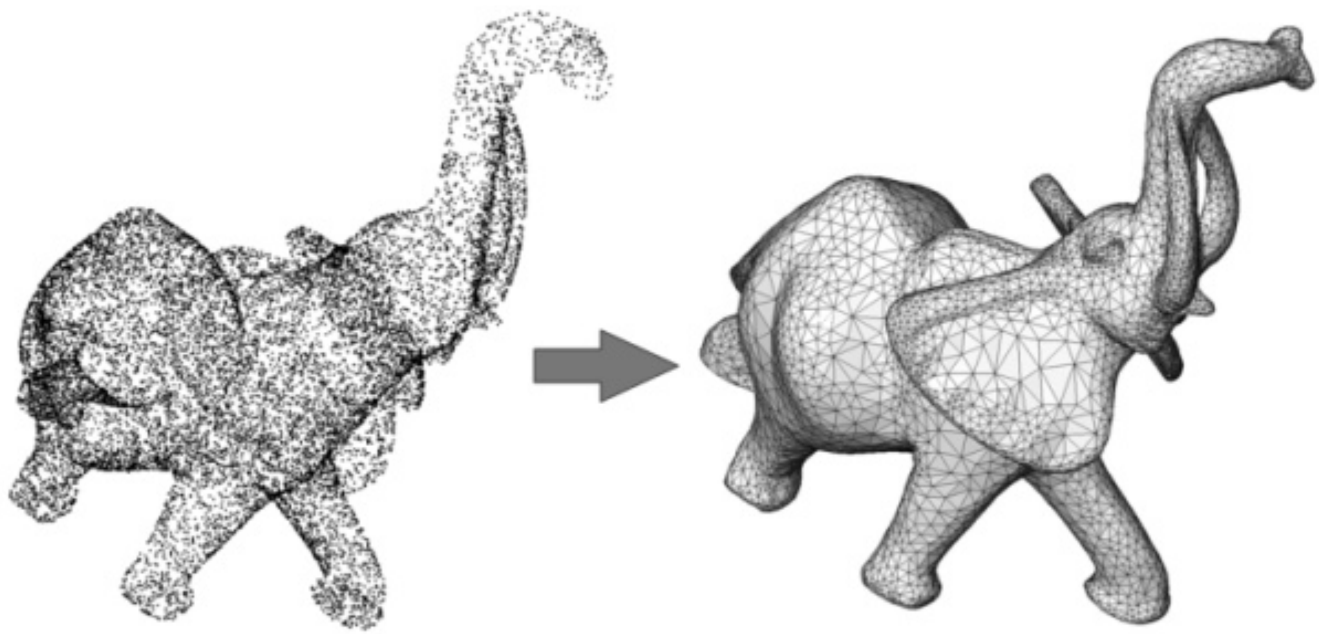
Ordinary Least-Squares

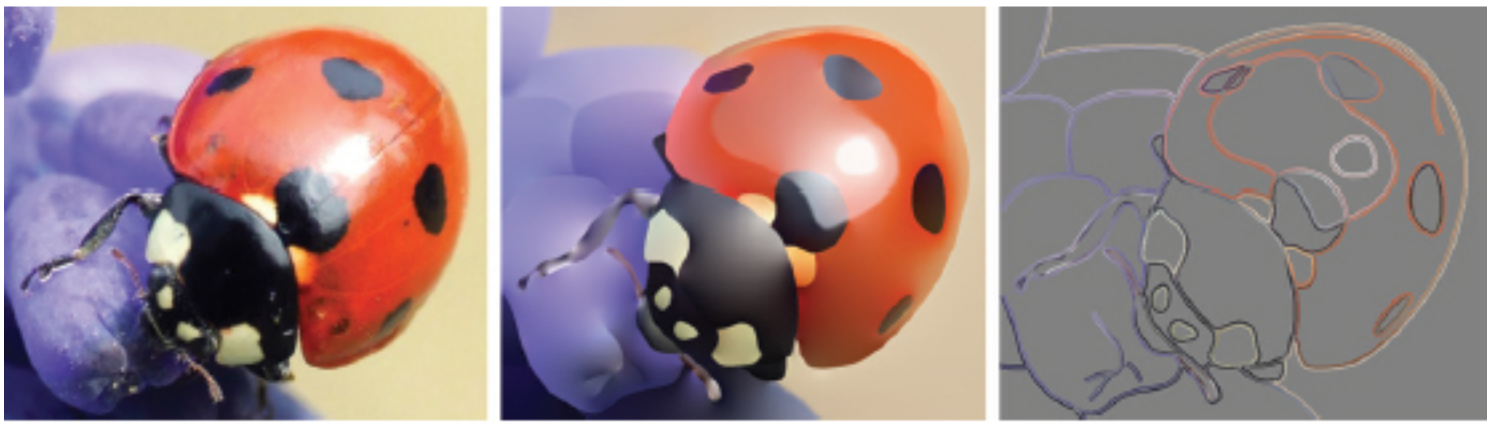
Emmanuel Iarussi

Many graphics problems can be seen as finding the best set of ***parameter****s for a* ***model****, given some* ***data***



Surface reconstruction

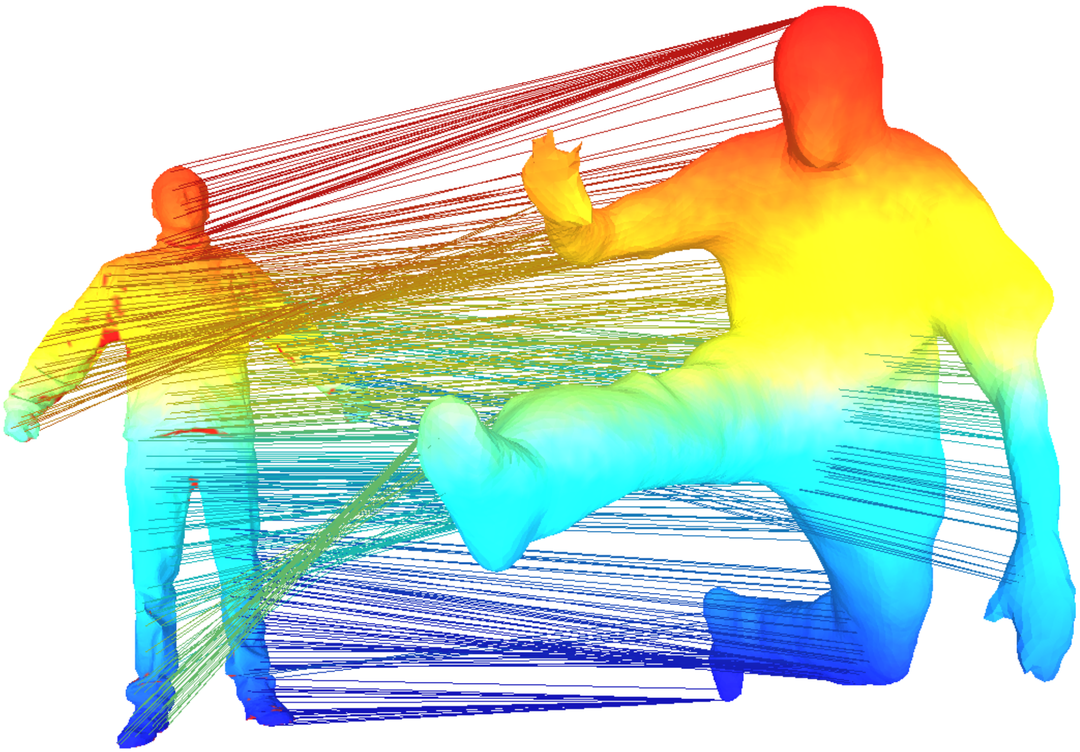
Many graphics problems can be seen as finding the best set of ***parameter****s for a* ***mode****l, given some* ***data***



(a) (b) (c)

Color diffusion

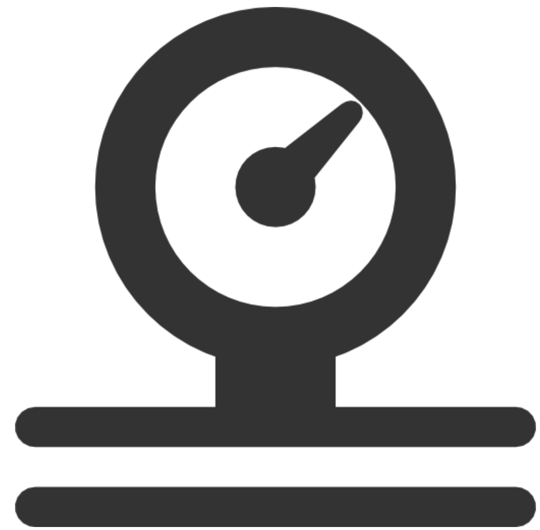
*Many graphics problems can be seen as finding the best set of **parameter**s for a **mode**l, given some **data***



Shape registration

Example: estimation of gas pressure for a given temperature sample

Temperature (a)



Gas pressure (b)

Assuming linear relationship

$$b = xa$$

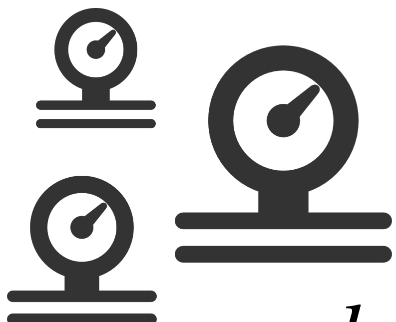
Assuming linear relationship

$$b = \boxed{x}a$$

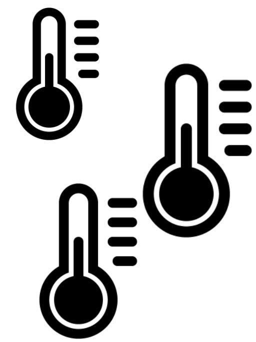
Assuming linear relationship

$$b = \boxed{x}a$$



b_i

Pressure measurements (m)

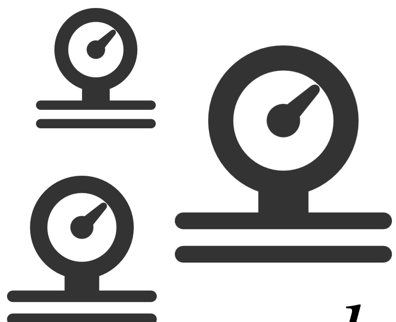


a_i

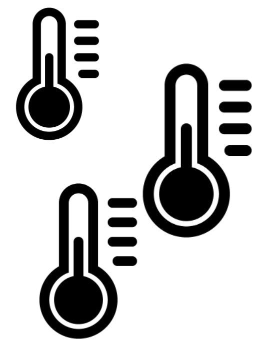
Temperature measurements (m)

Assuming linear relationship

$$b = \boxed{x}a$$



$$= x$$



b_i

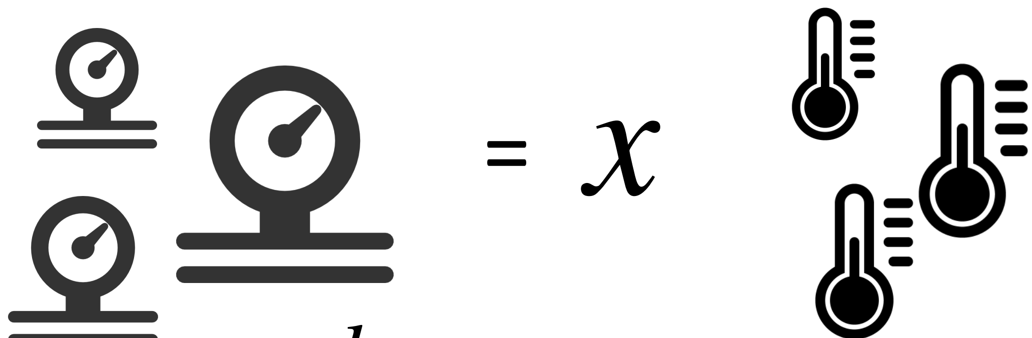
Pressure measurements (*m*)

a_i

Temperature measurements (*m*)

Assuming linear relationship

$$b = \boxed{x}a$$



$$= x$$

b_i

Pressure measurements (m)

a_i

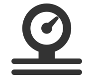
Temperature measurements (m)

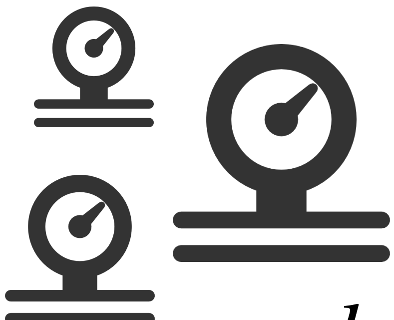
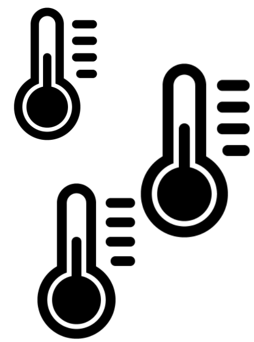
$$\hat{b} = \boxed{x}a$$

The relationship might not be exact

Assuming linear relationship

$$b = \boxed{x}a$$

 $= x$ 

b_i

Pressure measurements (m)

a_i

Temperature measurements (m)

$$\hat{b} = \boxed{x}a$$

The relationship might not be exact

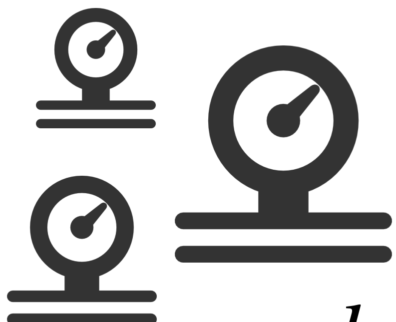
$$(b_0 - a_0 x)^2 \quad (b_1 - a_1 x)^2$$

$$(b_2 - a_2 x)^2 \quad \ldots$$

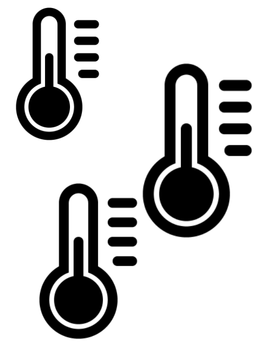
Match “as best as possible” the observations

Assuming linear relationship

$$b = \boxed{x}a$$



$$= x$$



Pressure measurements (*m*) b_i

Temperature measurements (*m*) a_i

$$\hat{b} = \boxed{x}a \longrightarrow e(x) = \sum_i^m (b_i - a_i x)^2$$

The relationship might not be exact

Match "as best as possible" the observations

$$e(x) = \sum_i (b_i - a_i x)^2$$

or

$$\arg\min_x \sum_i (b_i - a_i x)^2$$

Many of CG problems can be formulated as ***minimizing the sum of squares of the residuals*** *between some features in the model and the data.*

Matrix notation (observations)

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

We can rewrite the residual function using linear algebra as:

$$e(x) = \sum_i (b_i - a_i x)^2$$

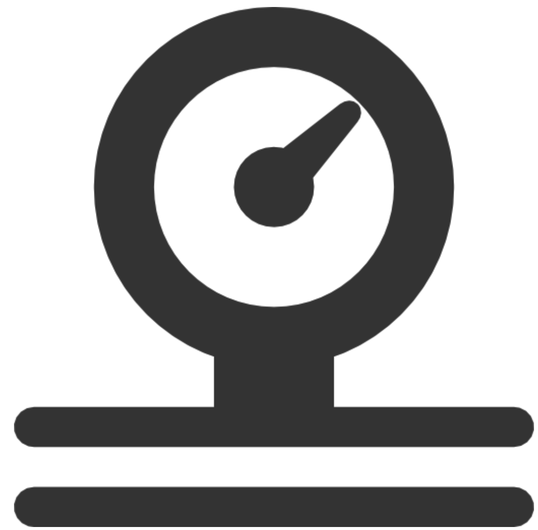
$$= (\boldsymbol{b} - x\boldsymbol{a})^T (\boldsymbol{b} - x\boldsymbol{a})$$

$$e(x) = \left\| \boldsymbol{b} - x\boldsymbol{a} \right\|^2$$

Example: estimation of gas pressure for a given temperature and altitude samples

Temperature

Altitude



Gas pressure

Multidimensional linear regression, using a model with *n* parameters

$$b = a_1 x_1 + \ldots + a_n x_n = \sum_j a_j x_j$$

Multidimensional linear regression, using a model with *n* parameters

$$\boxed{b = xa}$$

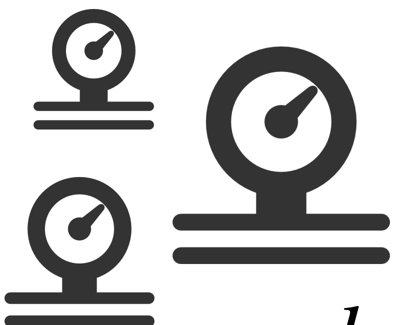
before

$$b = a_1 x_1 + \ldots + a_n x_n = \sum_j a_j x_j$$

+  +...

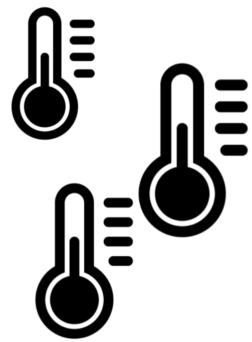
Multidimensional linear regression, using a model with *n* parameters

$$b = a_1 x_1 + \ldots + a_n x_n = \sum_j a_j x_j$$



b_i

Pressure measurements (*m*)



a_{i1}

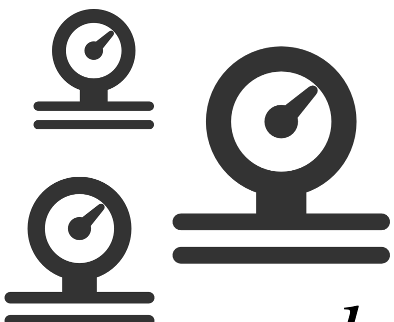
Temperature measurements (*m*)

a_{i2}

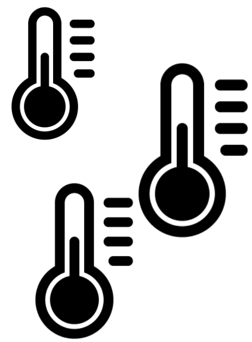
Altitude measurements (*m*)

