joseph-Louis Lagrange* developed systematic methods for solving first- and second-order ODEs, as well as studying oscillations, planetary motion, and mechanical systems.

- *19th Century (Theory):* With contributions from *Augustin-Louis Cauchy*, *Sophie Kovalevskaya*, and others, the rigorous theory of existence and uniqueness of solutions was established. The study of linear differential equations and series solutions flourished.

- *20th Century (Modern Applications):* Differential equations became essential in physics, engineering, biology, and economics. The qualitative theory of dynamical systems (initiated by *Henri Poincaré*) emphasized studying the behavior of solutions without necessarily finding exact formulas.

- *Today:* ODEs remain central in mathematical modeling across disciplines: population dynamics, control systems, epidemiology, neuroscience, climate modeling, finance, etc.

Ordinary differential equations provide the mathematical language for describing change. From Newton's laws of motion to modern systems biology, ODEs are the foundation of mathematical modeling in the natural and applied sciences.

Section 1.3: Mathematical modelling with Differential Equations

Example 1.3.1 (Newton's Law of Cooling). *Newton's law of cooling states that the rate of change of temperature $T(t)$ of an object is proportional to the difference between the object's temperature and the ambient temperature T_a.*

$$\frac{dT}{dt} = -k\big(T - T_a\big), \quad k > 0.$$

Solution: *This is a first-order linear ODE. We solve it using separation of variables:*

$$\frac{dT}{T - T_a} = -k\,dt.$$

Integrating both sides:

$$\ln|T - T_a| = -kt + C.$$

Exponentiating:

$$T - T_a = Ce^{-kt}.$$

Hence the general solution is

$$T(t) = T_a + (T_0 - T_a)e^{-kt},$$

where $T_0 = T(0)$ is the initial temperature.

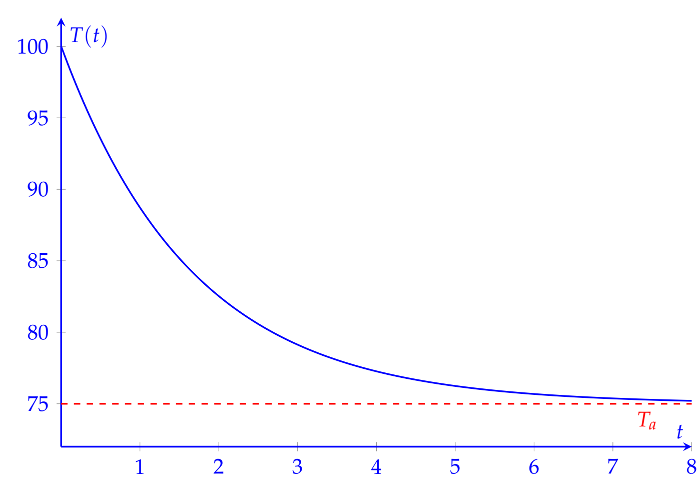


Figure 1.1: Exponential decay of temperature in Newton's Law of Cooling.

Example 1.3.2 (Simple Harmonic Oscillator). *The motion of a mass attached to a spring (without damping) is governed by*

$$m\frac{d^2x}{dt^2} + kx = 0,$$

where m is the mass and k is the spring constant.
Solution: *Dividing through by m gives*

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \quad \text{where } \omega = \sqrt{\frac{k}{m}}.$$

This is a second-order linear homogeneous ODE with constant coefficients. The characteristic equation is

$$r^2 + \omega^2 = 0 \quad \Rightarrow \quad r = \pm i\omega.$$

Hence, the general solution is

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t),$$

which describes oscillatory motion with angular frequency ω.

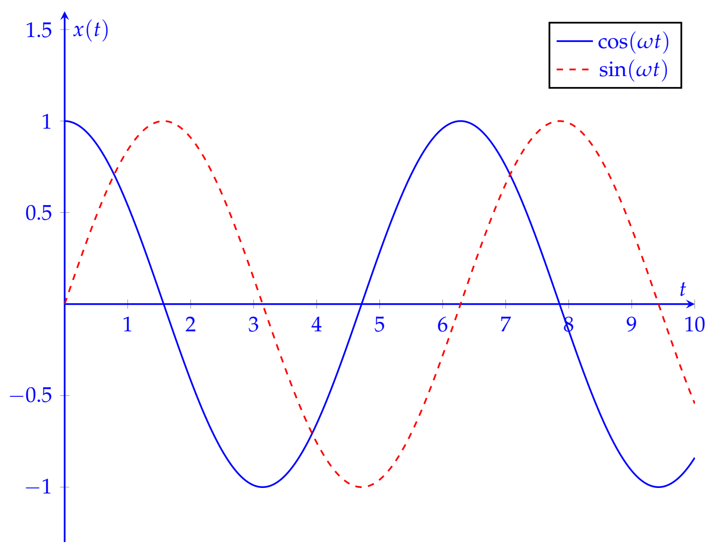


Figure 1.2: Solutions of the simple harmonic oscillator.

Example 1.3.3 (Logistic Population Growth). *A simple model for population growth is given by the logistic equation:*

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where r is the intrinsic growth rate and K is the carrying capacity.
Solution: *Separating variables:*

$$\frac{dP}{P(1-P/K)} = r\,dt.$$

Using partial fractions:

$$\frac{1}{P(1-P/K)} = \frac{1}{P} + \frac{1}{K-P}\cdot\frac{1}{K}.$$

So,

$$\int\left(\frac{1}{P} + \frac{1}{K-P}\right)dP = r\int dt.$$

This gives

$$\ln|P| - \ln|K - P| = rt + C.$$

Simplifying:

$$\ln\left|\frac{P}{K-P}\right| = rt + C.$$

$$\Rightarrow P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-rt}},$$

where $P_0 = P(0)$.

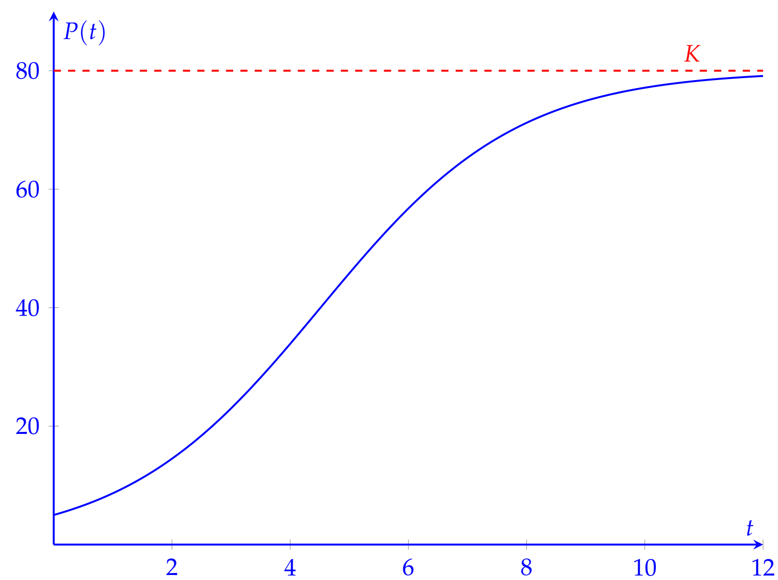


Figure 1.3: Logistic growth approaching the carrying capacity $K = 80$.

Section 1.4: Slope (Direction) Fields

Definition 1.4.1. *A **slope field** is a graphical representation of a first-order differential equation*

$$\frac{dy}{dx} = f(x, y).$$

At each point (x, y) in the plane, a small line segment is drawn with slope $f(x, y)$.

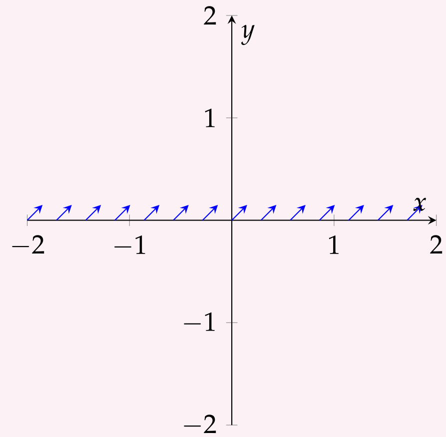
Definition 1.4.2. *The graph of a solution of a differential equation is called a* solution curve*. A curve C is called an* integral curve *of a differential if every solution curve is a part of it.*

- The slope field provides a qualitative view of solutions without explicitly solving the ODE.
- Solution curves are tangent everywhere to the slope field.

Example 1.4.3 (Constant Slope Field)**.** *Consider*

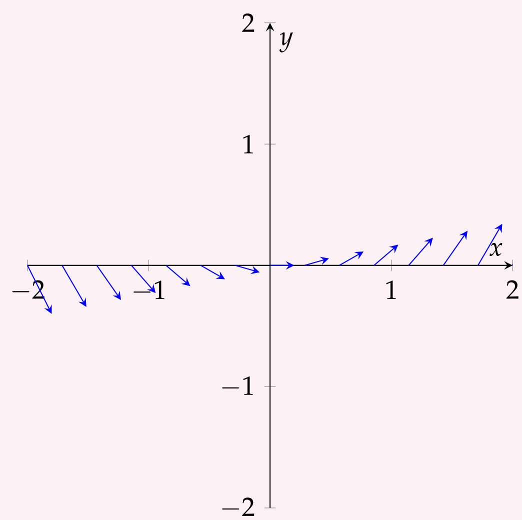
$$\frac{dy}{dx} = 1.$$

Every line segment has slope 1 (a 45° line). Solution curves are lines of the form $y = x + C$.



Example 1.4.4 (Non-constant Slope Field). *We visualize the slope field for*

$$\frac{dy}{dx} = x.$$



The general solution is:

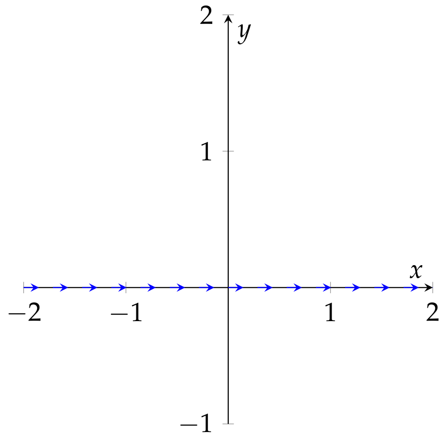
$$y = \frac{x^2}{2} + C$$

which agrees with the slope field.

