Elementary Linear Algebra Notes Part I
MATH 1890
Spring 2025

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Outline

Linear Equations

Systems of Linear Equations

- A **real number** is a number that can be used to measure a continuous one-dimensional quantity such as a distance, duration or temperature. In short, any number you encounter in real world measurements or transactions is a real number. For example; $4, 2.8, \pi, 1.333..., -\frac{2}{7}, e, \sqrt{2}$, etc.

Consider a set of n variables $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$.
A **linear equation** is a combination of scalar multiples of the n variables to yield an output. This can be written in the form

$$a_1\mathbf{x_1} + a_2\mathbf{x_2} + \cdots + a_n\mathbf{x_n} = b \tag{1}$$

where b and the scalars $a_1, a_2, \ldots, a_n$ are real numbers.
A **system of linear equation** collection of equations of the form in equation (1).

Solutions to Systems of Linear Equations

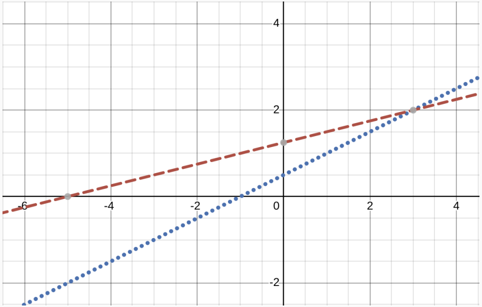
The **solution to a linear system** is the point or points that satisfy each of the linear equations within the system. There can be one, or many, or none. If it has one or many solutions, the system is **consistent**. Otherwise it is inconsistent.

The linear system

$$-2x_1 + 4x_2 = 2 \tag{2}$$

$$-x_1 + 4x_2 = 5 \tag{3}$$

has a unique solution, $x_1 = 3$, $x_2 = 2$ because the two lines intersect at one and only one point.



Solutions to Systems of Linear Equations

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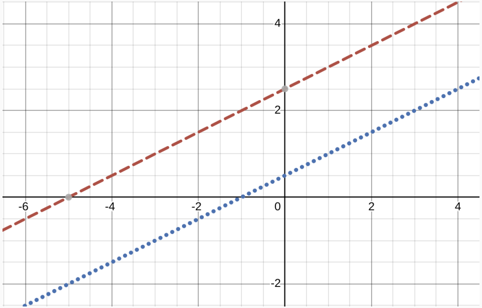
The linear system

$$-2x_1 + 4x_2 = 2 \tag{2}$$

$$-x_1 + 2x_2 = 5 \tag{3}$$

has no solution because the two lines are parallel and never intersect.



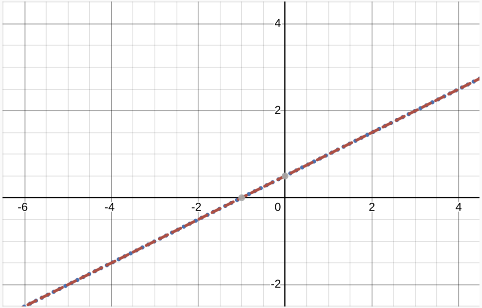
Solutions to Systems of Linear Equations

The **solution to a linear system** is the point or points that satisfy each of the linear equations within the system. There can be one, or many, or none. If it has one or many solutions, the system is **consistent**. Otherwise it is inconsistent.

The linear system

$$-2x_1 + 4x_2 = 2 \tag{2}$$
$$-x_1 + 2x_2 = 1 \tag{3}$$

has infinitely may solutions because the lines intersect everywhere (at infinite number of points).



Solving Systems of Equations

A linear system may be represented as a matrix and solved using a series of the following row operations:

- **Replacement**: Replace one row by the sum of itself and a multiple of another row.
- **Interchange**: Interchange two rows.
- **Scaling**: Multiply all entries in a row by a nonzero real number.

For instance, the linear system

$$\begin{aligned}-2x_1 + 4x_2 + 6x_3 =& 2\\ -x_1 + 7x_2 + x_3 =& 5\\ 3x_1 - 7x_2 + 2x_3 =& 1\end{aligned}$$

can be written as

$$\left[\begin{array}{ccc|c} -2 & 4 & 6 & 2\\ -1 & 7 & 1 & 5\\ 3 & -7 & 2 & 1\end{array}\right]$$

Complete this row operation so that you have an upper triangular matrix (all values below the main diagonal are zeros) as your **practice**.

Solving Systems of Equations

We then perform the following row operations:

$$\left[\begin{array}{ccc|c} -2 & 4 & 6 & 2 \\ -1 & 7 & 1 & 5 \\ 3 & -7 & 2 & 1 \end{array}\right] -\frac{1}{2}R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ -1 & 7 & 1 & 5 \\ 3 & -7 & 2 & 1 \end{array}\right]$$

$$R_1 + R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 5 & -2 & 4 \\ 3 & -7 & 2 & 1 \end{array}\right] -3R_1 + R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 5 & -2 & 4 \\ 0 & -1 & 11 & 4 \end{array}\right]$$

Complete this row operation so that you have an upper triangular matrix (all values below the main diagonal are zeros) as your **practice**.

Homework 1

Question Q1 Given the linear system

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{aligned}
$$

a Rewrite the system in matrix notation.

b Using elementary row operations only, reduce the system to an upper triangular matrix.

c Using the upper triangular matrix, solve for x_1, x_2, and x_3.
Note: *for an inconsistent system, you will not be able to solve the system*

d Is the system consistent?

Homework 1

Question Q2 Given the linear system

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

a Rewrite the system in matrix notation.

b Using elementary row operations only, reduce the system to an upper triangular matrix. **Hint:** *You may need to swap rows to achieve this.*

c Using the upper triangular matrix, solve for x_1, x_2, and x_3. **Note:** *for an inconsistent system, you will not be able to solve the system.*

d Is the system consistent?

Row Reduction and Echelon Forms

A rectangular matrix is in **echelon form** if it has the following properties:

- Any row with all zeros should be moved to the last row.
- Each leading entry should have only zeros below it in that column.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Note: a leading entry is the first nonzero term in a row.

Theorem (Uniqueness of the Reduced Echelon Form)

Each matrix is equivalent to one and only one reduced echelon matrix.

Row Reduction and Echelon Forms

A rectangular matrix is in **echelon form** if it has the following properties:

- Any row with all zeros should be moved to the last row.
- Each leading entry should be **1** and have only zeros below and above it in that column.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Note: a leading entry is the first nonzero term in a row.

