

MA2860 Elementary Differential Equations Notes and Problems

Emmanuel Atindama, PhD Mathematics

Copyright © 2025

PUBLISHED BY

<http://emmanuelatindama.github.io>

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License:

<http://creativecommons.org/licenses/by-nc-sa/4.0/>

v.0.1 Updated on September 18, 2025

*This text provides notes and problems that build on concepts in calculus for MA2860.
It is open source and meant for students and instructors to use, adapt, and contribute.
The computational component primarily consists of Sage code, and online symbolic
calculators are provided for supplemental study.*

Contents

Preface

THIS TEXT PROVIDES NOTES AND PROBLEMS that build on concepts in calculus for *MA2860 Elementary Differential Equations* at the University of Toledo. The book is a companion to various textbooks including:

- Zill, Dennis G. *A First Course in Differential Equations with Modeling Applications*, 10th ed. Cengage Learning, 2012.
- Boyce, William E., DiPrima Richard C. *Elementary Differential Equations*, 8th ed. John Wiley & Sons Inc., 2004.

Although it can be used with other texts as well.

IN MAKING THIS STARTING RESOURCE OF LESSON PLANNING **open source**, I hope that each user (with code access) will contribute with notes, corrections, and exercises (with solutions). Users without code access may also email me any corrections. *Professors may email to request code access*. Each week has a portion of the assignment reserved for preparatory homework and another portion for further discussion or exam preparation.

KNOWING THAT ALL PEOPLE ARE MADE IN THE IMAGE OF GOD WITH A POTENTIAL FOR THE HIGHEST GOOD, I hope that all users of this material will work to make this work of highest quality for all educators.

The idea for this work came from my professor in Dynamical Systems during my grad school years, Dr. Marko Budišić at Clarkson University. He encouraged students to compile class notes into one document for easy review. All credit to him.

This text uses a Differential Equations workbook compiled by Clarkson University faculty as a starting point, although the content is mostly different: K. Black, D. White, G. Yao, J. Martin, and M. Budišić.

THE COMPUTATIONAL COMPONENT of this book primarily consists of **SAGEMATH** code. No prerequisite experience is needed, as problems can be completed by following links and editing code in the browser. Essential manual for **SAGE-MATH** is the textbook:

- Bard, G.V. *Sage for Undergraduates*. American Mathematical Society, 2015. Available at no cost in PDF form: <http://www.gregorybard.com/Sage.html>.

ONLINE TOOLS for studying involve symbolic calculators at links: **CAUTION: do not rely heavily on them while studying!**

- <https://www.symbolab.com/solver/>
- <http://www.emathhelp.net/calculators/>

THE GOAL of this class is to teach the **principles** behind the calculations, so they can be built on and extended in future courses or practice, not convert people into human calculators. Therefore, emphasis is on algorithms, and memorization or **reliance on formulas and calculators is strongly discouraged**.

Contributing Authors:

- (a) Emmanuel Atindama, PhD Mathematics
- (b) Second Author

- (c) Third Author
- (d) Fourth Author

EMMANUEL ATINDAMA
[emmanuelatindama.github.io](https://github.com/emmanuelatindama)
 Toledo, OH
 September 2025

Chapter 1: Introduction to Differential Equations

We begin our journey into the world of elementary differential equations by looking at some review.

Section 1.1: Review: Precalculus and Calculus

In this section, you will review the techniques from Calculus classes that you should absolutely know in and out: sketching functions, implicit derivatives, and three integration techniques: substitution, by-parts, and partial fractions.

a) Properties of exponents and logarithms For $a > 0$, $a \neq 1$, and integers m, n and $x, y > 0$. Then

- $a^m \cdot a^n = a^{m+n}$
- $a^{-n} = \frac{1}{a^n}$
- $\ln x^r = r \cdot \ln x$
- $\frac{a^m}{a^n} = a^{m-n}$, ($m > n$)
- $\ln 1 = 0$
- $\ln(e^x) = x, e^{\ln(x)} = x$
- $(a^m)^n = a^{mn}$
- $\ln e = 1$
- Should be familiar with graphs of the natural exponential and natural logarithm.
- $(ab)^n = a^n b^n$
- $\ln xy = \ln x + \ln y$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

b) Trigonometric Properties

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
- $1 + \tan^2 \theta = \sec^2 \theta$
- $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$
- $1 + \cot^2 \theta = \csc^2 \theta$
- $\sin(2\theta) = 2 \sin \theta \cos \theta$
- $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$
- $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
- $\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
- Need to know the values of the trig functions for multiples of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
- Need to be familiar with polar coordinates: $x = r \cos(\theta), y = r \sin(\theta)$.

c) **Product rule:** If $f(x)$ and $g(x)$ are differentiable functions:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Example:

$$f(x) = x^2 \sin x$$

$$f'(x) = (2x)(\sin x) + (x^2)(\cos x) = 2x \sin x + x^2 \cos x$$

d) **Chain rule:** If $y = f(g(x))$, then:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Example:

$$y = (3x^2 + 1)^5$$

$$\frac{dy}{dx} = 5(3x^2 + 1)^4 \cdot (6x) = 30x(3x^2 + 1)^4$$

e) **Integration by Substitution** If $u = g(x)$, then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

Example:

$$\int 2x \cos(x^2) dx$$

Let $u = x^2$, so $du = 2x dx$:

$$\int 2x \cos(x^2) dx = \int \cos(u) du = \sin(u) + C = \sin(x^2) + C$$

f) **Integration by Parts** From the product rule:

$$\int u dv = uv - \int v du$$

Example:

$$\int xe^x dx$$

Let $u = x \implies du = dx$ and $dv = e^x dx \implies v = e^x$:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = (x - 1)e^x + C$$

g) **Partial Fraction Decomposition:** If the integrand is a rational function $\frac{P(x)}{Q(x)}$, decompose into simpler fractions.

Example:

$$\int \frac{1}{x^2 - 1} dx$$

Factor denominator:

$$\frac{1}{x^2 - 1} = \frac{1}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

Multiply through:

$$1 = A(x + 1) + B(x - 1)$$

Setting $x = 1 \implies 1 = 2A \implies A = \frac{1}{2}$ Setting $x = -1 \implies 1 = -2B \implies B = -\frac{1}{2}$

So:

$$\frac{1}{x^2 - 1} = \frac{1}{2} \cdot \frac{1}{x - 1} - \frac{1}{2} \cdot \frac{1}{x + 1}$$

Now integrate:

$$\begin{aligned}\int \frac{1}{x^2 - 1} dx &= \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C \\ &= \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C\end{aligned}$$

In-Class Exercise

Exercise 1.1.1

Simplify the following expressions:

(a) $e^{4 \ln |x+1|} =$

(b) $\exp \left[-\frac{1}{2} \ln(x^2) \right] =$

Exercise 1.1.2

Evaluate the following integrals using substitution, integration by parts, and partial fractions, as appropriate.

(a) $\int_0^x x e^{x^2} dx$ ¹

¹ Hint: Substitution

(b) $\int_1^2 x \sin(\pi x) dx$ ²

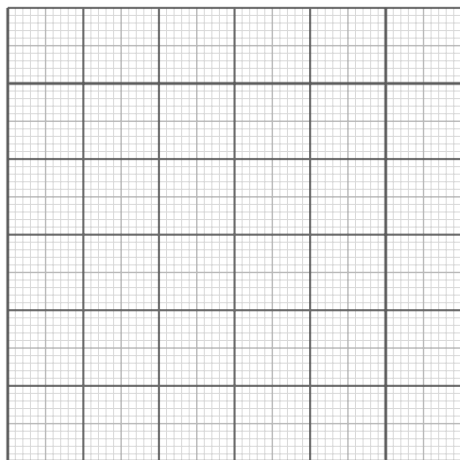
² Hint: Integration by parts

(c) $\int \frac{x}{x^2 + x - 6x} dx$

Exercise 1.1.3

Sketch the graph of the function $f(x) = (x^2 - 4x + 3)e^{-2x}$ ³

³ Determine any minima/maxima, x-intercepts and y-intercepts, and asymptote $x \rightarrow +\infty$ for an accurate sketch.

**Exercise 1.1.4**

If $y(x) = e^x \sin(2x)$, evaluate the expression $y''(x) - 2y'(x) + 5y(x)$. Compute the required derivatives⁴ first, and then substitute.

⁴ We use y' to mean the derivative with respect to the argument, in this case dy/dx . When the function depends on time, say $x(t)$, then we sometimes additionally use the dot-notation $\dot{x} = dx/dt$.

Discussion Problems**Exercise 1.1.5**

Compute the derivative $y'(x)$ if the following relation between y and x holds:

$$\ln|y^2 + 2| + x^3 \sin(2x) = C, \text{ a constant.}$$

Exercise 1.1.6

What is the domain of definition of each of the two functions:

(a) $\ln(2x + 1)$

(b) $\ln|2x + 1|$

Exercise 1.1.7

Split the following rational functions into partial fractions:

(a) $\frac{1}{x^2+4x+3}$

(b) $\frac{2x}{x^2+4x+3}$

(c) $\frac{x^2+4x}{x^2+4x+3}$

Exercise 1.1.8

A curve is defined by the implicit expression $\cos(2x) + \frac{1}{2}y^2 = 1$. Calculate **all** the points at which the tangent⁵ to this curve is

⁵ Hint: find the implicit derivative $y'(x)$

(a) $y'(x) \rightarrow \infty$, (b) $y'(x) \rightarrow 0$

(a) Vertical.

(b) Horizontal.