Multidimensional linear regression, using a model with *n* parameters

$$b = a_1 x_1 + \ldots + a_n x_n = \sum_j a_j x_j$$



$$= x_1$$



$$+ x_2$$

b_i

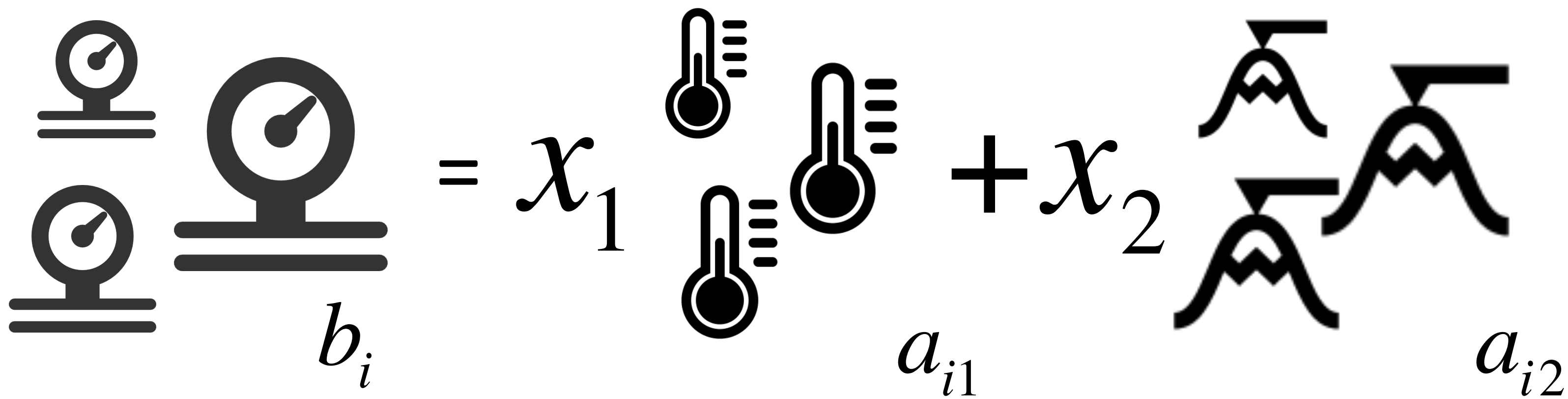
Pressure measurements (m)

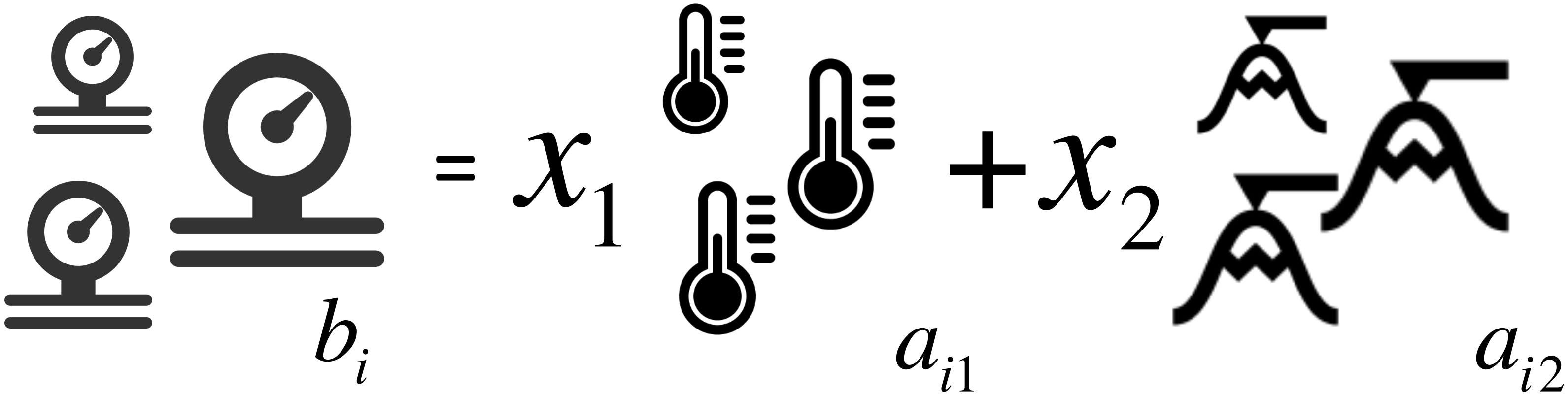
a_{i1}

Temperature measurements (m)

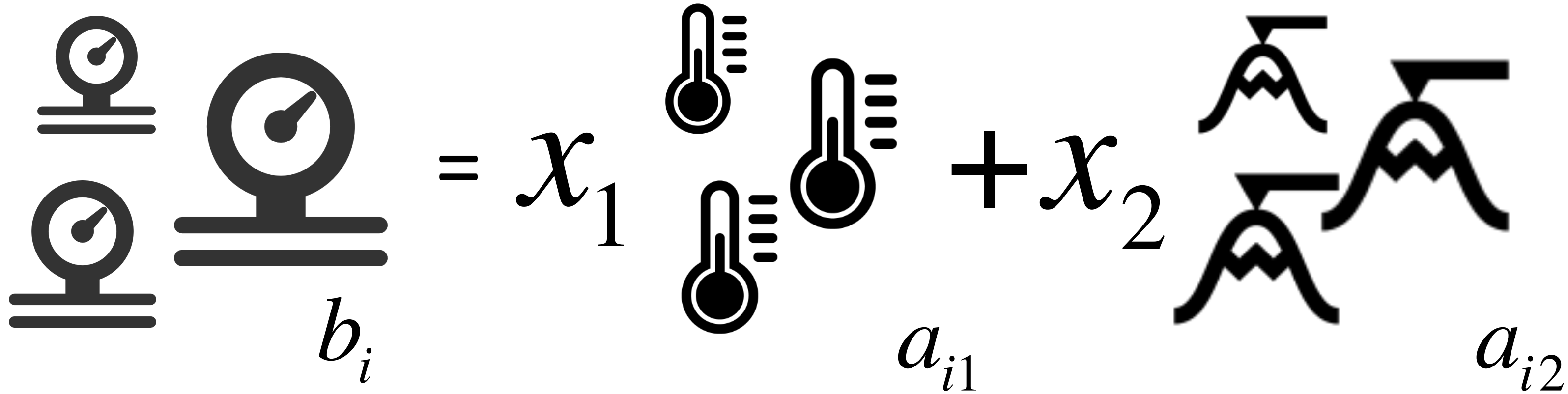
a_{i2}

Altitude measurements (m)


$= x_1$
$+ x_2$
b_i
a_{i1}
a_{i2}



$$e(\boldsymbol{x}) = \sum_{i=1}^{m} \left(b_i - \sum_{j=1}^{n} a_{i,j} x_j\right)^2$$



$$e(\boldsymbol{x}) = \sum_{i=1}^{m}\left(b_i - \sum_{j=1}^{n} a_{i,j}x_j\right)^2$$

$$= \left\| \boldsymbol{b} - \left[\sum_{j=1}^{n} a_{i,j}x_j \right] \right\|^2$$

$$\boxed{e(\boldsymbol{x}) = \left\| \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \right\|^2}$$

Objective: find ***x*** subject to minimize:

$$e(\boldsymbol{x}) = \left\| \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \right\|^2$$

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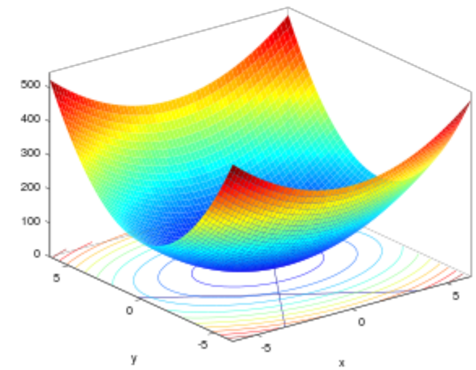
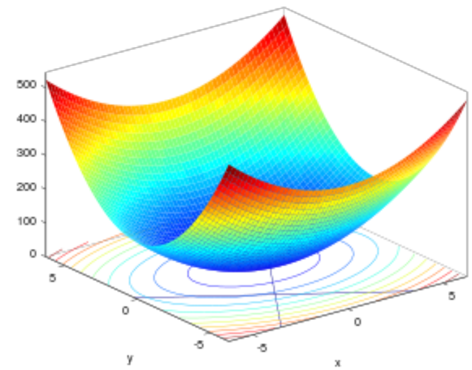


Objective: find ***x*** subject to minimize:

$$e(\boldsymbol{x}) = \left\| \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \right\|^2$$



Convex bowl function has minima when:

$$\nabla e(\boldsymbol{x}) = \nabla \left\| \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \right\|^2 = 0$$

Objective: find $\boldsymbol{x}$ subject to minimize:

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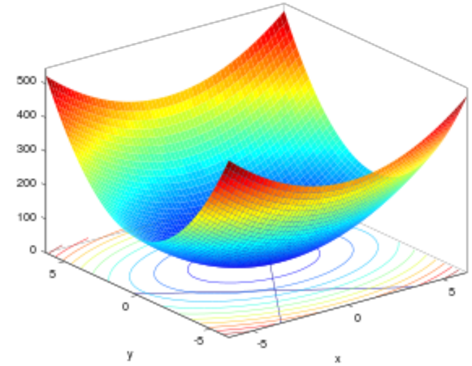

Convex bowl function has minima when:

$$\nabla e(\boldsymbol{x}) = \nabla \left\| \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \right\|^2 = 0$$

gradient

$$\nabla e(\boldsymbol{x}) = \left[\frac{\partial e}{\partial x_1}, \frac{\partial e}{\partial x_2}, \ldots, \frac{\partial e}{\partial x_n} \right]$$

Expanding square term

$$e(\boldsymbol{x}) = \left\| \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} \right\|^2$$

Expanding square term

$$e(\boldsymbol{x}) = \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|^2 = \boldsymbol{b}^T\boldsymbol{b} - 2\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{b} + \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}$$

Expanding square term

$$e(\boldsymbol{x}) = \left\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\right\|^2 = \cancel{\boldsymbol{b}^T}\boldsymbol{b} - 2\cancel{\boldsymbol{x}^T}\boldsymbol{A}^T\boldsymbol{b} + \underline{\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}}$$

$$= \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}^2$$

Differentiating with respect to $\boldsymbol{x}$ (gradient)

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}$$

Expanding square term

$$e(\boldsymbol{x}) = \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|^2 = \cancel{\boldsymbol{b}^T\boldsymbol{b}} - \cancel{2\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{b}} + \underline{\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}}$$

$$= \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}^2$$

Differentiating with respect to $\boldsymbol{x}$ (gradient)

$$\nabla e(\boldsymbol{x}) = \boxed{-2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0}$$

Expanding square term

$$e(\boldsymbol{x}) = \left\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\right\|^2 = \cancel{\boldsymbol{b}^T\boldsymbol{b}} - \cancel{2\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{b}} + \underline{\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}}$$

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Differentiating with respect to $\boldsymbol{x}$ (gradient)

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

$$\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b} \longrightarrow \text{Normal equation}$$

```
x = (A.transpose()*A).solve(A.transpose()*b);
```

Expanding square term

$$e(\boldsymbol{x}) = \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|^2 = \cancel{\boldsymbol{b}^T\boldsymbol{b}} - 2\cancel{\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{b}} + \underline{\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}}$$

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Differentiating with respect to $\boldsymbol{x}$ (gradient)

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

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```
x = (A.transpose()*A).solve(A.transpose()*b);
```

Issue: matrix multiplication

Expanding square term

$$e(\boldsymbol{x}) = \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|^2 = \cancel{\boldsymbol{b}^T\boldsymbol{b}} - \cancel{2\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{b}} + \underline{\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}}$$

$$= \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}^2$$

Differentiating with respect to ***x***

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

$$\underline{\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}} \longrightarrow \text{Normal equation}$$

```
x = (A.transpose()*A).solve(A.transpose()*b);
```

Issue: matrix multiplication → Solution: find expression

Let's go back to gradient = 0

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

Let's go back to gradient = 0

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

Differentiating again with respect to ***x***

$$\nabla\nabla e(\boldsymbol{x}) = \cancel{-2\boldsymbol{A}^T\boldsymbol{b}} + 2\boldsymbol{A}^T\boldsymbol{A}\cancel{\boldsymbol{x}} = 0$$

Let's go back to gradient = 0

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

Differentiating again with respect to ***x***

$$\nabla\nabla e(\boldsymbol{x}) = \cancel{-2\boldsymbol{A}^T\boldsymbol{b}} + 2\boldsymbol{A}^T\boldsymbol{A}\cancel{\boldsymbol{x}} = 0$$

$$= 2\boldsymbol{A}^T\boldsymbol{A}$$

Let's go back to gradient = 0

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

Differentiating again with respect to ***x***

$$\nabla\nabla e(\boldsymbol{x}) = \cancel{-2\boldsymbol{A}^T\boldsymbol{b}} + 2\boldsymbol{A}^T\boldsymbol{A}\cancel{\boldsymbol{x}} = 0$$

$$H_e\boldsymbol{x} = 2\boldsymbol{A}^T\boldsymbol{A}$$

hessian

Let's go back to gradient = 0

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

Differentiating again with respect to $\boldsymbol{x}$

$$\nabla\nabla e(\boldsymbol{x}) = \cancel{-2\boldsymbol{A}^T\boldsymbol{b}} + 2\boldsymbol{A}^T\boldsymbol{A}\cancel{\boldsymbol{x}} = 0$$

$$H_e\boldsymbol{x} = 2\boldsymbol{A}^T\boldsymbol{A}$$

hessian

$$H_e\boldsymbol{x} = \begin{bmatrix} \frac{\partial^2 e}{\partial x_1^2} & \frac{\partial^2 e}{\partial x_1 \partial x_2} & & \frac{\partial^2 e}{\partial x_1 \partial x_n} \\ \frac{\partial^2 e}{\partial x_2 \partial x_1} & \frac{\partial^2 e}{\partial x_2^2} & & \frac{\partial^2 e}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 e}{\partial x_n \partial x_1} & \frac{\partial^2 e}{\partial x_n \partial x_2} & & \frac{\partial^2 e}{\partial x_n^2} \end{bmatrix}$$

Let's go back to gradient = 0

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

Differentiating again with respect to ***x***

$$\nabla\nabla e(\boldsymbol{x}) = \cancel{-2\boldsymbol{A}^T\boldsymbol{b}} + 2\boldsymbol{A}^T\boldsymbol{A}\cancel{\boldsymbol{x}} = 0$$

$$H_e\boldsymbol{x} = 2\boldsymbol{A}^T\boldsymbol{A}$$

Normal equation:

$$\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$$

```
x = (A.transpose()*A).solve(A.transpose()*b);
```

Let's go back to gradient = 0

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

Differentiating again with respect to ***x***

$$\nabla\nabla e(\boldsymbol{x}) = \cancel{-2\boldsymbol{A}^T\boldsymbol{b}} + 2\boldsymbol{A}^T\boldsymbol{A}\cancel{\boldsymbol{x}} = 0$$

$$H_e\boldsymbol{x} = 2\boxed{\boldsymbol{A}^T\boldsymbol{A}}$$

Normal equation:

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x = (A.transpose()*A).solve(A.transpose()*b);
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Let's go back to gradient = 0

$$\nabla e(\boldsymbol{x}) = -2\boldsymbol{A}^T\boldsymbol{b} + 2\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = 0$$

Differentiating again with respect to ***x***

$$\nabla\nabla e(\boldsymbol{x}) = \cancel{-2\boldsymbol{A}^T\boldsymbol{b}} + 2\boldsymbol{A}^T\boldsymbol{A}\cancel{\boldsymbol{x}} = 0$$

$$H_e\boldsymbol{x} = 2\boldsymbol{A}^T\boldsymbol{A}$$

Two alternatives for solving:

$$\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$$

```
x = (A.transpose()*A).solve(A.transpose()*b);
```

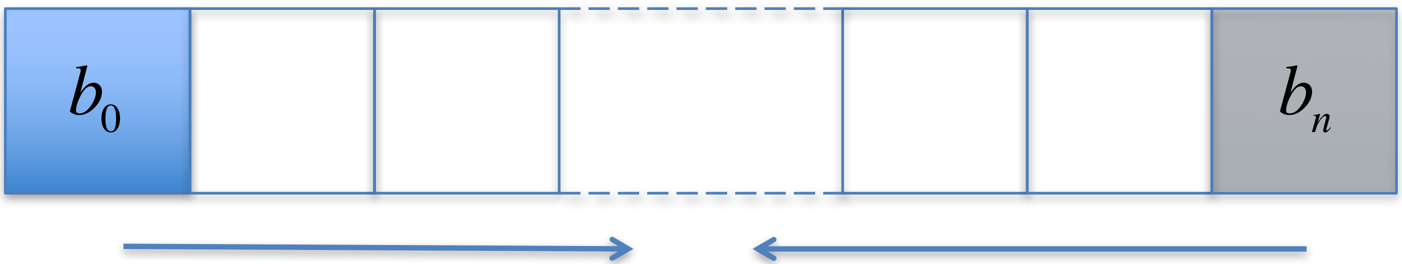
$$H_e\boldsymbol{x} = \boldsymbol{c}$$