Example 1.4.5 (Logistic Growth). *Consider the logistic ODE:*

$$\frac{dy}{dx} = y(1 - y).$$

- *When $0 < y < 1$, slopes are positive (growth).*
- *When $y > 1$, slopes are negative (decay).*
- *$y = 0$ and $y = 1$ are equilibrium solutions.*



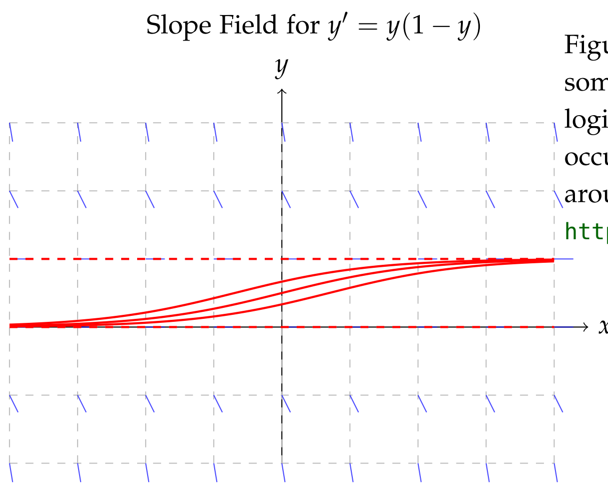


Figure 1.4: Slope field and some solution curves for the logistic equation. Equilibria occur at $y = 0$ and $y = 1$. Play around with the sage code here https://shorturl.at/K9f7l

Example 1.4.6. *For the differential equation,*

$$\frac{dy}{dx} = \frac{x}{y},$$

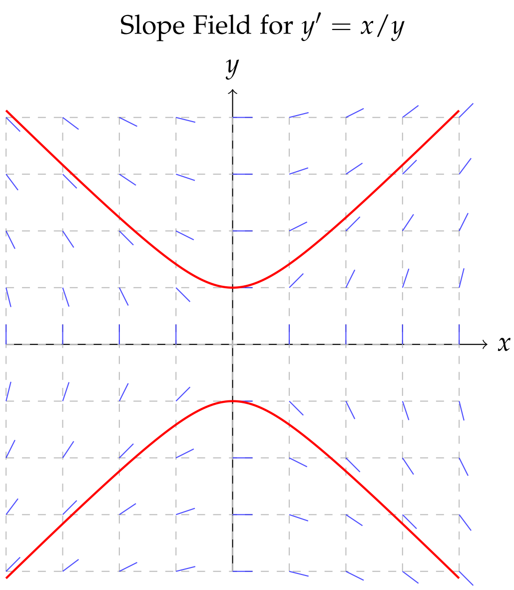


Figure 1.5: An *integral curve* is more general term for the graph of a solution, including those that are not functions. For example, the full hyperbola $y^2 - x^2 = K$ is an integral curve, but the upper branch ($y = \sqrt{x^2+K}$) and lower branch ($y = -\sqrt{x^2+K}$) are the individual solution curves.

- Slope fields help visualize the qualitative behavior of solutions.
- Equilibrium solutions correspond to horizontal slopes ($f(x, y) = 0$).
- They are especially useful when an ODE cannot be solved analytically.

Practice Problems

Exercise 1.4.1

Give definitions of the following terms and/or answer the questions[8]:

[8] research it if we did not discuss it in class.

(a) **Define** an equilibrium of a ODE.

(b) **Explain** the procedure for calculating equilibria.

(c) **Explain** the difference between a *general* and *particular* solution.

Exercise 1.4.2
Show that out of three functions Provided two are solutions of the ODE

$x^2 y''(x) - x y'(x) + y(x) = 0$, and the third is not.

(a) $y_1(x) = \ln(x)$ (b) $y_2(x) = x \ln(x)$ (c) $y_3(x) = x - x \ln(x)$

Summary

1. A differential equation relates a function to its derivatives.
2. The order of a differential equation is the highest derivative present.
3. A solution of a differential equation is a function that satisfies the equation.
4. Initial value problems specify conditions at a single point, while boundary value problems specify conditions at multiple points.
5. Mathematical modeling uses differential equations to describe real-world phenomena.
6. Historical development of differential equations is closely tied to calculus and physics.
7. Slope fields visualize solutions qualitatively.

Chapter 2: First Order Ordinary Differential Equations (ODEs)

Section 2.1: Introduction

As mentioned in the previous chapter, the order is determined by the highest derivative appearing in the equation. For a first order differential equation, only y' appears.

A first-order ordinary differential equation is an equation involving an unknown function $y(x)$ and its first derivative $y'(x)$. The general form is:

$$y' = f(x, y).$$

We will explore several common types of first-order ODEs and the methods used to solve them.

First-order differential equations can be classified into several types:

Type	General Form	Method
Separable	$\frac{dy}{dx} = g(x)h(y)$	Rewrite as $\frac{dy}{h(y)} = g(x)\,dx$ and integrate both sides.
Linear	$y' + P(x)y = Q(x)$	Use an integrating factor $\mu(x) = e^{\int P(x)\,dx}$. Multiply the equation by $\mu(x)$, then integrate.
Exact	$M(x,y)\,dx + N(x,y)\,dy = 0$	Check if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If so, find a potential function $\Phi(x,y)$ such that $d\Phi = M\,dx + N\,dy$. The solution is $\Phi(x,y) = C$.
Special Forms		
Bernoulli	$y' + P(x)y = Q(x)y^n$	Substitute $u = y^{1-n}$ to convert it to a linear equation.
Homogeneous	$y' = f(y/x)$	Substitute $v = y/x$ to convert it to a separable equation.

Section 2.2: Existence and Uniqueness

Before we proceed, we need to establish the conditions under which a different equation will have a solution.

Theorem 2.2.1 (Existence and Uniqueness for 1st Order Nonlinear Equations (Picard–Lindelöf))**.** *Suppose $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : \alpha < x < \beta, \delta < x < \gamma$ containing (x_0, y_0). Then the initial value problem*

$$y' = f(x,y), \quad y(x_0) = y_0$$

has a unique solution in some interval around x_0.

Theorem 2.2.2 (Existence and Uniqueness for 1st Order Linear Equations)**.** *If the functions P and Q are continuous on an open interval $I : \alpha < x < \beta$ containing the point $x = x_0$, then there exists a unique function $y = \phi(x)$ that satisfies the differential equation*

$$y' + P(x)y = Q(x) \tag{2.1}$$

for each $x \in I$, and that also satisfies the initial condition $y(x_0) = y_0$.

Section 2.3: Separable Equations

A first-order differential equation is **separable** if it can be written in the form:

$$\frac{dy}{dx} = g(x)h(y)$$

where $g(x)$ is a function of x alone and $h(y)$ is a function of y alone.

Solution Method

1. Separate the variables by rearranging the equation:

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

2. Integrate both sides of the equation:

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx + C$$

3. Solve the resulting algebraic expression for y to get the general solution.

Example 2.3.1. *Solve:*

$$y' = xy.$$

Solution:

$$\frac{dy}{dx} = xy \quad \Rightarrow \quad \frac{dy}{y} = x\,dx.$$

Integrate:

$$\ln|y| = \tfrac{1}{2}x^2 + C \quad \Rightarrow \quad y(x) = Ce^{x^2/2}.$$

Example 2.3.2. *Solve $y' = \dfrac{x^2}{y^2}$.*

Solution:

$$y^2\,dy = x^2\,dx \quad \Rightarrow \quad \int y^2\,dy = \int x^2\,dx$$

.

Integrate both sides

$$\frac{y^3}{3} = \frac{x^3}{3} + C \quad \Rightarrow \quad y^3 = x^3 + 3C$$

Since 3C is an arbitrary constant, we can write $y^3 = x^3 + K$.

$$y = \sqrt[3]{x^3 + K}$$

We should note that this solution method for certain kinds of differential equations may lead to an Integral-defined function (i.e the integral cannot be written in terms of any elementary function).

Practice Problems

Exercise 2.3.1
Solve the following separable ODEs:

a) $(2y)\, e^{y^2} y' = 1$

b) $y' = \dfrac{\cos(3x)}{\sin^2(3x)}$

c) $y' = \dfrac{1}{1+x^2}$

d) $y' = \dfrac{2x}{x^2+4}$

e) $y' = \dfrac{1}{\sqrt{1-x^2}}$

f) $y' = \dfrac{e^{-y}}{y}$

g) $y' = \dfrac{1}{y^2 \ln(y)}$

h) $y' = e^x \cos(x)$

i) $y' = -\ln(y)$

j) $y' = \dfrac{1}{x^2-1}$.

k) $y' = \dfrac{3x+5}{(x+1)(x-2)}$.

l) $y' = \dfrac{2x}{x^2+3x+2}$.

m) $y' = \dfrac{x^2+1}{x(x^2+1)}$.

n) $y' = \dfrac{1}{(x+1)^2(x-2)}$.

Exercise 2.3.2
Solve the following separable ODE

a) $y' = xe^{x^2}$.

b) $y' = \dfrac{\ln(x)}{x^2}$.

c) $y' = \sqrt{4-y^2}$.

d) $y' = \dfrac{\arctan(x)}{1+x^2}$.

Exercise 2.3.3
Solve the following separable ODEs:

a) $y' = y^2 \sin(x)$

b) $y' = y^3 x$

c) $y' = (y^2-1)x$

d) $y' = (y+2)(y-1)$

e) $y' = (y-1)(y-3)x^2$

f) $y' = (y+1)(y-2)xe^x$

g) $y' = (y^2-4)xe^{x^2}$

h) $y' = (y^2-9)\cos(x)$

i) $y' = (y-1)(y+3)\sin(x)$

j) $y' = (y+4)(y-2)x^3$

k) $y' = (y-5)(y+1)e^{-x}$

l) $y' = (y^2-16)x^4$

Exercise 2.3.4
Identify the separable ODEs out of those given below. Demonstrate that it can be written in the standard form $y'(x) = F(x)G(y)$:

(a) $y' + xy - x = 0$ (b) $y' - 3(y^2 - 5y - 6) = 0$

Exercise 2.3.5
Solve $\dfrac{dy}{dx} = y^2 - 4$.

Exercise 2.3.6
Find the implicit formula $F(x, y) =$ const. for the **solution** of the following separable ODE:

$$y'(x) = (y^2 - 6y + 8)xe^x$$

Exercise 2.3.7
Find the solution to the separable ODE $y'(x) = y(3 - y)$.

Exercise 2.3.8
Consider a population of bacteria that grows at a rate proportional to the size of the population.

(a) Determine a ODE to describe the growth of bacteria at any time t.

(b) Sketch the direction field, and make a prediction of what the solution is based on the sketch.

(c) Determine the solution analytically.

(d) The number of bacteria in a colony at 10:00am is approximately five million. At noon the number is approximately seven million. How many bacteria were in the colony at 9:00am?

Exercise 2.3.9
Determine the equilibrium solutions for each ODE and decide if they are stable (an attractor), unstable (a repellor) or semistable[1]. Draw a 1-D phase portrait for each.

[1] Do some research if you need to.

(a) $y' = (y - 2)(y + 1)(y + 2)$ (c) $y' = y^3 + 2y^2 + y$
(b) $y' = y^3 - 9y$ (d) $y' = -y - y^3$

Exercise 2.3.10
A model for the population $P(t)$ in a suburb of a large city is given by the Initial Value Problem (IVP)[2]

[2] IVPs are ODEs with additional condition that the solution or its derivatives has to take certain values at a specific point.

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P), \qquad P(0) = 5000,$$

where t is measured in months. What is the limiting value of the population[3]? At what time will the population be equal to one-half of this limiting value?

[3] Hint: recall equilibrium solutions

Exercise 2.3.11
Solve these IVPs:

(a) $e^x \frac{dy}{dx} = xy^2$, $y(0) = 2$ (b) $\frac{dy}{dx} = y^2 + 1$, $y(0) = 3$

Exercise 2.3.12
Find the 1-parameter family of solutions for each ODE:

(a) $xy' + 2y = e^{-x}$

(b) $x\dfrac{dy}{dx} = 2y + x^4\sin(x^2)$

(c) $\dfrac{dy}{dx} + xy = xy^3$

(d) $(1+x^2)y' + 4xy = \dfrac{4x}{(1+x^2)^3}$

Exercise 2.3.13

Solve $y' = e^{-x^2}, \quad y(3) = 5.$

See how to solve here[4]

[4] https://math.stackexchange.com/questions/154968/is-there-really-no-way-to-integrate-e-

Section 2.4: Linear Equations

Linear[5] ODEs are those that can be written in the form

$$\underbrace{a_0(x)y(x) + a_1(x)\frac{dy}{dx} + a_2(x)\frac{d^2y}{dx^2} + \dots a_n(x)\frac{d^ny}{dx^n}}_{\textbf{Linear combination}\text{ of derivatives}} = g(x)$$

[5] The name comes from the **linear combination**, which is a more-general expression that appears in various parts of mathematics.

are called **linear** ODEs. Notice: the **only way** the **dependent** variable appears in the equation is in expressions of the sort[6]

$$\dots + a_k(x)y^{(k)}(x) + \dots,$$

[6] $y^{(k)}$ denotes the k-th derivative. So $y^{(2)}(x) = y''(x)$.

that is y or its derivatives are never squared, multiplied with each other, or appear in more complicated formulas. The **coefficient functions** $a_k(x)$ may be very complicated themselves, but they never involve y or its derivatives.[7]

[7] For more on linear equations, see the beginning of Zill §2.3.

The **standard form** of a linear ODE is obtained by dividing through by the highest coefficient, to result in expression that has a constant ($= 1$) as its highest coefficient:

$$\underbrace{P_0(x)y(x) + P_1(x)\frac{dy}{dx} + P_2(x)\frac{d^2y}{dx^2} + \dots + \mathbf{1}\cdot\frac{\mathbf{d^ny}}{\mathbf{dx^n}}}_{\textbf{Linear combination}\text{ of derivatives}} = \underbrace{Q(x)}_{\textbf{Input}\text{ or }\textbf{Inhomogeneity}}.$$

The **standard form** is the starting point for many solution algorithms.

The standard[8] form for **1st order linear ODEs** is:

$$P(x)y(x) + y'(x) = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions of x.

[8] **If you want to show that an ODE is linear**, rearrange the terms to match the standard form. If you can find such functions $P(x)$ and $f(x)$ that **do not depend on** y, y' then you are looking at a linear ODE.

Here are several examples of 1st order linear ODEs. Rearrange them into a standard form and identify their coefficient function $P(x)$ and input function $Q(x)$ for each of them.

(a) $y'(x) = -0.5y(x) + 8$

(b) $4y'(x) = -2y(x) + 32 + 4\sin(x)$

(c) $x^2y'(x) + 0.5y(x) = x^3$

(d) $y'(x) = 3y(x)$

Exercise 2.4.1

Which of the following differential equations are linear?

(a) $y' + 2xy = 0$ (b) $y' = y + 5$ (c) $y' + y = x$

Is any of them separable?

Solution Method using Variation of parameters

To solve 1st order linear ODEs, we will use the **variation of parameters** method: [9]

Step (1) Convert a linear ODE into its standard form $y' + P(x)y = Q(x)$.

Step (2) Create the homogeneous subproblem by zeroing out the input function:

$$y'_H + P(x)y_H = 0$$

Step (3) Find the general solution of the homogeneous subproblem.[10]

Step (4) To form **a candidate solution** y_I for the inhomogeneous problem

$$y'_I + P(x)y_I = Q(x), \tag{4.1}$$

replace the integration parameter C in y_H by an unknown function $I(x)$. [11]

Step (5) **Substitute the candidate**[12] y_I into the **full (original)** ODE. You will need to take its derivative first, which will contain the unknown functions $I(x)$ and $I'(x)$. Simplify until you get an expression $I'(x) = \ldots$.

$I(x)$ will not appear on the right hand side (and neither will y).

Step (6) Integrate $I'(x)$ to calculate $I(x)$ function[13], which then gives you formula for $y_I(x)$.

Step (7) Now you have two functions: y_H and y_I. y_H contains a constant C, while y_I has no free constant. The general solution to the original problem is:

$$y(x) = y_H(x) + y_I(x)$$

[9] Textbook explains the **integrating factor** technique instead. The two methods are different, but yield equivalent results. Through variation of parameters we will learn skills that will transfer over to higher-order ODEs.

[10] For 1st order ODEs, this is always possible by separation of variables.

[11] Variation of constant $C \to$ function $I(x)$ is what gives the method its name.

[12] Anytime something is called "solution candidate" your instinct should be to "plug it into" the ODE.

[13] The integration constant inside $I(x)$ is not important, so pick a constant that makes $I(x)$ simple. Say 0.

Example 2.4.1. *Find general solution to*

$$xy' - 4y = x^6 e^x$$

(this is the same problem as textbook §2.3.Ex.3). Additionally, find the particular solution that has $y(2) = 0$.

General solution:

Step (1) *Standard form:* $y' + \underbrace{\frac{-4}{x}}_{P(x)} y = \underbrace{x^5 e^x}_{f(x)}$

Step (2) *Homogeneous subproblem:*

$$y'_H + \frac{-4}{x} y_H = 0 \tag{4.2}$$

Step (3) *Solving for y_H:*

$$\begin{aligned} \frac{dy_H}{dx} &= \frac{4y_H}{x} \quad \textit{(Separate variables)} \\ \frac{dy_H}{y_H} &= \frac{4dx}{x} \\ \int \frac{dy_H}{y_H} &= \int \frac{4dx}{x} \\ \ln|y_H| &= 4\ln|x| + K \quad \textit{(Exponentiate both sides)} \\ y_H(x) &= Cx^4 \end{aligned}$$

Step (4) ***Candidate solution*** *for the inhomogeneous problem:* $y_I(x) = I(x)x^4$

Step (5) *Substitute $y_I = I(x)x^4$ into $y_I' + \frac{-4}{x}y_I = x^5e^x$*

$$y_I'(x) = I'(x)x^4 + 4I(x)x^3$$
$$I'(x)x^4 + 4I(x)x^3 + \frac{-4}{x}I(x)x^4 = x^5e^x$$
$$I'(x)x^4 = x^5e^x \;\Rightarrow\; I'(x) = xe^x$$

Step (6) *Integrate $I'(x)$ (in this case, integration by parts):*

$$I(x) = xe^x - e^x + K \quad \textit{(in this step only, constant is arbitrary)}$$
$$I(x) = xe^x - e^x$$
$$y_I(x) = I(x)x^4 = (xe^x - e^x)x^4$$

Step (7) *General solution:*

$$\boxed{y(x) = \underbrace{Cx^4}_{y_H(x)} + \underbrace{x^5e^x - x^4e^x}_{y_I(x)}}$$

(Particular) solution to the IVP: *Start with the general solution (terms collected): $y(x) = x^4(C + xe^x - e^x)$, and plug into it the time-point $x = 2$:*

$$y(2) = 16C + 32e^2 - 16e^2 = 16(C + e^2)$$

At the same time, Initial Value says $y(2) = 0$, therefore

$$16(C - e^2) = y(2) = 0 \Rightarrow C = e^2.$$

So the particular solution to the IVP is:

$$y(x) = x^4(e^2 + xe^x - e^x)$$

Example 2.4.2. *Solve*

$$y' - 2y = e^x.$$

Step (1) *Standard form: $y' - 2y = e^x$., with $P(x) = -2$, $Q(x) = e^x$.*
Step (2) *Homogeneous subproblem: $y' - 2y = 0$.*
Step (3) *Homogeneous solution:*

$$y_H(x) = Ce^{-\int -2dx} = Ce^{2x}.$$

Step (4) *Candidate solution (Variation of parameters): let*

$$y_I(x) = I(x)e^{2x}.$$

Step (5) *Substitute into the full ODE:*

$$I'(x) = Q(x)e^{\int P(x)\,dx} = e^xe^{\int -2dx} = e^xe^{-2x} = e^{-x}.$$

Step (6) *Integrate $I'(x)$:*

$$I(x) = \int e^{-x}\,dx = -e^{-x}.$$

Thus,

$$y_I(x) = -e^{-x}e^{2x} = -e^x.$$

Step (7) *General solution:*

$$y(x) = Ce^{2x} - e^x.$$

Exercise 2.4.2
Find general solution to

$$\dot{y} = -\frac{1}{2}y + 8 + \sin(t)$$

Exercise 2.4.3
Solve:

$$y' + y = e^x.$$

Exercise 2.4.4
Solve

$$y' - y = e^{2x}.$$

Exercise 2.4.5
Solve the IVP:

$$y' + \frac{2}{x}y = \frac{\cos(x)}{x^2}, \quad y(\pi) = 0.$$

Solution Method using an Integrating Factor

The method of **integrating factor** is another way to solve 1st order linear ODEs. The goal of this method is to rewrite the differential equation in a separable form so that we can integrate both sides.

Multiplying the entire equation by a certain function $\mu(x)$ (the integrating factor), will allow us to rewrite the left-hand side as a single derivative of a product.

Step (1) Rewrite the linear ODE in its standard form: $y' + P(x)y = Q(x)$.

Step (2) Using $P(x)$ from the standard form, compute the **integrating factor**, $\mu(x)$:

$$\mu(x) = e^{\int P(x)\,dx}$$

Step (3) Multiply the entire differential equation by the integrating factor:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

The left side will become the derivative of a product:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

To see why, apply the product rule to the left-hand side:

$$\frac{d}{dx}\left[\overbrace{\mu(x)y}^{\text{product}}\right] = \overbrace{\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y}^{\text{product rule}} = \overbrace{\mu(x)\frac{dy}{dx} + \mu(x)P(x)y}^{\text{LHS of linear eq multiplied by}\,\mu(x)}.$$

This is only true if $\frac{d\mu}{dx} = \mu(x)P(x)$.