Exercise 1.1.9

Evaluate the integrals:⁶

(a) $\int \ln(x) dx$

(c) $\int \frac{x}{x+2} dx$

(e) $\int \frac{x^2}{x^2+x-6x} dx$

(b) $\int xe^x dx$

(d) $\int \frac{2x}{x^2-4} dx$

(f) $\int e^x \sin(\pi x) dx$

⁶ Hint: (a) integration by parts, (b) substitution, (d) substitution/partial fraction, (f) by parts

Section 1.2: Basic Terminology & History**Basic Terminology**

Definition 1.2.1. A differential equation is an equation that contains one or more derivatives of an unknown function.

A differential equation is an ordinary differential equation (ODE) if it involves an unknown function of only one independent variable. Example:

$$\frac{dy}{dt} = -ky.$$

A differential equation involving partial derivatives of a function of two or more independent variables is a partial differential equation. Example:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}.$$

Differential equations describe how one quantity changes in relation to other variables.

For a standard algebraic equation, the solution is a number, but for a differential equation, the solution is a function.

- $y(x)' = y + x^2 + 3$ is a differential equation. The unknown function is y depending on the variable x and the derivative of y is involved in the equation.
- $\cos x = -1$ is not a differential equation. There is no unknown function.
- $y = x^2 - 2x + 1$ is not a differential equation since there is no derivative involved.
- $y'(x) = y$ is a differential equation, with unknown function y with its derivative depending on the variable x .
- $\frac{d}{dx}(\tan x) = \sec^2 x$ is not a differential equation since there is no unknown function

Definition 1.2.2. The order of a differential equation is the highest derivative that appears in it.

- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ is a second-order ODE.
- $y' + 3x^2 = 1$ is a first-order ODE.
- $y'' - x^2y' + y = \cos x$ is a second-order ODE.
- $y^4 - xy' + y^2 = 3$ is a fourth-order ODE.

Definition 1.2.3 (Linearity). An ODE is linear if the dependent variable y and its derivatives y', y'', \dots, y^n all appear only to the first power and not multiplied together. Example:

$$a_2(x)y'' + a_1(x)y' + P(x)y = Q(x).$$

An ODE is nonlinear if it contains products, powers, or nonlinear functions of the dependent variable or its derivatives. Example:

$$\frac{dy}{dx} = y^2 + \sin(y).$$

Definition 1.2.4. A differential equation is homogeneous if all terms depend on the unknown function or its derivatives. If there is a term that does not, it is nonhomogeneous.

- **Homogeneous:** $\frac{dy}{dx} + p(x)y = 0$
- **Nonhomogeneous:** $\frac{dy}{dx} + p(x)y = q(x)$

Definition 1.2.5. An Initial Value Problem (IVP) of an n -th order differential equation is the differential equation together with an initial condition of y and the first $n - 1$ derivatives specified at a single point. Example:

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0.$$

Definition 1.2.6. A Boundary Value Problem (BVP): is a differential equation together with conditions specified at two or more points. Example:

$$\frac{d^2y}{dx^2} + y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

Definition 1.2.7. A solution of a differential equation is any function that satisfies the differential equation and its initial conditions on some open interval.

Note that solutions should be accompanied by intervals, since they can provide important information about the solution.⁷ The solution is general if no initial condition is provided.

⁷ If y is a solution to the differential equation, differentiate to obtain y' , then substitute y and y' into the differential equation to verify.

Example 1.2.8.

$$\frac{dy}{dx} = 2x \quad \Rightarrow \quad y = x^2 + C$$

is the general solution. If $y(0) = 3$, then $y = x^2 + 3$ is the particular solution.

Exercise 1.2.1

Determine whether the given function is a solution of the differential equation.

- a) Determine whether $y = e^{2x} + 1$ is a solution of

$$\frac{dy}{dx} - 2y = -2.$$

b) Determine whether $y = \sin(x) + \cos(x)$ is a solution of

$$\frac{d^2y}{dx^2} + y = 0.$$

c) Determine whether $y = x^2$ is a solution of

$$\frac{dy}{dx} - 2x = 0.$$

d) Determine whether $y = e^{2x}$ is a solution of

$$\frac{dy}{dx} - 2y = 0.$$

e) Is $x = e^{4t}$ a solution to

$$x''' - 12x'' + 48x' - 64x = 0?$$

f) Is $y = \sin t$ a solution to

$$\left(\frac{dy}{dt}\right)^2 = 1 - y^2?$$

g) Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.

h) Determine whether $y = x^{-\frac{3}{2}}$, $y = x^{-\frac{1}{2}}$, $y = -9x^{-\frac{3}{2}}$, and $y = 7x^{-\frac{1}{2}} - 9x^{-\frac{3}{2}}$ are solutions of the differential equation $4x^2y'' + 12xy' + 3y = 0$, for $x > 0$.

Historical Background

The study of ODEs is closely tied to the development of calculus and mathematical physics:

- *17th Century (Foundations)*: The birth of differential equations coincides with the invention of calculus by *Isaac Newton* and *Gottfried Wilhelm Leibniz* in the late 1600s. Newton formulated equations of motion, such as

$$F = ma \quad \Rightarrow \quad m \frac{d^2x}{dt^2} = F(x, t),$$

which is a second-order ODE.

The Bernoulli brothers also developed methods to solve differential equations to extend the range of their applications

- *18th Century (Development)*: Mathematicians like *Leonhard Euler*, a student of Johann Bernoulli and *Joseph-Louis Lagrange* developed systematic methods for solving first- and second-order ODEs, as well as studying oscillations, planetary motion, and mechanical systems.
- *19th Century (Theory)*: With contributions from *Augustin-Louis Cauchy*, *Sophie Kovalevskaya*, and others, the rigorous theory of existence and uniqueness of solutions was established. The study of linear differential equations and series solutions flourished.

- *20th Century (Modern Applications):* Differential equations became essential in physics, engineering, biology, and economics. The qualitative theory of dynamical systems (initiated by *Henri Poincaré*) emphasized studying the behavior of solutions without necessarily finding exact formulas.
- *Today:* ODEs remain central in mathematical modeling across disciplines: population dynamics, control systems, epidemiology, neuroscience, climate modeling, finance, etc.

Ordinary differential equations provide the mathematical language for describing change. From Newton's laws of motion to modern systems biology, ODEs are the foundation of mathematical modeling in the natural and applied sciences.

Section 1.3: Mathematical modelling with Differential Equations

Example 1.3.1 (Newton's Law of Cooling). *Newton's law of cooling states that the rate of change of temperature $T(t)$ of an object is proportional to the difference between the object's temperature and the ambient temperature T_a .*

$$\frac{dT}{dt} = -k(T - T_a), \quad k > 0.$$

Solution: This is a first-order linear ODE. We solve it using separation of variables:

$$\frac{dT}{T - T_a} = -k dt.$$

Integrating both sides:

$$\ln |T - T_a| = -kt + C.$$

Exponentiating:

$$T - T_a = Ce^{-kt}.$$

Hence the general solution is

$$T(t) = T_a + (T_0 - T_a)e^{-kt},$$

where $T_0 = T(0)$ is the initial temperature.

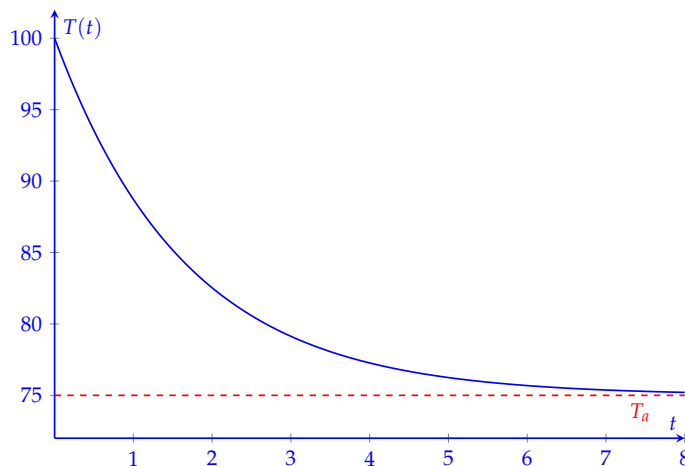


Figure 1.1: Exponential decay of temperature in Newton's Law of Cooling.

Example 1.3.2 (Simple Harmonic Oscillator). *The motion of a mass attached to a spring (without damping) is governed by*

$$m \frac{d^2x}{dt^2} + kx = 0,$$

where m is the mass and k is the spring constant.

Solution: Dividing through by m gives

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \quad \text{where } \omega = \sqrt{\frac{k}{m}}.$$

This is a second-order linear homogeneous ODE with constant coefficients. The characteristic equation is

$$r^2 + \omega^2 = 0 \quad \Rightarrow \quad r = \pm i\omega.$$

Hence, the general solution is

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t),$$

which describes oscillatory motion with angular frequency ω .

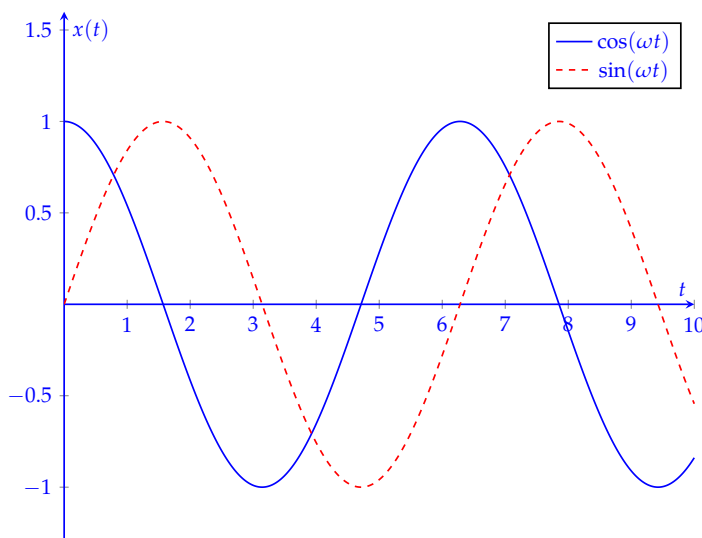


Figure 1.2: Solutions of the simple harmonic oscillator.