```
x = (H).solve(c);
```

$$= 2\boldsymbol{A}^T\boldsymbol{b}$$

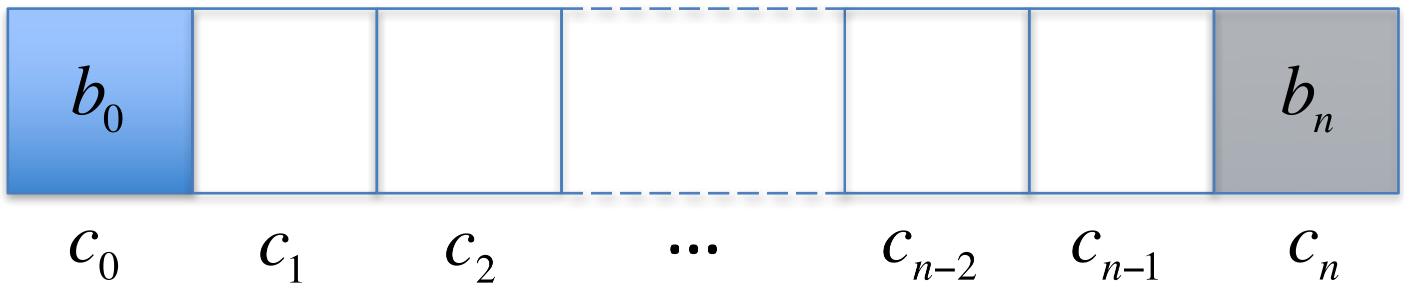
Constant terms in gradient

Example: Diffusion



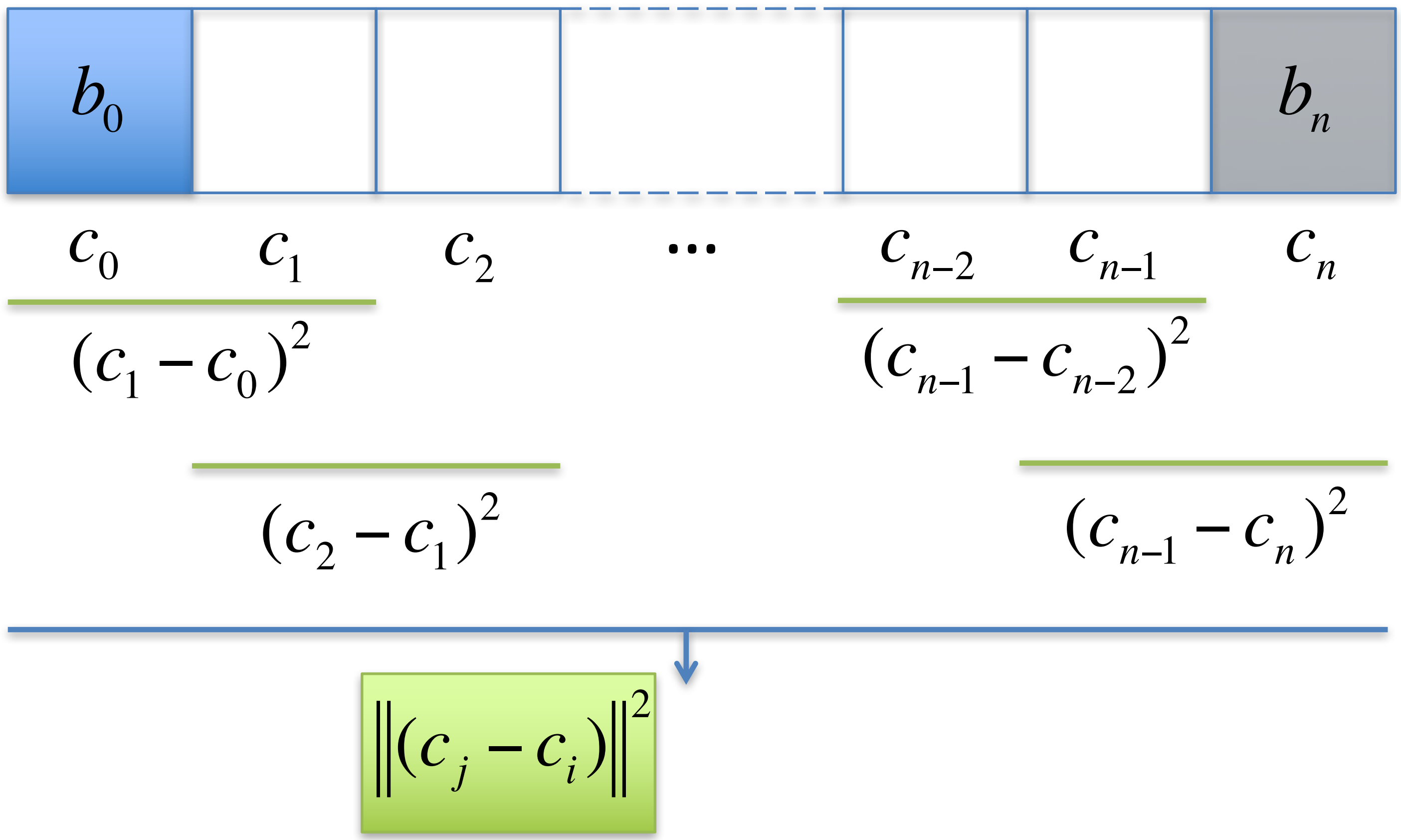
Example: Diffusion

Minimize difference between neighbors



Example: Diffusion

Minimize difference between neighbors



Example: Diffusion

Minimize difference between neighbors

c_0 c_1 c_2 … c_{n-2} c_{n-1} c_n

$$(b_0 - c_0)^2 + (c_1 - c_0)^2$$

$$(c_{n-1} - c_{n-2})^2$$

$$(c_2 - c_1)^2$$

$$(c_{n-1} - c_n)^2 + (b_n - c_n)^2$$

$$\left\|(c_j - c_i)\right\|^2$$

Example: Diffusion

Minimize difference between neighbors

$c_0 \quad c_1 \quad c_2 \quad \cdots \quad c_{n-2} \quad c_{n-1} \quad c_n$

$$(b_0 - c_0)^2 + (c_1 - c_0)^2$$

$$(c_{n-1} - c_{n-2})^2$$

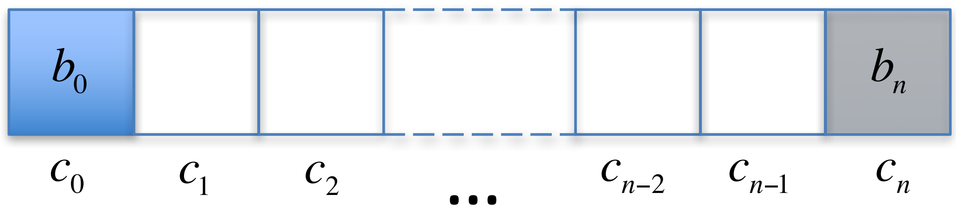
$$(c_2 - c_1)^2$$

$$(c_{n-1} - c_n)^2 + (b_n - c_n)^2$$

$$\boxed{\left\|(c_j - c_i)\right\|^2} + \boxed{\left\|(b_i - c_i)\right\|^2}$$

Solving with normal equation: $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$

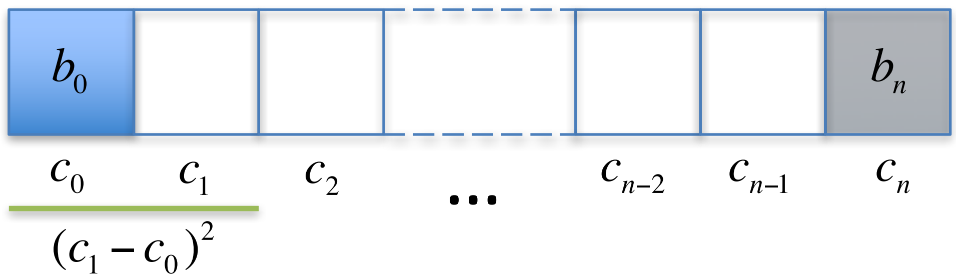
1. Build A $\boxed{\left\|(c_j - c_i)\right\|^2}$



$$\boldsymbol{A} = \begin{array}{c} \begin{array}{ccc} c_0 & c_1 & c_2 \end{array} \\ \left[\begin{array}{ccc} \; & \; & \; \\ \; & \; & \; \end{array}\right] \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \cdots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} \; \\ \; \end{bmatrix}$$

Solving with normal equation: $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$

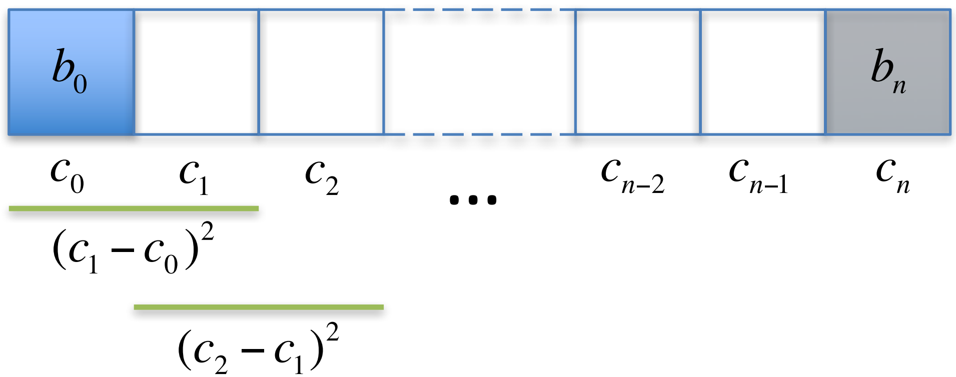
1. Build A $\boxed{\left\|(c_j - c_i)\right\|^2}$



$$\boldsymbol{A} = \begin{array}{c} \begin{matrix} c_0 & c_1 & c_2 & \end{matrix} \\ \begin{bmatrix} -1 & 1 & 0 & \dots \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ \\ \\ \\ \end{bmatrix}$$

Solving with normal equation: $\boldsymbol{A}^T \boldsymbol{A} \boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{b}$

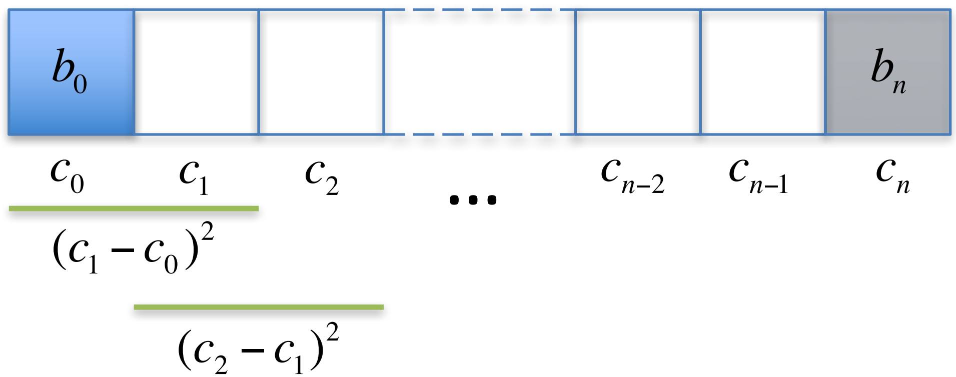
1. Build A $\boxed{\left\| (c_j - c_i) \right\|^2}$



$$\boldsymbol{A} = \begin{array}{c} \begin{array}{cccc} c_0 & c_1 & c_2 & \\ \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 0 \\ & & & \\ & & & \end{bmatrix} \end{array} \quad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \cdots \\ c_{n-1} \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \\ \end{bmatrix}$$

Solving with normal equation: $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$

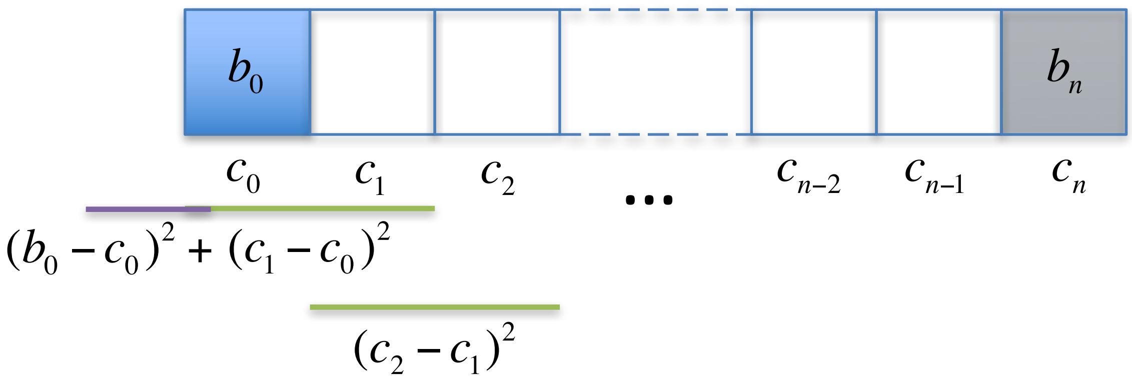
1. Build A $\boxed{\left\|(c_j - c_i)\right\|^2}$



$$\boldsymbol{A} = \begin{array}{c} \begin{array}{cccc} c_0 & c_1 & c_2 & \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ & & & \end{bmatrix} \end{array} \quad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \cdots \\ c_{n-1} \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ \end{bmatrix}$$

Solving with normal equation: $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$

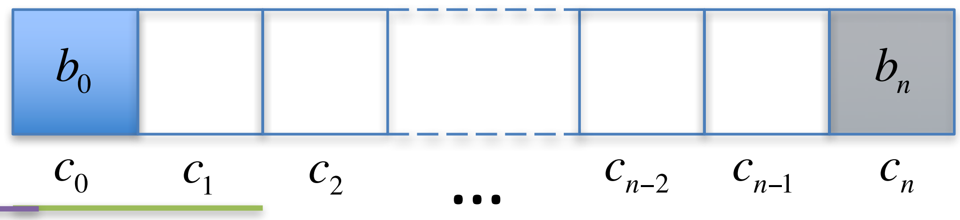
1. Build A $\boxed{\left\|(b_i - c_i)\right\|^2}$



$$\boldsymbol{A} = \begin{array}{cccc} c_0 & c_1 & c_2 & \\ \multicolumn{4}{c}{\left[\begin{array}{cccc} -1 & 1 & 0 & \ldots \\ 0 & -1 & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ & & & \end{array}\right]} \end{array} \quad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \ldots \\ c_{n-1} \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ \end{bmatrix}$$

Solving with normal equation: $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$

1. Build A $\boxed{\|(b_i - c_i)\|^2}$



$$(b_0 - c_0)^2 + (c_1 - c_0)^2$$

$$c_0 = b_0$$

$$(c_2 - c_1)^2$$

$$\boldsymbol{A} = \begin{array}{c} \begin{array}{cccc} c_0 & c_1 & c_2 & \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & \dots \\ 0 & -1 & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ b_0 \end{bmatrix}$$

Solving with normal equation: $\boldsymbol{A}^T \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T \boldsymbol{b}$

2. Compute $\boldsymbol{A}^T \boldsymbol{A}$ and $\boldsymbol{A}^T \boldsymbol{b}$

Solving with normal equation: $\boldsymbol{A}^T \boldsymbol{A x} = \boldsymbol{A}^T \boldsymbol{b}$

3. Solve using:
   - Cholesky decomposition
   - Conjugate Gradient
   - …

in Eigen (i.e):

```
x = (A.transpose()*A).solve(A.transpose()*b);
```

Solving with normal equation: $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$

3. Solve using:
   - Cholesky decomposition
   - Conjugate Gradient
   - …

in Eigen (i.e):

```
x = (A.transpose()*A).solve(A.transpose()*b);
```

x wil have:

$$\boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ .. \\ c_{n-1} \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\nabla e(\boldsymbol{x}) = \left[\frac{\partial e}{\partial x_1}, \frac{\partial e}{\partial x_2}, \ldots, \frac{\partial e}{\partial x_n} \right]$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\nabla e(\boldsymbol{x}) = \left[\frac{\partial e}{\partial x_1}, \frac{\partial e}{\partial x_2}, \ldots, \frac{\partial e}{\partial x_n}\right]$$

$$\boxed{\left\|(c_j - c_i)\right\|^2}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\nabla e(\boldsymbol{x}) = \left[\frac{\partial e}{\partial c_i}, \frac{\partial e}{\partial c_j}\right]$$

$$\boxed{\left\|(c_j - c_i)\right\|^2}$$

$$\frac{\partial}{\partial c_i} = -2(c_j - c_i)$$

$$\frac{\partial}{\partial c_j} = 2(c_j - c_i)$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\boxed{\left\|(c_j - c_i)\right\|^2}$$

$$H_e = \begin{bmatrix} & \\ & \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} c_i \\ c_j \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\frac{\partial}{\partial c_i} = -2(c_j - c_i)$$

$$\frac{\partial}{\partial c_j} = 2(c_j - c_i)$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\left\| (c_j - c_i) \right\|^2$$

$$H_e = \begin{bmatrix} & \\ & \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} c_i \\ c_j \end{bmatrix} \quad \boxed{\boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\frac{\partial}{\partial c_i} = -2(c_j - c_i) \longrightarrow -const\left(\frac{\partial}{\partial c_i}\right) = 0$$

$$\frac{\partial}{\partial c_j} = 2(c_j - c_i) \longrightarrow -const\left(\frac{\partial}{\partial c_j}\right) = 0$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

2. **Hessian matrix**: second-order partial derivatives of energy terms

$$\left\|(c_j - c_i)\right\|^2$$

$$H_e = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} c_i \\ c_j \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

2. **Hessian matrix**: second-order partial derivatives of energy terms

$$\boxed{\left\|(c_j - c_i)\right\|^2}$$

$$H_e = \begin{bmatrix} \frac{\partial^2 e}{\partial c_i^2} & \frac{\partial^2 e}{\partial c_i \partial c_j} \\ \frac{\partial^2 e}{\partial c_j \partial c_i} & \frac{\partial^2 e}{\partial c_j^2} \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} c_i \\ c_j \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

2. **Hessian matrix**: second-order partial derivatives of energy terms

$$\left\|(c_j - c_i)\right\|^2$$

$$H_e = \begin{bmatrix} \frac{\partial^2 e}{\partial c_i^2} & \frac{\partial^2 e}{\partial c_i \partial c_j} \\ \frac{\partial^2 e}{\partial c_j \partial c_i} & \frac{\partial^2 e}{\partial c_j^2} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} c_i \\ c_j \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial^2}{\partial c_i^2} = \frac{\partial}{\partial c_i}\big(-2(c_j - c_i)\big) = 2 \qquad \frac{\partial^2}{\partial c_i \partial c_j} = \frac{\partial}{\partial c_j}\big(-2(c_j - c_i)\big) = -2$$

$$\frac{\partial^2}{\partial c_j^2} = \frac{\partial}{\partial c_j}\big(2(c_j - c_i)\big) = 2 \qquad \frac{\partial^2}{\partial c_j \partial c_i} = \frac{\partial}{\partial c_i}\big(2(c_j - c_i)\big) = -2$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

2. **Hessian matrix**: second-order partial derivatives of energy terms

$$\left\| (c_j - c_i) \right\|^2$$

$$H_e = \begin{bmatrix} \boxed{\begin{matrix} 2 & -2 \\ -2 & 2 \end{matrix}} \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} c_i \\ c_j \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial^2}{\partial c_i^2} = \frac{\partial}{\partial c_i}\left(-2(c_j - c_i)\right) = \boxed{2} \qquad \frac{\partial^2}{\partial c_i \partial c_j} = \frac{\partial}{\partial c_j}\left(-2(c_j - c_i)\right) = \boxed{-2}$$

$$\frac{\partial^2}{\partial c_j \partial c_i} = \frac{\partial}{\partial c_i}\left(2(c_j - c_i)\right) = \boxed{-2} \qquad \frac{\partial^2}{\partial c_j^2} = \frac{\partial}{\partial c_j}\left(2(c_j - c_i)\right) = \boxed{2}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$$\boxed{\left\|(c_j - c_i)\right\|^2} \quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \ \boldsymbol{x} = \begin{bmatrix} c_i \\ c_j \end{bmatrix} \ \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\nabla e(\boldsymbol{x}) = \left[\frac{\partial e}{\partial c_1} \right]$$

$$\left\| (b_i - c_i) \right\|^2$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\nabla e(\boldsymbol{x}) = \left[\frac{\partial e}{\partial c_1} \right]$$

$$\left\| (b_i - c_i) \right\|^2$$

$$\frac{\partial}{\partial c_i} = 2(c_i - b_i)$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\boxed{\|(b_i - c_i)\|^2}$$

$$H_e = \begin{bmatrix} \quad \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} c_i \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} \quad \end{bmatrix}$$

$$\frac{\partial}{\partial c_i} = 2(c_i - b_i)$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

1. **Gradient vector**: first-order partial derivatives of energy terms

$$\|(b_i - c_i)\|^2$$

$$H_e = \begin{bmatrix} \quad \end{bmatrix} \quad \boldsymbol{x} = [c_i] \quad \boxed{\boldsymbol{c} = [2b_i]}$$

$$\frac{\partial}{\partial c_i} = 2(c_i - b_i) \longrightarrow -const\left(\frac{\partial}{\partial c_i}\right) = \boxed{2b_i}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

2. **Hessian matrix**: second-order partial derivatives of energy terms

$$\boxed{\left\|(b_i - c_i)\right\|^2}$$

$$H_e = \begin{bmatrix} \\ \quad\quad\quad \\ \\ \end{bmatrix} \quad \boldsymbol{x} = \left[c_i\right] \quad \boldsymbol{c} = \left[2b_i\right]$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

2. **Hessian matrix**: second-order partial derivatives of energy terms

$$\boxed{\left\|(b_i - c_i)\right\|^2}$$

$$H_e = \left[\frac{\partial^2 e}{\partial c_i^2}\right] \quad \boldsymbol{x} = [c_i] \quad \boldsymbol{c} = [2b_i]$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

2. **Hessian matrix**: second-order partial derivatives of energy terms

$$\boxed{\left\|(b_i - c_i)\right\|^2}$$

$$H_e = \left[\frac{\partial^2 e}{\partial c_i^2}\right] \qquad \boldsymbol{x} = \left[c_i\right] \qquad \boldsymbol{c} = \left[2b_i\right]$$

$$\frac{\partial^2}{\partial c_i^2} = \frac{\partial}{\partial c_i}\left(2(c_i - b_i)\right) = 2$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

2. **Hessian matrix**: second-order partial derivatives of energy terms

$$\|(b_i - c_i)\|^2$$

$$\boxed{H_e = [2]} \qquad \boldsymbol{x} = [c_i] \qquad \boldsymbol{c} = [2b_i]$$

$$\frac{\partial^2}{\partial c_i^2} = \frac{\partial}{\partial c_i}\left(2(c_i - b_i)\right) = \boxed{2}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$$\left\|(b_i - c_i)\right\|^2$$

$$H_e = [2] \qquad \boldsymbol{x} = [c_i] \qquad \boldsymbol{c} = [2b_i]$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

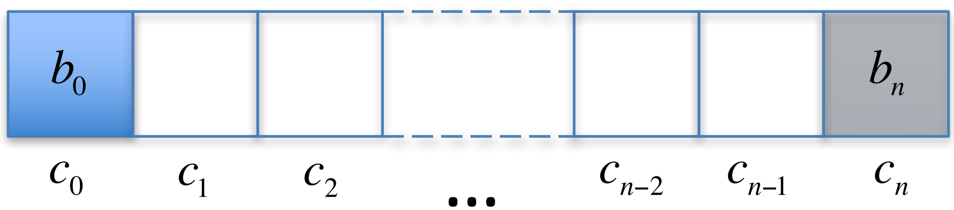
To recap:

$\boxed{\left\|(c_j - c_i)\right\|^2}$ $H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ $\boldsymbol{x} = \begin{bmatrix} c_i \\ c_j \end{bmatrix}$ $\boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{\left\|(b_i - c_i)\right\|^2}$ $H_e = \begin{bmatrix} 2 \end{bmatrix}$ $\boldsymbol{x} = \begin{bmatrix} c_i \end{bmatrix}$ $\boldsymbol{c} = \begin{bmatrix} 2b_i \end{bmatrix}$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