Thus, solving for $\mu(x)$ gives

$$\mu(x) = e^{\int P(x)\,dx}$$

Step (4) Integrate both sides with respect to x:

$$\mu(x)y = \int \mu(x)Q(x)\,dx + C$$

Step (5) Solve for y to find the general solution:

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)Q(x)\,dx + C\right)$$

Example 2.4.3. *Solve:*

$$y' + y = e^x.$$

Solution: *Here* $P(x) = 1$, *so*

$$\mu(x) = e^{\int 1dx} = e^x.$$

Multiply through:

$$e^x y' + e^x y = e^{2x}.$$

LHS is derivative:

$$\frac{d}{dx}(e^x y) = e^{2x}.$$

Integrate:

$$e^x y = \tfrac{1}{2}e^{2x} + C \quad \Rightarrow \quad y(x) = \tfrac{1}{2}e^x + Ce^{-x}.$$

Example 2.4.4. *Solve*

$$y' - y = e^{2x}.$$

Solution: *Here,* $P(x) = -1$ *and* $Q(x) = e^{2x}$. *So integrating Factor,*

$$\mu(x) = e^{\int -1\,dx} = e^{-x}$$

Multiply by $\mu(x)$*:*

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}e^{2x} = e^x \quad \Rightarrow \quad \frac{d}{dx}(e^{-x}y) = e^x$$

Integrating,

$$\int \frac{d}{dx}(e^{-x}y)\,dx = \int e^x\,dx \quad \Rightarrow \quad e^{-x}y = e^x + C$$

Thus,

$$y = e^x(e^x + C) = e^{2x} + Ce^x.$$

Practice Problems

Exercise 2.4.6
Each 1st order linear ODEs can be rewritten in the standard form with components
• input function; • coefficient function; • linear combination of derivatives.

(a) Rewrite the following linear ODE in the standard form and identify the listed components:

$$t^2\dot{x} + \sin(t)x + \cos(t) = 0.$$

(b) Write out the corresponding homogeneous subproblem.

Exercise 2.4.7
Solve

$$y' - 3y = 6$$

Exercise 2.4.8
Solve

$$xy' - 4y = x^6 e^x.$$

Exercise 2.4.9
Given a linear ODE.

$$t^2 \frac{dx}{dt} + x(t) - 3 = 0. \tag{4.3}$$

(a) Use *variation of parameters* technique to find the general solution to the ODE.

Do not use any formulas from memory or the textbook. Show every step of the process.

(b) Find the solution that additionally satisfies the Initial Value $x(1) = 1$.

Exercise 2.4.10
Find the **general** solution of the equation $\dot{x} + \sin(t)x = 3$.

Exercise 2.4.11
Solve the IVP $y' + \frac{2}{x}y = \frac{\cos(x)}{x^2}, \quad y(\pi) = 0$.

Exercise 2.4.12
The equation below could be linear, separable, or exact. Identify its type and support your answer by either:

- putting the linear into its standard form,
- performing the exactness test,
- identifying the separated F and G components.

Do not erase attempts that are unsuccessful. Showing that an ODE is **not** of a certain class is still a useful result.

$$(y - \ln x) \cdot y'(x) = 1 + \ln x + \frac{y}{x}$$

Exercise 2.4.13

A tank initially contains 2,000 litres of water with a concentration of mercury of 6.0×10^{-5} grams per litre. Water that contains 3.0×10^{-5} grams per litre is pumped into the tank at 100 litres per hour. The well mixed solution is pumped out of the tank at 100 litres per hour. How long will it take for the concentration in the tank to reach 4.5×10^{-5} grams per litre?

Exercise 2.4.14

A vat with capacity of 500 gallons initially has 10 lbs. of salt dissolved in 100 gallons of brine. A brine solution with 0.5 lbs. of salt per gallon is pumped into the tank at a rate of 4 gallons per minute, and the well mixed solution is pumped out at a rate of 2 gallons per minute. How much salt will be in the vat when the vat is full? (Hint: first find the solution for the volume of the fluid in the tank, before tackling the amount of salt in it.)

Exercise 2.4.15

A car's fuel tank holds 15 gallons of gasoline. When the tank is full, the gasoline contains 0.015 gallons of ethanol. Suppose that pure gasoline is pumped into the tank at a rate of 1 gallon per minute, and the well mixed solution is pumped out at a rate of 1 gallon per minute. How long will it take for the amount of ethanol in the tank to be 0.003 gallons?

Exercise 2.4.16

Solve the IVP $y' + y\tan(x) = \sin(2x), \quad y(0) = 1.$

Exercise 2.4.17

Solve the linear ODE $y' + y\cot(x) = 2x\csc(x).$

Exercise 2.4.18

Solve the ODE, $y' + yln(x^(2)) = 1.$

Exercise 2.4.19

Solve the following linear IVPs:

(a) $(t+1)\dot{x}(t) + x(t) = \ln t, \quad x(1) = 10$

(b) $\dot{P} + 2tP = P + 4t - 2, \quad P(1) = 1$

Section 2.5: Exact Equations

Definition 2.5.1. *A differential expression written in the form* $M(x,y)\,dx + N(x,y)\,dy$ *is an* ***exact differential*** *in a region* R *of the* $xy-$*plane if it corresponds to the differential of some function* $F(x,y)$*, that is defined in* R*. A first order differential equationof the form*

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if the following condition holds in R*:*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Theorem 2.5.2. *If* $M(x,y)$ *and* $N(x,y)$ *have continuous first partial derivatives in a simply connected region* R *of the* $xy-$*plane, then the differential equation*

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for all points (x,y) *in* R*.*

Solution Method

Step (1) Write the differential equation in the form $M(x,y)\,dx + N(x,y)\,dy = 0$:

Step (2) Check the exactness condition: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Step (3) If exact, find a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$.

Step (4) Integrate $M(x,y)$ with respect to x, treating y as a constant:

$$F(x,y) = \int M(x,y)\,dx + g(y)$$

The function $g(y)$ is a "constant" of integration that depends on y.

Step (5) Differentiate this expression for $F(x,y)$ with respect to y and set it equal to $N(x,y)$:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'(y) = N(x,y)$$

Step (6) Solve for $g'(y)$ and integrate to find $g(y)$.

Step (7) Substitute $g(y)$ back into the expression for $F(x,y)$. The general solution is $F(x,y) = C$.

Example 2.5.3. *Solve* $(2x+y)\,dx + (x+2y)\,dy = 0$*.*
Solution:

Step (1) ***Write the differential equation in the form*** $M(x,y)\,dx + N(x,y)\,dy = 0$*:*

$$(2x+y)\,dx + (x+2y)\,dy = 0$$

Step (2) ***Check Exactness:*** *$M(x, y) = 2x + y$ and $N(x, y) = x + 2y$. $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$. The equation is exact.*

Step (3) ***Integrate M w.r.t x:***

$$F(x, y) = \int (2x + y)\, dx + g(y) = x^2 + xy + g(y)$$

Step (4) ***Differentiate w.r.t y and compare to N:***

$$\frac{\partial F}{\partial y} = x + g'(y)$$

We set this equal to $N(x, y) = x + 2y$:

$$x + g'(y) = x + 2y \implies g'(y) = 2y$$

Step (5) ***Integrate to find $g(y)$:***

$$g(y) = \int 2y\, dy = y^2$$

Step (6) ***General Solution:*** *$F(x, y) = x^2 + xy + y^2 = C$.*

Example 2.5.4. *Solve:*

$$(2xy + 3)\, dx + (x^2 + 4y)\, dy = 0.$$

Check exactness:

$$M = 2xy + 3, \quad N = x^2 + 4y,$$
$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x \quad \Rightarrow \text{Exact}.$$

Find potential $\Phi(x, y)$:

$$\frac{\partial \Phi}{\partial x} = M = 2xy + 3 \quad \Rightarrow \quad \Phi(x, y) = x^2 y + 3x + g(y).$$

Differentiate wrt y:

$$\frac{\partial \Phi}{\partial y} = x^2 + g'(y) = N = x^2 + 4y \quad \Rightarrow \quad g'(y) = 4y.$$

So $g(y) = 2y^2$. Solution:

$$\Phi(x, y) = x^2 y + 3x + 2y^2 = C.$$

Integration Factor for Non-Exact Equations

Recall that the left-hand side of a linear ODE can be made into a product derivative by multiplying it by an integrating factor $\mu(x)$. Similarly, a non-exact equation can sometimes be made exact by multiplying it by an appropriate integrating factor $\mu(x, y)$.

Sometimes it is possible to find an integrating factor $\mu(x, y)$, a function of only one variable (either x or y) such that

$$\mu(x, y) M(x, y)\, dx + \mu(x, y) N(x, y)\, dy = 0$$

is exact.

In this case, the exactness condition becomes:

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N].$$

By the product rule, this expands to:

$$\mu M_y + M\mu_y = \mu N_x + N\mu_x.$$

Rearranging gives:

$$N\mu_x - M\mu_y = \mu(M_y - N_x). \tag{5.1}$$

If μ is a function of x only, then $\mu_y = 0$ and (**??**) becomes:

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}.$$

If the right-hand side is a function of x only, then we can integrate to find $\mu(x)$:

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}.$$

Example 2.5.5. *The following ODE is not exact. Find an integrating factor that makes it exact, and then solve it.*

$$xy\,dx + (x^2 - y^2)\,dy = 0.$$

Solution:

Step (1) *Write in standard form:*

$$xy\,dx + (x^2 - y^2)\,dy = 0.$$

Step (2) *Check exactness:*

$$M = xy, \quad N = x^2 - y^2,$$

$$\frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = 2x \quad \Rightarrow \textit{Not exact.}$$

Step (3) *Find integrating factor:*

$$\frac{M_y - N_x}{N} = \frac{x - 2x}{x^2 - y^2} = \frac{-x}{x^2 - y^2}.$$

This is not a function of x only, but it is a function of y only if we divide by x:

$$\frac{M_y - N_x}{-M} = \frac{x - 2x}{-xy} = \frac{1}{y}.$$

So we can take $\mu(y) = e^{\int \frac{1}{y}\,dy} = e^{\ln|y|} = |y|$.

Step (4) *Multiply the original equation by $\mu = y$:*

$$y(xy\,dx + (x^2 - y^2)\,dy) = 0 \quad \Rightarrow \quad (xy^2)\,dx + (x^2y - y^3)\,dy = 0.$$

Step (5) *Check exactness:*

$$M = xy^2, \quad N = x^2y - y^3,$$

$$\frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = 2xy \quad \Rightarrow \textit{Exact.}$$

Step (6) *Find potential function:*

$$\frac{\partial F}{\partial x} = M = xy^2 \quad \Rightarrow \quad F(x,y) = \frac{1}{2}x^2y^2 + g(y).$$

$$\frac{\partial F}{\partial y} = x^2y + g'(y) = N = x^2y - y^3 \quad \Rightarrow \quad g'(y) = -y^3.$$

$$g(y) = -\frac{1}{4}y^4.$$

Step (7) *General solution:*

$$F(x,y) = \frac{1}{2}x^2y^2 - \frac{1}{4}y^4 = C.$$

Exercise 2.5.1
The following ODE is not exact. Find an integrating factor that makes it exact, and then solve it.