Example 1.3.3 (Logistic Population Growth). *A simple model for population growth is given by the logistic equation:*

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right),$$

where r is the intrinsic growth rate and K is the carrying capacity.

Solution: Separating variables:

$$\frac{dP}{P(1 - P/K)} = r dt.$$

Using partial fractions:

$$\frac{1}{P(1 - P/K)} = \frac{1}{P} + \frac{1}{K - P} \cdot \frac{1}{K}.$$

So,

$$\int \left(\frac{1}{P} + \frac{1}{K - P} \right) dP = r \int dt.$$

This gives

$$\ln |P| - \ln |K - P| = rt + C.$$

Simplifying:

$$\ln \left| \frac{P}{K - P} \right| = rt + C.$$

$$\Rightarrow P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1 \right) e^{-rt}},$$

where $P_0 = P(0)$.

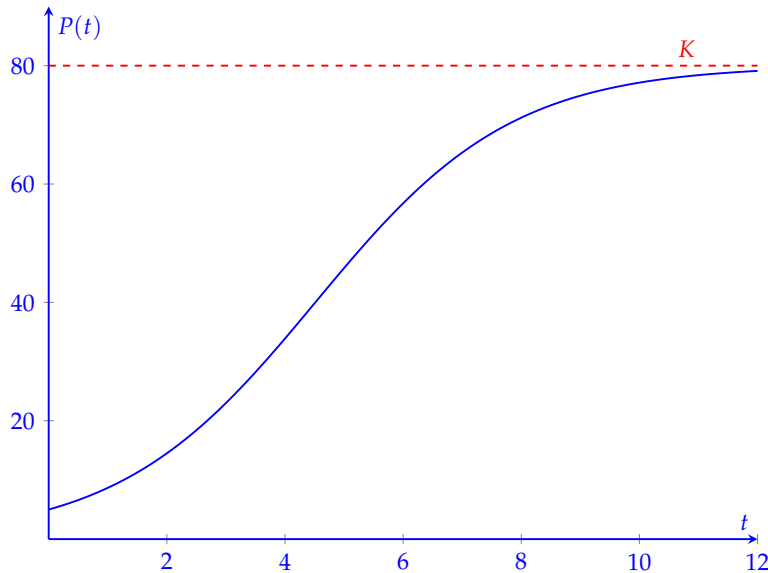


Figure 1.3: Logistic growth approaching the carrying capacity $K = 80$.

Section 1.4: Slope (Direction) Fields

Definition 1.4.1. A *slope field* is a graphical representation of a first-order differential equation

$$\frac{dy}{dx} = f(x, y).$$

At each point (x, y) in the plane, a small line segment is drawn with slope $f(x, y)$.

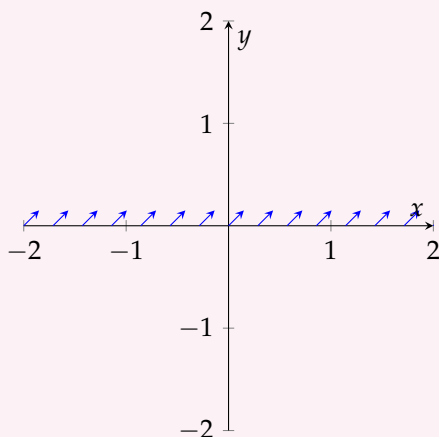
Definition 1.4.2. The graph of a solution of a differential equation is called a *solution curve*. A curve C is called an *integral curve* of a differential if every solution curve is a part of it.

- The slope field provides a qualitative view of solutions without explicitly solving the ODE.
- Solution curves are tangent everywhere to the slope field.

Example 1.4.3 (Constant Slope Field). Consider

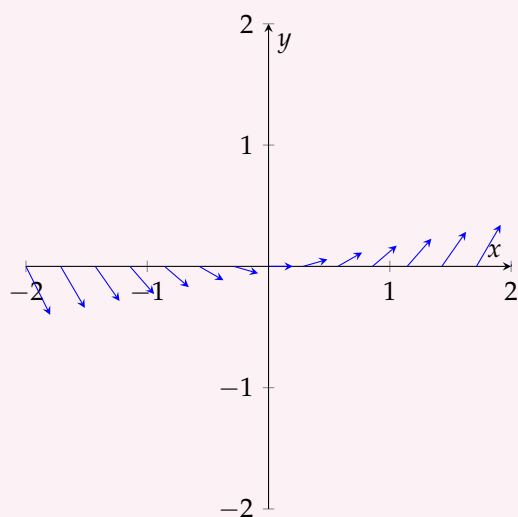
$$\frac{dy}{dx} = 1.$$

Every line segment has slope 1 (a 45° line). Solution curves are lines of the form $y = x + C$.



Example 1.4.4 (Non-constant Slope Field). We visualize the slope field for

$$\frac{dy}{dx} = x.$$



The general solution is:

$$y = \frac{x^2}{2} + C$$

which agrees with the slope field.

Example 1.4.5 (Logistic Growth). Consider the logistic ODE:

$$\frac{dy}{dx} = y(1 - y).$$

- When $0 < y < 1$, slopes are positive (growth).
- When $y > 1$, slopes are negative (decay).
- $y = 0$ and $y = 1$ are equilibrium solutions.

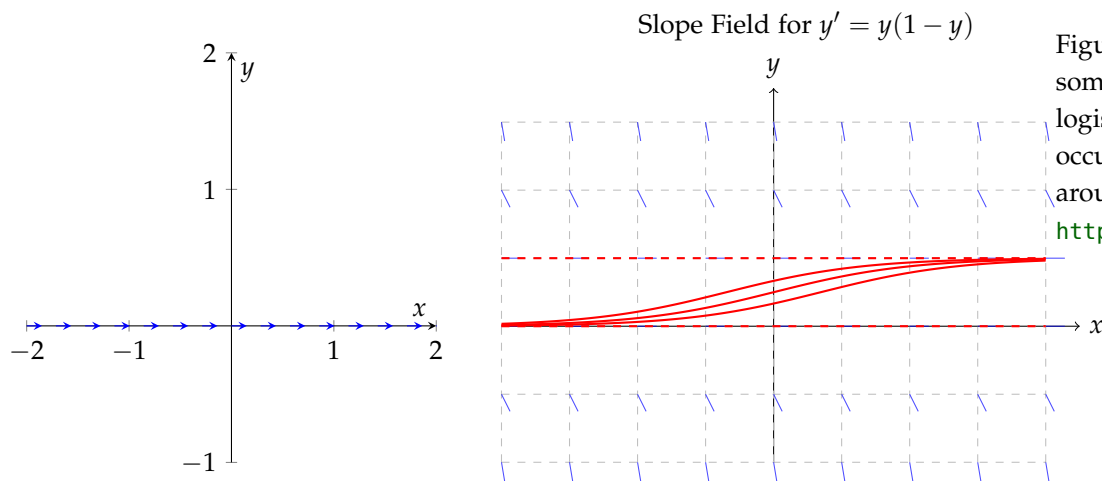


Figure 1.4: Slope field and some solution curves for the logistic equation. Equilibria occur at $y = 0$ and $y = 1$. Play around with the sage code here <https://shorturl.at/K9f7l>

Example 1.4.6. For the differential equation,

$$\frac{dy}{dx} = \frac{x}{y},$$

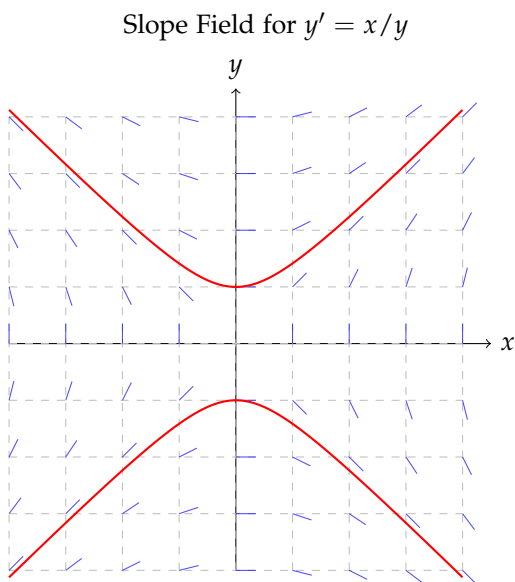


Figure 1.5: An *integral curve* is more general term for the graph of a solution, including those that are not functions. For example, the full hyperbola $y^2 - x^2 = K$ is an integral curve, but the upper branch ($y = \sqrt{x^2 + K}$) and lower branch ($y = -\sqrt{x^2 + K}$) are the individual solution curves.

- Slope fields help visualize the qualitative behavior of solutions.
- Equilibrium solutions correspond to horizontal slopes ($f(x, y) = 0$).
- They are especially useful when an ODE cannot be solved analytically.

Practice Problems

Exercise 1.4.1

Give definitions of the following terms and/or answer the questions⁸:

- Define an equilibrium of a ODE.
- Explain the procedure for calculating equilibria.

⁸ research it if we did not discuss it in class.

(c) **Explain** the difference between a *general* and *particular* solution.