Theorem (Uniqueness of the Reduced Echelon Form)

Each matrix is equivalent to one and only one reduced echelon matrix.

Example

Row reduce the matrix below to **reduced** echelon form

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$

Echelon form:

$$A = \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solving Linear Systems with Row Reductions

Suppose your row reduction yields the following, we solve using back substitution;

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We say that x_3 is a free variable, and hence

$$x_3 = x_3,$$

$$\begin{aligned} x_2 - 2x_3 &= 4 \\ \text{so, } x_2 &= 4 + 2x_3, \text{ and} \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -1 \\ \text{so, } x_1 &= -1 + 2x_2 + 3x_3. \\ \text{Thus, } x_1 &= -1 + 2(4 + 2x_3) + 3x_3 \\ &= 7 + 7x_3. \end{aligned}$$

Homework 2

Determine whether each of the following matrices is in *echelon form*, *reduced echelon form*, or *neither*. Justify your answers.

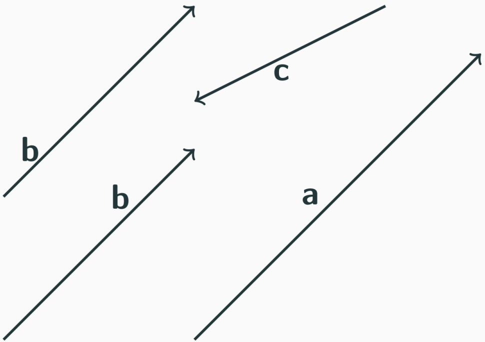
Given Matrices:

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Vectors

Linear algebra is a branch of mathematics that focuses on the study of **vectors**, the space within which they exist (**vector spaces**) as well as their characteristics and operations that can be performed on them. Provides a mathematical framework for analyzing systems of linear equations and their representations in higher dimensions.



Vectors have **magnitude** and **direction** (arrows pointing in space)

Another View is that Vectors are **ordered lists of numbers**. Consider the model for the price of gold.

$$\begin{bmatrix} 1\text{oz} \\ \$1200 \end{bmatrix} \neq \begin{bmatrix} \$1200 \\ 1\text{oz} \end{bmatrix}$$

Vectors

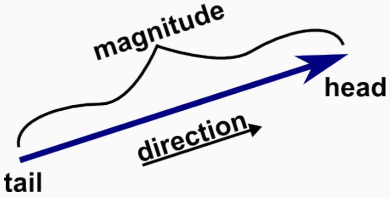
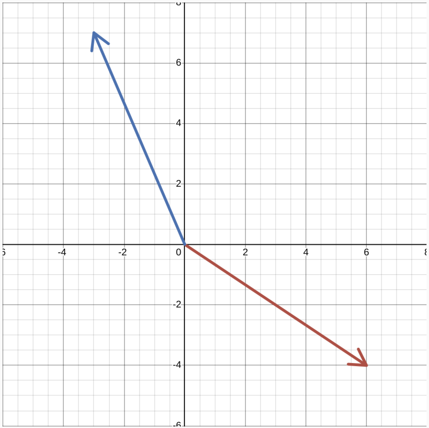


Figure 1: https://mathinsight.org/vector_introduction

In this course when we say vectors we will be referring to a directed line segment sitting at the origin.

- The two numbers (coordinates) are a pair of numbers that tell you how to get from the tail(origin) to the head(tip)

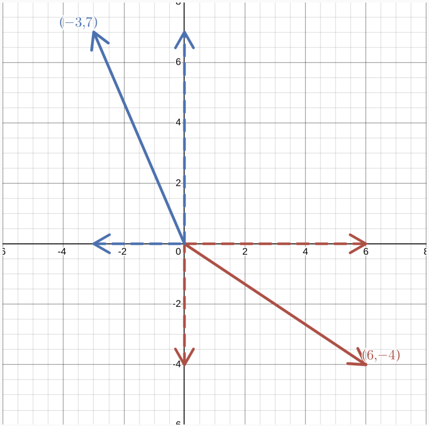
Vectors



Consider the vectors $\begin{bmatrix} -3 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$

- The first number tells you how far to walk on the $x-$axis. **Positive** numbers mean **right**, and **negative** numbers mean **left**.
- The second number tells you how far to walk parallel to the $y-$axis. **Positive** numbers mean **up**, and **negative** numbers mean **down**.
- Every pair of numbers is associated with only one vector (arrow), and every vector (arrow) is associated with only one pair of numbers.

Vectors



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- Every pair of numbers is associated with only one vector (arrow), and every vector (arrow) is associated with only one pair of numbers.

Since vectors are ordered (component-wise), addition is also component-wise.

- **Addition**: Subtraction is also a form of addition

$$\begin{bmatrix} -3 \\ 7 \end{bmatrix} + \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

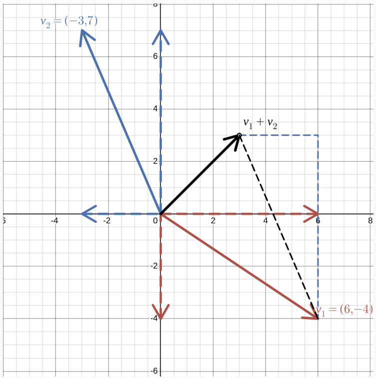

Figure 1: Vector Addition

Addition of Vectors and Scalar Multiplication

Since vectors are ordered (component-wise), addition is also component-wise.

- **Scalar Multiplication**

$$-v_1 = -\begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}-3\\-6\end{bmatrix}$$

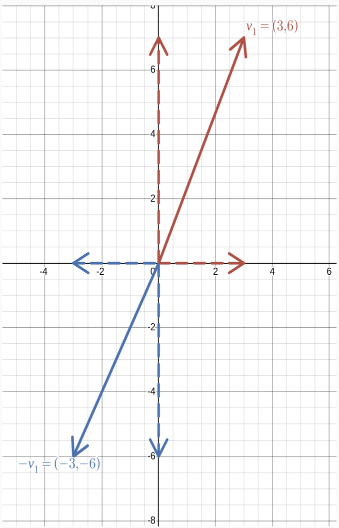

Figure 1: Scalar Multiplication

Addition of Vectors and Scalar Multiplication

Since vectors are ordered (component-wise), addition is also component-wise.

- **Scalar Multiplication**

$$3v_2 = 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}6\\9\end{bmatrix}$$

Note: a vector can have more than 2 dimensions.

$\begin{bmatrix}2\\3\\5\end{bmatrix}$ is a 3-dimensional vector, and $\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ is n-dimensional

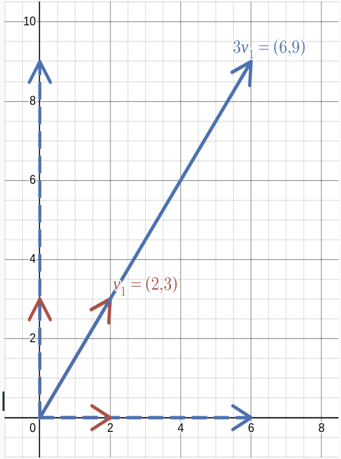

Figure 1: Scalar Multiplication

Special Vectors (Basis)

Consider the special vectors $\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Observe that the vectors $\begin{bmatrix} -3 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$ can be written as a sum and scalar multiplication of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

$\begin{bmatrix} -3 \\ 7 \end{bmatrix} = -3\hat{\mathbf{i}} + 7\hat{\mathbf{j}}$, and $\begin{bmatrix} 6 \\ -4 \end{bmatrix} = 6\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$.

All the vectors you can reach using only vector addition and scalar multiplication is called the **span** of these special vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. This is also known as the **line**ar combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ (Desmos illustration `https://www.desmos.com/calculator/apimvn3ln9`)

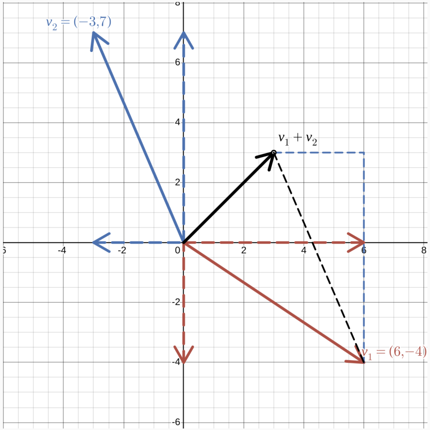

Figure 2: Vector Addition

Algebraic Properties of Vectors in $\mathbb{R}^n$ and Matrix Equations

- **Commutative Property:**
  $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
- **Associative Property:**
  $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
- **Additive Identity:** $\mathbf{u}+\mathbf{0}=\mathbf{u}$
- **Additive Inverse:** $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
- **Distributive Property:**
  $r(\mathbf{u}+\mathbf{v})=r\mathbf{u}+r\mathbf{v}$

Where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors. r is a scalar

A vector equation

$$x_1\mathbf{v_1}+x_2\mathbf{v_2}+\cdots+x_n\mathbf{v_n}=\mathbf{w}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} & \mathbf{w}\end{bmatrix}.$$

In the vector equation, we say that $\mathbf{w}$ is a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$.

Definition ($Span$ of a Set of Vectors)

The *span* of a set of vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_p} \in \mathbb{R}^n$ is the set of all possible linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_p}$.

That is, if a vector $\mathbf{b}$ is in $Span\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_p}\}$, then $\mathbf{b}=c_1\mathbf{v_1}+c_2\mathbf{v_2}+\cdots+c_p\mathbf{v_p}$.

Matrix Multiplication

Here are some examples to illustrate multiplication of matrices and vectors with different dimensions, showcasing various matrix equations:
Example 1: Multiplication of a Matrix and a Column Vector

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Matrix Multiplication

Matrix dimensions: A (2×2), $\mathbf{v}$ (2×1)
The product is:

$$A\mathbf{v} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 + 3 \cdot 6 \\ 4 \cdot 5 + 1 \cdot 6 \end{bmatrix} = \begin{bmatrix} 28 \\ 26 \end{bmatrix}$$

Matrix Multiplication

Example 2: Multiplication of Two Matrices**

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

Matrix Multiplication

Matrix dimensions: B (2×3), C (3×2) The product is:

$$BC = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 & 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 \\ 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 & 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Example 3: Multiplication Involving a Row Vector and a Matrix

$$\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 5 \\ 6 & 7 \\ 8 & 9 \end{bmatrix}$$

Matrix Multiplication

Matrix dimensions: $\mathbf{u}$ (1×3), D (3×2) The product is:

$$\mathbf{u}D = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4 & 5\\6 & 7\\8 & 9\end{bmatrix} = \begin{bmatrix}1\cdot 4+2\cdot 6+3\cdot 8 & 1\cdot 5+2\cdot 7+3\cdot 9\end{bmatrix} = \begin{bmatrix}40 & 46\end{bmatrix}$$

Example 4: Non-Conformable Matrices

$$E = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Matrix Multiplication

Matrix dimensions: E (2×3), F (2×2)
You cannot multiply EF because the number of columns in E does not match the number of rows in F. However, if their roles are reversed (FE), multiplication is possible, resulting in another 2×3 matrix.

Matrix Equations

Recall: A vector equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n} = \mathbf{w}. \tag{4}$$

Equation (4) may be rewritten (with columns as vectors) as

$$\begin{aligned} \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} &= \mathbf{w} \\ \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} &= \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\ A\mathbf{x} &= \mathbf{w}. \end{aligned}$$

Matrix Equation $A\mathrm{x} = \mathbf{w}$

Definition

Matrix Equation If A is and $m \times n$ matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and if $\mathbf{x} \in \mathbb{R}^n$, then the product of A and $\mathbf{x}$, is denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; i.e,,

$$A\mathbf{x} = [\mathbf{v_1}\ \mathbf{v_2}\ \cdots\ \mathbf{v_n}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n}$$

That is, we view a linear combination of vectors as the product of a matrix and a vector.

Formulating Matrix equations

For example consider the system of linear equations,

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

The corresponding Matrix equation is

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

which is equivalent to the vector equation

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Solution to Matrix Equations

Theorem
Existence of Solutions The equation $A\mathbf{x} = \mathbf{w}$ *has a solution if and only if* $\mathbf{w}$ *is a linear combination of the columns of* A.

How is this related to the $Span$?

Lemma
Let A *be an* $m \times n$ *matrix. Then the following statements are logically equivalent;*

(a) *For each* $\mathbf{w} \in \mathbb{R}^m$, *the equation* $A\mathbf{x} = \mathbf{w}$ *has a solution.*

(b) *Each* $\mathbf{w} \in \mathbb{R}^m$ *is a linear combination of the columns of* A.

(c) *The columns of* A *span* $\mathbb{R}^m$.

(d) A *has a pivot in every row after row reduction.*

Examples

Example 1

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$$

- Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for all possible $\mathbf{b} \in \mathbb{R}^3$?
- In other words, do the columns of A span $\mathbb{R}^3$?
- Alternatively, can any vector, $\mathbf{b} \in \mathbb{R}^3$ be written as a linear combination of the 3 columns of A?

