I will not violate the Clarkson University Code of Ethics or assist others in doing so, especially by presenting others' work as mine, or allow them to present my work as theirs. **I am better than that** and I take pride in, and responsibility for, my work. I understand that violations of the Code may result in loss of credit for the exam, the course, or even jeopardize my academic standing.

Signed:

Problem	Max	Scored
Exam grade		

- Start the exam only at the proctor's signal.
- Closed books and notes, no brought-in summary sheets, formula sheets, or any such accessories.
- No external paper allowed; if you need extra paper, please ask the proctor for it.
- Only basic sci. calculators allowed; no graphing, matrix, or CAS calculators.
- If needed, use both sides of each sheet for your answers. Clearly indicate where the answer is written, if it is not in the space provided for it.

 Name (Last, First): ____________________

Question 1

(i) For what type of ODEs is the **phase portrait** a useful summary of the behavior of solutions?

(ii) What is an equilibrium?

(iii) Calculate everything needed to sketch the phase portrait of the ODE:

$$\frac{dy}{dx} = y(y-2)^2(y^2+1)e^{-y}$$

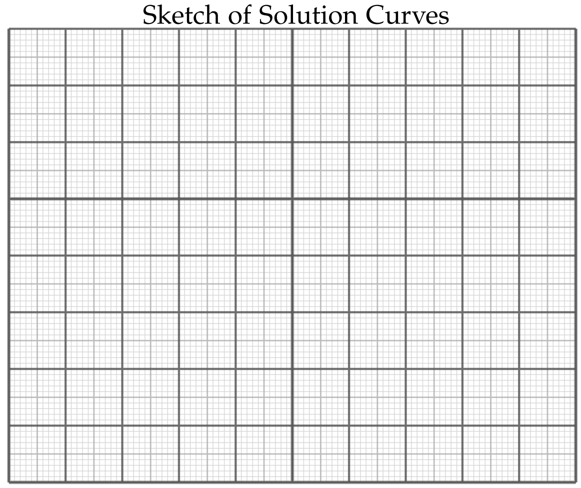
and then sketch the phase portrait and **classify the type of fixed points**.

(iv) **Sketch and label** the solution curves started at the following initial conditions as they extend into $x \to \infty$:

(a) $y(0) = 2$ (b) $y(0) = 1$ (c) $y(0) = -1$ (d) $y(1) = 3$

Sketch of Phase Portrait

Sketch of Solution Curves



Label the axes and the initial conditions.

Question 2

(i) Show that one of the following ODEs is exact and the other is linear.
Show your work for **both** ODEs.

(a) $\dfrac{1}{t^2}\dfrac{dx}{dt} + x - e^{t^3} = 0,$

(b) $[5x - 2t]\dfrac{dx}{dt} - 2x = 0.$

(ii) Solve **one** of the two ODEs given above (whichever you personally prefer). **Show your work.**

Question 3

The following ODE has a positive damping parameter $c \geq 0$ in it:

$$y''(x) + cy'(x) + 4y(x) = 16$$

(i) Calculate the particular solution for $c = 5$, $y(0) = 1$, $y'(0) = 1$.

(ii) Sketch the solution from (i) into the grid, according to the guidelines.

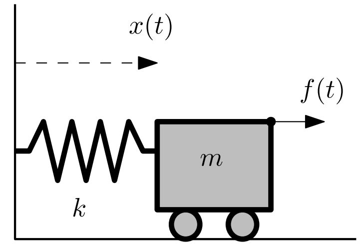
(iii) Use the characteristic equation and its roots to determine the **range of positive values** of c for which the solutions **do not** have any oscillations.

USE EMPTY SPACE BELOW AS NEEDED, BUT LABEL IMPORTANT PARTS OF YOUR CALCULATIONS.

Space for the solution sketch.

Guidelines. Accurately represent: • initial value • sign of the initial slope, • presence/absence of oscillations, • limit as $x \to \infty$. **Label the axes.**

Question 4



The sketch shows a mechanical system. $x(t)$ is the displacement of the body at time t, $f(t)$ is the force that is applied at the center of the mass of the body.

(i) **Use a free-body diagram and Newton's laws of motion** to derive the ODE for the position of the body $x(t)$. **Make sure to show your work.**

(ii) Explain in your words, what is **resonance**?

(iii) If the external force is $f(t) = \cos(2t)$, what value of the spring constant k is needed for the body of mass $m = 2$ to be in resonance with the input?

USE EMPTY SPACE BELOW AS NEEDED, BUT LABEL IMPORTANT PARTS OF YOUR CALCULATIONS.

Question 5

Use **any technique** to calculate the **general** solution of:

$$\ddot{x}(t) + 2x(t) + 10x(t) = t + e^{-2t}$$

Table of Laplace transform is on the last page if you need it.

Question 6

Use Laplace transform to calculate the solution to the ODE below. :

$$y''(t) + 4y'(t) + 3y(t) = \delta(t-1), \quad y'(0) = 0,\ y(0) = 3.$$

Note: $\delta(t)$ stands for the Dirac δ-impulse. Table of Laplace transform is on the last page.

Question 7

(i) Compute and **sketch**: $g(t) = \mathscr{L}^{-1}\left\{ \frac{1}{s} + \frac{2}{s^2}e^{-s} - \frac{2}{s^2}e^{-2s} \right\}$

(ii) Compute: $F(s) = \mathscr{L}\left\{ (t^2 - t)\, \mathscr{U}(t-1) \right\}$

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Question 8

The ODE system is given by the following equations:

$$\dot{x}(t) = 2x(t) - 5y(t)$$
$$\dot{y}(t) = 4x(t) - 2y(t)$$

(i) Calculate everything needed to sketch a phase portrait.

(ii) Based on your calculations in (i) sketch the phase portrait.

(iii) (Extra credit) Calculate the implicit formula $E(x, y) = \text{const.}$ for the solution curves using the procedure analogous to the one in Project 3.

USE EMPTY SPACE BELOW AS NEEDED, BUT LABEL IMPORTANT PARTS OF YOUR CALCULATIONS.

Space for phase portrait sketch.

Guidelines. Accurately represent: • location of the fixed point, • presence/absence of oscillations, • growth/decay (or neither) of solutions, • characteristic directions (if applicable). **Label the axes.**

Question 9

Given is the matrix ODE:

USE EMPTY SPACE BELOW AS NEEDED, BUT LABEL IMPORTANT PARTS OF YOUR CALCULATIONS.

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & 6 & -10 \\ 0 & 4 & -6 \end{bmatrix}}_{\boldsymbol{M}} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix}$$

(i) Verify that the eigenvalues of matrix $\boldsymbol{M}$ are $\lambda_1 = -1, \lambda_{2,3} = \pm 2i$.

(ii) Verify that the vector $\underline{v} = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \cdot \alpha$ is an eigenvector of the matrix $\boldsymbol{M}$.

(iii) Calculate the remaining two eigenvectors.

(iv) Write out a general solution of the matrix ODE. (For full credit, use real-valued form of the characteristic solutions associated to complex eigenvalues).

Laplace integral:
$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$
Properties, given that
$\mathscr{L}\{f(t)\} = F(s)$, $\mathscr{L}\{g(t)\} = G(s)$:

- Linearity:
  $\mathscr{L}\{c_1 f(t) + c_2 g(t)\}$
  $= c_1 F(s) + c_2 G(s)$
- s-shift:
  $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$
- t-shift:
  $\mathscr{L}\{\mathscr{U}(t-a) f(t-a)\} = e^{-sa} F(s)$
- Derivative in t: If $\mathscr{L}\{y(t)\} = Y(s)$, then
  $\mathscr{L}\{y'(t)\} = sY(s) - y(0)$,
  $\mathscr{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
$\delta(t)$	1
1	$\frac{1}{s}$
$\mathscr{U}(t)$	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{at}\sin(\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$