Exercise 1.4.2

Show that out of three functions Provided two are solutions of the ODE

$$x^2 y''(x) - x y'(x) + y(x) = 0, \text{ and the third is not.}$$

(a) $y_1(x) = \ln(x)$ (b) $y_2(x) = x \ln(x)$ (c) $y_3(x) = x - x \ln(x)$

Summary

1. A differential equation relates a function to its derivatives.
2. The order of a differential equation is the highest derivative present.
3. A solution of a differential equation is a function that satisfies the equation.
4. Initial value problems specify conditions at a single point, while boundary value problems specify conditions at multiple points.
5. Mathematical modeling uses differential equations to describe real-world phenomena.
6. Historical development of differential equations is closely tied to calculus and physics.
7. Slope fields visualize solutions qualitatively.

Chapter 2: First Order Ordinary Differential Equations (ODEs)

Section 2.1: Introduction

As mentioned in the previous chapter, the order is determined by the highest derivative appearing in the equation. For a first order differential equation, only y' appears.

A first-order ordinary differential equation is an equation involving an unknown function $y(x)$ and its first derivative $y'(x)$. The general form is:

$$y' = f(x, y).$$

We will explore several common types of first-order ODEs and the methods used to solve them.

First-order differential equations can be classified into several types:

Type	General Form	Method
Separable	$\frac{dy}{dx} = g(x)h(y)$	Rewrite as $\frac{dy}{h(y)} = g(x) dx$ and integrate both sides.
Linear	$y' + P(x)y = Q(x)$	Use an integrating factor $\mu(x) = e^{\int P(x) dx}$. Multiply the equation by $\mu(x)$, then integrate.
Exact	$M(x, y) dx + N(x, y) dy = 0$	Check if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If so, find a potential function $\Phi(x, y)$ such that $d\Phi = M dx + N dy$. The solution is $\Phi(x, y) = C$.
Special Forms		
Bernoulli	$y' + P(x)y = Q(x)y^n$	Substitute $u = y^{1-n}$ to convert it to a linear equation.
Homogeneous	$y' = f(y/x)$	Substitute $v = y/x$ to convert it to a separable equation.

Section 2.2: Existence and Uniqueness

Before we proceed, we need to establish the conditions under which a differential equation will have a solution.

Theorem 2.2.1 (Existence and Uniqueness for 1st Order Nonlinear Equations (Picard–Lindelöf)). Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : \alpha < x < \beta, \delta < y < \gamma$ containing (x_0, y_0) . Then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

has a unique solution in some interval around x_0 .

Theorem 2.2.2 (Existence and Uniqueness for 1st Order Linear Equations). *If the functions P and Q are continuous on an open interval $I : \alpha < x < \beta$ containing the point $x = x_0$, then there exists a unique function $y = \phi(x)$ that satisfies the differential equation*

$$y' + P(x)y = Q(x) \quad (2.1)$$

for each $x \in I$, and that also satisfies the initial condition $y(x_0) = y_0$.

Section 2.3: Separable Equations

A first-order differential equation is **separable** if it can be written in the form:

$$\frac{dy}{dx} = g(x)h(y)$$

where $g(x)$ is a function of x alone and $h(y)$ is a function of y alone.

Solution Method

1. Separate the variables by rearranging the equation:

$$\frac{1}{h(y)} dy = g(x) dx$$

2. Integrate both sides of the equation:

$$\int \frac{1}{h(y)} dy = \int g(x) dx + C$$

3. Solve the resulting algebraic expression for y to get the general solution.

Example 2.3.1. *Solve:*

$$y' = xy.$$

Solution:

$$\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x dx.$$

Integrate:

$$\ln |y| = \frac{1}{2}x^2 + C \Rightarrow y(x) = Ce^{x^2/2}.$$

Example 2.3.2. *Solve $y' = \frac{x^2}{y^2}$.*

Solution:

$$y^2 dy = x^2 dx \Rightarrow \int y^2 dy = \int x^2 dx$$

Integrate both sides

$$\frac{y^3}{3} = \frac{x^3}{3} + C \Rightarrow y^3 = x^3 + 3C$$

Since $3C$ is an arbitrary constant, we can write $y^3 = x^3 + K$.

$$y = \sqrt[3]{x^3 + K}$$

We should note that this solution method for certain kinds of differential equations may lead to an Integral-defined function (i.e the integral cannot be written in terms of any elementary function).

Practice Problems

Exercise 2.3.1

Solve the following separable ODEs:

a) $(2y)e^{y^2}y' = 1$

f) $y' = \frac{e^{-y}}{y}$

k) $y' = \frac{3x+5}{(x+1)(x-2)}$

b) $y' = \frac{\cos(3x)}{\sin^2(3x)}$

g) $y' = \frac{1}{y^2 \ln(y)}$

l) $y' = \frac{2x}{x^2 + 3x + 2}$

c) $y' = \frac{1}{1+x^2}$

h) $y' = e^x \cos(x)$

m) $y' = \frac{x^2+1}{x(x^2+1)}$

d) $y' = \frac{2x}{x^2+4}$

i) $y' = -\ln(y)$

e) $y' = \frac{1}{\sqrt{1-x^2}}$

j) $y' = \frac{1}{x^2-1}$

n) $y' = \frac{1}{(x+1)^2(x-2)}$

Exercise 2.3.2

Solve the following separable ODE

a) $y' = xe^{x^2}$

b) $y' = \frac{\ln(x)}{x^2}$

c) $y' = \sqrt{4-y^2}$

d) $y' = \frac{\arctan(x)}{1+x^2}$

Exercise 2.3.3

Solve the following separable ODEs:

a) $y' = y^2 \sin(x)$

e) $y' = (y-1)(y-3)x^2$

i) $y' = (y-1)(y+3) \sin(x)$

b) $y' = y^3 x$

f) $y' = (y+1)(y-2)xe^x$

j) $y' = (y+4)(y-2)x^3$

c) $y' = (y^2-1)x$

g) $y' = (y^2-4)xe^{x^2}$

k) $y' = (y-5)(y+1)e^{-x}$

d) $y' = (y+2)(y-1)$

h) $y' = (y^2-9) \cos(x)$

l) $y' = (y^2-16)x^4$

Exercise 2.3.4

Identify the separable ODEs out of those given below. Demonstrate that it can be written in the standard form $y'(x) = F(x)G(y)$:

(a) $y' + xy - x = 0$

(b) $y' - 3(y^2 - 5y - 6) = 0$

Exercise 2.3.5Solve $\frac{dy}{dx} = y^2 - 4$.**Exercise 2.3.6**Find the implicit formula $F(x, y) = \text{const.}$ for the **solution** of the following separable ODE:

$$y'(x) = (y^2 - 6y + 8)xe^x$$

Exercise 2.3.7Find the solution to the separable ODE $y'(x) = y(3 - y)$.**Exercise 2.3.8**

Consider a population of bacteria that grows at a rate proportional to the size of the population.

- (a) Determine a ODE to describe the growth of bacteria at any time t .
- (b) Sketch the direction field, and make a prediction of what the solution is based on the sketch.
- (c) Determine the solution analytically.
- (d) The number of bacteria in a colony at 10:00am is approximately five million. At noon the number is approximately seven million. How many bacteria were in the colony at 9:00am?

Exercise 2.3.9Determine the equilibrium solutions for each ODE and decide if they are stable (an attractor), unstable (a repeller) or semistable¹. Draw a 1-D phase portrait for each.¹ Do some research if you need to.

(a) $y' = (y - 2)(y + 1)(y + 2)$

(c) $y' = y^3 + 2y^2 + y$

(b) $y' = y^3 - 9y$

(d) $y' = -y - y^3$

Exercise 2.3.10A model for the population $P(t)$ in a suburb of a large city is given by the Initial Value Problem (IVP)²

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P), \quad P(0) = 5000,$$

² IVPs are ODEs with additional condition that the solution or its derivatives has to take certain values at a specific point.where t is measured in months. What is the limiting value of the population³? At what time will the population be equal to one-half of this limiting value?³ Hint: recall equilibrium solutions**Exercise 2.3.11**

Solve these IVPs:

(a) $e^x \frac{dy}{dx} = xy^2, y(0) = 2$

(b) $\frac{dy}{dx} = y^2 + 1, y(0) = 3$

Exercise 2.3.12

Find the 1-parameter family of solutions for each ODE:

(a) $xy' + 2y = e^{-x}$

(b) $x \frac{dy}{dx} = 2y + x^4 \sin(x^2)$

(c) $\frac{dy}{dx} + xy = xy^3$

(d) $(1 + x^2)y' + 4xy = \frac{4x}{(1 + x^2)^3}$

Exercise 2.3.13Solve $y' = e^{-x^2}$, $y(3) = 5$.See how to solve here⁴⁴ <https://math.stackexchange.com/questions/154968/is-there-really-no-way-to-integrate-e>**Section 2.4: Linear Equations**Linear⁵ ODEs are those that can be written in the form

$$\underbrace{a_0(x)y(x) + a_1(x)\frac{dy}{dx} + a_2(x)\frac{d^2y}{dx^2} + \dots + a_n(x)\frac{d^ny}{dx^n}}_{\text{Linear combination of derivatives}} = g(x)$$

are called **linear** ODEs. Notice: the **only way** the **dependent** variable appears in the equation is in expressions of the sort⁶

$$\dots + a_k(x)y^{(k)}(x) + \dots,$$

that is y or its derivatives are never squared, multiplied with each other, or appear in more complicated formulas. The **coefficient functions** $a_k(x)$ may be very complicated themselves, but they never involve y or its derivatives.⁷The **standard form** of a linear ODE is obtained by dividing through by the highest coefficient, to result in expression that has a constant ($= 1$) as its highest coefficient:

$$\underbrace{P_0(x)y(x) + P_1(x)\frac{dy}{dx} + P_2(x)\frac{d^2y}{dx^2} + \dots + 1 \cdot \frac{d^ny}{dx^n}}_{\text{Linear combination of derivatives}} = \underbrace{Q(x)}_{\text{Input or Inhomogeneity}}.$$

The **standard form** is the starting point for many solution algorithms.The standard⁸ form for **1st order linear ODEs** is:

$$P(x)y(x) + y'(x) = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions of x .Here are several examples of 1st order linear ODEs. Rearrange them into a standard form and identify their coefficient function $P(x)$ and input function $Q(x)$ for each of them.

