Elementary Linear Algebra Notes Part II
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Emmanuel Atindama, PhD Mathematics

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Vector Spaces

Vector Spaces and Subspaces

Definition (Vector Space)

A nonempty set V, is a **vector space** if it satisfies the following:

(a) For every $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v} \in V$.

(b) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(c) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, where $\mathbf{w} \in V$.

(d) There is a **zero** vector $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(e) For every $\mathbf{u} \in V$, there is a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + -\mathbf{u} = \mathbf{0}$.

(f) For every $c \in \mathbb{R}$, $c\mathbf{u} \in V$.

(g) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(h) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

(i) $c(d\mathbf{u}) = (cd)\mathbf{u}$.

(j) $1\mathbf{u} = \mathbf{u}$.

Example1: For $n \geq 0$, the set $\mathbb{P}_n$ of polynomials of degree at most n consists of all polynomials of the form

$$\mathbf{P}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

each form a vector space.

Example2: The space of n dimensional real number vectors, $\mathbb{R}^n$ is a vector space.

Vector Spaces and Subspaces

Definition (Subspace)

A **subspace** of $\mathbb{R}^n$ is any set $S \in \mathbb{R}^n$ that has 3 properties:

i) The zero vector is in S.

ii) For each $\mathbf{u}$, $\mathbf{v} \in S$, $\mathbf{u} + \mathbf{v} \in S$.

iii) For each $\mathbf{u} \in S$, and scalar $c \in \mathbb{R}$, $c\mathbf{u} \in S$.

Vector Spaces and Subspaces

Example1: The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{\mathbf{0}\}$.

Example2: The set of all 3D vectors whose first component is $\mathbf{0}$ is a subspace of $\mathbb{R}^3$.

Example3: The $Span(\{\mathbf{v}_1, \mathbf{v}_2\})$, where $\mathbf{v}_1, \mathbf{v}_2 \in V$ of an arbitrary vector space is a subspace of V.

Theorem (Subspace Theorem)

If $\mathbf{v_1}, \ldots, \mathbf{v_p}$ are vectors in a vector space V, then $Span(\{\mathbf{v_1}, \ldots, \mathbf{v_p}\})$ is a subspace of V.

Example: Show that H is a subspace of $\mathbb{R}^3$.

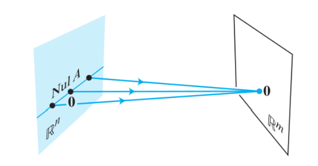
$$H = \left\{ s \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} : \text{ for } s, t \in \mathbb{R} \right\}$$

Null Spaces and Column/Range Spaces

Definition (Null Space)

A **null space** of matrix A is the set, $Nul A$ - of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Null } A = \{\mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = 0\}$$



Theorem

A **null space** *of and $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is also a subspace of $\mathbb{R}^n$.*

Null Spaces and Column/Range Spaces

Definition (Column Space)

A **column space** or **range space** of matrix A is the set, $Col A = Span(\{\text{Columns of } A\})$. That is, the set of all linear combinations of the columns of A.

Theorem

The **column space** *of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$. Equivalently, the set of all $\mathbf{b} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is the column space or range space (the range of the linear transformation $A\mathbf{x}$).*

Null Spaces and Column/Range Spaces

Example 1: Find the null space, (Nul$A$) of the matrix below

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ -3 & -2 & 1 & 0 & -1 \\ 2 & -5 & 1 & 8 & -3 \end{bmatrix}$$

Example 2: Find the null space, (Nul$A$) of the matrix below

$$B = \begin{bmatrix} 1 & -3 & 5 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 7 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Definition (Linear Transformation)

A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear if it satisfies the following:
For any input vector $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalar $c \in \mathbb{R}$,

(i) **Additivity**: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

(ii) **Scalar Multiplication**: $T(c\mathbf{u}) = cT(\mathbf{u})$, for any scalar c.