If it does, write out the form of each $\mathbf{b}$.
If not, describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ does have a solution.

Examples

Solution: according to the lemma in the previous page, each $A\mathbf{x} = \mathbf{b}$ will have a solution for each $\mathbf{b} \in \mathbb{R}^3$ if A has a pivot in every row. So perform a row reduction to get:

$$\begin{bmatrix} 1 & -3 & -4 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since row 3 does not have a pivot (no nonzero number in the row), we conclude that $A\mathbf{x} = \mathbf{b}$ does not have a solution for all $\mathbf{b} \in \mathbb{R}^3$.

To find the form of $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, use row reductions with the b's in an augmented matrix form.

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_1 \end{array}\right] \begin{array}{c} R_2 + 3R_1 \to R_2 \\ R_3 - 5R_1 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{array}\right]$$

Examples

$$R_3 + 2R_2 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & 2b_2 + b_1 + b_3 \end{array}\right]$$

This matrix equation only has a solution if the entire 3rd row is 0. That is, $2b_2 + b_1 + b_3 = 0$ otherwise the system will be inconsistent. So, $b_3 = -b_1 - 2b_2$.

$$\text{Thus, } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b2 \\ -b_1 - 2b2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ -b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 2b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{b} = \left\{ b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \text{for } b_1, b_2 \in \mathbb{R} \right\}.$$

Example 2

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$. Do these 3 vectors span $\mathbf{R}^3$?

Solution: Yes! Reduce to echelon form and observe that each row has a pivot. So, by the previous lemma, the columns span $\mathbb{R}^3$.

Example 3

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Do these 3 vectors span $\mathbf{R}^4$?

Solution: No! Reduce to echelon form and observe that the last row has no a pivot. So, by the previous lemma, the columns do not span $\mathbb{R}^4$.

Example 4 Construct a 3×3 matrix, A and vectors $\mathbf{w}, \mathbf{b} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ has a solution, but $A\mathbf{x} = \mathbf{w}$ does not have a solution.
Solution: You can use the matrix in example 1. Then choose a vector that meets the condition of $\mathbf{b}$, and another vector that does not meet that condition.

Homework3

Question 1
If A is an 3×2 matrix, and $\mathbf{u}$, $\mathbf{w} \in \mathbb{R}^2$ are vectors, show that $A(\mathbf{u} + c\mathbf{w}) = A\mathbf{u} + cA\mathbf{w}$, where c is a scalar.
That is, $A = [\mathbf{v_1}\ \mathbf{v_2}]$ and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

Question 2
Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.
Show that the equation $A\mathbf{x} = \mathbf{b}$ does not a have a solution for all possible $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution. In other words, do the columns of A span $\mathbb{R}^2$?

Linear Independence and Solution Sets of Linear Systems I

Recall: (The *Span* of a set of vectors $S = \{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3}\}$, is the set of all the possible linear combinations, $c_1\mathbf{v_1}, +c_2\mathbf{v_2} + c_3\mathbf{v_3}$).
Thus, if we can obtain $\mathbf{v_3}$ by a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$, then $\mathbf{v_3}$ is linearly dependent on $\mathbf{v_1}$ and $\mathbf{v_2}$. Otherwise, we say $\mathbf{v_3}$ is linearly independent of $\mathbf{v_1}$ and $\mathbf{v_2}$.
If linear independence applies for all the elements within the set, we say S is linearly independent.

Example 1 Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + \ x_2 - 8x_3 &= 0 \end{aligned}$$

Is $\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ as solution vector?

Example 2 Determine the solution of the following homogeneous system

$$10x_1 - 3x_2 - 2x_3 = 0.$$

Write the solution as a parametric vector equation.

A linear system is homogeneous if it can be written as $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix.

Theorem
Suppose that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous solution $A\mathbf{x} = \mathbf{0}$.

That is, if $\mathbf{Ax} = \mathbf{b}$ has a solution, then the solution is obtained by translating the solution set of $\mathbf{Ax} = \mathbf{0}$.

Example 1b Determine if the following nonhomogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

Definition (Linear Dependence)

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is said to be *linearly dependent* if there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Definition (Linear Independence)

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is said to be *linearly independent* if the only scalars $c_1, c_2, \ldots, c_n$ that satisfy:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

are $c_1 = c_2 = \cdots = c_n = 0$.

Linear Independence and Solution Sets of Linear Systems Part IV

Lemma (Linear Independence and Homogeneous Systems)
The columns of a matrix A are linearly independent if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Proof.

- **($\Rightarrow$)** Suppose the columns of A are linearly independent. Then, by definition, the only scalars $c_1, c_2, \ldots, c_n$ that satisfy:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

  are $c_1 = c_2 = \cdots = c_n = 0$, where $\mathbf{a}_i$ are the columns of A. This implies that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
- **($\Leftarrow$)** Suppose $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. Then there are no nontrivial combinations of the columns of A that sum to $\mathbf{0}$, implying that the columns of A are linearly independent.

$\square$

Example 1: Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

Example 2: Determine if the vectors are $\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ are linearly independent.

Example 3: Show that the set of two vector $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

Linear Independence and Solution Sets of Linear Systems Part VI

Theorem

Characterization of Linearly Dependent Sets An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *of two or more vectors is linearly dependent if and only if at least one of the vectors in* S *is a linear combination of the others. That is, each* $\mathbf{v}_i \in S$ *does not have to be a linear combination of the other vectors. Only one of them needs to satisfy the condition.*

Lemma

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *whose elements* $\mathbf{v}_i \in \mathbb{R}$ *is linearly dependent if* $p > n$.

Lemma

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ *whose elements* $\mathbf{v}_i \in \mathbb{R}$ *is linearly dependent if one of the elements,* $\mathbf{v}_j = \mathbf{0}$ *is the zero vector.*

Homework 4

Question Q1

Construct a 3×3 nonzero matrix such that the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

Question Q2

Are the following vectors linearly dependent? Justify your answer.

$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Introduction to Linear Transformations

A linear transformation is a mathematical operation that maps vectors from one vector space to another (or to itself). This function can be represented as a matrix.
A linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be written as:

$$T(\mathbf{x}) = A\mathbf{x},$$

where: A is an $m \times n$ matrix (the transformation matrix), $\mathbf{x} \in \mathbb{R}^n$ is the input vector.
$T(\mathbf{x}) \in \mathbb{R}^m$ is the transformed vector.
Note: A **transformation** is just a function
A transformation is linear if it has the following properties:

- All lines remain lines.
- The origin stays the same.