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0.$$

Practice Problems

Exercise 2.5.2
Determine whether the following ODE is exact. If it is, find its general solution.

$$(3x^2y - 6x)\,dx + (x^3 + 2y)\,dy = 0.$$

Exercise 2.5.3
Determine whether the following ODE is exact. If it is, find its general solution.

$$y'(x) = \frac{3x + 4y}{x}.$$

Exercise 2.5.4
Solve the following IVPs:

1. $(y + x)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1.$
2. $(e^x + y)\,dx + (x + \cos(y))\,dy = 0, \quad y(0) = 0.$
3. $(e^x + 2y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1.$
4. $$\left(\frac{3y^2 - t^2}{y^5}\right)\frac{dy}{dt} + \frac{t}{2y^4} = 0, \quad y(1) = 1.$$

Exercise 2.5.5
Determine whether the following ODE is exact. If it is, find its general solution.

$$(2xy + 3)\,dx + (x^2 + 4y)\,dy = 0.$$

Exercise 2.5.6
Find the solution and interval of validity for the IVP:

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0, \quad y(5) = 0.$$

Exercise 2.5.7
Determine whether the following ODE is exact. If it is, find its general solution:

$$y'(x) = \frac{xy^2 - \cos(x)\sin(x)}{y(1-x^2)}$$

Section 2.6: Substitution

Many first-order ODEs can be solved by making an appropriate substitution that transforms the equation into a separable or linear form. Two common substitution methods are for Homogeneous equations and Bernoulli equations.

Homogeneous Equations

Definition 2.6.1. *A function $F(x,y)$ is **homogeneous of degree** $n \in \mathbb{R}$ if for all t:*

$$F(tx, ty) = t^n F(x,y)$$

For example, $F(x,y) = x^3 + y^3$ is a homogeneous function of degree 3 because

$$F(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 F(x,y).$$

whereas $F(x,y) = x + y^3$ is not homogeneous because

$$F(tx, ty) = tx + (ty)^3 = tx + t^3y^3 \neq t^n F(x,y)$$

for any n.

Definition 2.6.2. *A first-order ODE of the form*

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{or} \quad \frac{dy}{dx} = F(x,y) = -\frac{M(x,y)}{N(x,y)} \tag{6.1}$$

*is **homogeneous** of degree n if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree n.*

So we can perform the substitution $v = \frac{y}{x}$ to transform the equation into a separable form. Since $M(x,y)$ and $N(x,y)$ are homogeneous of degree n, we have:

$$M(x,y) = x^n M(1,v) \quad \text{and} \quad N(x,y) = x^n N(1,v).$$

And equation (**??**) becomes:

$$x^n M(1,v)\,dx + x^n N(1,v)\,dy = 0 \quad \Rightarrow \quad M(1,v)\,dx + N(1,v)\,dy = 0 (\text{for 0 degree}) \tag{6.2}$$

where $v = \frac{y}{x} \to y = vx \Rightarrow dy = u\,dx + x\,du$ which reduces equation (**??**) to a separable form in u and x

$$\frac{dx}{x} = -\frac{N(1,u)}{M(1,u) + uN(1,u)}\,du.$$

Solution Method for Homogeneous Equations

Step (1) Make the substitution $v = \frac{y}{x}$, which implies $y = vx$.

Step (2) Differentiate $y = vx$ using the product rule to get $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Step (3) Substitute v and the new expression for $\frac{dy}{dx}$ into the original equation. The result will be a separable equation in v and x.

Step (4) Solve the separable equation for v.

Step (5) Substitute back $v = \frac{y}{x}$ to get the solution in terms of x and y.

Example 2.6.3. *Solve* $\frac{dy}{dx} = \frac{x+y}{x}$.
Solution:

1. ***Check Homogeneity:*** *Divide numerator and denominator by x:*

$$\frac{dy}{dx} = \frac{1 + y/x}{1} = 1 + \frac{y}{x}$$

This is a homogeneous equation.

2. ***Substitution:*** *Let $v = y/x$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.*
3. ***New Equation:*** *$v + x\frac{dv}{dx} = 1 + v$. This simplifies to $x\frac{dv}{dx} = 1$, which is separable.*
4. ***Solve Separable Equation:*** *$dv = \frac{1}{x}\,dx$.*

$$\int dv = \int \frac{1}{x}\,dx \implies v = \ln|x| + C$$

5. ***Substitute back:*** *$\frac{y}{x} = \ln|x| + C$.*
6. ***General Solution:*** *$y = x(\ln|x| + C)$.*

Example 2.6.4. *Solve* $\dfrac{dy}{dx} = \dfrac{x+y}{x}$.
Answer:

$$y = x\ln|x| + Cx$$

Example 2.6.5. *Solve:*

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Answer:

$$\frac{y}{x} = \ln|x| + C \quad \Rightarrow \quad y = x\ln|x| + Cx$$

Example 2.6.6. *Solve:*

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

Answer:

$$\left(\frac{y}{x}\right)^2 = 2\ln|x| + C$$

Bernoulli Equations

Definition 2.6.7. *A first-order ODE of the form*

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

*is a **Bernoulli equation** if n is any real number other than 0 or 1.*

The substitution $v = y^{1-n} \quad \Rightarrow \quad \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ transforms the equation into a linear ODE in v.

Solution Method for Bernoulli Equations

Step (1) Identify the Bernoulli equation in the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$.

Step (2) Make the substitution $v = y^{1-n}$, which implies $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$.

Step (3) Rearrange the original equation to express $\frac{dy}{dx}$ in terms of y, $P(x)$, and $Q(x)$.

Step (4) Substitute v and the new expression for $\frac{dy}{dx}$ into the rearranged equation. The result will be a linear equation in v and x.

Step (5) Solve the linear equation for v using an integrating factor if necessary.

Step (6) Substitute back $v = y^{1-n}$ to get the solution in terms of x and y.

Example 2.6.8. *Solve* $\dfrac{dy}{dx} + y = y^2$.
Solution:
Here $n = 2$, so $v = y^{-1}$. Compute $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. Original equation: $\frac{dy}{dx} = y^2 - y$.
Substitute:

$$\frac{dv}{dx} = -(1 - \frac{1}{y}) = -1 + v$$

So:

$$\frac{dv}{dx} - v = -1$$

This is linear. Integrating factor e^{-x}:

$$\frac{d}{dx}(ve^{-x}) = -e^{-x}$$

Integrate:

$$ve^{-x} = e^{-x} + C \implies v = 1 + Ce^x$$

Back-substitute $v = 1/y$:

$$y = \frac{1}{1 + Ce^x}$$

Example 2.6.9. *Solve* $\dfrac{dy}{dx} + \dfrac{2}{x}y = xy^3$.
Solution:
Here $n = 3$, so $v = y^{-2}$. Compute $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$.
$\vdots$

$$y = \frac{1}{\sqrt{x^2 + Cx^4}}$$

Example 2.6.10. *Solve:*

$$\frac{dy}{dx} + y = y^2$$

Solution:
Here $P(x) = 1$, $Q(x) = 1$, $n = 2$.
Substitute: $v = y^{1-2} = y^{-1}$.
$\vdots$

$$\frac{1}{y} = 1 + Ce^x \quad \Rightarrow \quad y = \frac{1}{1 + Ce^x}$$

Example 2.6.11. *Solve:*

$$\frac{dy}{dx} + \frac{2}{x}y = xy^3$$

Solution:
Here $P(x) = \frac{2}{x}$, $Q(x) = x$, $n = 3$.
Substitute: $v = y^{1-3} = y^{-2}$.
$\vdots$

$$y^{-2} = x^2 + Cx^4 \quad \Rightarrow \quad y = \frac{1}{\sqrt{x^2 + Cx^4}}$$

Practice Problems

Exercise 2.6.1
Solve $(x^2 + y^2)\,dx + (x^2 - 2xy)\,dy = 0$.

Exercise 2.6.2
Solve $(x + y)\,dx - (x - y)\,dy = 0$.

Exercise 2.6.3
Solve the following homogeneous ODEs:

1. $\frac{dy}{dx} = \frac{x+3y}{3x+y}$
2. $(y^2 + xy)\,dx + x^2\,dy = 0$
3. $-y\,dx + (x + \sqrt{xy})\,dy = 0$
4. $xy' = y + \sqrt{x^2 - y^2}, \quad x > 0$

Exercise 2.6.4
Solve the following Bernoulli equations:

1. $xy' + y = y^{-2}$
2. $y' - y = e^x y^2$
3. $y' = y(xy^3 - 1)$

4. $xy' - (1+x)y = xy^2$

Exercise 2.6.5
Solve $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$ using the homogeneous substitution.

Exercise 2.6.6
Solve $\frac{dy}{dx} + \frac{y}{x} = x^2y^3$ using the Bernoulli substitution.

Exercise 2.6.7
Solve $\frac{dy}{dx} = \frac{y^2 + xy}{x^2}$ using the homogeneous substitution.

Exercise 2.6.8
Solve:

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

Exercise 2.6.9
Solve:

$$\frac{dy}{dx} = \frac{y + x}{y - x}$$

Exercise 2.6.10
Solve:

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^3$$

Exercise 2.6.11
Solve:

$$\frac{dy}{dx} - 2y = 3y^2$$

Section 2.7: Qualitative Analysis of First-Order ODEs

A first-order ODE can be written in the form

$$\frac{dy}{dx} = f(x, y).$$

If f does not depend on x, then the equation is called **autonomous**:

$$\frac{dy}{dx} = f(y).$$

Autonomous equations are often used to model the evolution of a system over time, where x represents time and y represents the state of the system.

Equilibria and Stability

Definition 2.7.1. *An **equilibrium** (or **fixed point**) of an autonomous ODE is a value $y = y_0$ such that $f(y_0) = 0$.*

At an equilibrium, the state of the system does not change over time.

Definition 2.7.2. *An equilibrium y_0 is* ***stable*** *if solutions that start close to y_0 remain close to y_0 as $x \to \infty$.*

Example 2.7.3. *Consider the autonomous ODE:*

$$\frac{dy}{dx} = y(1-y).$$

The equilibria are found by setting $f(y) = y(1-y) = 0$, which gives $y_0 = 0$ and $y_0 = 1$.

If solutions that start close to y_0 move away from y_0 as $x \to \infty$, then the equilibrium is **unstable**.

Phase Portraits and Direction Fields

A **phase portrait** is a graphical representation of the trajectories of a dynamical system in the phase plane. Each point in the phase plane represents a state of the system, and the trajectories show how the state evolves over time. To construct a phase portrait for an autonomous ODE, we can use a **direction field** (or **slope field**). A direction field is a grid of arrows that indicate the slope of the solution curve at each point in the phase plane. The slope at a point (x, y) is given by $f(y)$. To sketch a direction field:

1. Choose a grid of points in the xy-plane.
2. At each point (x, y), compute the slope $f(y)$
3. Draw a small line segment with the computed slope at each point.

Example 2.7.4. *Consider the autonomous ODE:*

$$\frac{dy}{dx} = y(1-y).$$

The equilibria are $y_0 = 0$ and $y_0 = 1$. To sketch the direction field, we compute the slope at various points:

- *For $y < 0$, $f(y) < 0$, so the slope is negative.*
- *For $0 < y < 1$, $f(y) > 0$, so the slope is positive.*
- *For $y > 1$, $f(y) < 0$, so the slope is negative.*

The direction field shows that solutions starting below $y = 0$ move downward, solutions starting between $y = 0$ and $y = 1$ move upward, and solutions starting above $y = 1$ move downward. Thus, $y = 0$ is an unstable equilibrium and $y = 1$ is a stable equilibrium.