Null Spaces and Column/Range Spaces

Example 1: Find the column/range space, (Col$A$) of the matrix below

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ -3 & -2 & 1 & 0 & -1 \\ 2 & -5 & 1 & 8 & -3 \end{bmatrix}$$

Example 2: Find the column/range space, (Col$A$) of the matrix below

$$B = \begin{bmatrix} 1 & -3 & 5 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 7 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Contrast between Range/Column space and Null space/Kernel of an $m \times n$ matrix, A

1. Nul$A$ is a subspace of $\mathbb{R}^n$
2. Nul$A$ is the solution to $A\mathbf{x} = \mathbf{0}$. You have to find $\mathbf{x}$ in order to find the null space.
3. There is no obvious relation between Nul$A$ and the columns or row of A.
4. Any vector $\mathbf{v}$ is in the null space of A if and only if $A\mathbf{v} = \mathbf{0}$.
5. Nul $A = \{\mathbf{0}\}$ if and only if the only solution to $A\mathbf{x} = \mathbf{0}$ is the zero vector.

1. Col$A$ is a subspace of $\mathbb{R}^m$
2. Col$A$ is a linear combination of some of the columns of A. **Select the columns that have pivots after row reduction**.
3. The column vectors of A are in Col$A$.
4. Any vector $\mathbf{b}$ is in the columns space of A if and only if $A\mathbf{x} = \mathbf{b}$ is a consistent system. (You need to row reduce).
5. Col $A = \mathbb{R}^m$ if and only if $A\mathbf{x} = \mathbf{b}$ has a pivot in each row.

Linearly Independent Sets and Bases

Definition (Linear Independence)

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is said to be **linearly independent** if the only scalars $c_1, c_2, \ldots, c_n$ that satisfy:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_n = 0$.
In other words, the solution to the homogeneous solution $A\mathbf{x} = \mathbf{0}$ is the trivial solution.

Basis

Definition (Basis)

Let H be a subspace of a vector space V. A set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p\}$ is a basis in V for H if

(i) $\mathcal{B}$ is a linearly independent set

(ii) $Span(\mathcal{B}) = H$

Theorem (The Basis Theorem)

Let V be an n-dimensional vector space V. Any linearly independent set $\mathcal{L} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V.

That is, any linearly independent set with exactly n elements is a basis.

Examples

Example 1: Find the basis of Col$A$ and Nul$A$ of the matrix below

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ -3 & -2 & 1 & 0 & -1 \\ 2 & -5 & 1 & 8 & -3 \\ -1 & 0 & 3 & 1 & 0 \end{bmatrix}$$

Example 2: Find the basis of Col$A$ and Nul$A$ of the matrix below

$$B = \begin{bmatrix} 1 & -3 & 5 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 7 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Example 3: Are $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 7 \end{bmatrix} \right\}$ and $\mathcal{D} = \left\{ \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \right\}$ bases?

Coordinate Systems

Theorem (Unique Representation)
Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ be a basis for a vector space V. Then for each $\mathbf{x} \in V$, there exists a unique set of scalars $c_1, c_2, \ldots, c_p$ such that $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p = \mathbf{x}$.

The **basis** imposes a coordinate system for that vector space.

Definition (Basis)
Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ be a basis for a vector space V. *The coordinates of* $\mathbf{x}$ *relative to the basis* $\mathcal{B}$(or the $\mathcal{B}-$coordinates of $\mathbf{x}$) are a unique set of scalars $c_1, c_2, \ldots, c_p$ such that $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p = \mathbf{x}$.

If $c_1, c_2, \ldots, c_n$ are the $\mathcal{B}-$coordinates of $\mathbf{x} \in \mathbb{R}^n$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the $\mathcal{B}-$coordinate vector of $\mathbf{x}$.

Examples

Example 1: Let a basis $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}$. Suppose that $\mathbf{x} \in \mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

Example 2: Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ from example 1 relative to the **standard basis** $\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

Example 3: Let a basis $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} \right\}$. $H = Span(\mathcal{B})$ is a subspace. Determine whether $\mathbf{x} = \begin{bmatrix} -3 \\ 3 \\ 15 \end{bmatrix}$ is in the subspace H, and find it's coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

Change of Basis

A **change of basis** allows us to represent vectors and linear transformations with respect to any coordinate systems of our choice.

Definition (Change of Basis)
Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be two different bases of an n-dimensional vector space V.
Let $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ represent the $\mathcal{B}$ - coordinate and $\mathcal{C}$ - coordinate of the vector $\mathbf{v}$ in bases $\mathcal{B}$ and $\mathcal{C}$ respectively, then its coordinate change form $[\mathbf{v}]_\mathcal{B}$ and $[\mathbf{v}]_\mathcal{C}$ is defined as:

$$[\mathbf{v}]_\mathcal{C} = P_{\mathcal{B}\to\mathcal{C}}[\mathbf{v}]_\mathcal{B}$$

where $P_{\mathcal{B}\to\mathcal{C}}$ is the **change of basis matrix** from basis $\mathcal{B}$ to basis $\mathcal{C}$

Finding the Change of Basis Matrix

The matrix $P_{\mathcal{B}\to\mathcal{C}}$ is found by expressing each vector in $\mathcal{B}$ in terms of the basis $\mathcal{C}$.

$$\text{If} \quad \mathbf{b}_i = a_{1i}\mathbf{c}_1 + a_{2i}\mathbf{c}_2 + \cdots + a_{ni}\mathbf{c}_n, \qquad \text{then } [\mathbf{b}_i]_{\mathcal{C}} = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}$$

$$\text{and} \quad P_{\mathcal{B}\to\mathcal{C}} \quad \text{has columns} \quad P_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} | & | & & | \\ [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \dots & [\mathbf{b}_n]_{\mathcal{C}} \\ | & | & & | \end{bmatrix}.$$

Alternatively, if we know $P_{\mathcal{C}\to\mathcal{B}}$, then

$$P_{\mathcal{B}\to\mathcal{C}} = P_{\mathcal{C}\to\mathcal{B}}^{-1}.$$

Examples

Example 1: Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. Express the vector $\mathbf{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ in the $\mathcal{B}$-basis.

Solution: Find scalars c_1, c_2 such that

$$\mathbf{v} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

This gives the system:

$$\begin{cases} c_1 + 2c_2 = 5, \\ c_1 + 3c_2 = 7. \end{cases} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & 3 & 7 \end{array}\right]$$

Solving, we get $c_1 = 1$, $c_2 = 2$, so $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}}$.

Examples

Example 2: Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$ find the **change of basis matrix** $P_{\mathcal{B}\to\mathcal{C}}$.

Solution: Express each vector in $\mathcal{B}$ in terms of $\mathcal{C}$:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = a_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & 4 & 1 \\ 1 & 3 & 2 \end{array}\right], \text{ solve for } a_1 \text{ and } a_2$$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = b_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & 4 & 3 \\ 1 & 3 & 5 \end{array}\right], \text{ solve for } b_1 \text{ and } b_2.$$

Solving both systems, we get $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ \frac{3}{2} \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -\frac{11}{2} \\ \frac{7}{2} \end{bmatrix}$. So, $P_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} -\frac{5}{2} & -\frac{11}{2} \\ \frac{3}{2} & \frac{7}{2} \end{bmatrix}$.

Alternatively, solve

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 3 \\ 1 & 3 & 2 & 5 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{5}{2} & -\frac{11}{2} \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \end{array}\right]$$

Examples

Try your hands on the following problems:

1. Given $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, describe the steps to compute the change of basis matrix $P_{\mathcal{B}\to\mathcal{C}}$.
2. If the transformation matrix of T in the standard basis is:

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix},$$

   what is its representation in a new basis $\mathcal{B}$ with basis vectors $\mathbf{b}_1 = (3, 1)$ and $\mathbf{b}_2 = (2, -1)$?
   Hint: to get the representation of T in the basis $\mathcal{B}$, you need to *transform your vector from the $\mathcal{B}$-coordinate to the standard coordinate system $\mathcal{E}$*, *then apply the transformation*, *and then return the vector back to the $\mathcal{B}$-coordinate system.* Now multiply all 3 in that order to get the new transformation matrix in the $\mathcal{B}$-coordinate system.
3. Let $\mathcal{B} = \{(2, 1), (1, 3)\}$ and $\mathcal{C} = \{(3, 2), (4, 1)\}$. Find the matrix $P_{\mathcal{B}\to\mathcal{C}}$.

Homework 10

Question 1: Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be the basis: $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$

Express the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ in the $\mathcal{B}$-basis.

Question 2: Let: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Find the change of basis matrix $P_{\mathcal{B} \to \mathcal{C}}$.

Question 3: If the transformation matrix of T in the basis $\mathcal{H}$ with basis vectors $\mathbf{h}_1 = (3, 1)$ and $\mathbf{h}_2 = (0, 1)$ is:

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix},$$

what is its representation in a new basis $\mathcal{B}$ with basis vectors $\mathbf{b}_1 = (3, 1)$ and $\mathbf{b}_2 = (2, -1)$?

Eigenvalues and Eigenvectors

Eigenvectors and Eigenvalues

Definition (Eigenvectors)

Given a square matrix A, a nonzero vector $\mathbf{v}$ is an eigenvector of A if there exists a scalar λ such that:

$$A\mathbf{v} = \lambda\mathbf{v}$$

where λ is called the **eigenvalue** corresponding to $\mathbf{v}$.

Example 1: Given the matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, are $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ eigenvectors of A?

Example 2: Given $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, are $\mathbf{u} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ eigenvectors of B?

Computing Eigenvalues and Eigenvectors

To find the eigenvalues of a matrix A, solve the characteristic equation:

$$\det(A - \lambda I) = 0 \quad \text{where } I \text{ is the identity matrix of the same dimension as } A.$$

Once an eigenvalue λ is found, we obtain it's corresponding eigenvector $\mathbf{v}$ by solving:

$$(A - \lambda I)\mathbf{v} = 0$$

for each eigenvalue λ.

Computing Eigenvalues and Eigenvectors

Example 1: Find the eigenvalues and eigenvectors for the matrix: $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

1. **Compute the Characteristic Equation** $\det\left(\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$$\det\left(\begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0 \quad \Rightarrow \quad (4-\lambda)(3-\lambda) - (2)(1) = 0$$

$$12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0 \quad \Rightarrow \quad \lambda^2 - 7\lambda + 10 = 0 \quad \Rightarrow \quad \lambda = 5, 2$$

2. **Finding the Eigenvectors**
   for $\lambda = 5$,

$$(A - 5I)\mathbf{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

   For $\lambda = 2$,

$$(A - 2I)\mathbf{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Computing Eigenvalues and Eigenvectors

Example 2: Find its eigenvalues and eigenvectors for the matrix $B = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & -1 & 6 \end{bmatrix}$

Compute the Characteristic Equation $\det(B - \lambda I) = \begin{vmatrix} 6-\lambda & -1 & 0 \\ -1 & 6-\lambda & -1 \\ 0 & -1 & 6-\lambda \end{vmatrix} = 0$

Ans: $\lambda_1 = 6,\ \mathbf{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $\quad \lambda_2 = 6 - \sqrt{2},\ \mathbf{v_2} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$; $\quad \lambda_3 = 6 + \sqrt{2},\ \mathbf{v_3} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$

Example 3: Find its eigenvalues and eigenvectors of $D = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & 0 & 6 \end{bmatrix}$

Ans: $\lambda_1 = 5,\ \mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$; $\quad \lambda_2 = 6,\ \mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $\quad \lambda_3 = 7,\ \mathbf{v_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$;

Computing Eigenvalues and Eigenvectors

Additional Practice Problems (Homework)

1. Find the eigenvalues and eigenvectors of

$$C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

2. Compute the eigenvalues and eigenvectors of the 3×3 matrix:

$$D = \begin{bmatrix} -26 & -33 & -2 \\ 31 & 42 & 23 \\ -11 & -15 & -4 \end{bmatrix}$$

3. Prove that the eigenvalues of an upper triangular matrix correspond to the values on the main diagonal.
4. **Extra-if you want to try**: Show that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

More on Eigenvectors and Eigenvalues

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\lambda = 0$ is **not** *an eigenvalue of A.*

Definition (Similarity)

An $n \times n$ matrix A, is **similar** to an $n \times n$ matrix B if there exists an invertible matrix P such that $P^{-1}AP = B$.

If A and B are **similar**, then they have the same characteristic polynomial, and hence the same eigenvalues (with the same multiplicities.)

Diagonalization

An $n \times n$ matrix D is **diagonal** if all its entries off the main diagonal are zero.
Diagonalization is the process of converting a square matrix A into a diagonal matrix D by means of a *similarity transformation*.

Definition (Diagonalizability)

A square matrix A is **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that:

$$A = PDP^{-1}$$

where D is a diagonal matrix with eigenvalues of A on the diagonal. The columns of P are the eigenvectors of A.
In other words, A is similar to D.

Diagonalization

Theorem (Conditions for Diagonalizability)

An $n \times n$ matrix, A is diagonalizable if and only if:

1. *It has n linearly independent eigenvectors.*
2. *The algebraic multiplicity of each eigenvalue equals its geometric multiplicity (If an eigenvalue has a multiplicity of 2, then that eigenvalue has 2 corresponding eigenvectors).*
3. *It has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and the matrix of eigenvectors $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$ is invertible.*

Diagonalization

Corollary (Steps to Diagonalize a Matrix)

To diagonalize a diagonalizable matrix, A:

1. **Find the eigenvalues:** *Solve* $\det(A - \lambda I) = 0$.
2. **Find the eigenvectors:** *Solve* $(A - \lambda I)x = 0$ *for each eigenvalue* λ.
3. **Check independence:** *Ensure there are* n *linearly independent eigenvectors.*
4. **Form P and D:**
   - P *consists of the eigenvectors as column*
   - D *is a diagonal matrix with eigenvalues on the diagonal.*
5. *Compute* $P^{-1}AP$ *to confirm* D *to* **verify**.

Useful things to note:

- Symmetric Matrices are always diagonalizable, and their eigenvectors are orthogonal (we will talk about this in the next module).
- Simplifies matrix exponentiation ($A^n = PD^nP^{-1}$).science.

Examples

Theorem (The Diagonalization Theorem)

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

NOTE: this does not imply that, if a matrix does not have n distinct eigenvalues, then it is not diagonalizable.

Check **Conditions for Diagonalizability Theorem**

Example 1:

Diagonalize the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ if possible. And find a formula for A^k, given that $A = PDP^{-1}$.

Example 2:
Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}.$$

Solution
The eigenvalues of A are $\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -1$. So by the diagonalization theorem, A is diagonalizable since it has 3 distinct eigenvalues.

It has corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ respectively.

So, $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$,
where $A = PDP^{-1}$.

Example 3:

Diagonalize the matrix, $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. if possible.

Solution

The eigenvalues of A are $\lambda_1 = 2$, $\lambda_{2,3} = -1$. So the **diagonalization theorem** cannot guarantee that A is diagonalizable since it does not have 3 distinct eigenvalues.

It has corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

Continue the solution to see whether A is diagonalizable. If $\lambda_{2,3}$ yields two eigenvectors, then it is. And you can go ahead to find the corresponding P and D.

Examples

Example 4:
Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
Example 5:

Determine whether $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$ is diagonalizable.

Examples

Theorem
Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$.

a. *For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.*
b. *The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if*
    - *the characteristic polynomial factors completely into linear factors*
    - *the dimension of the eigenspace for each $lambda_k$ equals the multiplicity of λ_k*
c. *If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the set $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$*

Examples

Diagonalize the following matrices if possible

a) $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

d) $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

e) $A = \begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$

Homework 11

1. Determine whether the following are diagonalizable: If so, diagonalize them.
   a)
   $$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
   b)
   $$A_2 = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$
   c)
   $$A_3 = \begin{bmatrix} 5 & -8 & 1 \\ 0 & -3 & 7 \\ 0 & 0 & -2 \end{bmatrix}.$$
2. Determine whether $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ are similar.

Complex Eigenvalues

1. Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
2. Find the eigenvalues and corresponding eigenvectors

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{3}{5} \\ \frac{3}{4} & \frac{11}{10} \end{bmatrix}$$

Orthogonality and Least Squares

Inner Product, Length, and Orthogonality

The **inner product** provides *a way to measure the similarity between two vectors*.