(a) $y'(x) = -0.5y(x) + 8$

(b) $4y'(x) = -2y(x) + 32 + 4 \sin(x)$

(c) $x^2y'(x) + 0.5y(x) = x^3$

(d) $y'(x) = 3y(x)$

Exercise 2.4.1

Which of the following differential equations are linear?

(a) $y' + 2xy = 0$

(b) $y' = y + 5$

(c) $y' + y = x$

Is any of them separable?

⁵ The name comes from the **linear combination**, which is a more-general expression that appears in various parts of mathematics.⁶ $y^{(k)}$ denotes the k -th derivative. So $y^{(2)}(x) = y''(x)$.⁷ For more on linear equations, see the beginning of Zill §2.3.⁸ If you want to show that an ODE is **linear**, rearrange the terms to match the standard form. If you can find such functions $P(x)$ and $Q(x)$ that **do not depend on** y, y' then you are looking at a linear ODE.

Solution Method using Variation of parameters

To solve 1st order linear ODEs, we will use the **variation of parameters** method:⁹

Step (1) Convert a linear ODE into its standard form $y' + P(x)y = Q(x)$.

Step (2) Create the homogeneous subproblem by zeroing out the input function:

$$y'_H + P(x)y_H = 0$$

Step (3) Find the general solution of the homogeneous subproblem.¹⁰

Step (4) To form a **candidate solution** y_I for the inhomogeneous problem

$$y'_I + P(x)y_I = Q(x), \quad (4.1)$$

replace the integration parameter C in y_H by an unknown function $I(x)$.¹¹

Step (5) **Substitute the candidate**¹² y_I into the **full (original)** ODE. You will need to take its derivative first, which will contain the unknown functions $I(x)$ and $I'(x)$. Simplify until you get an expression $I'(x) = \dots$

$I(x)$ will not appear on the right hand side (and neither will y).

Step (6) Integrate $I'(x)$ to calculate $I(x)$ function¹³, which then gives you formula for $y_I(x)$.

Step (7) Now you have two functions: y_H and y_I . y_H contains a constant C , while y_I has no free constant. The general solution to the original problem is:

$$y(x) = y_H(x) + y_I(x)$$

⁹ Textbook explains the **integrating factor** technique instead. The two methods are different, but yield equivalent results. Through variation of parameters we will learn skills that will transfer over to higher-order ODEs.

¹⁰ For 1st order ODEs, this is always possible by separation of variables.

¹¹ Variation of constant $C \rightarrow$ function $I(x)$ is what gives the method its name.

¹² Anytime something is called "solution candidate" your instinct should be to "plug it into" the ODE.

¹³ The integration constant inside $I(x)$ is not important, so pick a constant that makes $I(x)$ simple. Say 0.

Example 2.4.1. Find general solution to

$$xy' - 4y = x^6 e^x$$

(this is the same problem as textbook §2.3.Ex.3). Additionally, find the particular solution that has $y(2) = 0$.

General solution:

Step (1) Standard form: $y' + \underbrace{\frac{-4}{x}}_{P(x)} y = \underbrace{x^5 e^x}_{f(x)}$

Step (2) Homogeneous subproblem:

$$y'_H + \frac{-4}{x} y_H = 0 \quad (4.2)$$

Step (3) Solving for y_H :

$$\begin{aligned} \frac{dy_H}{dx} &= \frac{4y_H}{x} && \text{(Separate variables)} \\ \frac{dy_H}{y_H} &= \frac{4dx}{x} \\ \int \frac{dy_H}{y_H} &= \int \frac{4dx}{x} \\ \ln|y_H| &= 4 \ln|x| + K && \text{(Exponentiate both sides)} \\ y_H(x) &= Cx^4 \end{aligned}$$

Step (4) Candidate solution for the inhomogeneous problem: $y_I(x) = I(x)x^4$

Step (5) Substitute $y_I = I(x)x^4$ into $y'_I + \frac{-4}{x}y_I = x^5e^x$

$$\begin{aligned}y'_I(x) &= I'(x)x^4 + 4I(x)x^3 \\I'(x)x^4 + 4I(x)x^3 + \frac{-4}{x}I(x)x^4 &= x^5e^x \\I'(x)x^4 &= x^5e^x \Rightarrow I'(x) = xe^x\end{aligned}$$

Step (6) Integrate $I'(x)$ (in this case, integration by parts):

$$\begin{aligned}I(x) &= xe^x - e^x + K \quad (\text{in this step only, constant is arbitrary}) \\I(x) &= xe^x - e^x \\y_I(x) &= I(x)x^4 = (xe^x - e^x)x^4\end{aligned}$$

Step (7) General solution:

$$y(x) = \underbrace{Cx^4}_{y_H(x)} + \underbrace{x^5e^x - x^4e^x}_{y_I(x)}$$

(Particular) solution to the IVP: Start with the general solution (terms collected):

$y(x) = x^4(C + xe^x - e^x)$, and plug into it the time-point $x = 2$:

$$y(2) = 16C + 32e^2 - 16e^2 = 16(C + e^2)$$

At the same time, Initial Value says $y(2) = 0$, therefore

$$16(C - e^2) = y(2) = 0 \Rightarrow C = e^2.$$

So the particular solution to the IVP is:

$$y(x) = x^4(e^2 + xe^x - e^x)$$

Example 2.4.2. Solve

$$y' - 2y = e^x.$$

Step (1) Standard form: $y' - 2y = e^x$, with $P(x) = -2$, $Q(x) = e^x$.

Step (2) Homogeneous subproblem: $y' - 2y = 0$.

Step (3) Homogeneous solution:

$$y_H(x) = Ce^{-\int -2dx} = Ce^{2x}.$$

Step (4) Candidate solution (Variation of parameters): let

$$y_I(x) = I(x)e^{2x}.$$

Step (5) Substitute into the full ODE:

$$I'(x)e^{2x} + 2I(x)e^{2x} - 2I(x)e^{2x} = e^xe^{2x} \Rightarrow I'(x) = e^{-x}.$$

Step (6) Integrate $I'(x)$:

$$I(x) = \int e^{-x} dx = -e^{-x}.$$

Thus,

$$y_I(x) = -e^{-x}e^{2x} = -e^x.$$

Step (7) General solution:

$$y(x) = Ce^{2x} - e^x.$$

Exercise 2.4.2

Find general solution to

$$\dot{y} = -\frac{1}{2}y + 8 + \sin(t)$$

Exercise 2.4.3

Solve:

$$y' + y = e^x.$$

Exercise 2.4.4

Solve

$$y' - y = e^{2x}.$$

Exercise 2.4.5

Solve the IVP:

$$y' + \frac{2}{x}y = \frac{\cos(x)}{x^2}, \quad y(\pi) = 0.$$

Solution Method using an Integrating Factor

The method of **integrating factor** is another way to solve 1st order linear ODEs. The goal of this method is to rewrite the differential equation in a separable form so that we can integrate both sides.

Multiplying the entire equation by a certain function $\mu(x)$ (the integrating factor), will allow us to rewrite the left-hand side as a single derivative of a product.

Step (1) Rewrite the linear ODE in its standard form: $y' + P(x)y = Q(x)$.

Step (2) Using $P(x)$ from the standard form, compute the **integrating factor**, $\mu(x)$:

$$\mu(x) = e^{\int P(x) dx}$$

Step (3) Multiply the entire differential equation by the integrating factor:

$$\mu(x) \frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

The left side will become the derivative of a product:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

To see why, apply the product rule to the left-hand side:

$$\frac{d}{dx} \left[\overbrace{\mu(x)y}^{\text{product}} \right] = \overbrace{\mu(x) \frac{dy}{dx} + \frac{d\mu}{dx} y}^{\text{product rule}} = \overbrace{\mu(x) \frac{dy}{dx} + \mu(x)P(x)y}^{\text{LHS of linear eq multiplied by } \mu(x)}.$$

This is only true if $\frac{d\mu}{dx} = \mu(x)P(x)$.

Thus, solving for $\mu(x)$ gives

$$\mu(x) = e^{\int P(x) dx}$$

Step (4) Integrate both sides with respect to x :

$$\mu(x)y = \int \mu(x)Q(x) dx + C$$

Step (5) Solve for y to find the general solution:

$$y = \frac{1}{\mu(x)} \left(\int \mu(x)Q(x) dx + C \right)$$

Example 2.4.3. Solve:

$$y' + y = e^x.$$

Solution: Here $P(x) = 1$, so

$$\mu(x) = e^{\int 1 dx} = e^x.$$

Multiply through:

$$e^x y' + e^x y = e^{2x}.$$

LHS is derivative:

$$\frac{d}{dx}(e^x y) = e^{2x}.$$

Integrate:

$$e^x y = \frac{1}{2}e^{2x} + C \Rightarrow y(x) = \frac{1}{2}e^x + Ce^{-x}.$$

Example 2.4.4. Solve

$$y' - y = e^{2x}.$$

Solution: Here, $P(x) = -1$ and $Q(x) = e^{2x}$. So integrating Factor,

$$\mu(x) = e^{\int -1 dx} = e^{-x}$$

Multiply by $\mu(x)$:

$$e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} e^{2x} = e^x \Rightarrow \frac{d}{dx}(e^{-x} y) = e^x$$

Integrating,

$$\int \frac{d}{dx}(e^{-x} y) dx = \int e^x dx \Rightarrow e^{-x} y = e^x + C$$

Thus,

$$y = e^x(e^x + C) = e^{2x} + Ce^x.$$

Practice Problems

Exercise 2.4.6

Each 1st order linear ODEs can be rewritten in the standard form with components

- input function;
- coefficient function;
- linear combination of derivatives.

- (a) Rewrite the following linear ODE in the standard form and identify the listed components:

$$t^2 \dot{x} + \sin(t)x + \cos(t) = 0.$$

(b) Write out the corresponding homogeneous subproblem.

Exercise 2.4.7

Solve

$$y' - 3y = 6$$

Exercise 2.4.8

Solve

$$xy' - 4y = x^6 e^x.$$

Exercise 2.4.9

Given a linear ODE.

$$t^2 \frac{dx}{dt} + x(t) - 3 = 0. \quad (4.3)$$

(a) Use *variation of parameters* technique to find the general solution to the ODE.

Do not use any formulas from memory or the textbook. Show every step of the process.

(b) Find the solution that additionally satisfies the Initial Value $x(1) = 1$.

Exercise 2.4.10

Find the **general** solution of the equation $\dot{x} + \sin(t)x = 3$.

Exercise 2.4.11

Solve the IVP $y' + \frac{2}{x}y = \frac{\cos(x)}{x^2}$, $y(\pi) = 0$.

Exercise 2.4.12

The equation below could be linear, separable, or exact. Identify its type and support your answer by either:

- putting the linear into its standard form,
- performing the exactness test,
- identifying the separated F and G components.

Do not erase attempts that are unsuccessful. Showing that an ODE is **not** of a certain class is still a useful result.

$$(y - \ln x) \cdot y'(x) = 1 + \ln x + \frac{y}{x}$$

Exercise 2.4.13

A tank initially contains 2,000 litres of water with a concentration of mercury of 6.0×10^{-5} grams per litre. Water that contains 3.0×10^{-5} grams per litre is pumped into the tank at 100 litres per hour. The well mixed solution is pumped out of the tank at 100 litres per hour. How long will it take for the concentration in the tank to reach 4.5×10^{-5} grams per litre?

Exercise 2.4.14

A vat with capacity of 500 gallons initially has 10 lbs. of salt dissolved in 100 gallons of brine. A brine solution with 0.5 lbs. of salt per gallon is pumped into the tank at a rate of 4 gallons per minute, and the well mixed solution is pumped out at a rate of 2 gallons per minute. How much salt will be in the vat when the vat is full? (Hint: first find the solution for the volume of the fluid in the tank, before tackling the amount of salt in it.)

Exercise 2.4.15

A car's fuel tank holds 15 gallons of gasoline. When the tank is full, the gasoline contains 0.015 gallons of ethanol. Suppose that pure gasoline is pumped into the tank at a rate of 1 gallon per minute, and the well mixed solution is pumped out at a rate of 1 gallon per minute. How long will it take for the amount of ethanol in the tank to be 0.003 gallons?

Exercise 2.4.16

Solve the IVP $y' + y \tan(x) = \sin(2x)$, $y(0) = 1$.

Exercise 2.4.17

Solve the linear ODE $y' + y \cot(x) = 2x \csc(x)$.

Exercise 2.4.18

Solve the ODE, $y' + y \ln(x^2) = 1$.

Exercise 2.4.19

Solve the following linear IVPs:

(a) $(t+1)\dot{x}(t) + x(t) = \ln t$, $x(1) = 10$

(b) $\dot{P} + 2tP = P + 4t - 2$, $P(1) = 1$

Section 2.5: Exact Equations

Definition 2.5.1. A differential expression written in the form $M(x, y) dx + N(x, y) dy$ is an **exact differential** in a region R of the xy -plane if it corresponds to the differential of some function $F(x, y)$, that is defined in R . A first order differential equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$

is exact if the following condition holds in R :

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Theorem 2.5.2. If $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives in a simply connected region R of the xy -plane, then the differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

for all points (x, y) in R .

Solution Method

Step (1) Write the differential equation in the form $M(x, y) dx + N(x, y) dy = 0$:

Step (2) Check the exactness condition: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Step (3) If exact, find a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = M(x, y)$ and $\frac{\partial F}{\partial y} = N(x, y)$.

Step (4) Integrate $M(x, y)$ with respect to x , treating y as a constant:

$$F(x, y) = \int M(x, y) dx + g(y)$$

The function $g(y)$ is a "constant" of integration that depends on y .

Step (5) Differentiate this expression for $F(x, y)$ with respect to y and set it equal to $N(x, y)$:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left(\int M(x, y) dx \right) + g'(y) = N(x, y)$$

Step (6) Solve for $g'(y)$ and integrate to find $g(y)$.

Step (7) Substitute $g(y)$ back into the expression for $F(x, y)$. The general solution is $F(x, y) = C$.

Example 2.5.3. Solve $(2x + y) dx + (x + 2y) dy = 0$.

Solution:

Step (1) Write the differential equation in the form $M(x, y) dx + N(x, y) dy = 0$:

$$(2x + y) dx + (x + 2y) dy = 0$$

Step (2) Check Exactness: $M(x, y) = 2x + y$ and $N(x, y) = x + 2y$. $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 1$. The equation is exact.

Step (3) Integrate M w.r.t x :

$$F(x, y) = \int (2x + y) dx + g(y) = x^2 + xy + g(y)$$

Step (4) Differentiate w.r.t y and compare to N :

$$\frac{\partial F}{\partial y} = x + g'(y)$$

We set this equal to $N(x, y) = x + 2y$:

$$x + g'(y) = x + 2y \implies g'(y) = 2y$$

Step (5) Integrate to find $g(y)$:

$$g(y) = \int 2y dy = y^2$$

Step (6) General Solution: $F(x, y) = x^2 + xy + y^2 = C$.

Example 2.5.4. Solve:

$$(2xy + 3) dx + (x^2 + 4y) dy = 0.$$

Check exactness:

$$M = 2xy + 3, \quad N = x^2 + 4y,$$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x \implies \text{Exact}.$$

Find potential $\Phi(x, y)$:

$$\frac{\partial \Phi}{\partial x} = M = 2xy + 3 \implies \Phi(x, y) = x^2y + 3x + g(y).$$

Differentiate wrt y :

$$\frac{\partial \Phi}{\partial y} = x^2 + g'(y) = N = x^2 + 4y \implies g'(y) = 4y.$$

So $g(y) = 2y^2$. Solution:

$$\Phi(x, y) = x^2y + 3x + 2y^2 = C.$$

Integration Factor for Non-Exact Equations

Recall that the left-hand side of a linear ODE can be made into a product derivative by multiplying it by an integrating factor $\mu(x)$. Similarly, a non-exact equation can sometimes be made exact by multiplying it by an appropriate integrating factor $\mu(x, y)$.

Sometimes it is possible to find an integrating factor $\mu(x, y)$, a function of only one variable (either x or y) such that

$$\mu(x, y)M(x, y) dx + \mu(x, y)N(x, y) dy = 0$$

is exact.

In this case, the exactness condition becomes:

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N].$$

By the product rule, this expands to:

$$\mu M_y + M\mu_y = \mu N_x + N\mu_x.$$

Rearranging gives:

$$N\mu_x - M\mu_y = \mu(M_y - N_x). \quad (5.1)$$

If μ is a function of x only, then $\mu_y = 0$ and (5.1) becomes:

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}.$$

If the right-hand side is a function of x only, then we can integrate to find $\mu(x)$:

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$

Example 2.5.5. The following ODE is not exact. Find an integrating factor that makes it exact, and then solve it.

$$xy \, dx + (x^2 - y^2) \, dy = 0.$$

Solution:

Step (1) Write in standard form:

$$xy \, dx + (x^2 - y^2) \, dy = 0.$$

Step (2) Check exactness:

$$\begin{aligned} M &= xy, & N &= x^2 - y^2, \\ \frac{\partial M}{\partial y} &= x, & \frac{\partial N}{\partial x} &= 2x \Rightarrow \text{Not exact.} \end{aligned}$$

Step (3) Find integrating factor:

$$\frac{M_y - N_x}{N} = \frac{x - 2x}{x^2 - y^2} = \frac{-x}{x^2 - y^2}.$$

This is not a function of x only, but it is a function of y only if we divide by x :

$$\frac{M_y - N_x}{-M} = \frac{x - 2x}{-xy} = \frac{1}{y}.$$

So we can take $\mu(y) = e^{\int \frac{1}{y} dy} = e^{\ln|y|} = |y|$.

Step (4) Multiply the original equation by $\mu = y$:

$$y(xy \, dx + (x^2 - y^2) \, dy) = 0 \Rightarrow (xy^2) \, dx + (x^2y - y^3) \, dy = 0.$$

Step (5) Check exactness:

$$\begin{aligned} M &= xy^2, & N &= x^2y - y^3, \\ \frac{\partial M}{\partial y} &= 2xy, & \frac{\partial N}{\partial x} &= 2xy \Rightarrow \text{Exact.} \end{aligned}$$

Step (6) Find potential function:

$$\frac{\partial F}{\partial x} = M = xy^2 \Rightarrow F(x, y) = \frac{1}{2}x^2y^2 + g(y).$$

$$\frac{\partial F}{\partial y} = x^2y + g'(y) = N = x^2y - y^3 \Rightarrow g'(y) = -y^3.$$

$$g(y) = -\frac{1}{4}y^4.$$

Step (7) General solution:

$$F(x, y) = \frac{1}{2}x^2y^2 - \frac{1}{4}y^4 = C.$$

Exercise 2.5.1

The following ODE is not exact. Find an integrating factor that makes it exact, and then solve it.

$$xy \, dx + (2x^2 + 3y^2 - 20) \, dy = 0.$$

Practice Problems

Exercise 2.5.2

Determine whether the following ODE is exact. If it is, find its general solution.

$$(3x^2y - 6x) \, dx + (x^3 + 2y) \, dy = 0.$$

Exercise 2.5.3

Determine whether the following ODE is exact. If it is, find its general solution.

$$y'(x) = \frac{3x + 4y}{x}.$$

Exercise 2.5.4

Solve the following IVPs:

1. $(y + x)^2 \, dx + (2xy + x^2 - 1) \, dy = 0, \quad y(1) = 1.$

2. $(e^x + y) \, dx + (x + \cos(y)) \, dy = 0, \quad y(0) = 0.$

3. $(e^x + 2y) \, dx + (2 + x + ye^y) \, dy = 0, \quad y(0) = 1.$

4.

$$\left(\frac{3y^2 - t^2}{y^5} \right) \frac{dy}{dt} + \frac{t}{2y^4} = 0, \quad y(1) = 1.$$

Exercise 2.5.5

Determine whether the following ODE is exact. If it is, find its general solution.

$$(2xy + 3) \, dx + (x^2 + 4y) \, dy = 0.$$

Exercise 2.5.6

Find the solution and interval of validity for the IVP:

$$\frac{2ty}{t^2 + 1} - 2t - (2 - \ln(t^2 + 1))y' = 0, \quad y(5) = 0.$$

Exercise 2.5.7

Determine whether the following ODE is exact. If it is, find its general solution:

$$y'(x) = \frac{xy^2 - \cos(x)\sin(x)}{y(1 - x^2)}$$

Section 2.6: Substitution

Many first-order ODEs can be solved by making an appropriate substitution that transforms the equation into a separable or linear form. Two common substitution methods are for Homogeneous equations and Bernoulli equations.

Homogeneous Equations

Definition 2.6.1. A function $F(x, y)$ is **homogeneous of degree** $n \in \mathbb{R}$ if for all t :

$$F(tx, ty) = t^n F(x, y)$$

For example, $F(x, y) = x^3 + y^3$ is a homogeneous function of degree 3 because

$$F(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 F(x, y).$$

whereas $F(x, y) = x + y^3$ is not homogeneous because

$$F(tx, ty) = tx + (ty)^3 = tx + t^3 y^3 \neq t^n F(x, y)$$

for any n .

Definition 2.6.2. A first-order ODE of the form

$$M(x, y) dx + N(x, y) dy = 0 \quad \text{or} \quad \frac{dy}{dx} = F(x, y) = -\frac{M(x, y)}{N(x, y)} \quad (6.1)$$

is **homogeneous of degree** n if both $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree n .

So we can perform the substitution $v = \frac{y}{x}$ to transform the equation into a separable form. Since $M(x, y)$ and $N(x, y)$ are homogeneous of degree n , we have:

$$M(x, y) = x^n M(1, v) \quad \text{and} \quad N(x, y) = x^n N(1, v).$$

And equation (6.1) becomes:

$$x^n M(1, v) dx + x^n N(1, v) dy = 0 \quad \Rightarrow \quad M(1, v) dx + N(1, v) dy = 0 \quad (\text{for } n \text{ degree}) \quad (6.2)$$

where $v = \frac{y}{x} \rightarrow y = vx \Rightarrow dy = v dx + x dv$ which reduces equation (??) to a separable form in v and x

$$\frac{dx}{x} = -\frac{N(1, u)}{M(1, u) + uN(1, u)} du.$$

Solution Method for Homogeneous Equations

Step (1) Make the substitution $v = \frac{y}{x}$, which implies $y = vx$.

Step (2) Differentiate $y = vx$ using the product rule to get $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

Step (3) Substitute v and the new expression for $\frac{dy}{dx}$ into the original equation. The result will be a separable equation in v and x .

Step (4) Solve the separable equation for v .

Step (5) Substitute back $v = \frac{y}{x}$ to get the solution in terms of x and y .

Example 2.6.3. Solve $\frac{dy}{dx} = \frac{x+y}{x}$.

Solution:

1. **Check Homogeneity:** Divide numerator and denominator by x :

$$\frac{dy}{dx} = \frac{1 + y/x}{1} = 1 + \frac{y}{x}$$

This is a homogeneous equation.

2. **Substitution:** Let $v = y/x$. Then $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

3. **New Equation:** $v + x \frac{dv}{dx} = 1 + v$. This simplifies to $x \frac{dv}{dx} = 1$, which is separable.

4. **Solve Separable Equation:** $dv = \frac{1}{x} dx$.

$$\int dv = \int \frac{1}{x} dx \Rightarrow v = \ln |x| + C$$

5. **Substitute back:** $\frac{y}{x} = \ln |x| + C$.

6. **General Solution:** $y = x(\ln |x| + C)$.

Example 2.6.4. Solve $\frac{dy}{dx} = \frac{x+y}{x}$.

Answer:

$$y = x \ln |x| + Cx$$

Example 2.6.5. Solve:

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Answer:

$$\frac{y}{x} = \ln |x| + C \Rightarrow y = x \ln |x| + Cx$$

Example 2.6.6. Solve:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

Answer:

$$\left(\frac{y}{x}\right)^2 = 2 \ln |x| + C$$

Bernoulli Equations

Definition 2.6.7. A first-order ODE of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is a **Bernoulli equation** if n is any real number other than 0 or 1.

The substitution $v = y^{1-n} \Rightarrow \frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}$ transforms the equation into a linear ODE in v .

Solution Method for Bernoulli Equations

Step (1) Identify the Bernoulli equation in the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$.

Step (2) Make the substitution $v = y^{1-n}$, which implies $\frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}$.

Step (3) Rearrange the original equation to express $\frac{dy}{dx}$ in terms of y , $P(x)$, and $Q(x)$.

Step (4) Substitute v and the new expression for $\frac{dy}{dx}$ into the rearranged equation. The result will be a linear equation in v and x .

Step (5) Solve the linear equation for v using an integrating factor if necessary.

Step (6) Substitute back $v = y^{1-n}$ to get the solution in terms of x and y .

Example 2.6.8. Solve $\frac{dy}{dx} + y = y^2$.

Solution:

Here $n = 2$, so $v = y^{-1}$. Compute $\frac{dv}{dx} = -y^{-2} \frac{dy}{dx}$. Original equation: $\frac{dy}{dx} = y^2 - y$.

Substitute:

$$\frac{dv}{dx} = -\left(1 - \frac{1}{y}\right) = -1 + v$$

So:

$$\frac{dv}{dx} - v = -1$$

This is linear. Integrating factor e^{-x} :

$$\frac{d}{dx}(ve^{-x}) = -e^{-x}$$

Integrate:

$$ve^{-x} = e^{-x} + C \implies v = 1 + Ce^x$$

Back-substitute $v = 1/y$:

$$y = \frac{1}{1 + Ce^x}$$

Example 2.6.9. Solve $\frac{dy}{dx} + \frac{2}{x}y = xy^3$.

Solution:

Here $n = 3$, so $v = y^{-2}$. Compute $\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$.

⋮

$$y = \frac{1}{\sqrt{x^2 + Cx^4}}$$

Example 2.6.10. Solve:

$$\frac{dy}{dx} + y = y^2$$

Solution:

Here $P(x) = 1$, $Q(x) = 1$, $n = 2$.

Substitute: $v = y^{1-2} = y^{-1}$.

\vdots

$$\frac{1}{y} = 1 + Ce^x \Rightarrow y = \frac{1}{1 + Ce^x}$$

Example 2.6.11. Solve:

$$\frac{dy}{dx} + \frac{2}{x}y = xy^3$$

Solution:

Here $P(x) = \frac{2}{x}$, $Q(x) = x$, $n = 3$.

Substitute: $v = y^{1-3} = y^{-2}$.

\vdots

$$y^{-2} = x^2 + Cx^4 \Rightarrow y = \frac{1}{\sqrt{x^2 + Cx^4}}$$

Practice Problems

Exercise 2.6.1

Solve $(x^2 + y^2) dx + (x^2 - 2xy) dy = 0$.

Exercise 2.6.2

Solve $(x + y) dx - (x - y) dy = 0$.

Exercise 2.6.3

Solve the following homogeneous ODEs:

1. $\frac{dy}{dx} = \frac{x+3y}{3x+y}$

2. $(y^2 + xy) dx + x^2 dy = 0$

3. $-y dx + (x + \sqrt{xy}) dy = 0$

4. $xy' = y + \sqrt{x^2 - y^2}, \quad x > 0$

Exercise 2.6.4

Solve the following Bernoulli equations:

1. $xy' + y = y^{-2}$

2. $y' - y = e^x y^2$

3. $y' = y(xy^3 - 1)$

4. $xy' - (1+x)y = xy^2$

Exercise 2.6.5

Solve $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$ using the homogeneous substitution.

Exercise 2.6.6

Solve $\frac{dy}{dx} + \frac{y}{x} = x^2y^3$ using the Bernoulli substitution.

Exercise 2.6.7

Solve $\frac{dy}{dx} = \frac{y^2 + xy}{x^2}$ using the homogeneous substitution.

Exercise 2.6.8

Solve:

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

Exercise 2.6.9

Solve:

$$\frac{dy}{dx} = \frac{y+x}{y-x}$$

Exercise 2.6.10

Solve:

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^3$$

Exercise 2.6.11

Solve:

$$\frac{dy}{dx} - 2y = 3y^2$$

Section 2.7: Qualitative Analysis of First-Order ODEs

A first-order ODE can be written in the form

$$\frac{dy}{dx} = f(x, y).$$

If f does not depend on x , then the equation is called **autonomous**:

$$\frac{dy}{dx} = f(y).$$

Autonomous equations are often used to model the evolution of a system over time, where x represents time and y represents the state of the system.