3. Build system



$$H_e = \begin{array}{c|c} & c_0 \quad c_1 \quad c_2 \\ \hline \begin{array}{c} c_0 \\ c_1 \\ c_2 \end{array} & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array}$$

$$\boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{bmatrix}$$

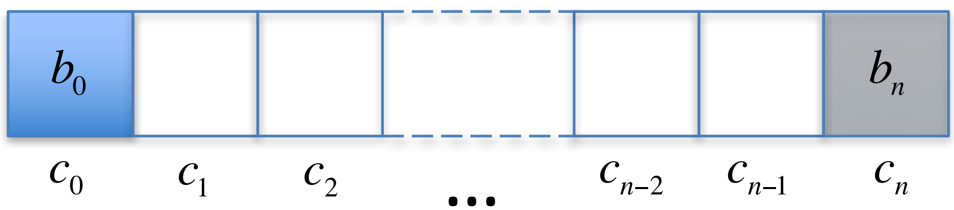
$$\boldsymbol{c} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$\|(c_j - c_i)\|^2$ $\quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\|(b_i - c_i)\|^2$ $\quad H_e = [2] \quad \boldsymbol{c} = [2b_i]$

3. Build system



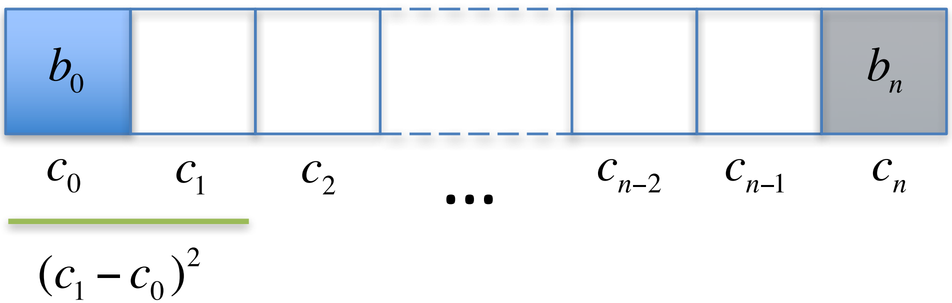
$$H_e = \begin{array}{c|ccc} & c_0 & c_1 & c_2 \\ \hline c_0 & & & \\ c_1 & & & \\ c_2 & & & \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \ldots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

3. Build system: add $\|(c_j - c_i)\|^2$ constraint

$\|(c_j - c_i)\|^2$ $\quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\|(b_i - c_i)\|^2$ $\quad H_e = [2] \quad \boldsymbol{c} = [2b_i]$



$$H_e = \begin{array}{c|c} & \begin{array}{ccc} c_0 & c_1 & c_2 \end{array} \\ \begin{array}{c} c_0 \\ c_1 \\ c_2 \end{array} & \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \end{array}$$

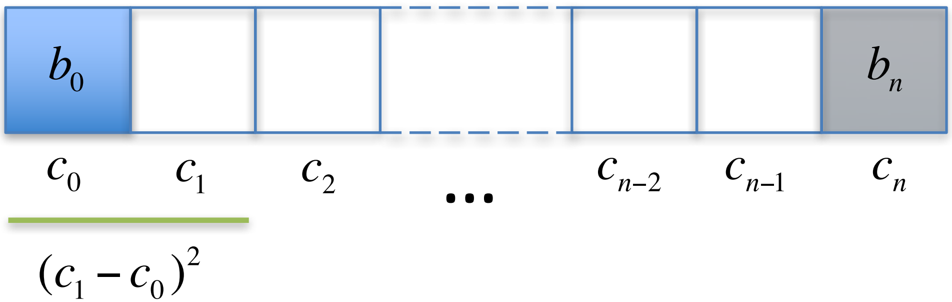
$$\boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \cdots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} \quad \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$\|(c_j - c_i)\|^2$ $\quad H_e = \boxed{\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}}$ $\quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\|(b_i - c_i)\|^2$ $\quad H_e = [2]$ $\quad \boldsymbol{c} = [2b_i]$

3. Build system: add $\|(c_j - c_i)\|^2$ constraint



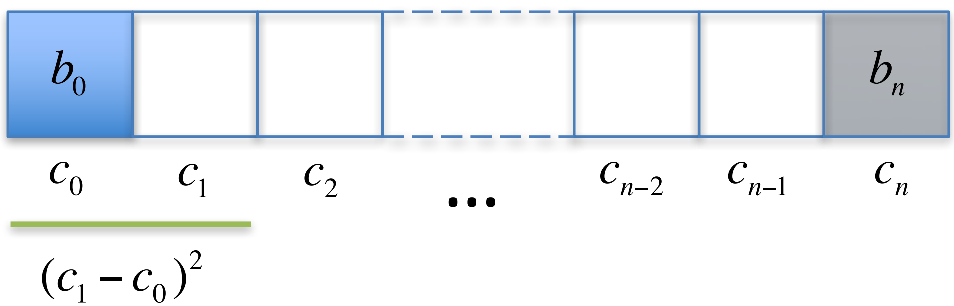
$$H_e = \begin{array}{c} \\ c_0 \\ c_1 \\ c_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} c_0 & c_1 & c_2 \end{array} \\ \begin{bmatrix} \boxed{\begin{matrix} 2 & -2 \\ -2 & 2 \end{matrix}} & & \\ & & \\ & & \end{bmatrix} \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$\|(c_j - c_i)\|^2$ $\quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ $\quad \boxed{\boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\|(b_i - c_i)\|^2$ $\quad H_e = [2]$ $\quad \boldsymbol{c} = [2b_i]$

3. Build system: add $\|(c_j - c_i)\|^2$ constraint



$$H_e = \begin{array}{c} \\ c_0 \\ c_1 \\ c_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} c_0 & c_1 & c_2 \end{array} \\ \begin{bmatrix} 2 & -2 & & \\ -2 & 2 & & \\ & & & \\ & & & \end{bmatrix} \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \cdots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} \\ \\ \\ \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$\|(c_j - c_i)\|^2$ $\quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \boldsymbol{c} = \boxed{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\|(b_i - c_i)\|^2$ $\quad H_e = [2] \quad \boldsymbol{c} = [2b_i]$

3. Build system: add $\|(c_j - c_i)\|^2$ constraint

$c_0 \quad c_1 \quad c_2 \quad \ldots \quad c_{n-2} \quad c_{n-1} \quad c_n$

$(c_1 - c_0)^2$

$(c_2 - c_1)^2$

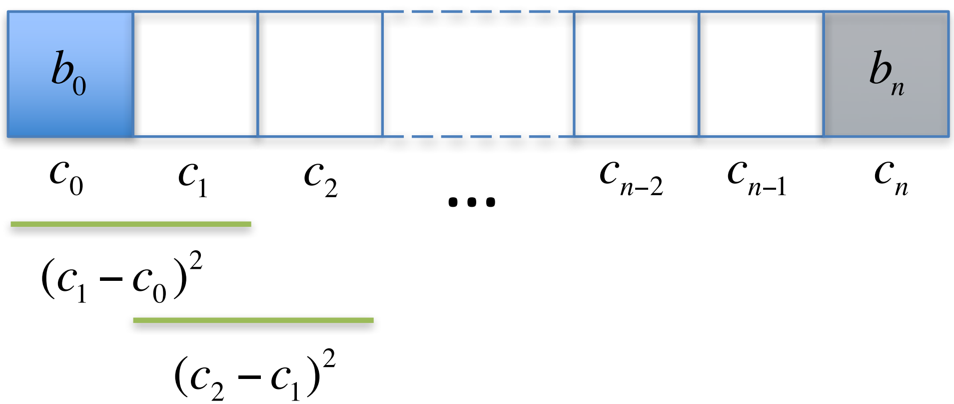
$$H_e = \begin{array}{c} \\ c_0 \\ c_1 \\ c_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} c_0 & c_1 & c_2 \end{array} \\ \begin{bmatrix} 2 & -2 & \\ -2 & 2 & \\ & & \\ & & \end{bmatrix} \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \ldots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} \\ \\ \\ \end{bmatrix}$$

Solving with Hessian: $H_e\boldsymbol{x} = \boldsymbol{c}$

$\left\|(c_j - c_i)\right\|^2$ $\quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left\|(b_i - c_i)\right\|^2$ $\quad H_e = \begin{bmatrix} 2 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 2b_i \end{bmatrix}$

3. Build system: add $\left\|(c_j - c_i)\right\|^2$ constraint



$$H_e = \begin{array}{cc} & \begin{array}{ccc} c_0 & c_1 & c_2 \end{array} \\ \begin{array}{c} c_0 \\ c_1 \\ c_2 \end{array} & \begin{bmatrix} 2 & -2 & \\ -2 & 4 & -2 \\ & -2 & 2 \\ & & \end{bmatrix} \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ \\ \\ \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$\|(c_j - c_i)\|^2$ $\quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ $\quad \boxed{\boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\|(b_i - c_i)\|^2$ $\quad H_e = [2]$ $\quad \boldsymbol{c} = [2b_i]$

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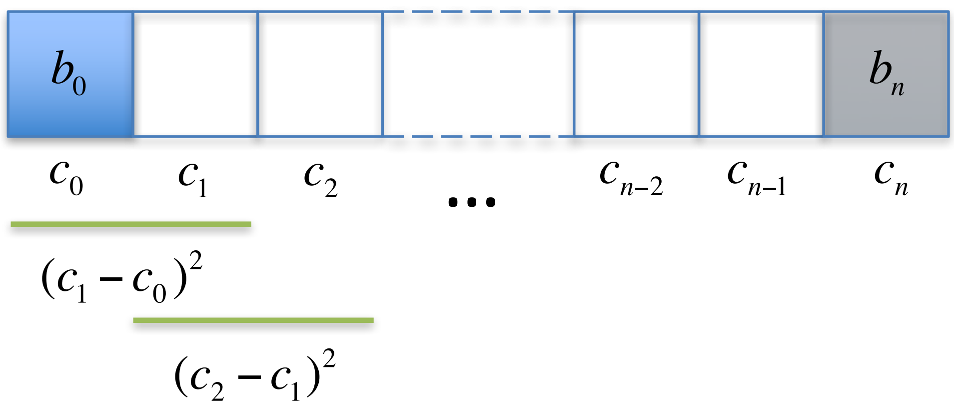


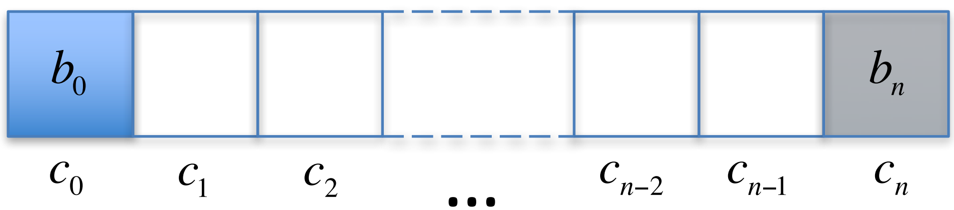
$$H_e = \begin{array}{c} \\ c_0 \\ c_1 \\ c_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} c_0 & c_1 & c_2 & \quad \end{array} \\ \begin{bmatrix} 2 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 2 & \\ & & & \end{bmatrix} \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \cdots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 0 \\ \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} \\ \\ \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$\boxed{\|(c_j - c_i)\|^2}$ $\quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{\|(b_i - c_i)\|^2}$ $\quad H_e = [2] \quad \boldsymbol{c} = [2b_i]$

3. Build system: add $\boxed{\|(b_i - c_i)\|^2}$ constraint



$$(b_0 - c_0)^2 + (c_1 - c_0)^2$$

$$(c_2 - c_1)^2$$

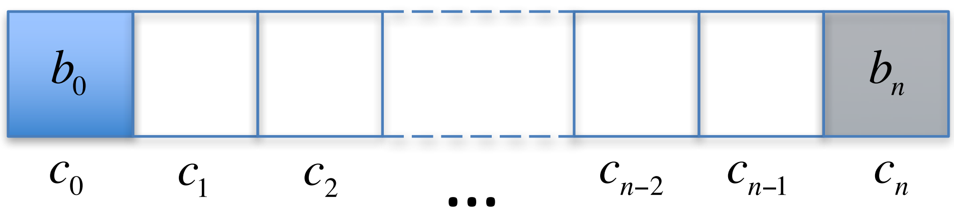
$$H_e = \begin{array}{c} \\ c_0 \\ c_1 \\ c_2 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} c_0 & c_1 & c_2 & & \end{array} \\ \begin{bmatrix} 2 & -2 & & & \\ -2 & 4 & -2 & & \\ & -2 & 2 & & \\ & & & & \end{bmatrix} \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

3. Build system: add $\|(b_i - c_i)\|^2$ constraint

$\|(c_j - c_i)\|^2 \quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\|(b_i - c_i)\|^2 \quad \boxed{H_e = [2]} \quad \boldsymbol{c} = [2b_i]$



$$(b_0 - c_0)^2 + (c_1 - c_0)^2$$

$$(c_2 - c_1)^2$$

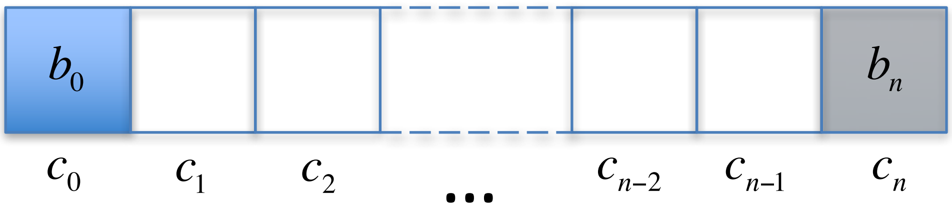
$$H_e = \begin{array}{c} \\ c_0 \\ c_1 \\ c_2 \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{ccc} c_0 & c_1 & c_2 \end{array} \\ \begin{bmatrix} \boxed{4} & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 2 & \\ & & & \end{bmatrix} \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

$\left\|(c_j - c_i)\right\|^2$ $\quad H_e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left\|(b_i - c_i)\right\|^2$ $\quad H_e = [2] \quad \boxed{\boldsymbol{c} = [2b_i]}$

3. Build system: add $\left\|(b_i - c_i)\right\|^2$ constraint



$$(b_0 - c_0)^2 + (c_1 - c_0)^2$$

$$(c_2 - c_1)^2$$

$$H_e = \begin{array}{c|ccc} & c_0 & c_1 & c_2 \\ \hline c_0 & 4 & -2 & \\ c_1 & -2 & 4 & -2 \\ c_2 & & -2 & 2 \end{array} \qquad \boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_{n-1} \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} \boxed{2b_i} \\ 0 \\ 0 \\ \\ \end{bmatrix}$$

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

4. Solve using:
   - Cholesky decomposition
   - Conjugate Gradient
   - …

Solving with Hessian: $H_e \boldsymbol{x} = \boldsymbol{c}$

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   - Cholesky decomposition
   - Conjugate Gradient
   - …

in Eigen (i.e):

```
x = (H).solve(b`);
```

x will have:

$$\boldsymbol{x} = \begin{bmatrix} c_0 \\ c_1 \\ .. \\ c_{n-1} \end{bmatrix}$$

References:

- *Solving Least Squares Problems. C. L.* Lawson and R. J. Hanson. 1974.
- *Practical Least Squares for Computer Graphics.* Fred Pighin and J.P. Lewis. Course notes 2007.