Definition

Inner Product For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the inner product (or dot product) is:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\top \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example 7: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 8: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$

Inner Product, Length, and Orthogonality

Theorem

Properties of Inner Product For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and scalar* c*:*

a) $\mathbf{u} \cdot \mathbf{v} \in \mathbb{R}$.
b) **Commutativity:** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
c) **Distributivity:** $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
d) **Scalar Multiplication:** $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
e) **Non-Negativity:** $\mathbf{u} \cdot \mathbf{u} \geq 0$, *with equality only if* $\mathbf{u} = 0$.

Example 7: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 8: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$

Inner Product, Length, and Orthogonality

Definition
Length (Norm) of a Vector The **length** (or **norm**) of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is given by:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

The **length** or **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ is the **length** of difference vector. That is,

$$\|u - v\|$$

Example 7: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 8: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$

Inner Product, Length, and Orthogonality

Example 1: Find the distance be the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 2: Find the length of the vector $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix}$.

Example 3: Find the inner product between the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$

Example 7: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 8: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$

Inner Product, Length, and Orthogonality

Definition
Orthogonal Vectors Two vectors $\mathbf{u}, \mathbf{v}$ are **orthogonal** (perpendicular) if their inner product is zero:

$$\mathbf{u} \cdot \mathbf{v} = 0$$

Example 4: Prove that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if the vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal**.

Example 5: Prove that $(\|\mathbf{u}\| + \|\mathbf{v}\|)^2 \geq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ for all the vectors $\mathbf{u}$ and $\mathbf{v}$.

Example 6: Prove that $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$ for all the vectors $\mathbf{v}$.

Example 7: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 8: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$

Inner Product, Length, and Orthogonality

Definition
Relations between Inner Product and Angles The inner product of two vectors $\mathbf{u}, \mathbf{v}$ can also be expressed in terms of their direction as:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

where θ is the angle between the two vectors.

Example 7: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Example 8: Find the angle between the vectors $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$

Orthogonal Sets

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **orthogonal** if each pair of distinct vectors within the set $\mathbf{v}_i$ and $\mathbf{v}_j$ are **orthogonal**. That is, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$

Example 1: Show that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are orthogonal.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Example 2: Are the following vectors, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ orthogonal?

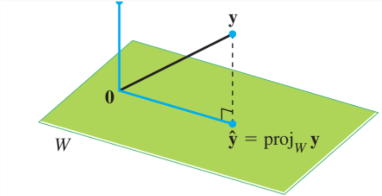
Example 3: Given that a set of nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are orthogonal, show that the set is linearly independent.

Orthogonal Projections

Given a nonzero vector $\mathbf{u} \in \mathbb{R}^n$, a different vector, $\mathbf{y} \in \mathbb{R}^n$ can always decomposed into a two components; one, a multiple of $\mathbf{u}$ (parallel to $\mathbf{u}$), and the orthogonal to $\mathbf{u}$.
The vector component of $\mathbf{y}$ that is a multiple of $\mathbf{u}$ or parallel to $\mathbf{u}$ is called the **Orthogonal Projection:** The projection of $\mathbf{y}$ onto $\mathbf{u}$ (if $\mathbf{u} \neq 0$) is:

$$\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}}\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$$

The vector, $\mathbf{y}$ can be written as a sum of $\mathbf{u}$ and $\text{proj}_{\mathbf{u}}\mathbf{y}$.



Alternatively if $W = Span(\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)\}$ where the $\mathbf{u}_i$'s are orthogonal, we say the **orthogonal projection** of $\mathbf{y}$ onto the subspace spanned by W is

$$\hat{\mathbf{y}} = \text{proj}_{\mathbf{u}}\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_n}{\mathbf{u}_n \cdot \mathbf{u}_n}\mathbf{u}_n.$$

Examples

Example 1:

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$. Find the orthogonal projection of $\mathbf{u}_1$ onto $\mathbf{u}_2$. Then write $\mathbf{u}_1$ as the sum of two orthogonal vectors, one being $\text{proj}_{\mathbf{u}_2}\mathbf{u}_1$, and the other being the orthogonal complement.

Selected References

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(ii) **Linear Algebra** by *Jim Hefferon*

(iii) **Introduction to Linear Algebra** by *Gilbert Strang*

(iv) **Linear Algebra with Applications** *W. Keith Nicholson*