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$,
and define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.
b. Find an $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$.
c. Is there more than one $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$?
d. Determine if $\mathbf{c}$ is in the range of the transformation T. In other words, is there an $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{c}$?

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$,

and define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$.

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix}$$

c. Is there more than one $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T. In other words, is there an $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{c}$?

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix},$

and define the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

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b. Find an $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$.

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c. Is there more than one $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T. In other words, is there an $\mathbf{x} \in \mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{c}$?

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

More Examples

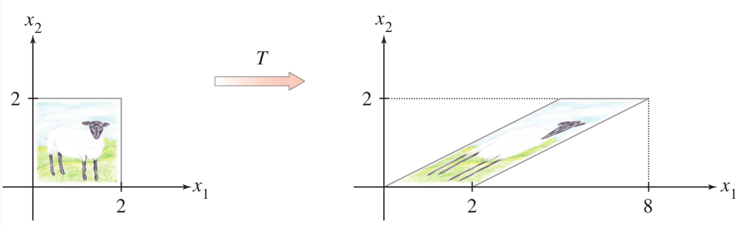
Example 2: Construct a transformation matrix that projects points in $\mathbb{R}^3$ to $\mathbb{R}^2$.
Can you construct another transformation to do a similar thing?

Example 3: Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a **shear transformation**.
What does the transformation do to a square?
Hint: apply the transformation to the points
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$. These four points form a square of length 2 on each side.

More Examples

Example 2: Construct a transformation matrix that projects points in $\mathbb{R}^3$ to $\mathbb{R}^2$. Can you construct another transformation to do a similar thing?

Example 3: Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a **shear transformation**.
What does the transformation do to a square?
Hint: apply the transformation to the points
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$. These four points form a square of length 2 on each side.



More Examples

Example 2: Construct a transformation matrix that projects points in $\mathbb{R}^3$ to $\mathbb{R}^2$. Can you construct another transformation to do a similar thing?

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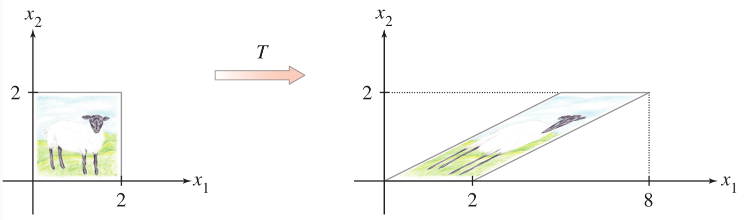


sheep

sheared sheep

Introduction to Linear Transformations

Definition (Linear Transformation)

A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear if it satisfies the following:
For any input vector $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalar $c \in \mathbb{R}$,

(i) **Additivity**: $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

(ii) **Scalar Multiplication**: $T(c\mathbf{u}) = cT(\mathbf{u})$, for any scalar c.

Geometrically, linear transformations can:
- **Scale (Enlarge or Shrink)** vectors (e.g., multiplication by a scalar matrix).
- **Rotate** vectors (e.g., multiplication by a rotation matrix).
- **Reflect** vectors (e.g., across an axis or plane).
- **Shear** vectors (e.g., skew transformations).

Matrix of Linear Transformation

Examples of Transformations

1. **Scaling**:

$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \quad T(\mathbf{x}) = A\mathbf{x}.$$

This scales all vectors by a factor k.

2. **Rotation in 2D**:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This rotates vectors by an angle θ.

3. **Reflection**:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This reflects vectors across the x-axis.

The Matrix of a Linear Transformation

Example: The columns of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}.$$

With no additional information, find a formula for the image of an arbitrary $\mathbf{x} \in \mathbb{R}^2$. And hence the matrix of the transformation A.

The Matrix of a Linear Transformation

Solution:

Any $\mathbf{x} \in \mathbb{R}^2$ can be written as: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

and $x_1, x_2 \in \mathbb{R}$.

Since T is a linear transformation,

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2).$$

Thus, the image of $\mathbf{x}$ under T is:

$$T(\mathbf{x}) = x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ -7x_1 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} -3x_2 \\ 8x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix}$$

Thus, the matrix transformation, is:

$$T(\mathbf{x}) = \underbrace{\begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The Matrix of a Linear Transformation

To find the matrix A of the linear transformation T, compute the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbb{R}^n$ under T. The i-th column of A is given by:

$$A_i = T(\mathbf{e}_i),$$

where A_i is the image of the i-th standard basis vector.
Thus, the matrix A is:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}.$$

Theorem (Matrix Transformation)
Let T from $\mathbb{R}^n$ to $\mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that:

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

The Matrix of a Linear Transformation

Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by:
$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+3y \\ -x+4y \end{bmatrix}.$
Applying T to the standard basis vectors, find A.
Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by:

$$T\mathbf{x} = 5\mathbf{x}.$$

Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that rotates each point counterclockwise about the origin through an angle θ.
Find the standard matrix A of this transformation.
Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that rotates each point clockwise about the origin through an angle θ.
Find the standard matrix A of this transformation.

The Matrix of a Linear Transformation

Definition (one-to-one)

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if each $\mathbf{b} \in \mathbb{R}^m$ is the image of **at most one** $\mathbf{x} \in \mathbb{R}^n$.
If $T(\mathbf{x}) = T(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$.
That is, if any two objects map to the same image, the objects must be the same. This does not guarantee that each image has a corresponding object.

Definition (unto)

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is **unto** if each $\mathbf{b} \in \mathbb{R}^m$ is the image of **at least one** $\mathbf{x} \in \mathbb{R}^n$.
That is, every image must have an object. This does not guarantee that each object has a corresponding image.

Example 1: One-to-One Transformation

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

Consider the matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. To check if $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one, we solve $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving for x_1 and x_2, we rewrite the system in augmented form and row reduce:

$$\begin{bmatrix} 1 & 2 & |0 \\ 3 & 4 & |0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & |0 \\ 0 & -2 & |0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & |0 \\ 0 & 1 & |0 \end{bmatrix}.$$

Since $x_1 = x_2 = 0$, A is **one-to-one**.

Example 2: Onto Transformation

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for every $\mathbf{b} \in \mathbb{R}^m$, there exists a solution $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$. This happens when the columns of A span $\mathbb{R}^m$.

Consider the matrix: $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Since B is a 2×3 matrix. so it is made up of 3 2-dimensional vectors. Thus, it can only span $\mathbb{R}^2$. We row reduce it to check if it has a pivot in each row (check theorem on page 19):

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}.$$

Since the matrix has a pivot in each row, the columns of B span all of $\mathbb{R}^2$, meaning B is **onto**.

The Matrix of a Linear Transformation

Example: Let T be the linear transformation whose standard matrix is

$$\begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Does T map $\mathbb{R}^4$ **onto** $\mathbb{R}^3$? Is T a **one-to-one** map?

Theorem
*A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.*

Theorem
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and A be the standard matrix for T. Then,

(i) *T is onto if and only if the columns of A span $\mathbb{R}^m$.*

(ii) *T is one-to-one if and only if the columns of A are linearly independent.*

*is **one-to-one** if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.*

Homework 5

Question Q1: Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$. Using the definition of linear transformation, show that

(i) $T(c_1\mathbf{u} + c_2\mathbf{v}) = c_1T(\mathbf{u}) + c_2T(\mathbf{v})$ for any scalars $c_1,\ c_2 \in \mathbb{R}$.

(ii) $T(\mathbf{0}) = \mathbf{0}$.

Question Q2: Define a a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Find the images under T of $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\mathbf{v} + \mathbf{w}$. Sketch the object represented by the four coordinates, and their corresponding image under the transformation.

Matrix Algebra

Matrix Operations

1. Matrix Addition and Subtraction: matrix addition and subtraction are performed element-wise. The matrices must have the same dimensions. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then:

$$A + B = [a_{ij} + b_{ij}], \quad A - B = [a_{ij} - b_{ij}].$$

Example:

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}.$$

$$A + B = \begin{bmatrix} 2+1 & 4+(-2) \\ 1+3 & 3+0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

$$A - B = \begin{bmatrix} 2-1 & 4-(-2) \\ 1-3 & 3-0 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ -2 & 3 \end{bmatrix}.$$

2. Scalar Multiplication: a matrix can be multiplied by a scalar c, which means each element of the matrix is multiplied by c. If $A = [a_{ij}]$, then:

$$cA = [c \cdot a_{ij}].$$

Example:

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}, \quad c = 4.$$

$$4A = \begin{bmatrix} 4 \cdot 3 & 4 \cdot (-1) \\ 4 \cdot 0 & 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 & -4 \\ 0 & 8 \end{bmatrix}.$$

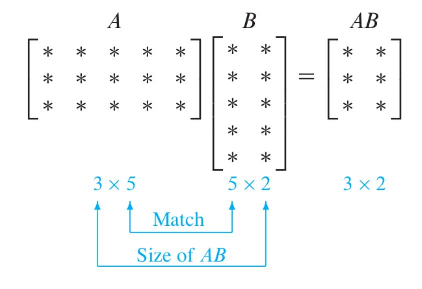
3. Matrix Multiplication: matrix multiplication is defined when the number of columns in the first matrix equals the number of rows in the second matrix.
If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{jk}]$ is an $n \times p$ matrix, then the product $C = AB$ is an $m \times p$ matrix where:

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}.$$

$$AB = \begin{bmatrix} (1 \cdot 2 + 2 \cdot 1) & (1 \cdot 0 + 2 \cdot (-1)) \\ (3 \cdot 2 + 4 \cdot 1) & (3 \cdot 0 + 4 \cdot (-1)) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 10 & -4 \end{bmatrix}.$$



4. Transpose of a Matrix: the transpose of a matrix is obtained by swapping its rows with its columns.
If $A = [a_{ij}]$ is an $m \times n$ matrix, then the transpose $A^T = [a_{ji}]$ is an $n \times m$ matrix.
Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}.$$

Theorem
Let A and B denote matrices whose sizes are appropriate for the following sums and products. The following is true:

(i) $(A^\top)^\top = A$
(ii) $(A+B)^\top = A^\top + B^\top$
(iii) $(AB)^\top = B^\top A^\top$
(iv) $(rA)^\top = rA^\top$

5. Determinant of a Square Matrix: the determinant is a scalar value that can be computed for square matrices.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is:

$$\det(A) = ad - bc.$$

Example:

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}.$$

$$\det(A) = (3)(2) - (5)(1) = 6 - 5 = 1.$$

5. Determinant of a Square Matrix

For 3×3 matrix) $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ the determinant is:

$$\det(A) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right) .$$

6. Inverse of a Matrix The inverse of a square matrix A (if it exists) is a matrix A^{-1} such that: $AA^{-1} = A^{-1}A = I$, where I is the identity matrix.

Formula (for 2×2 matrix): If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) \neq 0$, then:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

$$\det(A) = (2)(4) - (1)(3) = 8 - 3 = 5.$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.2 \\ -0.6 & 0.4 \end{bmatrix}.$$

7. Identity Matrix The identity matrix I_n is a square matrix with ones on the diagonal and zeros elsewhere. It acts as the multiplicative identity for matrices:

$$AI = IA = A.$$

Example:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix Operations Summary

Theorem
Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are define:

(i) ***Associativity:*** $A(BC) = (AB)C$
(ii) ***Left Distribution:*** $A(B + C) = AC + BC$
(iii) ***Right Distribution:*** $(A + B)C = AC + BC$
(iv) ***Scalar Distribution:*** $r(AB) = (rA)B = A(rB)$

Note: $AB \neq BA$.
Also, there are no cancellation laws. That is, if $AB = AC$, then it is not true in general that $B = C$.

Matrix Operations Summary

Theorem

If A and B are $n \times n$ invertible matrices, then A^{-1} and B^{-1} are also invertible, and:

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

(iii) ***Right Distribution:*** $(A + B)C = AC + BC$

(iv) $(A^{-1})^{\top} = (A^{\top})^{-1}$, *since* $A^{\top})$ *is also invertible.*

Algorithm for Finding the Inverse of a Matrix Using Row Reductions

Algorithm Steps

1. Write the augmented matrix $[A \mid I_n]$.
2. Use row operations to transform the left-hand side A into I_n.
3. Ensure all pivot elements (diagonal entries) are 1, and all other elements in the column are 0.
4. Once the left-hand side is I_n, the right-hand side will be A^{-1}.

Conditions for Invertibility The matrix A is invertible if and only if:

1. A is a square matrix.
2. $\det(A) \neq 0$ (i.e., A is full rank).

Important Notes

- If A cannot be row reduced to I_n (e.g., a row becomes zero during the process), A is not invertible.
- The algorithm is applicable only for square matrices.

Examples

Example 1 Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Find A^{-1}.

Solution: Start with the augmented matrix:

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right].$$

Step 1: Scale the first row by $\frac{1}{2}$:

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \end{array}\right].$$

Step 2: Subtract the first row from the second row:

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right].$$

Step 3: Scale the second row by 2:

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \end{array}\right].$$

Step 4: Subtract $\frac{1}{2}$ of the second row from the first row:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

Examples

Example 2 Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Find A^{-1}.

Solution: Start with the augmented matrix:

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right].$$

Step 1: Scale the first row by $\frac{1}{3}$:

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 4 & 2 & 0 & 1 \end{array}\right].$$

Step 2: Subtract 4 times the first row from the second row:

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & -\frac{4}{3} & 1 \end{array}\right].$$

Step 3: Scale the second row by $\frac{3}{2}$:

$$\left[\begin{array}{cc|cc} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -2 & \frac{3}{2} \end{array}\right].$$

Step 4: Subtract $\frac{1}{3}$ of the second row from the first row:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -2 & \frac{3}{2} \end{array}\right].$$

Thus, $A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$.

Examples

Example 3: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right].$$

Step 1: $-R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right].$$

Step 2: Swap R_1 andR_3

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \end{array}\right].$$

Step 3: $-R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right].$$

Step 3: $\frac{1}{2}R_3 \to R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right].$$

Continue until you obtain the inverse.

Thus, $A^{-1} = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} \\ -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

Homework 6

1. Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ if it exists.
2. Determine whether $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is invertible. Justify your response.
3. Verify that the inverse you found for $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies $AA^{-1} = I$.
4. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$.
5. Verify that the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 2 \\ 3 & -2 & 1 \end{bmatrix}$ is invertible and find its inverse.
6. Show that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is not invertible.

Characterization of Invertible Matrices

Theorem (Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either ***all TRUE*** *or* ***all FALSE****.*

(a) *A is an invertible matrix. $A^\top$ is also invertible.*

(b) *A is row equivalent to an $n \times n$ identity matrix.*

(c) *A has a pivot in each row after row reduction.*

(d) *The matrix equation, $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$.*

(e) *The columns of A span $\mathbb{R}^n$.*

(f) *The columns of A are linearly independent.*

(g) *The homogeneous equation, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(h) *The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one*

(i) *The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.*

Characterization of Invertible Matrices

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $T(\mathbf{x}) = A\mathbf{x}$, where A is the transformation matrix. Then T is invertible if and only if A is an invertible matrix. And the inverse of the transformation $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$

Homework 7

1. Is $A = \begin{bmatrix} 0 & 2 & -2 & 3 \\ 0 & 0 & 0 & 5 \\ 1 & -4 & 8 & 1 \end{bmatrix}$ one-to-one? Explain the reason for your answer.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ x + 4y \end{bmatrix}$.

   Write out T in the form, $T(\mathbf{x}) = A\mathbf{x}$.
   Is T is invertible? If so, find a formula for T^{-1}.

Subspaces of $\mathbb{R}^n$

Definition
A **subspace** of $\mathbb{R}^n$ is any set $S \in \mathbb{R}^n$ that has 3 properties:

i) The zero vector is in S.
ii) For each $\mathbf{u},\ \mathbf{v} \in S$, $\mathbf{u} + \mathbf{v} \in S$.
iii) For each $\mathbf{u} \in S$, and scalar $c \in \mathbb{R}$, $c\mathbf{u} \in S$.

Example: If $\mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^n$, and $S = Span(\{\mathbf{u}_1,\ \mathbf{u}_2\})$. Then S is a subspace of $\mathbb{R}^n$.

Definition
A **column space** of matrix A is the set, $Col A = Span(\{\text{Columns of } A\})$. That is, the set of all linear combinations of the columns of A.

Example: Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}$. Determine whether $\mathbf{b}$ is in the column space of A.

Subspaces of $\mathbb{R}^n$

Definition
A **null space** of matrix A is the set, $NulA$ - of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem
A ***null space*** *of and $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is also a subspace of $\mathbb{R}^n$.*

Basis

Definition
A **basis** of a subspace S of $\mathbb{R}^n$ is a linearly independent set in S that spans S.

Note: The number of pivots tells you the number of basis elements the **null space** and **column space**.

$$\# \text{ of basis elements in null space} + \# \text{ of basis elements in column space} = \# \text{ of columns}$$

Theorem
A ***pivot columns*** *of a matrix A form a basis for the columns space of A.*

Definition
A **basis** of a subspace S of $\mathbb{R}^n$ is a linearly independent set in S that spans S.

Note: The number of pivots tells you the number of basis elements the **null space** and **column space**.

$$\# \text{ of basis elements in null space} + \# \text{ of basis elements in column space} = \# \text{ of columns}$$

Example 1: Find the basis for the null space, and the basis of the column space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Basis

Definition
A **basis** of a subspace S of $\mathbb{R}^n$ is a linearly independent set in S that spans S.

Note: The number of pivots tells you the number of basis elements the **null space** and **column space**.

$$\# \text{ of basis elements in null space} + \# \text{ of basis elements in column space} = \# \text{ of columns}$$

Example 2: Find the basis for the null space, and the basis of the column space of the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Homework 8

Question 1: Find the basis for the null space, and the basis of the column space of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

Dimension of a Subspace

Definition
The **dimension** of a nonzero subspace S, denoted by dim$S$ is the number of vectors in a basis for the subspace S.
The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

Definition
The **rank** of a matrix A, denoted by rank$A$ is the dimension of the column space of A.
The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

Theorem (Rank Theorem)
If a matrix A has n columns, then rank$A$ + dim(Nul$A$).

Examples

Example 1: Determine the rank and dim(Nul$A$) of the matrix below

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Example 2: etermine the rank and dim(Nul$A$) of the matrix below

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Characterization of Invertible Matrices Cont'd

Theorem (Invertible Matrix Theorem Continued)

Let A be an $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either ***all TRUE*** *or* ***all FALSE****.*

(a) *A is an invertible matrix. $A^\top$ is also invertible.*

(c) *A has a pivot in each row after row reduction.*

(j) *Let A be and $n \times n$ matrix .*

(k) *The columns of A form a basis of $\mathbb{R}^n$.*

(l) *The column space of A, Col$A$=$\mathbb{R}^n$.*

(m) *The dimension of the column space of A, dim Col$A = n$.*

(n) *rank$A = n$.*

(o) *The null space of A, Nul$A = \{\mathbf{0}\}$.*

(p) *The dimension of the null space of A, dim Nul$A = 0$.*

Determinants

Introduction to Determinants

Intuition: Geometrically, the determinant of a matrix A measures the scaling factor after applying the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$.
In a 2×2 matrix, the scaling factor measures how much bigger/smaller the area of the unit square gets after the transformation has been applied. For a 3×3 matrix, how much bigger volume of the unit cube gets after the transformation has been applied.
If $\det(A) = 0$, the transformation collapses space into a lower dimension, implying that A is non-invertible (singular/ does not have an inverse).
For example:

- If a 2×2 matrix has a determinant of 5, it scales areas by a factor of 5.
- If a 3×3 matrix has a determinant of -2, it scales volumes by a factor of 2 and reverses orientation (flips it like a page).

Definition (Determinant)

The determinant of a square matrix A of order $n \times n$ is a scalar value that provides important information about the matrix, such as whether it is invertible and how it scales volumes in linear transformations. The determinant of a matrix A is denoted as $\det(A)$ or $|A|$.

For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{its determinant is given by:} \quad \det(A) = ad - bc.$$

For a 3×3 matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Example 1
Given the matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix},$$

its determinant is:

$$\det(A) = (3)(4) - (2)(1) = 12 - 2 = 10.$$

Example 2
Consider the matrix:

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

its determinant is:

$$\det(B) = (1)(4) - (2)(2) = 4 - 4 = 0.$$

Since the determinant is zero, B is singular and non-invertible.

Definition (Generalized Definition of Determinant)

The **determinant** of a square matrix $A = [a_{ij}]$ of order $n \times n$ is the sum of n terms of the form $\pm a \det(A_{ij})$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A (not entirely true). **Matrix A_{ij} is obtained by deleting the ith row and the jth column**.

$$\det(A) \text{ or } |A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$
$$= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - \cdots + (-1)^{1+n} a_{1n} \det(A_{1n})$$

Example: Compute the determinant of $A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 0 \\ 1 & 5 & 0 \end{bmatrix}$

Introduction to Determinants

Definition (Cofactor Definition of Determinant)

The **cofactor** of a matrix A is the number C_{ij} given by $C_{ij} = (-1)^{i+j}\det(A_{ij})$.
The **determinant** of a square matrix $A = [a_{ij}]$ of order $n \times n$ can be computed using a **cofactor expansion** across the first row of A (not entirely true).

$$\det(A) \text{ or } |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

Example: Compute the determinants of

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 0 \\ 1 & 5 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 2 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Introduction to Determinants

Recall: when we row reduce a matrix, it does not change the matrix.
Hence this can make the computation of the determinant easier!

Theorem (Determinant Theorem)
*If A **is a triangular matrix**, then $\det A$ is a product of the entries on the main diagonal of A).*

Theorem (Row Operations)
Let A a square matrix.

(a) *If a multiple of one row of A is added to another row to produce a matrix B, then $\det A = \det B$.*
(b) *If two rows of A are interchanged to produce a matrix B, then $\det A = -\det B$.*
(c) *If a row of A is multiplied by k to produce a matrix B, then $\det B = k \cdot \det A$.*

Properties of Determinants

Theorem (Properties of Determinants)

(a) ***Determinant of Identity:*** *The determinant of the identity matrix I_n is 1, i.e.,* $\det(I_n) = 1$.

(b) ***Row or Column Swap:*** *Swapping two rows or columns of a matrix A to produce B negates its determinant:*

$$\det(B) = -\det(A),$$

(c) ***Multiplication by a Scalar:*** *Multiplying a row or a column by a scalar k multiplies the determinant by k: Therefore multiplying all n rows of a matrix by k,*

$$\det(kA) = k^n \det(A).$$

(d) ***Triangular Matrices:*** *The determinant of a triangular (upper or lower) matrix is the product of its diagonal elements:*

$$\det(A) = a_{11}a_{22}\dots a_{nn}.$$

Theorem (Properties of Determinants)

(e) ***Determinant of a Product:*** *The determinant of the product of two matrices is the product of their determinants:*

$$\det(AB) = \det(A) \cdot \det(B).$$

(f) ***Effect of Row Operations:*** *Adding a multiple of one row to another does not change the determinant.*

(g) ***Invertibility:*** *A square matrix A is invertible if and only if $\det(A) \neq 0$.*

(h) ***Transpose Property:*** *The determinant of a matrix is equal to the determinant of its transpose:*

$$\det(A^T) = \det(A).$$

Cramer's Rule Cramer's rule is needed in a variety of theoretical calculations. It can be used to study how the solution of $A\mathbf{x} = \mathbf{b}$ is affected by changes in the entries of $\mathbf{b}$. However, the formula is inefficient for hand calculations, except for 2×2 or 3×3 matrices.
Cramer's Rule provides an explicit formula for solving a system of linear equations using determinants. Given a system of n linear equations in n variables represented in matrix form as:

$$A\mathbf{x} = \mathbf{b},$$

where A is an $n \times n$ invertible matrix, the solution for each variable x_i is given by:

$$x_i = \frac{\det(A_i)}{\det(A)},$$

where A_i is the matrix obtained by replacing the i-th column of A with the column vector $\mathbf{b}$.

Cramer's Rule, Volume, and Linear Transformations

Example 1: Consider the system,

$$\begin{aligned} 2x_1 + 3x_2 &= 5, \\ 4x_1 + x_2 &= 6. \end{aligned}$$

The coefficient matrix is:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix},$$

with determinant:

$$\det(A) = (2)(1) - (3)(4) = 2 - 12 = -10.$$

Replacing the first column with $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ gives $A_1 = \begin{bmatrix} 5 & 3 \\ 6 & 1 \end{bmatrix}$, with $\det(A_1) = (5)(1) - (3)(6) = 5 - 18 = -13$. Thus,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-13}{-10} = 1.3.$$

Repeating for A_2 yields $x_2 = 0.8$.

Definition (Cofactor Formula for Inverse A^{-1})

Let A be an invertible $n \times n$ matrix. Then

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} \cdot \operatorname{adj} A \\ &= \frac{1}{\det A} \cdot \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \end{aligned}$$

$C_{ij} = (-1)^{i+j} \det(A_{ij})$. **Matrix A_{ij} is obtained by deleting the ith row and the jth column**.

Theorem (Determinant-Area)
If A is a 2×2 matrix,

$$\text{the area of the parallelogram determined by the columns of } A = |\det A|.$$

Theorem (Determinant-Area)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation determined by 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

Example:

Let S be the parallelogram determined by $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$.
Compute the area of the image of S under the mapping $\mathbf{x} \to A\mathbf{x}$.

Selected References