Question 1

Given is the ODE $\dfrac{dy}{dt} = y^2(3-y)$.

Fill in the information about the ODE:

(a) Order: ________

(b) Independent variable: ________

(c) Dependent variable: ________

(d) Time-dependence (circle one):
dependent — independent

Question 2

The following Initial Value Problem involves a **linear** ODE:

(i)

$$t\frac{dx}{dt} + x(t) = e^t, \quad x(2) = 0.$$

(ii)

$$\frac{1}{t^2}\frac{dx}{dt} + x(t) = \exp\left(t^3\right), \quad x(1) = 0.$$

For one of the above IVPs, do the following:

(a) Convert the equation into the **standard form** and **label** all its components.

(b) Use **variation of parameters** to calculate the general solution.

(c) Compute the particular solution that satisfies the initial condition.

You will be graded both on **clarity of the process** to solution, and on **technical execution**. Therefore, tell us what you are calculating, label your (sub)solutions and any partial results, and **if you cannot complete the problem because of a calculation error**, explain in words what you would do to complete it.

Question 3

Solve the following separable ODE:

(i)

$$\frac{dy}{dx} = \frac{x^2}{y+1}$$

(ii)

$$\frac{dy}{dx} = \frac{\ln(x)}{x^2}$$

(iii)

$$\frac{dy}{dx} = xe^{x^2}$$

Question 4

Solve the following linear ODE:

(i) $y' + \frac{2}{x}y = \frac{\cos(x)}{x^2}$

(ii) $y' + y\tan(x) = \sin(2x)$

(iii) $y' + y\csc(x) = \cos(x)$

Question 5

The following ODE is exact. Calculate the **implicit** formula of its solution curves.

(i)
$$(x^2+y^2)y'(x)+2xy+1=0.$$

(ii)
$$\left(\frac{1}{2}x^2+xy\right)y'(x)+2x^3+xy+\frac{1}{2}y^2=0.$$

Show all steps in your calculation.

Question 6

The following ODE is linear, homogeneous 2nd order ODE. Using the characteristic equation, calculate the formula for its general solution

(i)
$$y''(x)-2y'(x)-3y(x)=0.$$

(ii)
$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}=0.$$

(iii)
$$y''(x)+3y'(x)+2y(x)=0.$$

(iv)
$$y''(x)-4y'(x)+4y(x)=0.$$

Show all steps in your calculation.

Question 7

Given a differential equation with a damping coefficient c in it
$$\frac{dy^2}{dt^2}+c\frac{dy}{dt}+25y(t)=0$$
perform both tasks below. Justify your work and explain your answers.

(a) For damping $c=0$ calculate the solution that satisfies $y(0)=1$, $y'(0)=10$.

(b) Calculate the general solution for damping $c=6$. **Explain:** do solutions here in general oscillate or not? Do they grow/decay/stay the same?

In all cases, use the real-valued form of solution functions (sines/cosines instead of complex exponentials).
Solutions to $a\lambda^2+b\lambda+c=0$ are $\lambda_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Question 8

Given is the differential equation:
$$y''(x)+16y(x)=2x-5e^{-2x}.$$

(a) Calculate its **general** solution.

(b) Calculate the particular solution that satisfies $y(0)=0$, $y'(0)=0$.

In all cases, use **real basis** (sines, cosines, etc.) where appropriate.

Question 9

For the ODE with the same left-hand side as in Question 1,

$$y''(x) + 16y(x) = f(x)$$

suggest an example of the input function $f(x)$ that would result in **resonant behavior**. (You do not need to solve for $y(x)$.)

Resonating input $f(x) =$

Explain your answer by describing, in short, what resonance is.