Equilibria and Stability

Definition 2.7.1. An *equilibrium* (or *fixed point*) of an autonomous ODE is a value $y = y_0$ such that $f(y_0) = 0$.

At an equilibrium, the state of the system does not change over time.

Definition 2.7.2. An equilibrium y_0 is **stable** if solutions that start close to y_0 remain close to y_0 as $x \rightarrow \infty$.

Example 2.7.3. Consider the autonomous ODE:

$$\frac{dy}{dx} = y(1 - y).$$

The equilibria are found by setting $f(y) = y(1 - y) = 0$, which gives $y_0 = 0$ and $y_0 = 1$.

If solutions that start close to y_0 move away from y_0 as $x \rightarrow \infty$, then the equilibrium is **unstable**.

Phase Portraits and Direction Fields

A **phase portrait** is a graphical representation of the trajectories of a dynamical system in the phase plane. Each point in the phase plane represents a state of the system, and the trajectories show how the state evolves over time. To construct a phase portrait for an autonomous ODE, we can use a **direction field** (or **slope field**). A direction field is a grid of arrows that indicate the slope of the solution curve at each point in the phase plane. The slope at a point (x, y) is given by $f(y)$. To sketch a direction field:

1. Choose a grid of points in the xy -plane.
2. At each point (x, y) , compute the slope $f(y)$
3. Draw a small line segment with the computed slope at each point.

Example 2.7.4. Consider the autonomous ODE:

$$\frac{dy}{dx} = y(1 - y).$$

The equilibria are $y_0 = 0$ and $y_0 = 1$. To sketch the direction field, we compute the slope at various points:

- For $y < 0$, $f(y) < 0$, so the slope is negative.
- For $0 < y < 1$, $f(y) > 0$, so the slope is positive.
- For $y > 1$, $f(y) < 0$, so the slope is negative.

The direction field shows that solutions starting below $y = 0$ move downward, solutions starting between $y = 0$ and $y = 1$ move upward, and solutions starting above $y = 1$ move downward. Thus, $y = 0$ is an unstable equilibrium and $y = 1$ is a stable equilibrium.

The direction field in Figure ?? illustrates the behavior of solutions near the equilibria $y = 0$ and $y = 1$.

The phase portrait in Figure ?? shows the stability of the equilibria. It is given in the diagram by arrows indicating the direction of movement of solutions near the equilibria.

Note: The stability of an equilibrium can often be determined by examining the

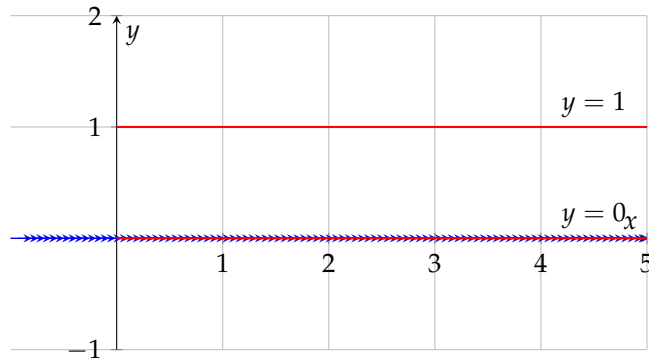


Figure 2.1: Direction field and equilibria for $\frac{dy}{dx} = y(1-y)$.

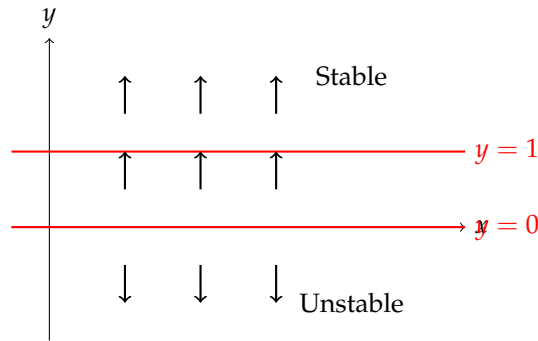


Figure 2.2: Phase portrait for $\frac{dy}{dx} = y(1-y)$.

sign of $f(y)$ near the equilibrium. If $f(y)$ changes from positive to negative as y increases through y_0 , then y_0 is stable. If $f(y)$ changes from negative to positive, then y_0 is unstable.

Section 2.8: A Numerical Method for ODEs

Euler's Method

Euler's method is a simple numerical technique for approximating the solution of an initial value problem (IVP) of the form:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

The idea is to use the slope of the solution at a known point to estimate the value of the solution at a nearby point. The method proceeds as follows:

1. Choose a step size h .
2. Compute the next point using the formula:

3.

$$y_{n+1} = y_n + hf(x_n, y_n), \quad x_{n+1} = x_n + h.$$

4. Repeat the process to compute subsequent points.

Example 2.8.1. Solve the IVP:

$$\frac{dy}{dx} = y - x^2 + 1, \quad y(0) = 0.5$$

1. Using Euler's method with step size $h = 0.2$ to approximate $y(1)$.
2. Using an analytical method to find the exact solution and compare it with the approximation.

Summary

- First-order ODEs appear in many forms; solution methods depend on type.
- Existence & uniqueness depend on continuity conditions.
- Separable \Rightarrow integrate directly.
- Linear \Rightarrow integrating factor.
- Exact \Rightarrow potential function.
- Substitution methods (homogeneous, Bernoulli) transform to solvable forms.
- Autonomous ODEs model time-evolving systems.
- Equilibria are found by setting $f(y) = 0$.
- Stability determined by sign of $f(y)$ near equilibria.
- Phase portraits visualize system behavior.
- Euler's method approximates solutions numerically.

Section 2.9: Practice for Test — First-Order ODEs

You can use the following step-by-step calculator to help you study: <https://goo.gl/sZEyVL>. Do not rely on it too much, though, as you will not have such help in an exam.

There is no homework due in this discussion. Instead, study from the previous weeks and additionally work through problems below.

Practice Problems

Exercise 2.9.1

For each equation below,

- Determine whether the given ODE is exact, linear, or separable by identifying the functions that demonstrate the type: M, N for exact, standard form for linear, F, G for separable and perform the appropriate tests (if needed).
- Find the general solution of equations using the appropriate technique.

(a) $(4x - 8y^3)y'(x) + (5x + 4y) = 0$

(b) $y' - e^{3x+2y} = 0$

(c) $(x - y^3 + y^2 \sin(x))y'(x) - 3xy^2 - 2y \cos(x) = 0$

Exercise 2.9.2

Find the value of k such that the given ODE is exact:

$$y^3 + kxy^4 - 2x + (3xy^2 + 20x^2y^3)y'(x) = 0$$

Exercise 2.9.3

Conceptual questions and definitions:

- What is the equilibrium of an ODE?
- What is the difference between general and particular solutions of an ODE?

- (c) What determines the angle of line segments used to plot direction fields?
 (d) What is an equilibrium of an ODE?
 (e) Define what it means for an equilibrium to be stable? Then explain how one can determine if an equilibrium is stable without finding a solution of an ODE.

Exercise 2.9.4

Find the fixed points, determine their stability, and sketch the phase portrait around them.

- (a) $\dot{x} - (x - 2)(x + 2)^2(x - 7)$ (c) $\dot{x} = \cos(x) + 1$
 (b) $\dot{x} = \cos(x)$ (d) $\dot{x} = \left(\frac{1}{5} - \frac{1}{x}\right) \left(\frac{1}{5} - \frac{2}{x}\right)^2$, for $x > 0$

Exercise 2.9.5

Use phase portrait information to determine what the solution does as $x \rightarrow \infty$ if $y(0) = 3$ for the ODE:

$$y'(x) + (y - 2)(y - 4)^2.$$

Exercise 2.9.6

Each of these ODEs could be linear, separable, or exact. Identify the type of each equation and support your answer by:

- putting the linear equation into its standard form and circling the input term,
- performing the exactness test on the exact equation,
- identifying the separated F and G components for the separable ODE.

- (a) $(4y + 2t - 5) + (6y + 4t - 1)\dot{y} = 0$ (f) $(x + 2)^2 y' - 5 + 8y + 4xy = 0$
 (b) $xy'(x) + (3x + 1)y - e^{-3x} = 0$ (g) $\sin(3x) + 2[\cos(3x)]^2 y' \cdot y = 0$
 (c) $\dot{N} + N - Nte^{t+2} = 0$ (h) $(e^x + y) + (2 + x + ye^y)y'(x) = 0$
 (d) $\sqrt{1 - y^2} - y'(x)\sqrt{1 - x^2} = 0$ (i) $y' + \tan(x)y - \cos^2(x) = 0$
 (e) $(2xy + x^2 - 1)y'(x) + (x + y)^2 = 0$

Exercise 2.9.7

(Challenge) Give an example of a first-order ODE that is neither linear, separable, or exact. Perform the tests as in the previous question that demonstrate this.

Exercise 2.9.8

Find the **general solution** of each of the following equations using any technique that you find appropriate. Then, find the **particular solution** that satisfies the given initial condition.

- (a) $x^2 y'(x) - y + xy, y(-1) = -1$
 (b) $xy'(x) + y - 4x - 1, y(1) = 8$
 (c) $(x - y^3 + y^2 \sin(x)) - (3xy^2 + 2y \cos(x))y'(x) = 0, y(0) = 2$

Exercise 2.9.9

The following equation is not exact (it fails the exactness test). Nevertheless, apply the technique for solving exact equations. At which point does the technique break down?

$$(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$$