The direction field in Figure **??** illustrates the behavior of solutions near the equilibria $y = 0$ and $y = 1$..

The phase portrait in Figure **??** shows the stability of the equilibria. It is given in the diagram by arrows indicating the direction of movement of solutions near the equilibria.

Note: The stability of an equilibrium can often be determined by examining the

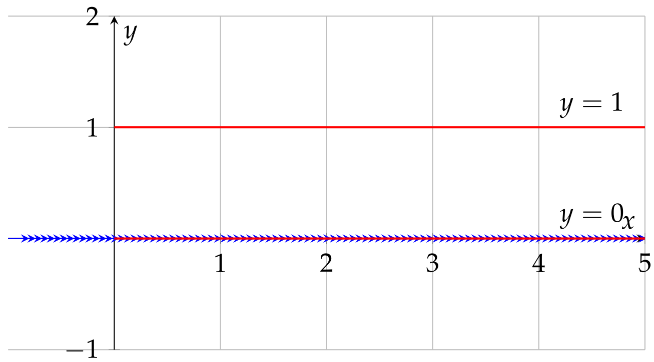


Figure 2.1: Direction field and equilibria for $\frac{dy}{dx} = y(1-y)$.

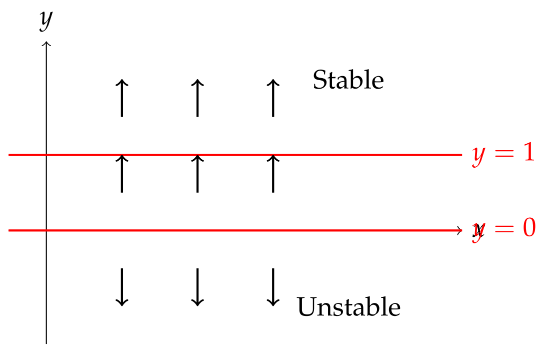


Figure 2.2: Phase portrait for $\frac{dy}{dx} = y(1-y)$.

sign of $f(y)$ near the equilibrium. If $f(y)$ changes from positive to negative as y increases through y_0, then y_0 is stable. If $f(y)$ changes from negative to positive, then y_0 is unstable.

Section 2.8: A Numerical Method for ODEs

Euler's Method

Euler's method is a simple numerical technique for approximating the solution of an initial value problem (IVP) of the form:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

The idea is to use the slope of the solution at a known point to estimate the value of the solution at a nearby point. The method proceeds as follows:

1. Choose a step size h.
2. Compute the next point using the formula:
3. $$y_{n+1} = y_n + hf(x_n, y_n), \quad x_{n+1} = x_n + h.$$
4. Repeat the process to compute subsequent points.

Example 2.8.1. *Solve the IVP:*

$$\frac{dy}{dx} = y - x^2 + 1, \quad y(0) = 0.5$$

1. *Using Euler's method with step size $h = 0.2$ to approximate $y(1)$.*
2. *Using an analytical method to find the exact solution and compare it with the approximation.*

Example 2.8.2. *Solve the IVP:*

$$y' = 0.1\sqrt{y} + 0.4x^2, \quad y(2) = 4$$

1. *Using Euler's method with step size $h = 0.1$ to approximate $y(2.5)$.*
2. *Using an analytical method to find the exact solution and compare it with the approximation.*

Example 2.8.3. *Solve the IVP:*

$$y' = 0.2xy, \quad y(1) = 1$$

1. *Using Euler's method with step size $h = 0.1$ to approximate $y(1.5)$.*
2. *Using Euler's method with step size $h = 0.05$ to approximate $y(1.5)$.*
3. *Using an analytical method to find the exact solution and compare it with the approximation.*

Summary

- First-order ODEs appear in many forms; solution methods depend on type.
- Existence & uniqueness depend on continuity conditions.
- Separable $\Rightarrow$ integrate directly.
- Linear $\Rightarrow$ integrating factor.
- Exact $\Rightarrow$ potential function.
- Substitution methods (homogeneous, Bernoulli) transform to solvable forms.
- Autonomous ODEs model time-evolving systems.
- Equilibria are found by setting $f(y) = 0$.
- Stability determined by sign of $f(y)$ near equilibria.
- Phase portraits visualize system behavior.
- Euler's method approximates solutions numerically.

Section 2.9: Practice for Test — First-Order ODEs

You can use the following step-by-step calculator to help you study: `https://goo.gl/sZEyvL`. Do not rely on it too much, though, as you will not have such help in an exam.

Exercise 2.9.1
For each equation below,

- Determine whether the given ODE is exact, linear, or separable by identifying the functions that demonstrate the type: M, N for exact, standard form for linear, F, G for separable and perform the appropriate tests (if needed).
- Find the general solution of equations using the appropriate technique.

(a) $(4x - 8y^3)y'(x) + (5x + 4y) = 0$

(b) $y' - e^{3x+2y} = 0$

(c) $(x - y^3 + y^2\sin(x))y'(x) - 3xy^2 - 2y\cos(x) = 0$

Exercise 2.9.2
Find the value of k such that the given ODE is exact:

$$y^3 + kxy^4 - 2x + (3xy^2 + 20x^2y^3)y'(x) = 0$$

Exercise 2.9.3
Conceptual questions and definitions:

(a) What is the equilibrium of an ODE?
(b) What is the difference between general and particular solutions of an ODE?
(c) What determines the angle of line segments used to plot direction fields?
(d) What is an equilibrium of an ODE?
(e) Define what it means for an equilibrium to be stable? Then explain how one can determine if an equilibrium is stable without finding a solution of an ODE.

Exercise 2.9.4
Find the fixed points, determine their stability, and sketch the phase portrait around them.

(a) $\dot{x} - (x-2)(x+2)^2(x-7)$
(b) $\dot{x} = \cos(x)$
(c) $\dot{x} = \cos(x) + 1$
(d) $\dot{x} = \left(\frac{1}{5} - \frac{1}{x}\right)\left(\frac{1}{5} - \frac{2}{x}\right)^2$, for $x > 0$

Exercise 2.9.5
Use phase portrait information to determine what the solution does as $x \to \infty$ if $y(0) = 3$ for the ODE:

$$y'(x) + (y-2)(y-4)^2.$$

Exercise 2.9.6
Each of these ODEs could be linear, separable, or exact. Identify the type of each equation and support your answer by:

- putting the linear equation into its standard form and circling the input term,
- performing the exactness test on the exact equation,
- identifying the separated F and G components for the separable ODE.

(a) $(4y + 2t - 5) + (6y + 4t - 1)\dot{y} = 0$
(b) $xy'(x) + (3x+1)y - e^{-3x} = 0$
(c) $\dot{N} + N - Nte^{t+2} = 0$
(d) $\sqrt{1-y^2} - y'(x)\sqrt{1-x^2} = 0$
(e) $(2xy + x^2 - 1)y'(x) + (x+y)^2 = 0$
(f) $(x+2)^2y' - 5 + 8y + 4xy = 0$
(g) $\sin(3x) + 2[\cos(3x)]^2y' \cdot y = 0$
(h) $(e^x + y) + (2 + x + ye^y)y'(x) = 0.$
(i) $y' + \tan(x)y - \cos^2(x) = 0$

Exercise 2.9.7

(Challenge) Give an example of a first-order ODE that is neither linear, separable, or exact. Perform the tests as in the previous question that demonstrate this.

Exercise 2.9.8

Find the **general solution** of each of the following equations using any technique that you find appropriate. Then, find the **particular solution** that satisfies the given initial condition.

(a) $x^2y'(x) - y + xy$, $y(-1) = -1$

(b) $xy'(x) + y - 4x - 1$, $y(1) = 8$

(c) $(x - y^3 + y^2\sin(x)) - (3xy^2 + 2y\cos(x))y'(x) = 0$, $y(0) = 2$

Exercise 2.9.9

The following equation is not exact (it fails the exactness test). Nevertheless, apply the technique for solving exact equations. At which point does the technique break down?

$$(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$$

Chapter 3: Modeling with First Order ODEs

A **model** is a mathematical description of a real-world situation.

- Many processes in nature, engineering, economics, and biology change at a rate that depends on the current state of the system.
- These are often modeled by first-order differential equations:

$$\frac{dy}{dt} = f(y,t)$$

  where

  - t: independent variable (usually time),
  - $y(t)$: dependent variable (population, temperature, amount, etc.),
  - $f(y,t)$: rule describing the rate of change.

Steps in Modeling

1. Identify the system (population, chemical reaction, motion, etc.).
2. Describe the rate of change in words (e.g., "rate proportional to the population").
3. Translate into a differential equation.
4. Solve the equation (analytically or numerically).
5. Interpret the solution in context (units, growth/decay behavior).

Section 3.1: linear Models

(a) Exponential Growth and Decay

Rate of change proportional to the current amount:

$$\frac{dy}{dt} = ky, \quad k \in \mathbb{R}, \, y(t_0) = y_0.$$

Solution:

$$y(t) = y_0 e^{kt}.$$

Applications: radioactive decay ($k < 0$), population growth ($k > 0$), carbon dating.

Example 3.1.1. ***Exponential Growth (Population)*** *A bacteria culture has initial population $P(0) = 500$. It doubles every 3 hours.*
Solution:

$$\frac{dP}{dt} = kP, \quad P(t) = P_0 e^{kt}.$$

At $t = 3$, $P(3) = 1000 = 500e^{3k}$, so $e^{3k} = 2 \Rightarrow k = \frac{\ln 2}{3}$.

$$P(t) = 500e^{(\ln 2/3)t}.$$

(b) Newton's Law of Cooling/Heating

The rate of cooling is proportional to the temperature difference between an object and its surroundings:

Rate proportional to difference from ambient temperature T_a:

$$\frac{dT}{dt} = -k(T - T_a).$$

Solution:

$$T(t) = T_a + (T_0 - T_a)e^{-kt}.$$

Exercise 3.1.1
Newton's Law of Cooling: Coffee at $90°C$ is placed in a $20°C$ room. After 10 minutes it cools to $60°C$. Find T after 20 minutes.

Solution:

$$T(t) = 20 + 70e^{-kt}.$$

At $t = 10$, $60 = 20 + 70e^{-10k} \Rightarrow e^{-10k} = \frac{4}{7}$. Thus $k = \frac{1}{10} \ln \frac{7}{4}$.

$$T(20) = 20 + 70\left(\frac{4}{7}\right)^2 \approx 42.9°\text{C}.$$

Exercise 3.1.2
A blacksmith removes a metal from the forge at 750°C. 5 minutes later, the temperature is 683°C. How long would it take the metal to cool down to 500°C?

(c) Mixing Problems

A mixing tank contains a volume of liquid with some solute (e.g., salt). If liquid flows in and out at certain rates, the concentration changes over time.

Let $Q(t)$ be the amount of solute (kg) at time t:

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}.$$

If inflow concentration is c_{in} and flow rate is r, and the tank volume is V, then:

$$\frac{dQ}{dt} = rc_{in} - \frac{r}{V}Q.$$

Example 3.1.2. ***Mixing Problem:*** *Tank: 100 L water with 20 g salt. Brine with 2 g/L enters at 5 L/min, mixture flows out at 5 L/min.*
Solution: *Let $A(t)$ = salt amount.*

$$\frac{dA}{dt} = 10 - \frac{A}{20}.$$

Solution:

$$A(t) = 200 - 180e^{-t/20}.$$

(d) Motion with Resistance

Falling object with air resistance proportional to velocity:

$$m\frac{dv}{dt} = mg - kv,$$

$$\frac{dv}{dt} = g - \frac{k}{m}v.$$

Terminal velocity occurs when $\frac{dv}{dt} \to 0$.

(e) Series Circuits

Series circuits are a common source of first-order linear differential equations in physics and engineering.

I - Components of a Series Circuit A typical circuit consists of:

- $i(t)$: current (A)
- A resistor with resistance R (measured in ohms Ω).
- An inductor with inductance L (measured in henries H).
- A capacitor with capacitance C (measured in farads F).
- A voltage source $E(t)$ (measured in volts).

II - Governing Equation (Kirchhoff's Voltage Law) Kirchhoff's Voltage Law states that the sum of voltage drops around a closed loop equals the applied voltage (The sum of potential differences around a closed loop is zero):

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i(t)\,dt = E(t).$$

Taking derivatives, this can also be expressed as:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = \frac{dE}{dt}.$$

III Special Cases

- **RL Circuit (no capacitor):**

  $$L\frac{di}{dt} + Ri = E(t).$$

  This is a first-order linear ODE in $i(t)$.

- **RC Circuit (no inductor):**

  $$Ri + \frac{1}{C}\int i(t)\,dt = E(t),$$

  or equivalently, in terms of charge $q(t)$ on the capacitor,

  $$R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

Example 3.1.3. ***RL Circuit:*** *Suppose $L = 1\,H$, $R = 2\,\Omega$, and $E(t) = 10\,V$. Then:*

$$\frac{di}{dt} + 2i = 10.$$

Solution:

$$i(t) = 5 + Ce^{-2t}.$$

Practice Exercises

1. Solve the RL circuit:
$$L\frac{di}{dt} + Ri = E_0,$$
with $L = 2$, $R = 4$, $E_0 = 20$, and $i(0) = 0$.

2. A tank initially contains 100 L of pure water. Brine containing 0.5 kg/L of salt enters at 5 L/min, and the mixture leaves at the same rate. Find the amount of salt after t minutes.

3. A cup of coffee at $90°C$ is left in a room at $20°C$. After 10 minutes the coffee has cooled to $70°C$. Find the temperature after 20 minutes.

Section 3.2: Nonlinear Models

Nonlinear first-order differential equations arise in many real-world applications such as biology, physics, and chemistry. Unlike linear models, nonlinear models often exhibit more complex behavior such as saturation, thresholds, or blow-up in finite time.

General form:
$$\frac{dy}{dt} = f(t, y), \quad \text{where } f \text{ is nonlinear in } y.$$

(a) Population Models (Logistic Growth)

A refinement of the exponential growth model that accounts for limited resources:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where

- $P(t)$ is the population at time t,
- $r > 0$ is the intrinsic growth rate,
- $K > 0$ is the carrying capacity.

Example 3.2.1. *Suppose a population grows logistically with $r = 0.5$, $K = 1000$, and initial population $P(0) = 100$.*

$$P(t) = \frac{1000}{1 + 9e^{-0.5t}}.$$

(b) Nonlinear Mixing Model

If concentration depends nonlinearly (e.g., saturation or reaction), the equation can become nonlinear.

Example 3.2.2.
$$\frac{dQ}{dt} = rc_{in} - \frac{r}{V}Q - kQ^2,$$
where kQ^2 models a nonlinear reaction term.

(c) Newton's Law of Cooling with Radiation

Newton's law (linear) is:
$$\frac{dT}{dt} = -k(T - T_s).$$
If radiation is included (Stefan–Boltzmann law), we obtain:
$$\frac{dT}{dt} = -k(T - T_s) - \sigma(T^4 - T_s^4),$$
which is nonlinear.

Example 3.2.3. *An object at 500 K cools in surroundings at 300 K with:*
$$\frac{dT}{dt} = -0.1(T - 300) - 10^{-8}(T^4 - 300^4).$$
This nonlinear ODE requires numerical methods (Euler, Runge–Kutta). The qualitative behavior: $T(t) \to 300$ as $t \to \infty$.

5. Practice Exercises

1. **Exponential Growth:** A population of 100 triples to 300 after 5 years. Find population after 10 years.
2. **Logistic Growth:** Fish population with carrying capacity 5000, initial 100, growth rate $k = 0.2$. Write explicit solution.
3. **Newton's Cooling:** A body at 37°C in a 20°C room cools to 30°C after 1 hour. Find temperature after 2 hours.
4. **Mixing:** Tank with 200 L pure water. Brine with 1 g/L enters at 4 L/min; mixture leaves at 2 L/min. Find salt content after 1 hour.
5. **Motion with Resistance:** A 2 kg mass falls under gravity ($g = 9.8$) with air resistance $k = 4$. Find terminal velocity as $t \to \infty$.

Section 3.3: Modeling with Systems of First Order ODEs

Predator-Prey Model (Lotka-Volterra)

A system of nonlinear equations describing interactions between two species:
$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$

where

- $x(t)$ = prey population,
- $y(t)$ = predator population,
- α = prey growth rate,
- β = predation rate,
- γ = predator death rate,
- δ = predator reproduction rate from consuming prey.

Chapter 4: Higher (Second) Order Differential Equations ODEs

Higher-order ODEs often serve as models for mechanical and electrical systems, as relationships between position/velocity/acceleration and charge/current/voltage require multiple derivatives.

Section 4.1: Linear Equations

A **linear differential equation of order** n is an equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x),$$

where the functions $a_0(x), a_1(x), \ldots, a_n(x)$ and $g(x)$ are given, and y is the unknown function.

If $g(x) = 0$, the equation is called **homogeneous**. Otherwise, it is called **inhomogeneous/nonhomogeneous**.

Definition 4.1.1 (Initial Value Problem). *A **higher-order initial value problem (IVP)** specifies the values of the function and its derivatives at a single point.*

$$\begin{aligned} \textit{That is,} \quad & a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x), \\ \textit{Subject to} \quad & y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \end{aligned}$$

For a 2nd-order equation, an IVP looks like

$$y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = y_0,\ y'(x_0) = y_1.$$

Theorem 4.1.2 (Existence of a Solution). *Let $a_0(x), a_1(x), \ldots, a_n(x)$ and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for all $x \in I$. If the initial condition $x = x_0 \in I$, then a unique solution $y(x)$ of the IVP exists.*

Definition 4.1.3 (Boundary Value Problems). *A second-order **boundary value problem (BVP)** specifies the values of the function at two different points.*

$$\begin{aligned} \textit{That is,} \quad & y'' + p(x)y' + q(x)y = g(x) \\ \textit{Subject to} \quad & y(a) = y_0,\ y(b) = y_1. \end{aligned}$$

Homogeneous Equations

A linear homogeneous n-th order equation has the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0.$$

Theorem 4.1.4 (Superposition Principle). *Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order differential equation on an interval I. Then the linear combination*

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x),$$

where c_i are arbitrary constants, is also a solution on the I.

Example 4.1.5. *Verify that the functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear equation $x^3 y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$. Show by the superposition principle that the linear combination is also a solution on the same interval*

Definition 4.1.6 (Linear Dependence/Independence). *A set of functions $\{f_1(x), f_2(x), \ldots, f_k(x)\}$ is said to be* ***linearly dependent*** *on an interval I if there exist constants $c_1 \neq c_2 \neq \ldots \neq c_k \neq 0$ such that*

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_k f_k(x) = 0$$

for every x in the interval. Otherwise, the set is ***linearly independent***

When finding solutions to linear differential equations, we are interested in linearly independent functions. We can use a more "indirect" definition to check whether a set of functions is linearly independent.

Definition 4.1.7 (Wronskian). *Suppose that the set of functions $\{f_1(x), f_2(x), \ldots, f_k(x)\}$ are $k-1$ times differentiable. The determinant*

$$W(f_1, f_2, \ldots, f_k)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** *of the functions.*

Theorem 4.1.8 (Criterion for Linearly Independent Solutions). *Let $y_1, y_2, \ldots, y_k$ be k solutions of the homogeneous linear nth-order differential equation*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0.$$

on an interval I. Then the set of solutions is linearly independent on I if $W(y_1, y_2, \ldots, y_k) \neq 0$ for every x in the interval.

Example 4.1.9. *Verify that*

$$y_1 = \frac{\cos(2\ln x)}{x^3} \quad \text{and} \quad y_2 = \frac{\sin(2\ln x)}{x^3}$$

are linearly independent solutions of the differential equation $x^2 y'' + 7xy' + 13y = 0$ on the interval $I = (0, \infty)$.

Definition 4.1.10 (Fundamental Set of Solutions)**.** *A fundamental set of solutions of a homogeneous linear kth-order differential equation*

$$a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = 0.$$

on an interval I is any set of k linearly independent solutions $\{y_1, y_2, \ldots, y_k\}$ on the interval I.

Theorem 4.1.11 (General Solution of Homogeneous Equations)**.** *Let $y_1, y_2, \ldots, y_k$ be a fundamental set of solutions of the homogeneous linear kth-order differential equation on the interval I. Then the general solution of the equation on the interval is*

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k,$$

where c_i are arbitrary constants.

Nonhomogeneous Equations

A **nonhomogeneous linear equation** has the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x).$$

The general solution is

$$y(x) = y_h(x) + y_p(x),$$

where y_h solves the homogeneous equation and y_p is any **particular solution**.

Methods for finding y_p include:

- Method of undetermined coefficients (for exponential, polynomial, sine, cosine $g(x)$).
- Variation of parameters (general but more involved).

Theorem 4.1.12 (General Solution of Nonhomogeneous Equations)**.** *Let y_p be any particular solution of the homogrneous linear kth-order differential equation on an interval I, and let $y_1, y_2, \ldots, y_k$ be a fundamental set of solutions of the associated homogeneous equation on the interval I. Then the general solution of the equation on the interval is*

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k + y_p,$$

where c_i are arbitrary constants.

Note: there is an equivalent superposition principle for nonhomogeneous equations

Example 4.1.13. *Is $y_p = -4x^2$ a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$?*

Section 4.2: Reduction of Order

The **reduction of order** method is a technique used to find a second linearly independent solution to a second-order linear homogeneous ODE when one solution is already known.

Consider the equation
$$y'' + p(x)y' + q(x)y = 0.$$
Suppose $y_1(x)$ is a known solution. To find another solution, assume
$$y_2(x) = v(x)y_1(x),$$
where $v(x)$ is an unknown function. Substituting into the equation gives a first-order equation for $v'(x)$, which can then be solved.

Example 4.2.1. *Solve*
$$y'' - y = 0,$$
given that $y_1(x) = e^x$ is a solution.
Solution. *Assume $y_2(x) = v(x)e^x$. Then*
$$y_2' = v'e^x + ve^x, \quad y_2'' = v''e^x + 2v'e^x + ve^x.$$
Substitute into the ODE:
$$(v''e^x + 2v'e^x + ve^x) - (ve^x) = e^x(v'' + 2v') = 0.$$
So
$$v'' + 2v' = 0.$$
Let $w = v'$, then $w' + 2w = 0$. Solution: $w = Ce^{-2x}$. Integrating:
$$v = \int Ce^{-2x}\,dx = -\tfrac{C}{2}e^{-2x} + D.$$
Thus
$$y_2(x) = v(x)e^x = -\tfrac{C}{2}e^{-x} + De^x.$$
A new independent solution is $y_2(x) = e^{-x}$.
So the general solution is
$$y(x) = C_1e^x + C_2e^{-x}.$$

Exercise 4.2.1
Given $y_1(x) = x$ is a solution of the ODE
$$x^2y'' - 2xy' + 2y = 0,$$
use reduction of order to find a second solution.

Section 4.3: Homogeneous Linear Equations with Constant Coefficients

Characteristic equations[1] help us solve higher-order linear ODEs whose parameters (coefficients) do not change. They convert a ODE into a polynomial equation, whose roots help us write down the solution. As polynomials can have complex numbers as roots, we will have to learn basic complex number arithmetic.

[1] "Characteristic" and "auxiliary" equations are used interchangeably here and in the textbook. We will see that another characteristic equations will appear in matrix ODEs, and its roots are going to carry a similar interpretation.

Complex Numbers

Complex numbers $z \in \mathbb{C}$ are specified using two (independent) real numbers (elements of $\mathbb{R}$). There are two different representations of any complex number z:

A complex number is of the form

$$z = x + iy, \quad x, y \in \mathbb{R}, \; i^2 = -1.$$

$$\mathrm{Re}(z) = x, \quad \mathrm{Im}(z) = y, \quad \mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

Section 4.4: Notations

Cartesian form: $z = \mathrm{Re}\, z + i \,\mathrm{Im}\, z$. Components are called real and imaginary parts.

Polar form: $z = |z|e^{i \arg(z)} = r(\cos\theta + i \sin\theta)$, with $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \arg(z)$. Components are called magnitude (or modulus) and angle (or argument).

Exponential form: $z = re^{i\theta}$, using Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$.

Trigonometric form: $z = r(\cos\theta + i \sin\theta)$.

Matrix form:

$$z \equiv \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

Vector form: $z = (x, y)$ in $\mathbb{R}^2$.

Operations

For $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \quad z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

$$\bar{z} = x - iy, \quad |z| = \sqrt{x^2 + y^2}.$$

Exponential Function:

The complex exponential function e^z can be defined in several equivalent ways:

1. Power Series Definition

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$

2. Limit Definition

$$e^z = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n.$$

3. **Differential Equation Definition** e^z is the unique function satisfying

$$\frac{d}{dz}e^z = e^z, \quad e^0 = 1.$$

4. **Euler's Formula (for purely imaginary arguments)**

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}.$$

These definitions are consistent and extend the exponential function naturally from $\mathbb{R}$ to $\mathbb{C}$.

De Moivre's Theorem

$$\big(r(\cos\theta + i\sin\theta)\big)^n = r^n(\cos(n\theta) + i\sin(n\theta)), \quad n \in \mathbb{Z}.$$

Roots of Complex Numbers The nth roots of $z = re^{i\theta}$ are

$$z_k = r^{1/n}e^{i\left(\frac{\theta+2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1.$$

A linear homogeneous differential equation with constant coefficients has the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where a_i are constants.

The method of solution:

Step (1) Form the **characteristic equation**:

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0.$$

Step (2) Solve for the roots r.

Step (3) Build the general solution from the roots:

- Real distinct roots $r_1, \ldots, r_n$: $y(x) = C_1 e^{r_1 x} + \cdots + C_n e^{r_n x}$.
- Repeated root r of multiplicity m: $y(x) = (C_1 + C_2 x + \cdots + C_m x^{m-1})e^{rx}$.
- Complex roots $r = \alpha \pm i\beta$: $y(x) = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

Example 4.4.1. *Solve*

$$y'' + 4y' + 4y = 0.$$

Solution. *Characteristic equation:*

$$r^2 + 4r + 4 = 0 \quad \Rightarrow \quad (r+2)^2 = 0.$$

Repeated root: $r = -2$ of multiplicity 2. So the solution is

$$y(x) = (C_1 + C_2 x)e^{-2x}.$$

Example 4.4.2. *Solve*

$$y''' - 3y'' + 3y' - y = 0.$$

Solution. *Characteristic equation:*

$$r^3 - 3r^2 + 3r - 1 = 0 \quad \Rightarrow \quad (r-1)^3 = 0.$$

So root $r = 1$ with multiplicity 3. General solution:

$$y(x) = (C_1 + C_2 x + C_3 x^2)e^x.$$

Exercise 4.4.1
Solve the following equations:

1. $y'' - 5y' + 6y = 0$
2. $y'' + 9y = 0$
3. $y''' + y'' - y' - y = 0$

Section 4.5: Method of Undetermined Coefficients: Superposition Approach

Consider a linear nonhomogeneous differential equation with constant coefficients:

$$L[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x),$$

where L is a linear differential operator with constant coefficients.

Idea

- Solve the **homogeneous equation** $L[y] = 0$ to get the complementary solution y_c.
- Find a **particular solution** y_p by guessing a form similar to $g(x)$ (with undetermined coefficients).
- The general solution is $y(x) = y_c + y_p$.

Superposition Principle

If

$$L[y] = g_1(x) + g_2(x) + \cdots + g_k(x),$$

then

$$y_p = y_{p1} + y_{p2} + \cdots + y_{pk},$$

where each y_{pi} corresponds to a particular solution of $L[y] = g_i(x)$.

—

Example 4.5.1. *Solve*

$$y'' - 3y' + 2y = e^x + \sin x.$$

Solution. *Homogeneous part:*

$$r^2 - 3r + 2 = 0 \quad \Rightarrow \quad (r-1)(r-2) = 0.$$

So $y_c = C_1 e^x + C_2 e^{2x}$.
For $g(x) = e^x$: since e^x already appears in y_c, guess

$$y_{p1} = Axe^x.$$

Substitute:

$$y'_{p1} = Ae^x + Axe^x, \quad y''_{p1} = 2Ae^x + Axe^x.$$

So

$$y''_{p1} - 3y'_{p1} + 2y_{p1} = (2Ae^x + Axe^x) - 3(Ae^x + Axe^x) + 2(Axe^x).$$

Simplify:

$$= (2A - 3A)e^x + (A - 3A + 2A)xe^x = -Ae^x.$$

Set equal to e^x: $-A = 1 \Rightarrow A = -1$. So $y_{p1} = -xe^x$.
For $g(x) = \sin x$: guess

$$y_{p2} = A\cos x + B\sin x.$$

Substitute:

$$y'_{p2} = -A\sin x + B\cos x, \quad y''_{p2} = -A\cos x - B\sin x.$$

So

$$y''_{p2} - 3y'_{p2} + 2y_{p2} = (-A\cos x - B\sin x) - 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x).$$

Simplify:

$$= (-A - 3B + 2A)\cos x + (-B + 3A + 2B)\sin x = (A - 3B)\cos x + (3A + B)\sin x.$$

Match with $\sin x$: system

$$A - 3B = 0, \quad 3A + B = 1.$$

From first: $A = 3B$. Substitute: $3(3B) + B = 1 \Rightarrow 10B = 1 \Rightarrow B = \frac{1}{10}$, $A = \frac{3}{10}$.
So

$$y_{p2} = \tfrac{3}{10}\cos x + \tfrac{1}{10}\sin x.$$

General solution:

$$y(x) = C_1e^x + C_2e^{2x} - xe^x + \tfrac{3}{10}\cos x + \tfrac{1}{10}\sin x.$$

—

Exercise 4.5.1

Solve the following using the superposition method:

1. $y'' + y = 5\cos(2x)$
2. $y'' - y = xe^x$
3. $y'' + 4y' + 4y = e^{-2x} + x$

Section 4.6: Method of Undetermined Coefficients: Annihilator Approach

The **annihilator method** uses differential operators to systematically find particular solutions.

Steps

1. Write the equation as $L[y] = g(x)$.
2. Find a differential operator A (called the annihilator) such that $Ag(x) = 0$.
3. Apply A to both sides: $AL[y] = 0$, giving a higher-order homogeneous equation.
4. Solve for the general solution of $AL[y] = 0$.
5. Extract the particular solution from the part not included in the original homogeneous solution.

—

Example 4.6.1. *Solve*

$$y'' - y = e^{2x}.$$

Solution. *Step 1: Homogeneous part:*

$$r^2 - 1 = 0 \quad \Rightarrow \quad r = \pm 1.$$

So $y_c = C_1 e^x + C_2 e^{-x}$.
Step 2: RHS is $g(x) = e^{2x}$. *Annihilator is* $(D-2)$, *since* $(D-2)e^{2x} = 0$.
Step 3: Apply $(D-2)$ *to both sides:*

$$(D-2)(D^2-1)y = 0.$$

Step 4: Characteristic equation:

$$(r-2)(r^2-1) = 0 \quad \Rightarrow \quad (r-2)(r-1)(r+1) = 0.$$

So roots: $2, 1, -1$. *General solution:*

$$y(x) = C_1 e^x + C_2 e^{-x} + C_3 e^{2x}.$$

Step 5: Since $C_1 e^x \; + \; C_2 e^{-x}$ *are already solutions of original homogeneous part, the new term* $C_3 e^{2x}$ *is the particular solution.*
Thus the general solution is

$$y(x) = C_1 e^x + C_2 e^{-x} + C_3 e^{2x}.$$

—

Example 4.6.2. *Solve*

$$y'' + y = \sin x.$$

Solution. *Homogeneous part:*

$$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i.$$

So $y_c = C_1 \cos x + C_2 \sin x$.

RHS $g(x) = \sin x$. *Annihilator:* $(D^2 + 1)$.
Apply to both sides:

$$(D^2 + 1)(D^2 + 1)y = 0.$$

Characteristic equation:

$$(r^2 + 1)^2 = 0,$$

roots $r = \pm i$ *of multiplicity* 2. *General solution:*

$$y(x) = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x.$$

The terms $C_1 \cos x + C_2 \sin x$ *come from homogeneous solution, while* $C_3 x \sin x + C_4 x \cos x$ *represent the particular solution.*
So general solution is

$$y(x) = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x.$$

—

Exercise 4.6.1
Use the annihilator method to solve:

1. $y'' - 4y = 3e^{2x}$
2. $y'' + y = \cos 2x$
3. $y'' - 2y' + y = xe^x$