Question 10

Solve the following ODE **using Laplace transform**: $y' + 2y = \frac{1}{3}e^{-t}, y(0) = -2$

Laplace integral:
$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
Properties, given that
$\mathscr{L}\{f(t)\} = F(s), \mathscr{L}\{g(t)\} = G(s)$:

- Linearity:
  $\mathscr{L}\{c_1 f(t) + c_2 g(t)\}$
  $= c_1 F(s) + c_2 G(s)$
- Convolution: $\mathscr{L}\{f * g\} = F(s)G(s)$, where
  $$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$
- s-shift: $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$
- t-shift: $\mathscr{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-sa}F(s)$
- Derivative in t: If $\mathscr{L}\{y(t)\} = Y(s)$, then
  $\mathscr{L}\{y'(t)\} = sY(s) - y(0)$,
  $\mathscr{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$

$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$

Question 11

Compute the inverse Laplace transform: $\mathscr{L}^{-1}\left\{\frac{6s}{s^2+2s+10}\right\}$

Laplace integral:
$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
Properties, given that
$\mathscr{L}\{f(t)\} = F(s), \mathscr{L}\{g(t)\} = G(s)$:

- Linearity:
  $\mathscr{L}\{c_1 f(t) + c_2 g(t)\}$
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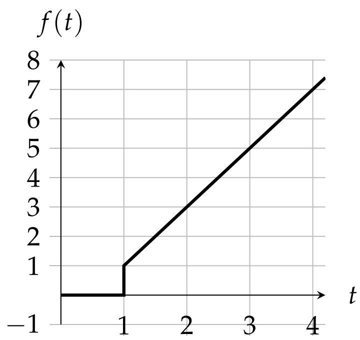
$f(t)$	$F(s) = \mathscr{L}\{f(t)\}$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$

Question 12
Sketch as accurately as possible the impulse $\delta(t-5)$ and give its Laplace transform.

Question 13
Compute and sketch the function $g(t)$ defined as: $g(t) = \mathscr{L}^{-1}\left\{\frac{2}{s-1}e^{-2s}\right\}$

Question 14
Compute the Laplace transform of the function given by the graph.



Question 15
Given the following system of ODEs:

$$\dot{x} = -5x + 6y$$
$$\dot{y} = -3x + 4y$$

(a) Determine the matrix $\boldsymbol{M}$ in the matrix ODE $\dot{\underline{v}}(t) = \boldsymbol{M}\underline{v}(t)$ that corresponds to the system above if $\underline{v}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

(b) Is the following candidate a solution of the matrix ODE from (a)? Show your work.

$$\underline{v}(t) = \begin{bmatrix} -2 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{t}$$

Question 16

Use row reduction (Gauss elimination) to solve the following system of linear equations. If needed, use $C_1, C_2, \ldots$ as free parameters in the solution. Label row operations in your solution.

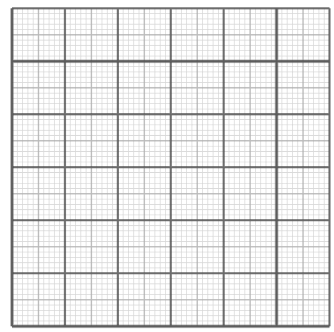
$$\begin{aligned} -y + z - 1 &= 0 \\ -x + 2y - 2z + 1 &= 0 \\ 2x - 3y + 2z - 1 &= 0 \end{aligned}$$

Question 17

Matrix ODE

$$\dot{\underline{x}} = \boldsymbol{M}\underline{x}$$

is given by the matrix $\boldsymbol{M}$, whose **eigenvalues and eigenvectors are already known**.



(i)

$$\boldsymbol{M} = \begin{bmatrix} 1/2 & -7/4 \\ -7 & 1/2 \end{bmatrix}$$

$$\lambda_1 = 4,\ \underline{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \lambda_2 = -3,\ \underline{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(ii)

$$\boldsymbol{M} = \begin{bmatrix} -4/3 & 2/3 \\ 1/3 & -5/3 \end{bmatrix}$$

$$\lambda_1 = -1,\ \underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda_2 = -2,\ \underline{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(a) Sketch the phase portrait in the empty grid. Make sure to draw main directions of the flow and sketch in several solution curves.

(b) Is the fixed point stable or not? **Explain why.**

(c) Write out the **general solution** of the matrix ODE.

Question 18

Given $\dot{\underline{x}} = \boldsymbol{A}\underline{x}$ with

$$\boldsymbol{A} = \begin{bmatrix} -7 & -13 \\ 2 & 3 \end{bmatrix}$$

(a) Compute eigenvalues and eigenvectors of $\boldsymbol{A}$.

(b) Based on your calculations, would the solutions of $\dot{\underline{x}} = \boldsymbol{A}\underline{x}$

- Oscillate — Not Oscillate? (circle correct)
  Explain:

- Grow — Decay — Neither (circle correct)
  Explain: